Competition for Procurement Contracts with Service Guarantees

Fernando Bernstein        Francis de Véricourt

The Fuqua School of Business, Duke University, Durham, NC 27708

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Abstract

We consider a market with two suppliers and a set of buyers in search of procurement contracts with one of the suppliers. In particular, each buyer needs to process a certain volume of work, and each supplier’s ability to process the customers’ requests is constrained by a production capacity. The procurement contracts include guarantees that the products will be available when needed, and the buyers select a supplier based on their service delivery offers. The suppliers are modeled as make-to-stock queues and compete for the buyers’ businesses. The main objective of the paper is to determine how the procurement contracts are established between buyers and suppliers. Since each buyer selects a single supplier to establish the sourcing relationship, the game fails to have a pure strategy Nash equilibrium. Instead, an equilibrium is defined as the limit equilibrium of some discrete action games.
1 Introduction

Given the coordination efforts and costs involved in the interaction between a company and its suppliers, there is a general trend towards supplier base rationalization. That is, firms interact with a smaller number of key suppliers who are prepared to work more closely with them, and in return get longer-term contracts. A broad range of firms have adopted the practice of strategic supply-base reduction realizing millionary savings (see Moore et al. 2002). In the automobile industry, McMillan (1990) reports that about 28% of parts procured by Toyota are single-sourced. For example, Aisin, a manufacturer of engine and brake parts, produces 99% of Toyota’s brake-fluid-proportioning valves. Toyota’s general manager of production control states that “depending on a single source […] is what keeps Toyota’s production lean; Aisin gets major economies of scale that it passes on to Toyota in lower prices.” (See Reitman 1997.) Sun Microsystems relies on single sourcing for displays and ASICs (Application-Specific Integrated Circuits). One company provides the vast majority of Sun’s monitors, which are a Sun customized version of a standard model. For ASICs, any one of a number of suppliers can supply the entire volume, such as Mitsubishi, LSI, or AT&T. Because of the customization required for this product, a single supplier is chosen to minimize development costs. (See Farlow et al., 1996.) In fact, many companies resort to single sourcing for customized components or parts that require high-technology production.

When specifying procurement contracts, firms need to stipulate service delivery guarantees. Even though price is an important consideration when selecting a supplier, the service delivery commitment should be a predominant factor since it affects the buyer’s production and inventory costs and customer service (see e.g., Cavatino 1994, Laseter 1998, Cachon and Zhang 2003). For example, in the semiconductor industry, the costs of delays are enormous since production lines need to be shut down if chips are not available (see Li 1992). In a recent survey, original-equipment manufacturers rated “ability to meet delivery schedules” the most important factor in selecting a contract manufacturer, while “price” ranked fifth (see Ansberry and Aeppel 2003). Anupindi and Bassok (1999) identify delivery commitments as key parameters in contracts between buyers and suppliers.

A procurement contract should be verifiable and enforceable. That is, both buyer and supplier should be able to enforce (and verify) that the other party comply with the terms of the contract. In particular, frequency of on-time deliveries, reserved capacity commitments, or stock allocation, are all difficult to enforce on the suppliers. Instead, it is common practice
in many industries to offer buyers a fixed penalty for unmet or delayed orders. For example, Fabtek, a company that provides titanium products for industrial use, establishes a penalty that is proportional to the amount of time that the delivery to the buyer is late, see Shapiro et al. (1992) and Chatterjee et al. (2002). Also, Holmes and France (2002) report on how Boeing’s late deliveries in 1997 triggered “enormous late fees.” (Due to their frustration over Boeing’s late deliveries, the company lost some of its most reliable customers to Airbus, see Bloomberg News 1999.) To some extent, the magnitude of the late fees a supplier commits to, provides a strong indication to the buyer of the supplier’s efforts to deliver on time.

In this paper, we consider a market with two suppliers and a set of buyers in search of procurement contracts with one of the suppliers. The suppliers are capable of manufacturing the same products. They both manufacture and stock a homogeneous product that can then be customized based on the specifications of the buyer. The prices paid by each buyer may depend on the particular product requirements they specify. We assume that customization is instantaneous.¹ Each buyer needs to process a certain volume of work, and each supplier’s ability to process the customers’ requests is constrained by a production capacity. The buyers do not know exactly when they will require the products, but they have information on the expected total volume of parts they will need over time. Given the specialized nature of the products they require, buyers look to single source their component in order to avoid the costs associated with dealing with multiple suppliers. Each supplier’s profit depends on which buyers establish a procurement contract with her and on the supplier’s production and stocking policy. At the same time, each buyer receives service delivery guarantee quotes from all the suppliers, in the form of late delivery penalties, and selects the supplier that offers the best service level commitment. The main objective of this paper is to determine how the procurement contracts are established between buyers and suppliers. We address this problem as a game among the suppliers, modeled as make-to-stock queues, and characterize the game’s Pareto optimal equilibrium, i.e., the equilibrium that maximizes both suppliers’ profits among all other equilibria.

We analyze the competition for procurement contracts from the perspective of the suppliers. That is, our focus is on how the suppliers determine their optimal production and stocking decisions when these decisions affect the level of service they can provide to the buyers and, in turn, the amount of business they can attract. Then, one feature of this

¹This situation is common in many industries. One example is the semiconductor industry, where the production process of integrated circuits involves two major stages, wafer fabrication and (customized) assembly, see Li (1992) and the references therein.

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paper is that it models in detail the suppliers’ production and inventory operations. The suppliers operate as make-to-stock and compete to attract buyers based on service delivery guarantees given by backorder (late delivery) penalty fees. Since a supplier selling to more than one buyer may quote them different late penalty fees, when a demand for parts from a buyer arrives, the supplier may choose to satisfy this volume requirement directly from stock (when available) or the order may be backordered to be satisfied later. An arriving order may then be delayed in consideration of future orders from a buyer with a higher late penalty commitment, even if there are parts available in stock. That is, the suppliers operate under stock-rationing policies.

Our analysis allows us to derive various insights as to how the procurement contracts are established and how they depend on the suppliers’ capacities. We also investigate how the volume of business and the prices of the products required by the different buyers affect the competition between the suppliers. We first derive these results for the case of two buyers, and then conduct a numerical study to investigate how they generalize to the case of more than two buyers.

To our knowledge, this represents one of the first papers in the operations management literature in which players engage in an all-or-nothing competition. This leads to discontinuous payoff functions, in this case for the suppliers, since a minor change in their action variables may determine whether they establish a procurement contract with a buyer, or not. Thus, there is no pure strategy Nash equilibrium in the game between the suppliers. Following the economics literature on games with discontinuous payoffs, an equilibrium in continuous strategies can, however, be regarded as the limit equilibrium of some discrete action set games (see Section 2 for a description on how this is defined in the economics literature for the Bertrand duopoly game with differentiated costs). We follow this approach to analyze the game between the suppliers.

Many papers consider the design of contracts in supply chains, and focus on how their specification affects the decisions made by the members of the chain. These contracts are specified in terms of wholesale prices, minimum purchase commitments, returns policies, etc. For instance, Cachon and Zipkin (1999) study a two-stage decentralized series system in which both firms implement base stock policies to manage their inventories. The authors propose a linear transfer payment scheme to coordinate the system that includes backorder penalties for the supplier’s late deliveries. We refer to Lariviere (1999) and Tsay, Nahmias and Agrawal (1999) for extensive reviews of the literature on contracting in supply chains.
A number of papers analyze service delivery competition for market share. Kalai et al. (1992) propose a model in which customers value delivery speed and where suppliers choose their respective service speeds. They model the suppliers as make-to-order queues and characterize the equilibrium along with the associated market shares. Li and Lee (1994) propose a similar model in which customers value delivery speed but also cost, quality and price. In both papers, customers’ time preferences are homogenous and servers treat them on a first-come-first-served basis. In our case, due to the (possibly) different penalty fees offered by the suppliers, priorities rule have to be applied to optimize the suppliers’ profits. Li (1992) considers a setting in which suppliers offer an identical product and charge equal prices but compete in terms of their production/inventory strategies. Customers arrive according to a Poisson process, and purchase with equal probability from any firm with positive inventory. If all firms are out of stock, the customer places the order with each firm, but buys from the firm which completes the order first. Anupindi and Akella (1993) consider a setting where buyers make quantity allocation decisions between two suppliers when supply is uncertain. Cachon and Zhang (2003a) consider a setting where buyers make quantity allocation decisions between two suppliers when supply is uncertain. Cachon and Zhang (2003a) consider a setting with a single buyer that sources from multiple make-to-order suppliers. The suppliers choose their capacities and the buyer allocates her demand to minimize her procurement costs subject to a lead time constraint.

The research on procurement auctions focuses on the design of bidding processes that optimize the profit of the buyer. Gallien (2001) analyzes a multi-item procurement auction mechanism with production capacity constraints. Unlike our model, competition is based on selling prices, and the procurement contracts are split among various suppliers. Anton and Yao (1989) show that with complete information and under a split-award procurement, suppliers can collude to extract all the gains from dual sourcing so that the buyer strictly prefers a single source only award auction. Cachon and Zhang (2003b) consider a procurement auction where the buyer seeks to source a good from a single supplier chosen from a pool of multiple potential suppliers, so as to minimize its procurement and operating costs. Chen (2001) considers a procurement auction problem where both the order quantity and the price need to be determined and shows that the optimal procurement strategy takes the form of an entry-fee auction. Elmaghraby (2000) presents a review on the literature on sourcing strategies, which also addresses the design of procurement auctions.

Stock and capacity allocation problems have been first introduced in the context of inventory control. Topkis (1968) provides one of the earliest formulations of an optimal dynamic rationing problem for an uncapacitated discrete time system. Nahmias and Demmy (1981) and, more recently, Desphande et. al (2003) also consider rationing problems for uncapacitated
systems. For limited production capacity, queuing-based models provide a framework to explicitly consider production capacity and randomness in the supply process (see Buzacott and Shanthikumar 1993).

There are several papers related to the underlying model we use to describe the suppliers' operations in the procurement game. This model was first studied by Ha (1997b). The author presents a setting in which each supplier produces to stock a single part-type that can satisfy any customer and holds a common stock that must be allocated among the customers, according to different backorder penalties. Demands are Poisson, and the production time is exponentially distributed. Ha partially characterized the optimal stock allocation policy, which minimizes the average holding and backorder costs, for the case of two customers. More recently, de Véricourt et al. (2002) fully characterize the optimal policy for the general case of $n$ customers. This policy specifies different inventory levels corresponding to the different customers. It backorders demands from a customer when the inventory level is below the corresponding threshold. In many situations, the optimal policy significantly outperforms more traditional allocation rules, see de Véricourt et al. (2001). These results allow us to derive the suppliers' profit functions, which provide the basis for the analysis in the paper.

Wein (1992) addresses the multi-product version of the make-to-stock system. In this setting, the supplier holds a separate inventory for each customer. A scheduling policy is specified in order to allocate the production capacity among the different products. A full characterization of the scheduling policy that minimizes the average holding and backorder costs has not been proposed so far, but Wein (1992), Veatch and Wein (1996) and Peña-Perez and Zipkin (1996) provide efficient heuristics. de Véricourt et al. (2000) partially characterize the optimal policy for two customers. Ha (1997a) studies the discounted case.

All of the models described above assume an infinite horizon setting and steady-state regime. In fact, this assumption is widely used in the inventory management literature and, in particular, in the context of make-to-stock queues. The characterization of the optimal inventory and production control policies for the finite-horizon case is far from trivial. While a finite-horizon model would be appropriate, for example, for settings where the set of buyers changes frequently, the present model allows us to provide analytical results regarding buyer/supplier selection and regarding the operational performance of the suppliers at the equilibrium procurement contracts.

The paper is structured as follows: Section 2 introduces the model and notation. In Section 3, we describe the operations for the make-to-stock suppliers and characterize a
supplier’s optimal choice of buyers. Section 4 provides an analysis of the procurement game. Section 5 offers insights and extensions to our model, and we present conclusions in Section 6. An Appendix contains all the proofs.

2 Model and Preliminaries

Consider a set of firms (buyers) that need to source a specialized product (part or component) over a long period of time, and there are several suppliers in the market able to manufacture this product. The buyers do not know exactly when they will need the parts, but they have information on the expected total volume required over time. Each buyer releases orders to the supplier randomly over the length of the contract, according to a Poisson process. The long-term order requirement of buyer $k$ consists of an arrival rate $\lambda_k$ and a unit price $p_k$. The unit prices may depend on buyer-specific requirements, and they are exogenous to the model. The suppliers manufacture and stock a homogeneous product that can then be customized based on the specifications of the buyer. We assume that customization is instantaneous.

Buyers are primarily concerned with product availability and also expect to receive the products in the same sequence in which the orders are released. All suppliers can provide the parts at the same price to each buyer, but they may distinguish themselves in terms of product availability and frequency of on-time deliveries. Thus, buyers call on tenders for a volume of product and request a measure for the service level each supplier is willing to provide. Ideally, each buyer would wish to establish a procurement contract with the supplier offering the highest level of service.

A procurement contract establishes a backorder penalty per unit of time the supplier is late. This makes the contract easy to verify and to enforce, and provides an incentive to the supplier to minimize the frequency of delivery delays. In addition, since the penalty is paid per unit of time that the delivery is delayed, the supplier has no incentive to cross orders. Suppliers then offer backorder penalty fees to ensure adequate availability, and each buyer selects the supplier that offers the highest backorder penalty. In Section 3.1 we show that selecting the supplier with the highest backorder penalty quote is equivalent to choosing the supplier with the highest percentage of on-time deliveries. In summary, when selecting the supplier based on late delivery penalty fees, a buyer makes sure that 1) the contract is enforceable and verifiable, 2) the frequency of on-time deliveries is the highest available, and 3) the supplier will provide the products in the sequence in which they were ordered.
We denote the set of buyers by $\Omega_C = \{1, \ldots, m\}$. Each supplier $i = 1, 2$, selects a late penalty offer $B^i_k$ to quote buyer $k$. This penalty represents a dollar amount to be paid per unit of time for each unit not delivered on time to the buyer. $B^i_k$ is a non-negative real-valued contract parameter. A vector $B^i \in \mathbb{R}_+^m$ represents the offers made by supplier $i$ to each of the buyers in $\Omega_C$.

Each supplier would like, in principle, to offer competitive service guarantees to establish as many procurement contracts as possible. However, the suppliers face capacity constraints in their production systems. The number of buyers they can serve then depends on production capacity constraints and on the way they allocate capacity and stock among their buyers. Given a subset of buyers $X \subset \Omega_C$, each supplier $i = 1, 2$ earns a profit for serving all of the buyers in $X$, given by a function $\pi^i_X : \mathbb{R}_+^m \rightarrow \mathbb{R}$, which depends on the vector of offers $B^i$. We define $d^i(B^i) := \text{argmax}_{X \subset \Omega_C} \pi^i_X(B^i) = \{X \subset \Omega_C : \pi^i_X(B^i) = \text{max}_{Y \subset \Omega_C} \pi^i_Y(B^i)\}$ as the subsets of buyers that maximize supplier $i$’s profit, for a given vector $B^i$. In Section 3, we describe in detail the suppliers’ production and stocking decisions and their corresponding profits. We also characterize the sets $d^i(B^i)$.

Each buyer selects the supplier that quotes the larger penalty offer. When both suppliers promise the same level of service to a buyer (as given by their late penalty quotes), the buyer randomly chooses one of the two suppliers with probability $1/2$. We refer to the competition among the suppliers to attract buyers based on the service guarantees as the ‘procurement game’.

Since each buyer chooses a single supplier to establish the sourcing relationship, the analysis of the procurement game has common elements with the Bertrand duopoly game with homogeneous products and differentiated costs, which has been extensively studied in the economics literature (see e.g., Tirole 1988, and Wolfstetter 1999). In this game, two firms offer homogenous products and compete in terms of their prices, so that consumers always buy from the firm that offers the lower price. In case of equal prices, consumers select any one of the two firms with probability $1/2$. Firms only differ in their unit production costs $c_i$, with $c_2 > c_1$. If total market demand is given by $q$, then each firm’s expected demand is:

$$D_i(p_i, p_j) = \begin{cases} 
0, & p_i > p_j \\
q/2, & p_i = p_j \\
q, & p_i < p_j 
\end{cases}$$

and the firms’ expected profit functions are given by $\pi_i = (p_i - c_i)D_i(p_i, p_j)$. This game fails to have a Nash equilibrium in pure strategies, see Tirole (1988). Consider, however, the
same setting with the assumption that $\epsilon > 0$ is the smallest monetary unit so that costs and prices have to be multiples of $\epsilon$. If $\epsilon$ is sufficiently small, it can be shown that the Bertrand duopoly game has a pure strategy equilibrium of the form $p_1 = c_2$ and $p_2 = c_2 + \epsilon$, and one of the form $p_1 = c_2 - \epsilon$ and $p_2 = c_2$ (see, for example, Erlei 2002). In both cases, $D_1(p_1, p_2) = q$ and $D_2(p_1, p_2) = 0$, whereas in the first case $\pi_1 = (c_2 - c_1)q$ and $\pi_2 = 0$, and in the second case $\pi_1 = (c_2 - \epsilon - c_1)q$ and $\pi_2 = 0$. In the economics literature, it is emphasized that one can regard the equilibrium in continuous strategies as the limit equilibrium of the discrete strategy models as $\epsilon \downarrow 0$ (see e.g., Tirole 1988, Exercise 5.1, and Wolfstetter 1999, Section 3.2.3). Thus, in the game with continuous strategies, the ‘equilibrium’ of the Bertrand duopoly game is defined as $p_1 = c_2$ and $p_2 = c_2$, leading to $D_1(p_1, p_2) = q, D_2(p_1, p_2) = 0, \pi_1 = (c_2 - c_1)q, \text{ and } \pi_2 = 0$. Note that, at this equilibrium, both firms’ prices coincide and all consumers buy from the firm with the lower cost. Note also that firm 2 could decrease its price slightly, gaining the entire market, but its profits would then be negative.

The procurement game studied in this paper requires a similar analysis to that of the Bertrand duopoly game: we will define the ‘equilibrium’ of the procurement game as the limit equilibrium of some discrete action set games.

3 The Make-to-Stock Suppliers

We now describe the suppliers’ profit functions in more detail. Suppliers produce to stock the same common component. We assume that this component can be very rapidly differentiated into the product required by the buyer who placed an order. Hence, each supplier holds a common inventory, which is shared among her buyers. Since the common component is the same for both suppliers, it is reasonable to assume that their unit holding costs are similar. Thus, for both suppliers, we take the common component’s unit holding cost to be equal to $h$. If the inventory is empty, demands are backordered. When it is not, an arriving demand can be either satisfied by the on-hand inventory, in which case the component is immediately customized, or backordered. The production time is exponentially distributed with mean $1/\mu^i$ for supplier $i, i = 1, 2$. We denote by $\rho_k^i = \lambda_k/\mu^i$ supplier $i$’s utilization rate from sourcing buyer $k, i = 1, 2$. We assume that $\sum_{k \in \Omega} \lambda_k < \mu^i, i = 1, 2$, i.e., both suppliers’ capacities are large enough to allow buyers to single source. The values of $\lambda_k, k = 1, 2$, and $\mu^i, i = 1, 2$, are exogenous. In Section 5, we explore how the outcome of the procurement game is affected by changes in the production capacities.
3.1 Supplier Profit Functions

We present first the profit function of supplier $i$ assuming she supplies all of the buyers (i.e., $\pi_{\Omega C}^i$). We then extend the result to the case where supplier $i$ serves a subset $X$ of the buyers (i.e., $\pi_{X}^i$). Given a fixed vector of late delivery penalty offers $B^i$, we need to specify the control policy applied by the supplier, and the associated holding and backorder costs. Without loss of generality, we re-index the offers of supplier $i$ such that $B^i_1 \geq \ldots \geq B^i_n$. Hence, backordering an arriving demand of buyer $k$ permits the supplier to reserve the current stock for future demands of customers $j < k$.

The state variable of this system can be described by the vector $s(t) = (s_0(t), \ldots, s_n(t))$, with $s_k \in Z_+$ for $1 \leq k \leq n$, where $s_0(t)$ and $s_k(t) \ (t > 0)$ are the on-hand inventory and the number of backorders of buyer $k$ at time $t$, respectively. Denote by $S(t) = (S_0(t), \ldots, S_n(t))$, the random variables corresponding to the state space $s$. Given a control policy $\varphi$ which states at any time when to produce and whether an arriving demand should be satisfied or backordered, the system incurs the following average cost:

$$\lim_{T \to \infty} \frac{E_{s_0}^\varphi \left[ \int_0^T c(S(t))dt \right]}{T},$$

where $s_0$ is the initial state and $c(s(t))$ the instantaneous cost equal to $hs_0(t) + \sum_{k=1}^n B^i_k s_k(t)$.

de Véricourt et al. (2002) establish a complete characterization of the policy minimizing (1): there is a threshold for each buyer, such that it is optimal to satisfy demands from this buyer if the on-hand inventory is above that threshold. More precisely, consider $n$ thresholds $z^i_1 \leq \ldots \leq z^i_n$, where $z^i_n$ corresponds to the base-stock level of the system and $z^i_{k-1}$ is the threshold associated with buyer $k$. The optimal policy for supplier $i$ is of the form:

- The policy allocates a produced item to a waiting demand of buyer $k$ if the stock level is at $z^i_{k-1}$;
- The policy backorders an arriving demand from buyer $k$ if the inventory level is less than $z^i_{k-1}$, while an arriving demand from buyer $k$ is satisfied otherwise.

(See de Véricourt et al. 2001 for more details.) Furthermore, the optimal thresholds $\{z^i_k\}$ that minimize (1), as well as the corresponding optimal cost denoted by $g^i(B^i)$, can be computed recursively, see de Véricourt et al. (2001, Property 3). This procedure provides integer thresholds, since the state space is discrete. We present here a heuristic which computes the thresholds as real values.
Procedure 1  (a) If $\sum_{k=1}^{n} \lambda_k \geq \mu^i$, the system is not stable and the average cost is infinite.

(b) If $\sum_{k=1}^{n} \lambda_k < \mu^i$, construct the sequences $z^i_k$, and $g^i_k$ as follows:

$$z^i_0 = g^i_0 = \tilde{\rho}^i_0 = B^i_{n+1} = 0,$$

$$\tilde{\rho}^i_k = \tilde{\rho}^i_{k-1} + \frac{\lambda_k}{\mu^i},$$

$$z^i_k - z^i_{k-1} = \log \tilde{\rho}^i_k \frac{h + B^i_{k+1}}{h + B^i_k},$$

$$g^i_k = (h + B^i_{k+1}) z^i_k.$$

The optimal rationing levels and the optimal cost $g^i(B^i)$ are taken equal to the $z^i_k$’s and $g^i_n$, respectively.

The exact algorithm computes the integer part of $z_k - z_{k-1}$ for each $k$. Procedure 1 only differs from the exact version in that step. This modification results in the continuity of $g^i(B^i)$. Also, the cost function $g^i(B^i)$ is strictly increasing in $B^i_k$ and $\lim_{B^i_k \to \infty} g^i(B^i) = \infty$, for all $k$. Supplier $i$’s profit equals $\pi^{\Omega_i}(B^i) = \sum_{k=1}^{n} p_k \lambda_k - g^i(B^i)$.

Hence, given $B^i$, supplier $i$ sets the rationing thresholds $z^i_k$ in order to minimize total inventory costs. In turn, these thresholds generate different fill rates for the buyers. The fill rate is the fraction of arriving demand from buyer $k$ satisfied directly from the stock or, in our context, the frequency of on-time deliveries. We denote by $f^i_k(B^i)$ the effective fill rate of buyer $k$ given the backorder penalty vector $B^i$. The following result provides analytical expressions for $f^i_k(B^i)$.

Proposition 1  Given the backorder penalty vector $B^i$, the frequency of on-time deliveries for buyer $k$ is given by

$$f^i_k(B^i) = 1 - \frac{h}{h + B^i_k}.$$  

It is worth noting that $f^i_k(B^i)$ does not depend on the other backorder penalties offered to buyers $j \neq k$ (which allows us to write $f^i_k(B^i) = f^i_k(B^i)$) or on the supplier’s capacity. A contract specifying backorder penalties then ensures the buyer a frequency of on-time deliveries regardless of the supplier’s commercial policy (future commitments with other buyers) or any capacity reduction/increase. Furthermore, $f^i_k(B^i_k)$ is increasing in $B^i_k$. Hence, when a buyer chooses a supplier based on the backorder penalty, it is also selecting the supplier with the higher percentage of on-time deliveries.
Now suppose supplier $i$, $i = 1, 2$ makes offers given by $B^i$, not necessarily ordered, and her customers are a subset $X = \{k_1, \ldots, k_q\} \subset \Omega_C$. We denote by $B^i(X)$, the ordered vector of offers for the set of buyers $X$, obtained by re-indexing the vector $(B^i_{k_1}, \ldots, B^i_{k_q})$ so that $(B^i(X))_1 \geq \ldots \geq B^i(X)_q)$. Applying Procedure 1 for the $q$ buyers in $X$, we can compute the optimal cost $g^i(B^i(X))$. The profit function is then equal to
\[
\pi^i_X(B^i) = \sum_{k \in X} p_k \lambda_k - g^i(B^i(X)), \quad X \subset \Omega_C.
\] (6)

Since $g^i(B^i(X))$ is continuous, strictly increasing in $B^i_k$ if $k \in X$, and independent of $B^i_k$ if $k \notin X$, $\pi^i_X$ satisfies the following properties:

(\(P_1\)) $\pi^i_X$ is continuous in $B^i$,

(\(P_2\)) $\pi^i_X$ strictly decreasing in $B^i_k$ for each $k \in X$ and constant in $B^i_k$ for each $k \notin X$,

(\(P_3\)) $\lim_{B^i_k \to +\infty} \pi^i_X = -\infty$ for any $k \in X$.

The profit functions in (6) provide the basis for the procurement game. The next step is to determine $d^i(B^i)$, the subset(s) of buyers that maximizes a supplier’s profit for any given vector of penalty offers $B^i$. Such characterization is rather complicated for the general case of $n$ buyers. On the other hand, analytical results, as well as graphic representations, can be obtained when there are two buyers in the market. We therefore now restrict attention to the case of two buyers, and we provide a discussion on the more general case in Section 5.5.

### 3.2 Optimal Supplier Choice with Two Buyers

In the case of two buyers, Procedure 1 can be greatly simplified. Throughout the rest of the paper, and to facilitate the analysis, consider the change of variables $\beta^i_k = \ln(h/(h + B^i_k)) = \ln(1 - f^i_k(B^i_k))$, where the offers $\beta^i_k$ are now negative. Properties (\(P_2\)) and (\(P_3\)) now become:

(\(P_2'\)) $\pi^i_X$ strictly increasing in $\beta^i_k$ for each $k \in X$ and constant in $\beta^i_k$ for each $k \notin X$,

(\(P_3'\)) $\lim_{\beta^i_k \to -\infty} \pi^i_X = -\infty$ for any $k \in X$,

where now $\pi^i_X : \mathbb{R}^n_+ \to \mathbb{R}$.

After some simple algebra, the following lemma results directly from the application of Procedure 1 for one, and then two buyers.
Lemma 1  The profit functions $\pi^i_X$ of supplier $i$, for all subsets $X \subset \{1, 2\}$, are

$$
\begin{align*}
\pi^i_{\{k\}}(\beta^i_k) &= v^i_k(\beta^i_k - r^i_k), \ k = 1, 2, \\
\pi^i_{\{1,2\}}(\beta^i_1, \beta^i_2) &= \pi^i_{\{1\}}(\beta^i_1) + \pi^i_{\{2\}}(\beta^i_2) - \gamma^i h \max\{\beta^i_1, \beta^i_2\},
\end{align*}
$$

where

$$
v^i_k = -\frac{h}{\ln \frac{\lambda_k}{\mu}} \geq 0, \ r^i_k = -\frac{p_k \lambda_k}{v^i_k} \leq 0, \ \gamma^i = \frac{1}{\ln \frac{\lambda_1 + \lambda_2}{\mu}} - \frac{1}{\ln \frac{\lambda_1}{\mu}} - \frac{1}{\ln \frac{\lambda_2}{\mu}}.
$$

Note that $-v^i_k r^i_k = p_k \lambda_k$ is the revenue rate earned by serving buyer $k$. At the same time, $v^i_k$ represents the relative increase in supplier $i$’s costs due to an increase in her service level offer, and it is directly proportional to this supplier’s holding cost $h$. On the other hand, $r^i_k$ is a lower bound for supplier $i$’s service offer $\beta^i_k$ or, equivalently, $f^i_k(B^i_k) \leq 1 - e^{r^i_k}$ is an upper bound on the supplier’s offered fill rate. Regarding the parameters $\gamma^i$, it is easy to check that $\gamma^i < 0$ for $\rho^i_k > 1/4$, $k = 1, 2$, $i = 1, 2$. Observe that the condition $\rho^i_k > 1/4$, $k = 1, 2$, $i = 1, 2$ implies that both buyers can provide significant business to any of the suppliers (or, alternatively, that none of the suppliers have an extremely large amount of excess capacity relative to the buyer’s needs). We are going to assume that this condition, and thus $\gamma^i < 0$, holds throughout the paper. The magnitude of $\gamma^i$ is related to the additional cost incurred by supplier $i$ for serving both buyers with a single production facility (as opposed to with two buyer-dedicated production facilities). It is easy to check that $\gamma^i$ is decreasing in each $\rho^i_k$, $k = 1, 2$. That is, the higher the load, the higher the cost of serving both buyers with a single production facility.

Define $\tilde{\beta}^i_k$ and $\bar{\beta}^i_k$ as follows:

$$
\tilde{\beta}^i_k := \frac{v^i_k r^i_k}{v^i_k - h \gamma^i} \text{ and } \bar{\beta}^i_k := \frac{h \gamma^i}{v^i_k} \tilde{\beta}^i_{3-k} + r^i_k.
$$

To understand these definitions, note that when $\tilde{\beta}^i_2 \geq \tilde{\beta}^i_1$, $\pi^i_{\{1\}}(\bar{\beta}^i_1, \beta^i_2) = \pi^i_{\{1,2\}}(\beta^i_1, \beta^i_2)$ for $\beta^i_2 = \bar{\beta}^i_2$ and all $\bar{\beta}^i_1 \leq \beta^i_1 \leq \bar{\beta}^i_2$ (see Figure 1). Similarly, when $\tilde{\beta}^i_1 \geq \tilde{\beta}^i_2$, $\pi^i_{\{2\}}(\beta^i_1, \bar{\beta}^i_2) = \pi^i_{\{1,2\}}(\beta^i_1, \beta^i_2)$ for $\beta^i_1 = \bar{\beta}^i_1$ and all $\bar{\beta}^i_1 \leq \beta^i_1 \leq \bar{\beta}^i_2$ (see Figure 2). The following result fully characterizes the set of buyers that maximizes supplier $i$’s profit, given a pair of service offers $(\bar{\beta}^i_1, \bar{\beta}^i_2)$.

Proposition 2

$$
\emptyset \in d^i(\bar{\beta}^i_1, \bar{\beta}^i_2) \iff \beta^i_k \leq r^i_k, k = 1, 2
$$

(8)
\{1\} \in d_i(\beta_i^1, \beta_i^2) \iff \max \left\{ r_i^1, \frac{v_i^1}{v_i^2} (\beta_i^2 - r_i^2) + r_i^1 \right\} \leq \beta_i^1 \leq \frac{v_i^1}{h_i} (\beta_i^2 - r_i^2) \beta_i^2 \leq \tilde{\beta}_i^2, \text{ if } \tilde{\beta}_i^2 \geq \tilde{\beta}_i^1 \tag{9}

\{2\} \in d_i(\beta_i^1, \beta_i^2) \iff \max \left\{ r_i^2, \frac{v_i^1}{v_i^2} (\beta_i^1 - r_i^1) + r_i^2 \right\} \leq \beta_i^2 \leq \frac{v_i^1}{h_i} (\beta_i^1 - r_i^1) \beta_i^1 \leq \tilde{\beta}_i^1, \text{ if } \tilde{\beta}_i^1 \geq \tilde{\beta}_i^2 \tag{10}

If \tilde{\beta}_k^i \geq \tilde{\beta}_{3-k}^i:

\{1, 2\} \in d_i(\beta_i^1, \beta_i^2) \iff \max \left\{ \frac{v_i^1}{h_i} (\beta_i^k - r_i^k), \frac{v_i^1}{h_i} (\beta_i^3-k - r_i^3-k), \tilde{\beta}_i^k \right\} \leq \beta_i^k, \text{ if } \beta_i^3-k \leq \beta_i^k \tag{11}

Figures 1 and 2 below show the sets \(d_i(\beta_i^1, \beta_i^2)\) for the cases when \(\tilde{\beta}_i^2 \geq \tilde{\beta}_i^1\) and \(\tilde{\beta}_i^1 \geq \tilde{\beta}_i^2\), respectively. In each case, the space of offers is divided into four regions corresponding to the cases where \(\emptyset, \{1\}, \{2\}\) and \(\{1, 2\}\) maximize the profit of supplier \(i\).

Figure 1: Optimal Supplier Choice with Two Customers, \(\tilde{\beta}_i^2 \geq \tilde{\beta}_i^1\)

4 The Procurement Game

In this section, we analyze the procurement game in a market with two suppliers and two buyers. The suppliers differ in their production capacities. We assume, without loss of generality, that supplier 1’s capacity is larger than that of supplier 2, i.e., \(\mu_1 > \mu_2\). The timing of the game is as follows. First, the suppliers set a late penalty rate for each buyer. Next, the
Figure 2: Optimal Supplier Choice with Two Customers, $\tilde{\beta}_1^i \geq \tilde{\beta}_2^i$

buyers select a supplier based on these service guarantees. Finally, the contracts are established and suppliers implement control policies for production and inventory management.

The procurement game is an all-or-nothing competition between the two suppliers. Each buyer selects a single supplier to establish the procurement contract based on the best service guarantee, so a slight change in the penalty offered by one supplier may fundamentally change the choices of the buyers and thus the profits for the suppliers. In this respect, the procurement game exhibits similarities to the Bertrand duopoly game described in Section 2, and like that game, it fails to have a Nash equilibrium in pure strategies. Indeed, since the action variables (late delivery penalties) are continuous, it is easy to check that any equilibrium of the procurement game would have equal service offers for each buyer, i.e., $\beta^1_k = \beta^2_k$ for $k = 1, 2$. However, any one of the suppliers could slightly decrease her value of $\beta_k$ (that is, increase her penalty offer) ensuring a procurement contract with buyer $k$. That may, in turn, be followed by a similar decrease in the other supplier’s $\beta$-offer, and so on.

As in the Bertrand duopoly game discussed in Section 2, we define an equilibrium of the procurement game as the limit equilibrium of related discrete action games. For all $\epsilon > 0$ sufficiently small, we consider a setting where $\epsilon$ is the smallest service offer increment. We refer to this game as the $\epsilon$-procurement game. We then define the equilibrium strategies, profits and buyers’ choices in the procurement game to be the limits of those in the $\epsilon$-procurement games as $\epsilon \to 0$.

**Definition** A pair $(\beta_1, \beta_2)$ is an *equilibrium of the procurement game* if, for each $\epsilon > 0$
sufficiently small, the $\epsilon$-procurement game has a pure strategy Nash equilibrium of the form 

$$\beta(\epsilon) = (\beta_1 + n_1^1 \epsilon, \beta_2 + n_2^1 \epsilon, \beta_1 + n_1^2 \epsilon, \beta_2 + n_2^2 \epsilon),$$

with $n_k^i = -1$, 0 or 1, $n_k^i \neq n_k^j$, $i = 1, 2$, $j \neq i$, $k = 1, 2$.\footnote{An equilibrium of the procurement game could, more generally, be defined as the limit of an arbitrary sequence of pure strategy Nash equilibria $\beta(\epsilon)$ of the $\epsilon$-procurement games. However, this more general definition would make the analysis of the procurement game unnecessarily complicated. The definition above allows for a more tractable analysis. In addition, the form of the $\beta(\epsilon)$ resembles the form of the equilibria of the discrete action games in the Bertrand duopoly competition introduced in Section 2 ($p_1 = c_2$ and $p_2 = c_2 + \epsilon$, or $p_1 = c_2 - \epsilon$ and $p_2 = c_2$, for $\epsilon$ small).}

Note that we define an equilibrium of the procurement game as a pair – rather than a quadruple – since both suppliers’ offers to a customer must be equal at equilibrium. For any pair $(\beta_1, \beta_2)$, the set $d^i(\beta_1, \beta_2) = \arg \max_{X \subset \Omega} \pi^i_X(\beta_1, \beta_2)$ can be thought of as supplier $i$’s best-response correspondence, as it contains the subsets of buyers that maximize this supplier’s profit. The next result provides a necessary condition for a pair of service offers $(\beta_1, \beta_2)$ to be an equilibrium of the procurement game in terms of $d^1$ and $d^2$.

**Proposition 3** Let $\beta = (\beta_1, \beta_2)$ be an equilibrium of the procurement game. Then, there are sets $X^1 \in d^1(\beta)$ and $X^2 \in d^2(\beta)$ such that $X^1 \cap X^2 = \emptyset$.
Lemma 2 Consider the procurement game in which two suppliers $i = 1, 2$ (with $\mu^1 > \mu^2$) compete for procurement contracts from two buyers $k = 1, 2$ (with $\lambda_1 \geq \lambda_2$). Define

$$\mu^* \overset{\text{def}}{=} \lambda_2 \exp \left( \frac{h}{v_2 - h \gamma^1} \right) = \lambda_2 \exp \left( \frac{\ln(\rho_1^2 + \rho_2^1) \ln \rho_1^1}{\ln(1 + \frac{\lambda_1}{\lambda_2})} \right),$$

and assume that

$$\frac{p_1 \lambda_1}{p_2 \lambda_2} \geq \frac{v_1^2 - h \gamma^1 + v_1^2 - v_2^2}{v_2^2 - h \gamma^1} = 1 + \frac{\ln \frac{\lambda_2}{\lambda_1} \ln \rho_2^2}{\ln \rho_2^1 \ln \rho_1^1}, \quad (12)$$

where recall that $\rho_k^i = \lambda_k / \mu^i$ and define $\rho_2^* = \lambda_2 / \mu^*$. Then,

(a) If $\mu_2 < \mu^*$, $d^1(\beta^*) = \{1, 2\}$ and $d^2(\beta^*) = \{\emptyset, \{1\}, \{2\}\}$ for $\beta^* = (\hat{\beta}_1^2, \tilde{\beta}_2^1)$.

(b) If $\mu_2 > \mu^*$, $d^1(\beta^*) = \{1\} \cup \{1, 2\}$ and $d^2(\beta^*) = \{\{1\}, \{2\}\}$ for $\beta^* = (\tilde{\beta}_1^2, \hat{\beta}_2^1)$.

Figure 3: Pareto Optimal Equilibrium

Condition (12) implies that $\tilde{\beta}_2^i \geq \hat{\beta}_1^i$ for $i = 1, 2$ (see the proof of Lemma 2), so that the optimal choice of buyers for each supplier is as in Figure 1. This condition also implies that the price of buyer 1’s product cannot be too low. Indeed, it is easy to verify that $v_1^2 - v_2^2 \geq 0$ since $\lambda_1 \geq \lambda_2$. Then, the right-hand-side of condition (12) is larger than or equal to 1, so that this condition implies that $p_1 \lambda_1 \geq p_2 \lambda_2$. In other words, not only does buyer 1 have a larger volume requirement than buyer 2, but (12) implies that the revenue rate from buyer 1 is also larger than that of buyer 2. If the products required by both buyers are identical and $p_1 = p_2$, then (12) introduces a lower bound on how much larger the average volume requirement of buyer 1 has to be compared to that of buyer 2 to guarantee the results in the lemma (and, in turn, the existence of a Pareto optimal equilibrium, see Theorem 1 below).

As shown in Lemma 2, the service offers given by $\beta^*$ satisfy the necessary conditions established in Proposition 3. We next show that these offers are indeed an equilibrium of
the procurement game, since they are the limit equilibrium of the \( \epsilon \)-procurement games. Furthermore, we show that they are Pareto optimal among all other possible equilibria.

**Theorem 1** Consider the procurement game in which two suppliers \( i = 1, 2 \) (with \( \mu^1 > \mu^2 \)) compete for procurement contracts from two buyers \( k = 1, 2 \) (with \( \lambda_1 \geq \lambda_2 \)). Define \( \mu^* \) as in Lemma 2, and assume (12) holds.

(a) If \( \mu^2 < \mu^* \), then there is a unique Pareto optimal equilibrium \( \beta^* = (r^1_2, r^2_2) \) under which supplier 1 establishes procurement contracts with both buyers.

(b) If \( \mu^2 > \mu^* \), then there is a unique Pareto optimal equilibrium \( \beta^* = (\hat{\beta}^2_1, \tilde{\beta}^1_2) \) under which supplier \( i \) establishes a procurement contract with buyer \( k = i \).

Both Lemma 2 and Theorem 1 omit the case \( \mu^2 = \mu^* \). In this case, \( r^1_2 = \hat{\beta}^2_1 \) and \( r^2_2 = \tilde{\beta}^1_2 \). Following a similar argument to that of the proof of Lemma 2, one can verify that \( d^1(\beta^*) = \{(1), \{1, 2\}\} \) and \( d^2(\beta^*) = \{\emptyset, \{1\}, \{2\}\} \), where \( \beta^* \overset{def}{=} (r^1_2, r^2_2) = (\hat{\beta}^2_1, \tilde{\beta}^1_2) \). In this case, any of the outcomes described in Theorem 1(a) or (b) lead to the same profits for both suppliers and to the same service guarantees for both buyers.

The two cases considered in Theorem 1 can be found in Figure 3. In the first case (\( \mu^2 < \mu^* \)), the (Pareto optimal) equilibrium is \( \beta^* = (r^1_2, r^2_2) \), and both buyers establish procurement contracts with supplier 1. For supplier 2, \( \pi^2_{\{k\}}(r^2_k) = 0, \ k = 1, 2, \) and \( \pi^2_{\{1, 2\}}(\beta^*) < 0 \). Thus, the situation described in part (a) of Theorem 1 is similar to the outcome of the Bertrand duopoly game of Section 2: in that game, all consumers buy from the lower-cost firm and the other firm makes zero profit – in this game, both buyers establish contracts with the supplier with the larger capacity and the other supplier makes zero profit. If the capacity of supplier 2 is sufficiently large, i.e., \( \mu^* < \mu^2 < \mu^1 \), then each supplier establishes a contract with a buyer and, in particular, supplier 1 retains the buyer with the larger volume requirement (buyer 1). In this case, Lemma 2 indicates that, at the (Pareto optimal) equilibrium \( \beta^* \), \( d^1(\beta^*) = \{(1), \{1, 2\}\} \) and \( d^2(\beta^*) = \{(1), \{2\}\} \). This implies that, at \( \beta^* \), buyer 2 would bring no additional benefit to supplier 1 (\( \pi^1_{\{1\}}(\beta^*) = \pi^1_{\{1, 2\}}(\beta^*) \)), and dealing with buyer 1 (in addition to buyer 2) would actually decrease profits for supplier 2, since \( \{1, 2\} \notin d^2(\beta^*) \).

Finally, note that when the volume requirement rates of both buyers coincide, i.e., \( \lambda_1 = \lambda_2 \), then the right-hand-side of condition (12) is equal to 1. That is, the assumption in this case, reduces to \( p_1 \geq p_2 \). Note that, if \( \lambda_1 = \lambda_2 \), the assumption \( p_1 \geq p_2 \) can be made
without loss of generality. Thus, when $\lambda_1 = \lambda_2$, there always exists a unique Pareto optimal equilibrium of the procurement game. The same is not true, in general, for the case of different volume requirements. Figure 4 below illustrates a setting where there are multiple Pareto optimal equilibria (the set of equilibria corresponds to the gray area). Then, when $\lambda_1 \neq \lambda_2$, Theorem 1 provides a simple and intuitive sufficient condition that guarantees the uniqueness of the Pareto optimal equilibrium.

Figure 4: Pareto Frontier

5 Insights and Extensions

5.1 Equilibrium Fill Rates

As discussed in Section 2, buyers look for a supplier that ensures the best possible service level. Based on Theorem 1 and Proposition 1, we now evaluate the fill rate experienced by the buyers as a result of the procurement game.

**Corollary 1** (a) If $\mu^2 < \mu^*$, then supplier 1’s fill rate with buyer $k = 1, 2$, is

$$f_k^* = 1 - (\rho_k^2)^{p_k \lambda_k/h} \text{ and } f_1^* > f_2^*.$$  

(b) If $\mu^2 > \mu^*$, then supplier 1’s fill rate with buyer 1 is

$$f_1^* = 1 - (\rho^*_2)^{(p_2 \lambda_2/h)(\ln \rho^*_2/\ln \rho^*_1)} \left(\rho^*_1\right)^{(p_1 \lambda_1-p_2 \lambda_2)/h},$$
supplier 2’s fill rate with buyer 2 is 

\[ f_2^* = 1 - (\rho_2^*)^{p_2 \lambda_2 / h}, \quad \text{and} \quad f_1^* > f_2^*. \]

(Recall that \( \rho_k^2 = \lambda_k / \mu^2, \ k = 1, 2, \) and \( \rho_2^* = \lambda_2 / \mu^*. \))

Observe from Corollary 1 that a buyer’s fill rate is increasing in its revenue rate, since the supplier is then willing to commit to a higher late delivery penalty. At the same time, when \( \mu^2 < \mu^* \), each buyer’s fill rate is independent of the other buyer’s revenue rate. This is rather intuitive since, in this case, the backorder penalties are such that supplier 2 does not make any profit when sourcing either buyer 1 or buyer 2, and supplier 2’s profit from sourcing buyer \( k \) is independent of buyer \( 3 - k \)’s characteristics. When \( \mu^2 > \mu^* \), buyer 2’s equilibrium backorder penalty (and hence buyer 2’s fill rate) is such that the benefit to supplier 1 of rationing her stock in order to serve buyer 2 is null. It turns out that when supplier 1 optimally allocates stock among the two buyers, this benefit depends on \( p_2 \lambda_2 \) exclusively, so that \( f_2^* \) depends only on buyer 2’s revenue rate. On the other hand, the equilibrium backorder penalty for buyer 1 (and hence buyer 1’s fill rate) is such that supplier 2’s profit from sourcing only buyer 1 is equal to her profit when sourcing only buyer 2. Thus, in this case, buyer 1’s fill rate depends on both buyers’ revenue rates. Finally note that, in either case, buyer 1’s equilibrium fill rate is larger than that of buyer 2.

5.2 Equilibrium Stock Allocation and Production Control

One of the features of this paper is that it models in detail the inventory and production control of the suppliers. It turns out that the presence of competition has interesting effects on how the suppliers’ stock allocation and production control policies depend on their capacities. Following Procedure 1, we now provide the policy parameters for each supplier that result from the procurement game.

Corollary 2 Consider the Pareto optimal equilibrium \( \beta^* \) that arises in the procurement game. Then,

(a) If \( \mu_2 < \mu^* \), then supplier 1’s base stock level is equal to

\[ z_2^1 = \frac{1}{h} \left( p_1 \lambda_1 \ln \rho_1^2 - p_2 \lambda_2 \ln \rho_2^2 \right). \tag{13} \]
Furthermore, supplier 1 backorders arriving demands from buyer 2 when the inventory level is less than the rationing threshold

\[ z_{1*}^1 = \frac{1}{h \ln \rho_1^1} \left( p_1 \lambda_1 \ln \rho_1^2 - p_2 \lambda_2 \ln \rho_2^2 \right). \]  \hspace{1cm} (14)

In this case, supplier 2 does not establish any procurement contract.

(b) If \( \mu^2 > \mu^* \), then supplier 1’s base stock level is equal to

\[ z_{1*}^1 = \frac{1}{h \ln \rho_1^1} \left( p_1 \lambda_1 \ln \rho_2^2 + p_2 \lambda_2 \ln \frac{\rho_2^*}{\rho_2^2} \right) \]  \hspace{1cm} (15)

and supplier 2’s base stock level is equal to

\[ z_{2*}^2 = \frac{p_2 \lambda_2 \ln \frac{\rho_2^*}{\rho_2^2}}{h \ln \rho_2^2}, \]  \hspace{1cm} (16)

where, recall, \( \rho_k^i = \lambda_k / \mu^i \) and \( \rho_2^* = \lambda_2 / \mu^* \).

(Note that when \( \mu^2 > \mu^* \), the suppliers do not ration their stocks, since each of them establishes a contract with only one buyer.)

Corollary 2 allows us to investigate the equilibrium inventory and production policies of the suppliers. As an example, consider the case where \( \lambda_1 = \lambda_2 = 1.2 \), \( p_1 = 10 \), \( p_2 = 5 \), \( h = 1 \), and \( \mu^2 = 2.5 \) are fixed. Figure 5 describes the impact of supplier 1’s production capacity \( \mu_1 \) on the base stock level \( z_{1*}^1 \) and rationing threshold \( z_{1*}^2 \) of supplier 1 when \( \mu_2 < \mu^* \), and on the base stock levels \( z_{1*}^1 \) of supplier 1 and \( z_{2*}^2 \) of supplier 2 when \( \mu^2 > \mu^* \). The horizontal axis represents the value of \( \mu^1 \) increasing from 2.5 and up to 5.

Note that at \( \mu^1 = 3.75 \) (shown with a vertical dotted line in Figure 5), the threshold \( \mu^* = 2.5 = \mu^2 \). To the left of \( \mu^1 = 3.75 \), \( \mu^2 > \mu^* \) and supplier \( i \) sources buyer \( k = i \). Without competition, it is easy to see that a supplier’s equilibrium base-stock level decreases as her production capacity increases. In contrast, in the presence of competition, each supplier’s equilibrium base-stock level may increase as supplier 1’s production capacity increases. Indeed, as \( \mu^1 \) increases, competition between the suppliers intensifies since supplier 1 is able to offer more competitive backorder penalties (i.e., service levels). Thus, both suppliers need to provide higher fill rates to the buyers, leading to higher base stock levels and, in turn, lower profits. In other words, when \( \mu^2 > \mu^* \), an increase in the production capacity of supplier 1 may lead to a decrease in her profit. This counterintuitive finding bears resemblance with the results in Carr et al. (1999), who present a model of price competition
where a strategic investment by one of the firms to increase its capacity may reduce its equilibrium profit (see also Bernstein and Federgruen 2004).

As shown in Theorem 1, when $\mu^2 < \mu^* (\mu^1 > 3.75$ in this example) supplier 2’s capacity is not enough to compete with supplier 1, who can source both buyers by appropriately choosing a base-stock level and a rationing threshold. Note that, as expected, supplier 1 can dramatically decrease the base-stock level (for common inventory) as $\mu^1$ increases. The rationing threshold level is, however, less sensitive to changes in the production capacity.

![Figure 5: Production Control Policy as Function of $\mu_1$](image_url)

**5.3 Delivery Time**

In this paper, we consider a setting where each buyer selects a supplier based on the quoted backorder penalties. According to Proposition 1, this ensures that the buyer receives the best corresponding fill rate. However, in the outcome of the procurement game, a buyer does not necessarily end up choosing the supplier that, a posteriori, provides the *shortest average delivery time*. That is, even if a buyer selects the supplier with the higher fill rate, the average time that it has to wait for the delivery of parts every time its supplier is out of stock may be larger than what the other supplier might have provided (even if this other supplier’s backorder penalty offer – and fill rate – was lower). This is not surprising since fill rate and average delivery time represent two different objectives for the supplier. For example, even for a given set of backorder penalties $B$, if the supplier allocates the stock...
among her customers in order to minimize costs, then the customer with the lowest backorder penalty $B_k$ (highest $\beta_k$) may not have to wait longer than the others.

Depending on the situation, a buyer may prefer the supplier with the higher fill rate (even if the supplier’s average delivery time turns out to be longer). This can be the case, for example, when the buyer experiences high costs in interrupting its production process, e.g., when setup costs are high. In any case, when a buyer has to wait for its order, the supplier partially compensates the losses this buyer incurs per unit of time through the late penalty fees.

As we noted in Section 2, it is difficult to enforce a contract based upon a metric like the fill rate, and the same holds for the average waiting time. In Proposition 1 we showed that contracting upon late penalty fees allows the buyers to select the supplier with the higher fill rate. In this section, we investigate in which cases selecting the supplier with the higher penalty offer guarantees, in addition, the lower possible average delivery time. Consider the contract allocation at the Pareto optimal equilibrium $\beta^*$ given in Theorem 1. For simplicity, we assume that $\lambda_1 = \lambda_2 = \lambda$ (in this case $\rho^i = \lambda/\mu^i$). The following result states necessary and sufficient conditions under which the buyers establish contracts with the supplier with the shortest average delivery time, when choosing based on the late penalty fees.

**Proposition 4** At the Pareto optimal equilibrium $\beta^*$:

(a) Buyer 1 always selects the supplier providing the shortest average delivery time.

(b) If $\mu^2 < \mu^*$ ($\mu^2 > \mu^*$), buyer 2 selects the supplier providing the shortest average delivery time if and only if

$$\frac{2\rho^1}{1 - 2\rho^1} \leq (\geq) \frac{\rho^1}{1 - \rho^1} + \frac{\rho^2}{1 - \rho^2}.$$

Proposition 4 shows that, by selecting a supplier based on the late penalty fees, buyer 2 may end up sourcing from the supplier with the longer average delivery time (even if the supplier’s offer guaranteed the best fill rate). We now estimate how frequent this situation may be. For the case of equal volume requirements ($\lambda_1 = \lambda_2 = \lambda$), the set of feasible utilization rates $\rho^1$ and $\rho^2$ is given by $F \overset{\text{def}}{=} \{(\rho^1, \rho^2) : \rho^1 < \rho^2 \text{ and } 1/4 < \rho^i < 1/2, \ i = 1, 2\}$, with an area of $1/32$. (Note that $\rho^i > 1/4$ by assumption, $\rho^i < 1/2$ since $\lambda_1 + \lambda_2 = 2\lambda < \mu_i$, and $\rho^1 < \rho^2$ since $\mu^1 > \mu^2$. ) We have then created a grid with step $(0.01, 0.01)$ within $F$ to numerically compute how often buyer 2 ends up sourcing
from the supplier with the longer average delivery time. We observe that the area of the set \( \{ (\rho_1, \rho_2) \in F : \mu_2 < \mu^* \text{ and } \frac{2\rho_1}{1-2\rho_1} - \frac{\rho_1}{1-\rho_1} > \frac{\rho_2}{1-\rho_2}, \text{ or } \mu_2 > \mu^* \text{ and } \frac{2\rho_1}{1-2\rho_1} - \frac{\rho_1}{1-\rho_1} < \frac{\rho_2}{1-\rho_2} \} \) is approximately 26% of the area of the set \( F \). Furthermore, it turns out that \( \{ (\rho_1, \rho_2) \in F : \mu_2 > \mu^* \text{ and } \frac{2\rho_1}{1-2\rho_1} - \frac{\rho_1}{1-\rho_1} < \frac{\rho_2}{1-\rho_2} \} \) is empty. Thus, from this numerical estimation, we conclude that when \( \mu_2 > \mu^* \), buyer 2 selects the supplier with the shorter average delivery time, and 26% of the parameter combinations when \( \mu_2 < \mu^* \) lead to instances when buyer 2 experiences a longer average delivery time by selecting the supplier based on the backorder penalty offers.

5.4 Effect of Production Capacities on the Procurement Game

We first investigate how the capacity threshold \( \mu^* \) introduced in Theorem 1 depends on \( \mu_1 \) and how this dependence affects the outcome of the procurement game. We consider a setting where the two buyers have equal volume requirements (\( \lambda_1 = \lambda_2 \)). In this case, the capacity threshold is \( \mu^* = \lambda \exp \left( \frac{\ln(\lambda/\mu_1)}{\ln(2/\lambda_1)} \right) \). Further, consider the (normalized) case with \( \lambda_1 = \lambda_2 = 1 \) and \( \mu_1 \) varying from slightly above 2 up to 4, which corresponds to an utilization \( \rho_1 = \lambda/\mu_1 \) varying from 25% to slightly under 50%. Figure 6 below shows how \( \mu^* \) changes as a function of \( \mu_1 \). The thick line represents the upper limit on \( \mu_2 \) (since \( \mu_2 < \mu_1 \)). The gray area represents, for each \( \mu_1 \), the range of \( \mu_2 \) for which supplier 1 earns the procurement contracts of both buyers. The area between the curve and the thick line corresponds to values of \( \mu_2 \) for which supplier 1 earns a contract with buyer 1 and supplier 2 with buyer 2. (These conclusions follow from Theorem 1.) As \( \mu_1 \) increases to 4, the threshold \( \mu^* \) approaches \( \mu_1 \). Then, the capacity of supplier 2 has to be very close to that of supplier 1 in order to be able to make a competitive offer and earn the contract with buyer 2. That is, when the capacity of supplier 1 is so high that her utilization is small even when serving both buyers, supplier 2 can earn a contract with a buyer only if her capacity is close to \( \mu_1 \); otherwise, supplier 2 presents little competition to supplier 1. The exact same conclusions are valid for the case of \( \lambda_1 > \lambda_2 \).

We now study the dependence of supplier 1’s profit \( (\pi_1) \) on the capacity of supplier 2 \( (\mu_2^0) \) and on the price of the product required by buyer 2 \( (p_2) \). Here, again, we consider the case where \( \lambda_1 = \lambda_2 = 1 \). In addition, we fix \( \mu_1 = 3 \), \( h = 1 \), and \( p_1 = 2.5 \). The threshold introduced in Theorem 1 is \( \mu^* \approx 1.9 \). The price \( p_2 \) takes two possible values: 1.5 and 2.3.

Figure 7 below shows the (equilibrium) profit of supplier 1 as a function of \( \mu_2^0 \). Following
Theorem 1, supplier 1 establishes procurement contracts with both buyers for any value of $\mu^2$ below $\mu^*$, and with buyer 1 only for values of $\mu^2$ above $\mu^*$. It is interesting to note that an increase in supplier 2’s capacity implies a decrease in the equilibrium profit of supplier 1, even when the capacity for supplier 2 remains below the threshold value $\mu^*$. That is, an increase in supplier 2’s capacity makes the competition for the procurement contracts more intense and consequently supplier 1 requires larger penalty guarantees in order to earn both contracts.

For values of $\mu^2$ below the threshold $\mu^*$, the profit of supplier 1 is reduced as a consequence of a lower value of $p_2$. This is to be expected as, for $\mu^2 < \mu^*$, supplier 1 establishes contracts with both buyers. More interestingly, the same conclusion holds for values of $\mu^2$ above the threshold $\mu^*$. Indeed, when faced with a lower value of $p_2$ (and keeping $p_1$ fixed), supplier 2 has less interest in establishing a contract with buyer 2 and more interest in buyer 1’s procurement contract. Competition for buyer 1 becomes then more intense. Consequently, supplier 1 needs to raise the penalty guarantee offered to this buyer, sacrificing some of her profit.

### 5.5 Menu-Based Contracts

Throughout the paper, we have assumed that the prices $p_1$ and $p_2$ are exogenously determined. An interesting extension corresponds to the situation where late penalty fees and purchasing prices are contracted jointly. In this section, we briefly discuss supply contracts
that offer a menu of different combinations of service levels (or backorder penalties) and component purchase prices. Faced with such menu-based contracts, each buyer can select the service level/price combination that maximizes its utility. Because the volume of business required by a buyer influences the equilibrium level of service that suppliers are willing to offer the buyer (see Lemma 2 and Theorem 1), we restrict attention to settings in which $\lambda_1 = \lambda_2$, i.e., both buyers have similar volume requirements. (A more general, volume-dependent menu-based contract may be required when buyers are of significantly different sizes.) Following similar arguments as in Section 4, the equilibrium menu-based contracts of both suppliers need to be equal. Note that if a supplier offered a penalty rate/purchase price combination different from any pair in the other supplier’s menu, then the latter supplier could add to her menu the same penalty rate/purchase price combination with any of two dimensions slightly more favorable to the buyer and thus possibly attract this buyer’s business (as long as that combination implies a non-negative profit for this supplier).

Consider first the case $\mu^2 < \mu^*$ (see Theorem 1(a)). In this environment, supplier 1 has significantly larger capacity than supplier 2 and captures, in equilibrium, both buyers’ businesses. The suppliers can establish a menu (with a possibly finite number) of penalty rate/purchase price combinations of the form $(\beta^* = -\frac{\lambda}{\mu^2} p, p)$. Note that $\beta^*$ decreases with $p$ or, equivalently, the fill rate increases with the price (for a given price $p$, the fill rate equals $1 - e^{-\frac{\lambda}{\mu^2} p}$). Thus, higher service levels are associated with higher prices. In addition, a buyer’s service level depends only on the price it is willing to pay and it does not depend on the price paid by the other buyer, although both buyers “compete” for the higher inventory allocation priority.
The case $\mu^2 > \mu^*$ (corresponding to Theorem 1(b)) is significantly more complex. Here, for any given pair of purchase prices, each supplier captures the business of one of the buyers. For any $p_1 \geq p_2$, the equilibrium penalties are $\beta^*_1 = -(p_1 - p_2)\frac{\lambda}{v^2} + \beta^*_2$ and $\beta^*_2 = p_2 \frac{\lambda}{h\gamma - v}$, and supplier $i$ establishes a contract with buyer $k = i$. In this setting, the buyer with the higher willingness to pay (or with the deeper pockets) – say, buyer 1 – would establish a contract with supplier 1. In contrast to the case $\mu^2 < \mu^*$, buyer 1’s service level now depends on the price paid by the other buyer. That is, contingent on that price $p_2$, buyer 1 could select a combination from a menu of the form $(p_2 \left(\frac{\lambda}{h\gamma - v} + \frac{\lambda}{v}\right) - \frac{\lambda}{\gamma} p_1, p_1)$, for a set of prices $p_1 \geq p_2$. Again, a higher price corresponds to a higher service level.

A complete characterization of the buyers' choice of penalty rate/purchase price combination would require an understanding of their sensitivities to prices and sourcing delays. In general, the buyer willing to pay more would receive the higher fill rate, and if both buyers source from the same supplier, it would receive priority in terms of inventory availability.

### 5.6 The Case of Multiple Buyers

In this section, we briefly explore the more general case of $n > 2$ buyers looking to establish a procurement contract with one of two suppliers. In this case, the equilibrium of the procurement game can again be defined as the limit equilibrium of $\epsilon$-procurement games as $\epsilon \to 0$. The result in Proposition 3 can be easily extended to the case of more than two buyers. As a result, for any equilibrium $\beta^* = (\beta^*_1, \ldots, \beta^*_n)$ of the procurement game the following condition is satisfied:

(C) There exist sets $X^1 \in d^1(\beta^*)$ and $X^2 \in d^2(\beta^*)$ such that $X^1 \cap X^2 = \emptyset$, where $X^i$ is a subset of $\Omega_C = \{1, \ldots, n\}$.

(Recall that this is only a necessary condition for an equilibrium of the procurement game.) For the general case of $n > 2$ buyers, we use the following heuristic to compute a Pareto optimal equilibrium of the procurement game. We create a finite grid in the space $\{\beta \in \mathbb{R}^n : \beta \leq 0\}$, and for each point in the grid we check whether (C) is satisfied, or not. Among all the points in the grid that satisfy (C), we then look for the one that maximizes both suppliers' profits, if such a point exists. (Note that the heuristic implicitly assumes that this point is the limit equilibrium of some $\epsilon$-procurement games, as for the case of $n = 2$ in Theorem 1.)
We now consider five different scenarios for the case of \( n = 3 \) buyers and use the above heuristic to compute the equilibrium of the procurement game. Table I below shows the parameters for the different scenarios. For each of them, the holding cost and the production capacity of supplier 2 are taken equal to one. We also fix \( \lambda_1 = 0.2, \lambda_2 = 0.3, \lambda_3 = 0.4 \) and \( p_1 = 20 \). The table also provides the equilibrium fill rates (corresponding to the Pareto optimal equilibrium offers) of the procurement game: \( f_k^*, k = 1, 2, 3 \). In addition, the table displays the set of buyers \( X^i \) that select supplier \( i \) under the offers given by the Pareto optimal equilibrium, and the suppliers’ equilibrium profits.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>( \mu_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
<th>( f_1^* )</th>
<th>( f_2^* )</th>
<th>( f_3^* )</th>
<th>( X^1 )</th>
<th>( X^2 )</th>
<th>( \pi_{X_1}(B^*) )</th>
<th>( \pi_{X_2}(B^*) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2</td>
<td>10</td>
<td>10</td>
<td>99.83%</td>
<td>93.92%</td>
<td>95.02%</td>
<td>{1, 2}</td>
<td>{3}</td>
<td>1.79</td>
<td>0.73</td>
</tr>
<tr>
<td>2</td>
<td>1.4</td>
<td>13.33</td>
<td>10</td>
<td>99.75%</td>
<td>98.50%</td>
<td>94.50%</td>
<td>{1, 2}</td>
<td>{3}</td>
<td>3.00</td>
<td>0.84</td>
</tr>
<tr>
<td>3</td>
<td>1.45</td>
<td>13.33</td>
<td>10</td>
<td>99.78%</td>
<td>98.77%</td>
<td>95.50%</td>
<td>{1, 2}</td>
<td>{3}</td>
<td>3.01</td>
<td>0.62</td>
</tr>
<tr>
<td>4</td>
<td>1.5</td>
<td>13.33</td>
<td>12</td>
<td>99.86%</td>
<td>98.99%</td>
<td>98.77%</td>
<td>{1, 3}</td>
<td>{2}</td>
<td>2.91</td>
<td>0.18</td>
</tr>
<tr>
<td>5</td>
<td>1.5</td>
<td>13.33</td>
<td>15</td>
<td>99.85%</td>
<td>98.34%</td>
<td>99.73%</td>
<td>{1, 3}</td>
<td>{2}</td>
<td>3.26</td>
<td>0.59</td>
</tr>
</tbody>
</table>

Table I: Parameters and Results for \( n = 3 \) Firms

In all of these examples, and as in the case of two buyers, supplier 1 earns the contract corresponding to the highest revenue rate \( p_k \lambda_k \) (the cost of offering high backorder penalties is less for supplier 1 since she possesses the higher production capacity). However, supplier 1 may not always earn all of highest revenue contracts. For instance, in scenario 1, supplier 1 earns contracts with buyers 1 and 2, even though \( 4 = p_3 \lambda_3 > p_2 \lambda_2 = 3 \). In scenarios 2 and 3 all buyers are characterized by the same revenue rates, but supplier 1 earns the contracts with the buyers having the lowest volume requirement - highest price combination.

Comparing scenarios 2 and 3, note that an increase in supplier 1’s capacity leads to a lower profit for supplier 2 due to the increased competition posed by supplier 1 (this was also observed in §5.4 for the case of two buyers). In addition, this increase in capacity leads to higher fill rates for all buyers. Finally, it is interesting to see that in scenarios 4 and 5 buyer 1’s fill rate with supplier 1 is larger than that of buyer 3’s, even though \( 4 = p_1 \lambda_1 < p_3 \lambda_3 = 4.8 \) in scenario 4 and \( 4 = p_1 \lambda_1 < p_3 \lambda_3 = 6 \) in scenario 5 (note that \( 20 = p_1 > p_3 = 12 \) in scenario 4 and \( 20 = p_1 > p_3 = 15 \) in scenario 5). On the operating level, this implies that part of supplier 1’s inventory is reserved for buyer 1’s orders.
6 Conclusions

In this paper, we introduce a new framework to address the competition for procurement contracts between suppliers. As it is common for customized components, each buyer seeks to establish a procurement relationship with a single supplier to avoid the duplicate efforts involved in finding and dealing with multiple suppliers. For the buyers, the most important factor determining the attractiveness of a supplier is her ability to deliver the components on time. The suppliers then compete for the buyers’ businesses by offering them service delivery commitment guarantees, giving rise to the procurement game. The suppliers operate as make-to-stock, and we provide a detailed model of how the suppliers manage their production and inventory operations.

Since the competition between the suppliers is all-or-nothing, the procurement game fails to have a pure strategy Nash equilibrium. Instead, following the economics literature on games with discontinuous payoffs, we show that an equilibrium in the procurement game can be defined as the limit equilibrium of discrete action games. This allows us to describe the equilibrium penalty offers and the resulting equilibrium choices of the buyers. We show that the outcome of the procurement game depends on which supplier possesses the larger production capacity: this supplier is always able to attract the buyer with the higher revenue rate. We also derive insights regarding the effect that changes in capacity have in the outcome of the procurement game. In addition, we show how competition affects the production control policies of the suppliers. Finally, the buyers select a supplier based on their service delivery commitment, as given by a penalty fee for late deliveries. This allows the buyers to obtain the highest possible fill rate and, in most cases, the lowest average delay time for deliveries.

Although we provide analytical results for two buyers only, we believe that our approach could be extended to the more general case of \( n \) buyers. The suppliers’ optimal choice of buyers would then be characterized by regions in the \( n \)-dimensional offer space. As mentioned in Section 5, the necessary condition for an equilibrium of the procurement game stated in Proposition 3 can be extended to the general case of \( n \) buyers. However, a formal proof identifying the Pareto optimal equilibrium (and the sequence of pure strategy Nash equilibria in the \( \epsilon \)-procurement games) would be overly extensive.

This paper assumes that buyers outsource the production of specialized components. For the case of non-specialized components, the buyers may want to source from multiple
suppliers and split their purchase requirements among the suppliers based on their service levels. Such an analysis would require a model for the buyers’ operations, in addition to the model for the suppliers’ production and inventory decisions. Finally, one could consider a discrete time multi-period problem, where some of the buyers change at the beginning of each period (new buyers arrive, while some existing contracts terminate). The suppliers’ profit functions within each period can be approximated by the long-run average profit of the steady-state regime. Based on these approximate profit functions, contracts are established in each period as a result of the procurement game we consider in this paper.

7 References


Appendix

Proof of Proposition 1. Consider the optimal allocation policy. The probability distribution of the on-hand inventory $S_0$ is then equal to (see de Véricourt et al. 2001, Appendix 7)

$$P(S_0 = s) = \prod_{q=k+1}^{n} (\tilde{\rho}_q)^{z^{i}_q - z^{i}_{q-1}} (1 - \tilde{\rho}_k)^{\gamma^i_k d^i_k - s},$$

for $z^i_{k-1} < s \leq z^i_k$, where $\tilde{\rho}_k$ is defined in (3). We can then compute the fill rate $f^i_k(B^i)$,

$$f^i_k(B^i) = 1 - P(S_0 \leq z^i_k)$$

$$= 1 - \prod_{q=k}^{n} (\tilde{\rho}_q)^{z^{i}_q - z^{i}_{q-1}}$$

and the result thus follows from (4). 

Proof of Proposition 2. The first part follows from Lemma 1 and the fact that $\gamma^i < 0$. We now present a detailed proof of the second and third parts.
By definition, \( \{k\} \in d^i(\beta_1, \beta_2^i) \) if and only if \( \pi_{\{k\}}^i(\beta_1, \beta_2^i) \) is larger than or equal to 0, \( \pi^i_{\{3-k\}}(\beta_1, \beta_2^i) \) and \( \pi^i_{\{1,2\}}(\beta_1, \beta_2^i) \). From Lemma 1, we have

\[
\pi^i_{\{k\}}(\beta_1, \beta_2^i) \geq 0 \iff \beta_k^i \geq r_k^i,
\]

(18)

\[
\pi^i_{\{k\}}(\beta_1, \beta_2) \geq \pi^i_{\{3-k\}}(\beta_1, \beta_2^i) \iff \beta_k^i \geq \frac{v_{3-k}^i}{v_k^i}(\beta^i_{3-k} - r_k^i) + r_k^i.
\]

(19)

On the other hand, since \( \gamma^i \) and \( h \gamma^i - v_k^i \) are negative, and using Lemma 1 again, we can derive the last inequality:

\[
\pi^i_{\{k\}}(\beta_1, \beta_2^i) \geq \pi^i_{\{1,2\}}(\beta_k^i, \beta^i_{3-k}) \iff 0 \geq \pi^i_{\{3-k\}}(\beta_1, \beta_2^i) - h \gamma^i \max(\beta_1^i, \beta_2^i),
\]

\[
\iff \begin{cases} 0 \geq v_{3-k}^i(\beta^i_{3-k} - r_k^i) - h \gamma^i \beta_k^i, & \text{if } \beta_k^i > \beta^i_{3-k} \\ 0 \geq (v_{3-k}^i - h \gamma^i)\beta^i_{3-k} - v_k^i r^i_{3-k}, & \text{if } \beta_k^i \leq \beta^i_{3-k} \end{cases}
\]

\[
\iff \begin{cases} \frac{v_{3-k}^i}{h \gamma^i}(\beta^i_{3-k} - r_k^i) \geq \beta_k^i, & \text{if } \beta_k^i > \beta^i_{3-k} \\ \beta^i_{3-k} \geq \beta_k^i, & \text{if } \beta_k^i \leq \beta^i_{3-k} \end{cases}
\]

(20)

When \( \beta_k^i > \beta^i_{3-k} \), we have from (20) that

\[
\frac{v_{3-k}^i}{h \gamma^i}(\beta^i_{3-k} - r_k^i) \geq \beta_k^i \iff \beta^i_{3-k} = \frac{v_{3-k}^i r^i_{3-k}}{v_k^i - h \gamma^i} \geq \beta_k^i.
\]

Similarly, it is easy to verify that \( v_{3-k}^i/(h \gamma^i)(\beta^i_{3-k} - r_k^i) \geq \beta_k^i \) when \( \beta_k^i \leq \beta^i_{3-k} \). From (18), (19) and (20) we can then deduce

\[
\{k\} \in d^i(\beta_1, \beta_2^i) \iff \begin{cases} \max[r_k^i, \frac{v_{3-k}^i}{v_k^i}(\beta^i_{3-k} - r_k^i) + r_k^i] \leq \beta_k^i \leq \frac{v_{3-k}^i}{h \gamma^i}(\beta^i_{3-k} - r_k^i) \\ \beta^i_{3-k} \leq \beta^i_{3-k} \end{cases}
\]

(21)

Now fix \( k \), and consider the case \( \beta_k^i \geq \beta^i_{3-k} \). From the first condition in (21), we have that

\[
\frac{v_{3-k}^i}{v_k^i}(\beta^i_{3-k} - r_k^i) + r_k^i \leq \frac{v_{3-k}^i}{h \gamma^i}(\beta^i_{3-k} - r_k^i) \iff \beta^i_{3-k} \leq \beta^i_{3-k}.
\]

On the other hand, \( \beta_k^i \geq \beta^i_{3-k} \) implies that \( \beta^i_{3-k} \geq \beta^i_{3-k} \). Then,

\[
\{k\} \in d^i(\beta^i_{3-k}, \beta^i_{3-k}) \iff \max \left\{ r_k^i, \frac{v_{3-k}^i}{v_k^i}(\beta^i_{3-k} - r_k^i) + r_k^i \right\} \leq \beta_k^i \leq \frac{v_{3-k}^i}{h \gamma^i}(\beta^i_{3-k} - r_k^i).
\]

(22)

The points \((\beta_1, \beta_2)\) such that \( \{1, 2\} \in d^i(\beta_1, \beta_2^i) \) are the ones that are not covered in any of the other three cases: \( d^i(\beta_1, \beta_2^i) = \emptyset, \{1\}, \) or \( \{2\} \). It is then straightforward to verify that those points are represented by the conditions in (11). ■
Proof of Proposition 3. Consider an \( \epsilon \)-procurement game and let \( \beta(\epsilon) = (\beta_1 + n_1^1 \epsilon, \beta_2 + n_2^1 \epsilon, \beta_1 + n_1^2 \epsilon, \beta_2 + n_2^2 \epsilon) = (\beta + n_1^1 \epsilon, \beta + n_2^2 \epsilon) \) be a pure strategy Nash equilibrium of the \( \epsilon \)-procurement game. It is easy to verify that \( |n_k^1 - n_k^2| \leq 1 \) for \( k = 1, 2 \). Define

\[
X^i = \{ k : n_k^i < n_k^j, j \neq i \}, \ i = 1, 2,
\]

and \( \pi^i(\beta|\beta') = \pi_Y^i(\beta') \), with \( Y = \{ k \in \Omega_C : \beta_k^i < \beta_k^j \} \). Note that \( \pi^i(\beta + n_1^1 \epsilon|\beta + n_2^2 \epsilon) = \pi_{X^i}(\beta + n_1^1 \epsilon) \), from the definition of \( X^i \). Since \( \beta(\epsilon) \) is a Nash equilibrium, we have that \( \pi_{X^i}(\beta + n_1^1 \epsilon) \geq \pi_{X^1}(\beta + n_1^1 \epsilon) \) for any vector \( n \). In particular, for each set \( Z = \emptyset, \{1\}, \{2\} \) or \( \{1, 2\} \subset \Omega_C \), we can find a vector \( n_Z \) such that \( \pi_1(\beta + n_2^2 \epsilon) = \pi_Z^1(\beta + n_2^2 \epsilon) \). Then, for any \( Z = \emptyset, \{1\}, \{2\} \) or \( \{1, 2\} \), \( \pi_{X^i}(\beta + n_1^1 \epsilon) \geq \pi_Z^i(\beta + n_2^2 \epsilon) \) for \( \epsilon \) small which, by (P1), implies that \( \pi_{X^i}(\beta) \geq \pi_Z^i(\beta) \). Therefore, \( X^1 \in d^i(\beta) \) and, by their definition, \( X^1 \cap X^2 = \emptyset \).

Proof of Lemma 2. First note the following:

- \( \mu^* < \mu^1 \), since \( \gamma^1 < 0 \);
- \( v_k^1 < v_k^2 \), \( k = 1, 2 \), since \( \mu^1 > \mu^2 \);
- \( r_k^1 < r_k^2 \), \( k = 1, 2 \), since \( \mu^1 > \mu^2 \);
- \( \gamma^1 > \gamma^2 \), since \( \mu^1 > \mu^2 \) and \( \lambda_1 \geq \lambda_2 \).

In addition, using simple algebra, it is easy to show that

\[
\mu^2 < (\geq) \mu^* \text{ if and only if } h\gamma^1 > (\leq) v_2^1 - v_2^2 \text{ if and only if } r_2^2 > (\leq) \beta_2^1. \tag{23}
\]

To show Lemma 2, we need to compute the sets \( d^i(\beta^*) \), \( i = 1, 2 \), following Proposition 2. To that end, we first show that \( \beta_1^1 \leq \beta_2^2 \), \( i = 1, 2 \). We then prove parts (a) and (b) of the lemma separately. We begin by showing that \( \beta_1^2 < \beta_2^2 \leq \beta_2^1 \):

\[
\beta_1^2 < \beta_2^1 \iff (v_2^1 - v_1^2) \beta_2^1 < p_1 \lambda_1 - p_2 \lambda_2 \iff \frac{v_2^1 - h\gamma^1 + v_1^2 - v_2^2}{v_2^1 - h\gamma^1} < \frac{p_1 \lambda_1}{p_2 \lambda_2},
\]

which holds from (12). At the same time,

\[
\beta_1^2 = \frac{h\gamma^1}{v_1^1} \beta_1^1 + r_1^1 < \frac{v_2^1}{v_1^1} (\beta_2^1 - r_2^2) + r_1^1 = \beta_2^1 \iff \left( h\gamma^1 - \frac{v_2^1}{v_1^1} \right) \beta_2^1 \leq r_2^2 - r_1^1 - \frac{v_2^1}{v_1^1} r_2^2.
\]

This implies

\[
\left( h\gamma^1 - \frac{v_2^1}{v_1^1} \right) \beta_2^1 \leq \frac{h\gamma^1 v_1^1 - v_2^1 v_1^1}{v_2^1 - h\gamma^1} \leq \frac{p_1 \lambda_1}{p_2 \lambda_2}.
\]
We then have that
\[
\frac{v_1^2(v_2^2 - v_1^2) + h \gamma^1 (v_1^3 - v_2^3)}{(v_2^3 - v_1^3)(v_1^3 - v_2^3)} < \frac{v_1^3(v_2^3 - v_1^3) + h \gamma^1 (v_1^4 - v_2^4)}{(v_2^4 - v_1^4)(v_1^4 - v_2^4)} =
\]
\[
\frac{v_1 - h \gamma^1}{v_2 - h \gamma^1} < \frac{v_1^3 - h \gamma^1 + v_2^2 - v_2^3}{v_2^4 - h \gamma^1} \leq \frac{p_1 \lambda_1}{p_2 \lambda_2},
\]
(24)

where the first and second inequalities follow since \(v_2^4 - v_1^4 < v_2^3 - v_1^3\), and the last inequality follows from (12). Thus,
\[
\hat{\beta}_1 < \hat{\beta}_2 < \hat{\beta}_2.
\]
(25)

Note that (24) also implies that
\[
\hat{\beta}_2 \geq \hat{\beta}_1.
\]
(26)

In addition, we have that
\[
\frac{v_1^2 - h \gamma^2}{v_2^2 - h \gamma^2} = \frac{\ln(1 + \frac{\lambda_1}{\lambda_2}) \ln \rho_2^2}{\ln(1 + \frac{\lambda_1}{\lambda_2}) \ln \rho_2^2} < \frac{\ln(1 + \frac{\lambda_1}{\lambda_2}) \ln \rho_1^2}{\ln(1 + \frac{\lambda_1}{\lambda_2}) \ln \rho_2^2} = \frac{v_1^3 - h \gamma^1}{v_2^4 - h \gamma^1} \leq \frac{p_1 \lambda_1}{p_2 \lambda_2},
\]
where the first inequality follows since \(\mu^2 < \mu^1\) and the second inequality follows from (24). This implies that
\[
\hat{\beta}_2 \geq \hat{\beta}_1.
\]
(27)

We now prove separately parts (a) and (b).

(a) Since \(\mu^2 < \mu^*\), we have that \(v_1^1 - v_1^2 < v_1^3 - v_2^2 < h \gamma^1\), which, in turn, implies that
\[
\frac{v_1^1 - h \gamma^1 + v_1^2}{v_1^3 - h \gamma^1} > \frac{v_1^1 - h \gamma^1}{v_1^3 - h \gamma^1} > \frac{v_1^1 - h \gamma^1}{v_1^3 - h \gamma^1} \frac{v_1^2}{v_1^3 - v_1^2}.
\]
(28)

Observe now that
\[
\hat{\beta}_1 < r_1^2 \text{ if and only if } -\frac{h \gamma^1}{v_1^3 - h \gamma^1} \frac{v_1^2}{v_1^3 - v_1^2} < \frac{p_1 \lambda_1}{p_2 \lambda_2},
\]
where the latter holds from (12) and (28). Define then \(\beta^* = (r_1^2, r_2^2)\). Note that
\[
r_1^2 < r_2^2 \iff \frac{p_1 \lambda_1}{p_2 \lambda_2} > \frac{v_1^2}{v_2^2}, \text{ and } \frac{v_2^2 - h \gamma^1 + v_1^2 - v_2^2}{v_1^3 - h \gamma^1} > \frac{v_1^2}{v_1^3 - v_1^2} \iff v_2^2 - v_2^2 < h \gamma^1.
\]
Then, the fact that \(\mu^2 < \mu^*\) together with (12) imply that \(r_1^2 < r_2^2\).
Consider first supplier $i = 1$. We now show that $\beta^*$ satisfies the second condition in (11) for $k = 2$ (from (26) and since $\beta^*_1 = r^1_1 < r^2_2 = \beta^*_2$). Observe that

$$
\frac{v^1_1}{h\gamma^1} (r^2_1 - r^1_1) = \frac{p_1 \lambda_1 v^1_2 - v^1_1}{h\gamma^1 v^1_1} = -\frac{v^1_2 - v^1_1}{v^1_1} - h\gamma^1 \frac{p_1 \lambda_1}{p_2 \lambda_2} \beta^*_1 < \beta^*_2,
$$

from (12), (28), and the fact that $\beta^*_1 < 0$. Then, it follows that

$$
\max \left\{ \frac{v^1_1}{h\gamma^1} (r^2_1 - r^1_1), \beta^*_1 \right\} = \beta^*_1 < r^2_2.
$$

This also shows that the second condition in (9) and the first condition in (10) are not satisfied. Finally, the condition in (8) is not satisfied since $r^1_k < r^2_k$, $k = 1, 2$. Therefore, $d^1(\beta^*) = \{1, 2\}$.

We now investigate supplier $i = 2$’s optimal choice for $\beta^*$. First, it is clear that $\emptyset \in d^2(\beta^*)$. Also,

$$
\max \left\{ r^1_1, \frac{v^2_2}{v^1_2} (r^2_2 - r^1_2) + r^2_2 \right\} = r^1_1 \leq \frac{v^2_2}{h\gamma^2} (r^2_2 - r^1_2) = 0, \quad \text{and} \quad r^2_2 \leq \beta^*_2 = \frac{v^2_2 r^2_2}{v^2_2 - h\gamma^2},
$$

thus satisfying the conditions in (9). Then, $\{1\} \in d^2(\beta^*)$. In addition,

$$
\max \left\{ r^2_2, \frac{v^2_2}{v^2_1} (r^2_1 - r^2_1) + r^2_2 \right\} = r^2_2 \leq \frac{v^2_1}{h\gamma^2} (r^2_1 - r^1_1) = 0,
$$

which implies $\{2\} \in d^2(\beta^*)$ (see (10)). Then, $d^2(\beta^*) = \emptyset, \{1, 2\}$.

(b) From the definitions of $\hat{\beta}^*_1$ and $\hat{\beta}^*_2$, note that $\pi^2_2(\hat{\beta}^*_2) = \pi^1_1(\beta^*_1)$. Define now $\beta^* = (\hat{\beta}^*_1, \hat{\beta}^*_2)$. Following the statement of Proposition 2, it is first clear that $\emptyset \not\in d^1(\beta^*)$ since $r^1_2 < r^2_2 < \beta^*_2$, where the first inequality follows since $\mu^1 > \mu^2$ and the second inequality from (23). This also shows that $\emptyset \not\in d^2(\beta^*)$.

Consider first supplier $i = 1$. Since $\beta^*_2 = \hat{\beta}^*_2$, the second part of (9) trivially holds. For the first part, note that

$$
\max \left\{ r^1_1, \frac{v^2_2}{v^1_1} (\hat{\beta}^*_2 - r^1_2) + r^1_2 \right\} = \frac{v^2_2}{v^1_1} (\hat{\beta}^*_2 - r^1_2) + r^1_1 = \frac{h\gamma^1}{v^1_1} \hat{\beta}^*_2 + r^1_1 = \hat{\beta}^*_2 \leq \hat{\beta}^*_1 = \beta^*_1 \leq \frac{v^1_2}{h\gamma^1} (\hat{\beta}^*_2 - r^2_2),
$$

where the first equality follows since $\hat{\beta}^*_2 > r^1_2$ (as $\gamma^1 < 0$), the second and last equalities follow from the definition of $\hat{\beta}^*_2$, and the inequalities follow from (25). Thus, $\{1\} \in d^1(\beta^*)$.

In addition, observe that

$$
\frac{v^1_1}{h\gamma^1} (\hat{\beta}^*_2 - r^1_1) < \hat{\beta}^*_1 \iff \hat{\beta}^*_1 < \beta^*_1 < \hat{\beta}^*_2,
$$

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where the latter holds from \(25\). This implies that \(\{2\} \not\in d^1(\beta^*)\) (see (10)), and that \(\{1, 2\} \in d^1(\beta^*)\) (see the second part of (11), as \(\beta^*_1 = \hat{\beta}^*_1 < \hat{\beta}^*_2 = \beta^*_2\)). Therefore, \(d^1(\beta^*) = \{\{1, 2\}\} \). Consider now supplier \(i = 2\). It is easy to verify that, since \(\mu^1 > \mu^2\), \(\hat{\beta}^1_2 < \hat{\beta}^2_2\). We now show that both parts of (9) hold (since \(\hat{\beta}^2_1 \geq \hat{\beta}^2_2\) from (27)). The second part is exactly \(\hat{\beta}^1_2 < \hat{\beta}^2_2\). For the first part, note that

\[
\max \left\{ r^2_1, \frac{v^2_2}{v^2_1} (\hat{\beta}^1_2 - r^2_2) + r^2_1 \right\} = \hat{\beta}^1_2 \leq \hat{\beta}^2_2 \leq \frac{v^2_2}{h\gamma^2} (\hat{\beta}^1_2 - \hat{\beta}^2_2) + \hat{\beta}^2_2 = \frac{v^2_2}{h\gamma^2} (\hat{\beta}^1_2 - r^2_2),
\]

where the first equality follows from the definition of \(\hat{\beta}^1_2\) and from (23), the second inequality since \(\hat{\beta}^1_2 < \hat{\beta}^2_2\) and \(\gamma^2 < 0\), and the last equality from the definitions of \(\hat{\beta}^1_2\) and \(\hat{\beta}^2_2\). Thus, \(\{1\} \in d^2(\beta^*)\). Similarly, to show that the first part of (10) holds, note that

\[
\max \left\{ r^2_2, \frac{v^2_2}{v^2_1} (\hat{\beta}^1_2 - r^2_1) + r^2_2 \right\} = \max \left\{ r^2_2, \hat{\beta}^1_2 \right\} = \hat{\beta}^1_2 < \frac{v^2_1}{h\gamma^2} (\hat{\beta}^1_2 - r^2_1) = \frac{v^2_2}{h\gamma^2} (\hat{\beta}^2_2 - r^2_2),
\]

where the first equality follows from the definition of \(\hat{\beta}^1_2\), the second equality from (23), and the inequality, again, since \(\hat{\beta}^1_2 < \hat{\beta}^2_2\) and \(\gamma^2 < 0\). This implies that \(\{2\} \in d^1(\beta^*)\). Finally, it is easy to verify from (11) that \(\{1, 2\} \not\in d^2(\beta^*)\) since \(\hat{\beta}^1_2 < \hat{\beta}^2_2\). Then, \(d^2(\beta^*) = \{\{1\}, \{2\}\}\).

**Proof of Theorem 1.** Observe first that

\[
\pi^i_{\{1\}}(\beta_k + x) = \pi^i_k(\beta_k) + v^i_k x, \\
\pi^i_{\{1, 2\}}(\beta_1 + x, \beta_2 + x) = \pi^i_{\{1, 2\}}(\beta_1, \beta_2) + x(v^i_1 + v^i_2 - \gamma^i h),
\]

and \(v^i_1 + v^i_2 - \gamma^i h > 0\).

For both parts (a) and (b) of the theorem, we first show that \(\beta^*\) is an equilibrium of the procurement game. We then show that it is also the only Pareto optimal equilibrium.

(a) Consider \(\beta^*\) as defined in Lemma 2(a) and recall that \(d^1(\beta^*) = \{\{1, 2\}\}\) and \(d^2(\beta^*) = \{\emptyset, \{1\}, \{2\}\}\). We first show that, for \(\epsilon > 0\) sufficiently small, \(\beta(\epsilon) = (\beta^*_1, \beta^*_2, \beta^*_1 + \epsilon, \beta^*_2 + \epsilon)\) is a pure strategy Nash equilibrium of the \(\epsilon\)-procurement game. (Supplier 1’s strategy is \(\beta^1(\epsilon) = (\beta^*_1, \beta^*_2)\) and supplier 2’s strategy is \(\beta^2(\epsilon) = (\beta^*_1 + \epsilon, \beta^*_2 + \epsilon)\).)

Observe that deviating any (or both) \(\beta^1_k(\epsilon) = \beta^*_k\) to any value \(\beta^*_k - n \epsilon\) for \(n \geq 0\) would only decrease the profit of supplier 1, by \((P^1)\). Deviating any (or both) \(\beta^2_k(\epsilon)\) to any value \(\beta^*_k + n \epsilon\) for \(n > 0\), would make supplier 1 lose buyer \(k\) and be worse off since \(d^1(\beta^*) = \{\{1, 2\}\}\). Also, it is clear that any combination of these deviations will lead to lower profit for supplier 1.
If supplier 1 chose a strategy of the form $\beta^1_k(\epsilon) = \beta^*_k + \epsilon$ and $\beta^1_{3-k}(\epsilon) = \beta^*_3 - k - n\epsilon$ with $n \geq 0$, then her expected profit would be $\frac{1}{2}\pi^1_{(1,2)}(\beta^*_k + \epsilon, \beta^*_3 - n\epsilon) + \frac{1}{2}\pi^1_{(3-k)}(\beta^*_3 - k - n\epsilon) \leq \frac{1}{2}\pi^1_{(1,2)}(\beta^*_k + \epsilon, \beta^*_3) + \frac{1}{2}\pi^1_{(3-k)}(\beta^*_3 - k) < \pi^1_{(1,2)}(\beta^*_k, \beta^*_3)$, where the first inequality follows by $(P'_2)$, and since $\{3-k\} \not\subseteq d^1(\beta^*) = \{(1,2)\}$, $\pi^1_X$ is continuous by $(P_1)$ and $\epsilon$ is small. Similarly, if supplier 1 deviated to a strategy of the form $\beta^1_k(\epsilon) = \beta^*_k + \epsilon$ and $\beta^1_{3-k}(\epsilon) = \beta^*_3 - k + n\epsilon$ with $n > 1$, then her expected profit would be $\frac{1}{2}\pi^1_{(1,k)}(\beta^*_k + \epsilon) + \frac{1}{2}\pi^1_{(3-k)}(\beta^*_3 - k - n\epsilon)$, which, as before, is clearly less than $\pi^1_{(1,2)}(\beta^*_k, \beta^*_3)$ for $\epsilon$ small. Finally, if supplier 1 chose $\beta^1_k(\epsilon) = \beta^*_k + \epsilon$, $k = 1, 2$, her expected profit, using (29), would be $\frac{1}{4}\pi^1_{(1,2)}(\beta^*_1 + \epsilon, \beta^*_2 + \epsilon) + \frac{1}{4}\pi^1_{(1,3)}(\beta^*_1 + \epsilon) + \frac{1}{4}\pi^1_{(2,3)}(\beta^*_2 + \epsilon) = \frac{1}{4}\pi^1_{(1,2)}(\beta^*_1, \beta^*_2) + \frac{1}{4}\pi^1_{(1,3)}(\beta^*_1) + \frac{1}{4}\pi^1_{(2,3)}(\beta^*_2) + \epsilon^2(2v_1^1 + 2v_2^1 - \gamma_1 h) < \frac{2}{4}\pi^1_{(1,2)}(\beta^*) + \epsilon^2(2v_1^1 + 2v_2^1 - \gamma_1 h) < \pi^1_{(1,2)}(\beta^*)$, for $\epsilon$ small, where the first inequality comes again from $d^1(\beta^*) = \{(1,2)\}$.

For supplier 2, deviating any (or both) $\beta^2_k(\epsilon) = \beta^*_k + \epsilon$ to any value $\beta^*_k + n\epsilon$ for $n \geq 1$ does not change her profit. If supplier 2 changed any (or both) $\beta^2_k(\epsilon)$ to any value $\beta^*_k - n\epsilon$ for $n \geq 1$, then her expected profit would decrease since $\emptyset \in d^2(\beta^*)$. Clearly, any combination of these deviations will lead to no higher profit for supplier 2.

If supplier 2 chose offers of the form $\beta^2_k(\epsilon) = \beta^*_k$ and $\beta^2_{3-k}(\epsilon) = \beta^*_3 - k + n\epsilon$ with $n \geq 1$, her expected profit would be $\frac{1}{2}\pi^2_{(1,k)}(\beta^*_k) \leq 0$, since $\emptyset \in d^2(\beta^*_1, \beta^*_2)$. If supplier 2 chose $\beta^2_k(\epsilon) = \beta^*_k$ and $\beta^2_{3-k}(\epsilon) = \beta^*_3 - k - n\epsilon$ with $n \geq 1$, her expected profit would be $\frac{1}{2}\pi^2_{(1,2)}(\beta^*_k, \beta^*_3 - k) + \frac{1}{2}\pi^2_{(3-k)}(\beta^*_3 - k - n\epsilon) \leq \frac{1}{2}\pi^2_{(1,2)}(\beta^*_k, \beta^*_3) + \frac{1}{2}\pi^2_{(3-k)}(\beta^*_3 - k) \leq 0$, where the first inequality follows by $(P'_2)$ and the second one, again, since $\emptyset \in d^2(\beta^*)$.

Finally, if supplier 2 deviated to $\beta^2_k(\epsilon) = \beta^*_k$, $k = 1, 2$, her expected profit would be $\frac{1}{4}\pi^2_{(1,2)}(\beta^*) + \frac{1}{4}\pi^2_{(1,3)}(\beta^*) + \frac{1}{4}\pi^2_{(2,3)}(\beta^*) \leq 0$, since $\emptyset \in d^2(\beta^*)$.

This proves that, for $\epsilon$ small, $\beta(\epsilon)$ is a pure strategy Nash equilibrium of the $\epsilon$-procurement game. Thus, by definition, $\beta^*$ is an equilibrium of the procurement game. Consider now the following four sets $\Omega_i = \{x \in \mathbb{R}^2_+: x \geq \beta^*, x \neq \beta^*\}$, $\Omega_{ii} = \{x \in \mathbb{R}^2_+: x_1 < \beta^*_1, x_2 > \beta^*_2\}$, $\Omega_{iii} = \{x \in \mathbb{R}^2_+: x_1 > \beta^*_1, x_2 < \beta^*_2\}$ and $\Omega_{iv} = \{x \in \mathbb{R}^2_+: x \leq \beta^*, x \neq \beta^*\}$. For $x \in \Omega_i$, $d^1(x) = \{(1,2)\}$ while $d^2(x) \subseteq \{(1,2), (1,2)\}$ for $x \in \Omega_{ii}$, $d^1(x) \subseteq \{(2)\}$ while $d^2(x) = \{(1)\}$. (The characterization of $d^1$ and $d^2$ on $\Omega_i$, $\Omega_{ii}$ and $\Omega_{iii}$ follow from similar arguments as those in the proof of Lemma 2. See also Figure 3.) Thus, Proposition 3 implies that no $x \in \Omega_i \cup \Omega_{ii} \cup \Omega_{iii}$ can be an equilibrium of the procurement game. On the other hand, take any possible equilibrium $x \in \Omega_{iv}$. Then $\pi^1_{(1,2)}(x) < \pi^1_{(1,2)}(\beta^*)$ by $(P'_2)$, and $\pi^2_{\hat{\Theta}}(x) = \pi^2_{\hat{\Theta}}(\beta^*) = 0$. Therefore, $\beta^*$ is the only Pareto optimal equilibrium.
(b) Consider now $\beta^*$ as defined in Lemma 2(b) and recall that $d^1(\beta^*) = \{1\}, \{1, 2\}$ and $d^2(\beta^*) = \{1\}, \{2\}$. Following the definition of an equilibrium of the procurement game, we first show that, for $\epsilon > 0$ sufficiently small, $\beta(\epsilon) = (\beta^*_1 - \epsilon, \beta^*_2, \beta^*_1, \beta^*_2 - \epsilon)$ is a pure strategy Nash equilibrium of the $\epsilon$-procurement game. (Supplier 1’s strategy is $\beta^*(\epsilon) = (\beta^*_1 - \epsilon, \beta^*_2)$ and supplier 2’s strategy is $\beta^2(\epsilon) = (\beta^*_1, \beta^*_2 - \epsilon)$.)

If supplier 1 deviated $\beta^*_1(\epsilon)$ to any value $\beta^*_1 + \epsilon$ for $n \geq 1$ and/or $\beta^*_2(\epsilon)$ to any value $\beta^*_2 - \epsilon$ for $n > 1$, her profit would decrease by $(P'_2)$ and since $\{1\} \in d^1(\beta^*)$. Deviating $\beta^*_1(\epsilon)$ to any value $\beta^*_1 + \epsilon$ for $n \geq 1$, would make supplier 1 lose buyer 1 and be worse off since $\emptyset \notin d^1(\beta^*)$, and deviating $\beta^*_2(\epsilon)$ to any value $\beta^*_2 + \epsilon$ for $n \geq 0$ would not change her profit. Any combination of these deviations will clearly lead to no higher profit for supplier 1.

If supplier 1 chose a strategy of the form $\beta^*_1(\epsilon) = \beta^*_1 - \epsilon$, $\beta^*_2(\epsilon) = \beta^*_2 - \epsilon$ with $n > 1$, her expected profit would be $\frac{1}{2} \pi_{1,2}(\beta^*_1 - \epsilon, \beta^*_2 - \epsilon) + \frac{1}{2} \pi_{2,1}(\beta^*_1 - \epsilon, \beta^*_2 - \epsilon) = \frac{1}{2} \pi_{1,1}(\beta^*_1 - \epsilon) + \pi_{1,2}(\beta^*_2 - \epsilon) - n \epsilon v_2 - \frac{1}{2} \pi_{1,2}(\beta^*_1 - \epsilon, \beta^*_2 - \epsilon) - n \epsilon v_2 = \frac{1}{2} \pi_{1,1}(\beta^*_1 - \epsilon) + \pi_{1,2}(\beta^*_2 - \epsilon) - n \epsilon v_2 - \frac{1}{2} \pi_{1,1}(\beta^*_1 - \epsilon) + \frac{1}{2} \pi_{1,2}(\beta^*_2 - \epsilon) - n \epsilon v_2 < \frac{1}{2} \pi_{1,1}(\beta^*_1 - \epsilon) - \pi_{1,2}(\beta^*_2 - \epsilon)$. But $\frac{1}{2} \pi_{1,2}(\beta^*_1 - \epsilon, \beta^*_2 - \epsilon) - n \epsilon v_2 < \pi_{1,1}(\beta^*_1 - \epsilon) - \pi_{1,2}(\beta^*_2 - \epsilon)$, since $d^1(\beta^*) = \{\{1\}, \{1, 2\}\}$, which is also equivalent to $\epsilon (v_1 - n \epsilon v_2) < \frac{1}{2} \pi_{1,1}(\beta^*_1 - \epsilon) - \pi_{1,2}(\beta^*_2 - \epsilon)$. This last inequality holds since $\{2\} \notin d^1(\beta^*)$, and either $v_1 - n \epsilon v_2 < 0$ or $\epsilon$ is small. Then, supplier 1 does not increase her profit.

If supplier 1 deviated to a strategy with $\beta^*_1(\epsilon) = \beta^*_1 - \epsilon$, $\beta^*_2(\epsilon) = \beta^*_2 - \epsilon$, her expected profit would be $\frac{1}{2} \pi_{1,2}(\beta^*_1 - \epsilon, \beta^*_2 - \epsilon) + \frac{1}{2} \pi_{1,1}(\beta^*_1 - \epsilon) - \epsilon v_1 + \frac{1}{2} \pi_{1,2}(\beta^*_2 - \epsilon) - \frac{1}{2} \epsilon v_1 - \frac{1}{2} \pi_{1,1}(\beta^*_1 - \epsilon, \beta^*_2 - \epsilon) \leq \pi_{1,1}(\beta^*_1 - \epsilon) - \epsilon v_1 + \pi_{1,2}(\beta^*_2 - \epsilon) - \frac{1}{2} \epsilon v_1 = \pi_{1,1}(\beta^*_1 - \epsilon) - \epsilon v_1 - \frac{1}{2} \epsilon v_1 = \pi_{1,1}(\beta^*_1 - \epsilon) - \epsilon v_1$. This last inequality holds since $\emptyset \notin d^1(\beta^*)$ and $\epsilon$ is small.

If supplier 1 chose a strategy of the form $\beta^*_1(\epsilon) = \beta^*_1 + \epsilon$ and $\beta^*_2(\epsilon) = \beta^*_2 + \epsilon$, her expected profit would be $\pi_{1,1}(\beta^*_1 + \epsilon) - \pi_{1,2}(\beta^*_2 - \epsilon) = \pi_{1,1}(\beta^*_1 + \epsilon) - \pi_{1,2}(\beta^*_2 - \epsilon) = \pi_{1,1}(\beta^*_1 - \epsilon) - \epsilon v_1$, where the second equality follows since $d^1(\beta^*) = \{\{1\}, \{1, 2\}\}$. Under a strategy of the form $\beta^*_1(\epsilon) = \beta^*_1 + \epsilon$, $\beta^*_2(\epsilon) = \beta^*_2 - \epsilon$, supplier 1’s expected profit would be $\pi_{1,1}(\beta^*_1 + \epsilon) - \pi_{1,2}(\beta^*_2 - \epsilon) < \pi_{1,1}(\beta^*_1 - \epsilon) - \epsilon v_1$, since $\{2\} \notin d^1(\beta^*)$ and $\epsilon$ is small.

Finally, if supplier 1 deviated to the strategy $\beta^*_1(\epsilon) = \beta^*_1$ and $\beta^*_2(\epsilon) = \beta^*_2 - \epsilon$, her expected profit would be $\pi_{1,2}(\beta^*_1, \beta^*_2 - \epsilon) + \pi_{2,1}(\beta^*_1, \beta^*_2 - \epsilon) = \pi_{1,1}(\beta^*_1, \beta^*_2 - \epsilon) - \frac{1}{2} \epsilon v_1 - \frac{1}{2} \pi_{1,2}(\beta^*_1, \beta^*_2 - \epsilon) - \frac{1}{2} \epsilon v_1 = \pi_{1,1}(\beta^*_1, \beta^*_2 - \epsilon) - \frac{1}{2} \epsilon v_1 < \pi_{1,1}(\beta^*_1, \beta^*_2 - \epsilon)$, where the first equality follows from (29), the first inequality follows since $\gamma^1 < 0$, and the last inequality is true for $\epsilon$ small.
For supplier 2, the arguments showing that $\beta^2(\epsilon)$ is the best response to $\beta^1(\epsilon)$ are similar to those for supplier 1, so we omit them for the sake of brevity.

This proves that, for $\epsilon$ small, $\beta(\epsilon)$ is a pure strategy Nash equilibrium of the $\epsilon$-procurement game. Thus, by definition, $\beta^*$ is an equilibrium of the procurement game. Consider now the sets $\Omega_i, \ldots, \Omega_{iv}$, as defined in (a). For $x \in \Omega_i$, $d^i(x) = \{(1,2)\}$ while $d^2(x) \subset \{(1,2), (1), (2)\}$; for $x \in \Omega_{ii}$, $d^1(x), d^2(x) \subset \{(2), (1,2)\}$; and for $x \in \Omega_{iii}$, $d^1(x), d^2(x) \subset \{(1), (1,2)\}$. (As in (a), the characterization of $d^i$ and $d^2$ on $\Omega_i, \Omega_{ii}$ and $\Omega_{iii}$ follow from similar arguments as those in the proof of Lemma 2. See also Figure 3.) Thus, once again, no $x \in \Omega_i \cup \Omega_{ii} \cup \Omega_{iii}$ can be an equilibrium. On the other hand, any equilibrium $x \in \Omega_{iv}$ satisfies $x \leq \beta^*$. If $x < \beta^*$, then $\pi^1_{(1)}(x) < \pi^1_{(1)}(\beta^*)$ and $\pi^2_{(2)}(x) < \pi^2_{(2)}(\beta^*)$, by $(P_2')$. If $x \leq \beta^*$, $x \not= \beta^*$, then either $x_1 < \beta^*_1$ or $x_2 < \beta^*_2$, which implies that either $\pi^1_{(1)}(x) < \pi^1_{(i)}(\beta^*)$ or $\pi^2_{(2)}(x) < \pi^2_{(2)}(\beta^*)$, again by $(P_2')$. Thus, $\beta^*$ is the only Pareto optimal equilibrium. }

**Proof of Corollary 1.** Note that, from Proposition 1 and the change of variables introduced in Section 3.2, the fill rates are $f_k^* = 1 - \exp(\beta_k^*)$, $k = 1, 2$, and each $f_k^*$ is decreasing in $\beta_k^*$. The result then follows by replacing $\beta^* = (\beta_1^*, \beta_2^*)$ in part (a) and $\beta^* = (\beta_1^*, \beta_2^*)$ in part (b), and noting that $r_1^2 < r_2^2$ and $\beta_1^* < \beta_2^*$ (see the proof of Lemma 2). 

**Proof of Proposition 4.** We first need to compute $E[T_k^i]$, the average delivery time experienced by buyer $k$ when sourced by supplier $i$, for $i, k \in \{1, 2\}$.

It follows from (3), (4), and the structure of the rationing policy, that if buyer $k$ has high priority with supplier $i$, then

$$E[T_k^i] = \frac{\lambda \rho^i}{1 - \rho^i} \exp(\beta_k^*).$$

If buyer $k$ has low priority with supplier $i$, then

$$E[T_k^i] = \lambda \left( \frac{2 \rho^i \rho^j - \rho^{i-j}}{1 - 2 \rho^i \rho^j} - \frac{\rho^{3-i}}{1 - \rho^{3-i}} \right) \exp(\beta_k^*).$$

It follows from Theorem 1 that, both when $\mu^2 > \mu^*$ and when $\mu^2 < \mu^*$, buyer 1 selects supplier 1 at the equilibrium and has the higher priority (since $\beta_1^* < \beta_2^*$ in both cases). Buyer 1’s average delivery delay with supplier 1 is $E[T_1^1]$. If buyer 1 selected supplier 2, it would still keep the higher priority, and its average delivery delay with supplier 2 would be $E[T_1^2]$. Since $\rho^1 < \rho^2$, we have that $E[T_1^1] < E[T_1^2]$, which proves part (a).

When $\mu^2 < \mu^*$, both buyers select supplier 1 at equilibrium, and buyer 2 has the lower priority. Its average delay time is then given by

$$E[T_2^1] = \lambda \left( \frac{2 \rho^i \rho^j - \rho^{i-j}}{1 - 2 \rho^i \rho^j} - \frac{\rho^{3-i}}{1 - \rho^{3-i}} \right) \exp(\beta_2^*).$$

If buyer
2 had, instead, selected supplier 2, it would be this supplier’s only customer and its average delay time would be $E[T^2_2] = \frac{\lambda \rho^2}{1 - \rho^2} \exp(\beta^*_2)$. It follows that buyer 2’s choice of supplier 1 leads to the lower average delay time if and only if $\frac{2 \rho^1}{1 - 2 \rho^1} - \frac{\rho^1}{1 - \rho^1} \leq \frac{\rho^2}{1 - \rho^2}$.

When $\mu^2 > \mu^*$, buyer 2 selects supplier 2 at equilibrium and has an average delivery time of $E[T^2_2] = \frac{\lambda \rho^2}{1 - \rho^2} \exp(\beta^*_2)$. In contrast, had it selected supplier 1, its average delay time would be $E[T^1_2] = \lambda \left( \frac{2 \rho^1}{1 - 2 \rho^1} - \frac{\rho^1}{1 - \rho^1} \right) \exp(\beta^*_2)$. Then, buyer 2 selects the supplier providing the lower average delivery time if and only if $\frac{2 \rho^1}{1 - 2 \rho^1} - \frac{\rho^1}{1 - \rho^1} \geq \frac{\rho^2}{1 - \rho^2}$. ■