Quality, Upgrades and Equilibrium in a Dynamic Monopoly Market

James J. Anton and Gary Biglaiser*

12 March, 2012

Abstract

We examine an infinite horizon model of quality growth for a durable goods monopoly. Quality improvements may be sold in any desired bundles. Consumers are identical and for a quality improvement to have value the buyer must possess previous qualities: goods are upgrades. Subgame perfect equilibrium seller payoffs range from capturing the full social surplus down to only the initial flow value of each good, as long as the value of all future quality growth exceeds the value of a single unit. Each of these payoffs is realized in a Markov perfect equilibrium that follows the socially efficient path. However, inefficient delay equilibria, with bundling, exist for innovation rates above a threshold.

JEL: C72, C73, D42, L15

Keywords: Upgrades, durable goods, monopoly, market power, coordination

We examine the commercialization process - pricing and adoption - of an upgrade good in a dynamic monopoly market. Prominent examples are provided by technology markets, such as those for software, where cycles of upgrades to existing products have become the

---

*Anton: Fuqua School of Business, Duke University, Durham NC 27705, james.anton@duke.edu; Biglaiser: Department of Economics, University of North Carolina, Chapel Hill NC 27599, gbiglais@email.unc.edu. We are grateful to the Fuqua Business Associates Fund and Microsoft for financial support. We thank Harvard Business School, the Portuguese Competition Authority, and UCSD for their hospitality where some of this research was conducted. We also thank the associate editor and referees, as well as Jacques Cremer, Leslie Marx, Larry Samuelson, Jean Tirole, Mike Waldman, Dennis Yao, many colleagues at conferences and seminars, and, especially, Joel Sobel for many helpful conversations. The views in this work are solely are own.
norm.\(^1\) Ongoing innovation implies that buyers face a sequence of purchasing decisions. Thus, rather than timing a single purchase and then exiting the market, buyers have an incentive to return to the market and ‘upgrade’ to a higher quality. Buyer expectations are pivotal for these decisions and, given the recurrent aspect of upgrading, bundling by the seller emerges as a critical aspect of the upgrade offers.

The Microsoft antitrust cases highlight a fundamental question regarding prices in an upgrade market. Fudenberg and Tirole (2000) observed that the expert witnesses all appeared to agree that Microsoft was pricing the Windows operating system well below the static monopoly price. There was, however, wide disagreement as to why. Prominent arguments included network formation with low prices spurring adoption, limit pricing where a low price deters rivals, and leverage to gain sales in markets for application programs. Implicit in all of these arguments is the presumption that prices would be higher in the absence of these forces. There is, however, no model of dynamic monopoly that provides a basis for this claim. We provide a game theoretic analysis of dynamic monopoly pricing for an upgrade good and establish that, in equilibrium, high prices are not a necessary outcome. Significantly, low prices, as measured by a seller who captures a small share of the social surplus, emerge in equilibrium.

Upgrade markets, by definition, regularly confront buyers with the choice of adopting a new higher-quality version or remaining with their current version. Microsoft’s recent introduction of Vista was an adoption failure as buyers overwhelmingly chose to stay with their existing XP version, echoing a previous episode with Windows Millennium in 2000. Microsoft moved quickly to introduce a new version. Windows 7 was launched in late October 2009 to a much more favorable buyer response. As early as May 2009, Microsoft CEO Steve Ballmer acknowledged that “If people want to wait [for Windows 7], they certainly can.” This simple observation, which implicitly takes the failure of Vista as a given, leads to a more subtle set of questions.

Consider the initial offer of Vista. An individual buyer has the option of remaining with XP. If most other buyers had purchased Vista then we can expect a concern about “falling behind” the market to be pivotal for an individual buyer’s willingness to pay. Given that others did not purchase Vista, an individual buyer who stayed with XP is in the position Ballmer described. By purchasing Vista a buyer would “jump ahead” of

\(^1\)Quality improvement is important in durable goods markets, as emphasized by Waldman (2003). In addition to software, upgrades to cellular networks often allow vendors to offer, for an added charge, new or improved services such as web browsing, e-mail access and text messaging. Many capital goods are regularly upgraded, including airports (terminals and runways) and oil refineries, among others.
the market and then be confronted with the choice of purchasing Windows 7 to “keep up” with market, assuming that Windows 7 is widely adopted. How does this recurrent interplay of individual and collective decisions with respect to incentives to “fall behind” or “jump ahead” of the market work to determine prices and adoption in an upgrade market? We argue that the ability of the seller to tempt an individual buyer to “jump ahead” is the critical factor and that this incentive provides the basis for a credible threat to reject an upgrade offer. Moreover, low prices can emerge even when buyers have a very strong incentive not to “fall behind” the market.

Our infinite horizon model of an upgrade market has a very simple economic structure. Innovation is exogenous but ongoing and in each period it is feasible for the seller to offer an additional quality increment. Buyers are homogeneous and have a fixed valuation per unit of quality; this corresponds to a horizontal demand curve in a static setting. Building on the recent literature, we assume ‘upgrade’ goods satisfy a downward complementarity property: an additional quality increment is valuable only if a buyer holds all previous quality increments. The seller is unconstrained with respect to bundling options and any combination of quality increments (a single bundle or a set of bundles) may be offered in each period. Bundling is thus endogenous.

This basic structure is arguably a very attractive setting for the seller - homogenous buyers with a fixed valuation per quality unit, unrestricted bundling, and a costless exogenous flow of upgrade innovations. It is natural to expect a “perfect” monopoly outcome in which the seller captures all of the social surplus, and this is exactly what happens in several benchmark cases. For example, when the seller can only offer a single good of fixed quality and buyers are homogeneous, the seller can make an offer that induces buyers to purchase now. This “speed-up” argument, for which an elegant version was developed by Fudenberg, Levine, and Tirole (1985) for a sequential offer game, is quite powerful and it undermines the credibility of buyers to reject offers with high prices.\(^2\) In sharp contrast, we find that surplus growth due to rising quality in an upgrade market provides buyers with an option to return to the market for future purchases, rather than exiting permanently after a single purchase, and that this option leads each buyer in a

\(^2\)The standard incentive (Coase (1972)) to cut price over time and move down the demand curve is not present with identical buyers. Papers on the Coase conjecture with a single good and heterogeneous buyers include Stokey (1981), Balow (1982), and Gul, Sonnenschein, and Wilson (1986). Ausubel and Deneckere (1989), Fehr and Kuhn (1995) and Sobel (1991) provide folk theorems. Bond and Samuelson (1984) examine a rational expectations equilibrium with depreciation and replacement sales. Methodologically, we are closest to Sobel (1991). In both cases, the market never closes, due to new demand in the case of Sobel and to quality growth in our case.
group to reject an offer that a single buyer would not.

The primary intuition is as follows. Suppose that buyers expect to receive a positive share of the surplus on future quality improvements. Further, imagine that the seller offers a price above the candidate equilibrium for today’s upgrade. Is it credible for buyers to refuse the offer? Consider the willingness to pay of an individual buyer when other buyers are expected to refuse the offer. When others refuse, we have delay and the next period will have the larger surplus due to quality growth and a market state in which buyers lack the previous upgrade. When the typical buyer’s share of this surplus is significant, a solitary individual buyer who purchased the high priced upgrade in the last period will wish to purchase again; while this may require the buyer to “re-purchase” some quality increments, the significant buyer share of future surplus makes it attractive to keep up with the market. But, then the initial upgrade purchase of a buyer who “jumps ahead” of the market reduces to a one-period flow of value. As a result, willingness to pay is limited to the one-period flow value. We can apply this result at any stage of the game provided only that the surplus generated by all future upgrades exceeds the surplus of one unit. The credible threat to reject a seller offer, given that other buyers also reject, leads to an implicit form of coordination among buyers and, in turn, to multiple equilibria.

We construct Markov perfect equilibria for this dynamic game. Two classes of equilibria are identified: efficient and generational. Efficient equilibria have buyers acquiring a new upgrade each period and payoffs span a significant economic range. At one extreme, the seller captures all surplus and each quality increment sells immediately for the full present discounted value. At the other extreme, each increment sells only for the one period flow value, leaving a buyer with the entire residual surplus. This is the range of subgame perfect payoffs and each is realized in a Markov perfect equilibrium.

In contrast, and despite the complete information setting, inefficient equilibria do exist. These “generational” equilibria exhibit cyclical delay in which multiple quality increments go unsold until they are bundled together for sale and, necessarily, the market returns to the “state of the art” with a new generation. The cycle length reflects a second type of equilibrium coordination in an upgrade market. Importantly, relative to the set of efficient equilibria, we find that generational equilibria compress the range of payoffs. Intuitively, delay requires that deviations to make early trades are unattractive, and this implies that the seller and the buyers share the joint surplus more equally.

The seller is free to offer any feasible collection of quality units. On the equilibrium path, it is sufficient to consider only upgrade offers with a contiguous set of quality units.
Equivalently, we show how to interpret these upgrade offers in terms of a full bundle (new version of the product) with pricing contingent on a buyer’s current product holding, much as the owner of an existing product faces an upgrade price to acquire a new version. Furthermore, we find no role for the commitment period (time between seller offers), in contrast to the literature on the Coase Conjecture. What matters for equilibrium outcomes is the frequency at which quality improves: allowing the seller to make offers more frequently has no impact with homogenous buyers.

There is a relatively small literature on upgrade models, with most of the work involving a finite horizon. Waldman (1996) and Nahm (2004) each examine a two period model, focusing on the incentive to invest in quality growth and R&D time inconsistency. Fudenberg and Tirole (1998) examine a two-period model where consumers are heterogeneous and the period two (new) good renders the period one (old) good obsolete; Hoppe and Lee (2003) extend this model to allow entry. Ellison and Fudenberg (2000) analyze a series of static and two period models that feature network externalities and a cost to consumers of upgrading the good. In the finite horizon version of our model, the monopolist captures all surplus, since a credible buyer threat is undermined by the terminal period. Fishman and Rob (2000) examine an infinite horizon upgrade model, focusing on innovation incentives, and analyze a rational expectations equilibrium in which the seller is assumed to offer only a single bundle consisting of all prior quality levels. We focus on pricing and adoption, taking innovation as exogenously given, and provide a game-theoretic analysis in which the seller choice of which bundles (and prices) to offer is endogenous.

In the next section, we present the model. In Section 2 we examine efficient equilibria and in Section 3 we examine generational equilibria. We discuss the upgrade structure of our model in Section 4 and consider directions for future research in Section 5. Proofs are in the Appendix; all omitted proofs are in Anton and Biglaiser (2010b).

1 The Model

We first describe the basic elements of the game. We next turn to strategies and payoffs, and then define and discuss Markov perfect equilibrium. We present the formal theoretic framework for bundling (strategies and equilibrium) in Appendix A.
1.1 Basic Elements

We examine an infinite horizon, discrete time model. Let $\tau = 1, 2, \ldots$ index periods. There is a continuum of identical buyers with a measure of 1 represented by the unit interval and a single seller. A new perfectly durable good, unit $\tau$, becomes available in each period $\tau$. All seller costs are 0. In period $\tau$, feasible offers for the seller consist of any collection of subsets of $\{1, 2, \ldots, \tau\}$ and associated prices. For example, the seller can offer the bundle of all feasible qualities $\{1, 2, \ldots, \tau\}$ for a price $p$, so that the new unit is made available only as part of a larger bundle. Alternatively, the seller can offer a collection of individual unit bundles, $\{1\}$ at price $p_1$, quality $\{2\}$ at a price $p_2$, and so on; a buyer could purchase every feasible quality or any subset of the available unit bundles. The seller can also withhold some qualities or even make no offer. Given a seller offer, buyers respond simultaneously with each choosing which bundle(s) to accept in period $\tau$.

An upgrade is a bundle that consists only of a set of contiguous qualities. For example, a “state of the art upgrade” from a status quo of 0 is the bundle $\{1, \ldots, \tau\}$; we also refer to this as a version. A partial upgrade is a bundle $\{\sigma, \ldots, \sigma + k\}$, where $1 \leq \sigma \leq \sigma + k \leq \tau$. We will show that, in equilibrium, a seller need only make upgrade offers.

A buyer holding contiguous units $1, \ldots, q$ but not $q + 1$ has a flow utility of $vq$ in a period. Thus, a buyer must have all lower quality units for quality $q$ to have value. This “downward complementarity” assumption is the upgrade payoff structure in our model.

Players are all risk neutral and have a common discount factor $\delta < 1$. Because a new unit of quality becomes available in each period, the discount factor reflects the rate of innovation as well as the rate of time preference for the players. Thus, we can interpret a large (small) $\delta$ in terms of rapid (slow) rate of innovation and assess limiting behavior.

Consider the payoff for a buyer. In each period, a buyer holds some subset of the feasible qualities. Let $q_\tau$ denote the maximal contiguous quality held by a buyer after any purchase in period $\tau$. That is, a buyer holds units 1 up through $q_\tau$ but does not hold unit $q_\tau + 1$. From any point in the game, the payoff of the buyer is the present discounted value from quality flows net of payments. From the start of the game this is given by

$$\sum_{\tau=1}^{\infty} \delta^{\tau-1} (vq_\tau - p_\tau),$$

We do not impose any arbitrage structure across bundles. For example, if the seller offers separate bundles for units 1 and 2, then there is no restriction on the price of a bundle of units 1 and 2. Rather, buyer choices determine which of these bundles will be purchased. Also, since buyers are identical, there are no possible gains in equilibrium for buyers from the possibility of resale.
where \( p_\tau \) is the payment made by the buyer in period \( \tau \). Similarly, the payoff of the seller from any point onward is the present discounted value of revenues, \( r_\tau \), from sales to buyers. From the start of the game, this is given by

\[
\sum_{\tau=1}^{\infty} \delta^{\tau-1} r_\tau.
\]

Consider efficient allocations. Payments and revenues are transfers that do not affect total surplus. Thus, for any path of quality holdings and payments, the sum of surplus for any given buyer and the seller from any period \( \tau_0 \) is

\[
\sum_{\tau=\tau_0}^{\infty} \delta^{\tau-\tau_0} v q_\tau.
\]

Thus, the realized joint surplus is fully determined by the quality path. Since \( q_\tau \leq \tau \) for any feasible path and \( q_0 \equiv 0 \), the joint surplus is maximized when each buyer holds the maximal quality, \( q_\tau = \tau \). The surplus in an efficient allocation from the start is

\[
S_1 = v + \delta v + \delta^2 v + \ldots = \frac{v}{(1 - \delta)^2}.
\]

Intuitively, \( S_1 \) is the surplus created when buyers acquire one new unit in each period, where each new unit has a present discounted value of \( \frac{v}{1 - \delta} \). Starting from any period \( \tau \), the maximal available surplus is

\[
S_\tau = v \tau + \delta v (\tau + 1) + \delta^2 v (\tau + 2) + \ldots = \frac{v(\tau - 1)}{1 - \delta} + S_1.
\]

Intuitively, the difference between \( S_\tau \) and \( S_{\tau+1} \) is the flow value of \( \tau \) units in period \( \tau \). Thus, we always have \( S_\tau > \delta S_{\tau+1} \), as delay necessarily involves lost surplus and hence inefficiency. However, because each unit generates surplus, we also have \( S_\tau < S_{\tau+1} \).

### 1.2 Markov Perfect Equilibrium

We examine Markov perfect equilibria (MPE) as defined by Maskin and Tirole (2001), with the natural modification for a continuum of agents. By definition, Markov strategies depend only on the payoff relevant aspects of a history of the game. In our model, the seller’s flow payoff depends only on revenues and each buyer’s flow payoff depends only on the maximal contiguous unit held and the payments in a period. Thus, past prices
and the timing of buyer acquisitions do not influence current period payoffs.

The simplest form of Markovian behavior is to focus on the distribution of maximal contiguous units across buyers and the gap relative to the current period \( \tau \), which indexes the seller’s feasible units. This allows us to generate all subgame perfect equilibrium seller payoffs. To proceed, consider any history in which all buyers enter period \( \tau \) with the same maximal quality level \( Q \) (units 1 through \( Q \)). We define this to be state \((\tau, Q)\) and refer to \( \tau - Q \) as the quality gap. Markovian behavior is defined by the condition that players’ strategies depend only on the size of the quality gap. Thus, if the seller offers an upgrade of \( \sigma \) units at a price \( p \) in state \((\tau, 0)\), then an upgrade from \( Q \) to \( Q + \sigma \) at the same price \( p \) must be offered in state \((\tau', Q)\), provided that the gaps coincide, \( \tau' - Q = \tau \). Furthermore, except for a translation of the index number on quality units, buyers’ accept/reject decisions are the same in states \((\tau, 0)\) and \((\tau', Q)\). This implies that the seller’s profits and buyers’ utilities satisfy

\[
\pi_\tau = \pi(\tau, 0) = \pi(\tau', Q),
\]

\[
u(\tau', Q) = \frac{vQ}{1 - \delta} + \nu(\tau, 0) \quad \text{and} \quad u_\tau = u(\tau, 0)
\]

for \( \tau' - Q = \tau \).\(^4\) Thus, buyer payoffs are always the sum of the PDV of current holdings, \( vQ/(1 - \delta) \), and the incremental utility, \( u_\tau \). When there is no risk of confusion, we will use \( \tau \) rather than \((\tau, 0)\) to refer to the state.

This definition of Markovian behavior implies that the same number of units are included in an upgrade bundle whenever the quality gaps coincide. In particular, the seller is not required to offer the full bundle of all feasible units; we discuss contingent pricing and versions in section 4. Henceforth, we use equilibrium to refer to a pure strategy buyer symmetric Markov perfect equilibrium in the quality gap.\(^5\)

We follow Gul, Sonnenschein, and Wilson (1986), Ausubel and Deneckere (1989), and Sobel (1991), among others, and restrict attention to equilibria that satisfy a zero-measure property: for any two histories (past seller offers and buyer acceptances) that differ only with respect to the actions of a set of buyers of measure zero, the strategies of the seller

\(^4\)Feasible payoffs have a simple stationary structure. Any subgame perfect equilibrium from state \((\tau + 1 - Q, 0)\) is also subgame perfect from state \((\tau + 1, Q)\) once we relabel units and translate payoffs.

\(^5\)Asymmetric buyer holdings are off-the-equilibrium path as are histories with multiple upgrade offers. Mixing by buyers in response to a seller offer would lead to asymmetric holdings and this is often needed for continuation equilibria in the durable goods literature. In our case, because buyers never exit the market, we are able to construct pure strategy continuation equilibria in all states. The Appendix provides a detailed analysis for any distribution of buyer holdings and for any seller offers.
and all other buyers are the same across the two histories. As a result, buyers act as price
takers: no buyer expects that their own decision will have any impact on subsequent play,
such as affecting future seller offers.

Finally, to streamline the equilibrium analysis, we specify strategies such that an in-
dividual buyer who deviates by not following other buyers in a purchase that increases
the maximal buyer quality will obtain no future additional surplus. Thus, if an individual
buyer has the first $k$ units of the good, when all other buyers also have additional con-
tiguous units, then this buyer’s continuation payoff is $vk/(1 - \delta)$. We can easily allow
for higher buyer catch-up continuation values as long as they do not exceed the equilib-
rium payoff. For the analysis, however, it is helpful to follow the above specification of a
zero increment in utility for a buyer who falls behind the market. This will highlight the
critical role played by the incentive for a buyer to jump ahead of the market.

2 Efficient Equilibria

We begin with a basic result on the necessary structure with respect to all equilibrium
payoffs: the seller can always induce buyers to make a purchase.

Lemma 1 (Flow Dominance) Consider any history such that, at the start of period $\tau$, all
buyers hold the first $Q$ quality units and no buyer holds unit $Q + 1$, where $\tau > Q$. Suppose
the seller makes an upgrade offer for units $\{Q + 1, \ldots, \tau\}$ at price $p$, where $p < v(\tau - Q)$. Then, in any continuation equilibrium, every buyer accepts the upgrade offer.

The intuition for “flow dominance” is simple. The upgrade from $Q$ to $\tau$ is priced suffi-
ciently low that that it pays for itself in the current period, since $v\tau - p > vQ$. Moreover,
even if all other buyers were to reject the offer, an individual buyer who accepts is always
weakly better off in the future. This follows from (1) the upgrade payoff structure, since
an accepting buyer has a flow surplus of at least $v\tau$ in future periods, and (2) all buyers
have the same opportunities for purchasing from the seller, so an accepting buyer always
has the option of making the same choices in the future as other buyers.

Lemma 1 implies that in state $\tau$ the continuation payoff of the seller is at least $v\tau +
\delta v/(1 - \delta)$. First, the seller can offer $\tau$ units at flow value. Second, in the future each new

---

6One can interpret this as (i) the seller (optimally) ignores individual buyers (measure zero) who di-
ffer from the market path, or (ii) the seller offers the necessary units but the upgrade price extracts all of
the continuation surplus. By the zero measure property, the seller must either completely refrain from
making “catch-up” offers, or always make such offers.
unit can be sold at \( v \). Lemma 1 and this flow dominance payoff bound are basic results. They apply to any subgame perfect payoff and do not depend on Markovian behavior or symmetric buyer strategies: they only rely on buyers acting as price takers.

In an efficient equilibrium, a good is sold in each period when it first becomes available. At the start of the game, the seller offers the first unit at price \( p_1 \) and all buyers accept. In the second period, the state is then \((2, 1)\) and the quality gap is again 1. Under Markovian behavior, the seller offers the second unit at price \( p_1 \) and all buyers accept, leading to state \((3, 2)\), and so on. Thus, the firm earns a profit of \( \pi_1 = \frac{p_1}{1 - \delta} \), buyers receive a payoff of \( u_1 = \frac{1}{1 - \delta} \left[ \frac{v}{1 - \delta} - p_1 \right] \), and they share the maximal social surplus, \( S_1 = \pi_1 + u_1 \).

To highlight the economic forces at work, we next discuss the extremal equilibria. We then develop sufficient conditions and characterize efficient equilibria.

### 2.1 Equilibria with extreme payoffs

Consider the two extremal equilibria. In the first, the seller captures all of the social surplus while the buyers receive a payoff of zero; each new upgrade unit is sold at the price \( p_1 = \frac{v}{1 - \delta} \). In the second, the seller payoff is held to the one-period flow value while the buyers capture the full residual value; each new upgrade unit is sold at the price \( p_1 = v \). Both equilibria follow the efficient path. Furthermore, each is supported by an immediate return to the efficient path in the event of deviations: in any state \( \tau \) the seller offers a state of the art bundle, a “cash-in” offer so that no feasible units remained unsold, and buyers choose to accept. Each equilibrium reflects implicit buyer coordination on the price \( p_1 \) and any higher price offer is rejected by all buyers. Thus, it is important to understand both how the low price equilibrium is supported by a credible (buyer) threat to reject prices above flow value and why such a threat is inoperative in the high price equilibrium.

Consider the low price equilibrium and suppose the seller offers an upgrade at a price above \( v \). If other buyers are expected to reject this offer, a buyer who deviates and accepts also has the option to make future purchases. In the low price equilibrium, the supporting cash-in offer for 2 units has a continuation payoff for a buyer who did not purchase last period that is strictly greater than \( \frac{v}{1 - \delta} \). Consequently, the deviating buyer will choose to accept this offer and rejoin the other buyers rather than fall behind for a continuation payoff of \( \frac{v}{1 - \delta} \). But this implies the same continuation payoff for the deviating buyer and for buyers who did not purchase last period. Returning to the initial deviation, given that other buyers are expected to reject, the deviating buyer will pay at most the flow value of \( v \) for the first unit. Hence, the equilibrium coordination of buyer decisions’ on the price...
$p_1$ supports the credible threat to reject higher prices.

We can extend this logic of a credible threat based on buyer coordination to support the entire range of buyer payoffs between 0 and $\delta S_1$ in equilibrium. As suggested by a willingness to pay equal to the flow value $v$ in the low price equilibrium, we are able to construct a credible threat for any discount factor greater than $1/2$. Essentially, $\delta > 1/2$ is a growth condition on social surplus as it implies that the incremental surplus from future units is larger than the value of the current unit ($\delta S_1 > \frac{v}{1-\delta}$). The future surplus is then sufficient to maintain the incentive of a deviating buyer to rejoin the support path and thus establish a credible threat to reject price increases.

By contrast, the high price equilibrium is very simple. Since other buyers accept any price up to $\frac{v}{1-\delta}$, a deviating buyer who rejects will fall behind the market and receive 0 (by construction). Thus, an individual buyer will accept any price up to $\frac{v}{1-\delta}$. This logic does not rely on surplus growth and the high price equilibrium exists for all discount factors. Comparing the low and high price equilibria, we see that the incentives of an individual buyer depend on other buyers’ accept/reject decisions, since future seller offers will vary with the market state. Thus, buyer coordination matters in an upgrade market.

### 2.2 Sufficient Conditions

In an efficient equilibrium, the seller offers the current unit at $p_1$, all buyers accept, and the market cycles. For these to be optimal actions, we must specify continuation payoffs that rule out deviations.\(^7\) Suppose that in state $\tau \geq 2$, the seller offers $\tau$ units at a price $p_\tau$ and this is accepted by all buyers. We define this to be a “cash-in” support. Thus, the next state is $(\tau + 1, \tau)$, where the quality gap has returned to 1. The continuation payoffs with this support are $\pi_\tau = p_\tau + \delta \pi_1$ and $u_\tau = \frac{v}{1-\delta} - p_\tau + \delta u_1$.

Buyers follow a simple cut-off rule: a buyer accepts the seller offer of price $p$ for $\sigma$ units in state $\tau$ if and only if $p \leq p(\sigma, \tau)$. On the acceptance side, it must be optimal for an individual buyer to accept any offer $p \leq p(\sigma, \tau)$, given that all other buyers are accepting (symmetric strategies) and the quality gap moves to $\tau + 1 - \sigma$. Rejecting when others all accept yields 0 by construction, as the buyer falls behind the market. Accepting along with other buyers yields a current flow of $v\sigma - p$ plus a future value of $\delta \left( \frac{v\sigma}{1-\delta} + \delta u_{\tau+1-\sigma} \right)$. Thus, it is optimal for all buyers to accept $p$ for $\sigma$ units in state $\tau$ if $\frac{v\sigma}{1-\delta} + \delta u_{\tau+1-\sigma} \geq p$.

---

\(^7\)We apply the one-stage-deviation principle to verify the proposed strategies constitute an equilibrium; our model conforms to the necessary requirement of “continuity at infinity,” since the limit of $\tau \delta^{\tau}$ is 0 as $\tau$ goes to infinity (see Fudenberg and Tirole (1991) pp. 108-110).
This reflects the incentive of a buyer to “keep up” with the market.

The rejection side of the cut-off rule reflects the incentive not to “jump ahead” of the market, and an offer of \( p > p(\sigma, \tau) \) must be rejected by all buyers. Rejecting when others reject yields a payoff of \( \delta u_{\tau+1} \). More subtly, accepting when all other buyers reject yields a flow of \( v\sigma - p \) plus the option of purchasing the cash-in offer for \( \tau + 1 \) units next period. Thus, an individual buyer optimally rejects when others reject if

\[
p > v\sigma + \delta \max \left\{ \frac{v\sigma}{1-\delta}, u_{\tau+1} \right\} - \delta u_{\tau+1} \equiv g(\sigma, u_{\tau+1}).
\]

Intuitively, when the other buyers purchase the cash-in offer in state \( \tau + 1 \), a deviating buyer who accepted in \( \tau \) has two options. If \( u_{\tau+1} > \frac{v\sigma}{1-\delta} \), it will be optimal to purchase with the other buyers. Thus, the deviating buyer is initially willing to pay at most the flow value of the units, \( v\sigma \), in state \( \tau \). Otherwise, the buyer will not purchase in \( \tau + 1 \) and is willing to pay up to \( \frac{v\sigma}{1-\delta} - \delta u_{\tau+1} \).

Thus, combining the acceptance and rejection sides of the cut-off strategy we have

\[
g(\sigma, u_{\tau+1}) \leq p(\sigma, \tau) \leq \frac{v\sigma}{1-\delta} + \delta u_{\tau+1-\sigma}
\]

for all \( 0 < \sigma \leq \tau \) and all \( \tau \geq 1 \). The cut-off strategies apply to full (\( \sigma = \tau \)) and partial (\( \sigma < \tau \)) cash-in offers. Since \( g \) is bounded above by \( \frac{v\sigma}{1-\delta} \), cut-off strategies exist for any non-negative utility sequence. The lower bound on \( g \) of \( v\sigma \) reflects flow dominance.

The distinct upper and lower bounds on the cut-off rule show that an individual buyer’s willingness to pay depends on the actions of other buyers. It is important to recall that there are no network externalities in our model, which is a standard reason for why buyers make their purchasing decisions based on expectations of other buyers’ choices. Instead, the linkage of decisions arises from i) quality growth and the resulting incentive for a buyer to return to the market for another upgrade, ii) the seller’s offer in the future depends on “the state of the market” and, iii) an individual buyer will be affected by his position relative to the market when making future purchasing decisions.

Given these buyer cut-off strategies, the seller must find it optimal to offer \( \tau \) units at price \( p_{\tau} \) in state \( \tau \). The seller has three ways of deviating: make no offer, a “delay;” offer an upgrade of less than \( \tau \) units, a “partial cash-in;” or offer an upgrade of \( \tau \) units but not at price \( p_{\tau} \). For partial cash-ins, \( p(\sigma, \tau) \) is the optimal price choice for any such offer and
it generates a payoff of $p(\sigma, \tau) + \delta \pi_{\tau+1-\sigma}$. Then equilibrium requires

$$\pi_\tau - \delta \pi_{\tau+1-\sigma} \geq p(\sigma, \tau)$$

(2)

for $\sigma = 1, \ldots, \tau - 1$. Delay, $\sigma = 0$, is not optimal if $\pi_\tau \geq \delta \pi_{\tau+1}$. Defining $p(0, \tau) \equiv 0$, (2) applies. Finally, for a cash-in offer of $\tau$ units, buyers will accept any price below $p(\tau, \tau)$, so we must have $p_\tau = p(\tau, \tau)$ or else the seller could successfully offer a price above $p_\tau$. Note that (2) is an equality at $\sigma = \tau$, by equilibrium construction. Similarly, the buyer condition (1) also applies for delay ($\sigma = 0$) and on the equilibrium path ($\tau = \sigma = 1$).

Combining the price bounds from (1) and (2), we must have

$$\pi_\tau - \delta \pi_{\tau+1-\sigma} \geq p(\sigma, \tau) \geq g(\sigma, u_{\tau+1}).$$

Since $S_\tau = \pi_\tau + u_\tau$ in an efficient equilibrium, the above condition is equivalent to

$$S_\tau - \delta S_{\tau+1-\sigma} \geq u_\tau - \delta u_{\tau+1-\sigma} + g(\sigma, u_{\tau+1}),$$

(3)

for $0 \leq \sigma \leq \tau$ and $\tau \geq 1$. The surplus difference on the left hand side is an increasing, exogenous sequence in $\tau$: as $\tau$ grows and more units are “on the table,” a larger set of utilities can be supported.\(^8\) Given utilities that satisfy (3), we can clearly construct the supporting prices $p(\sigma, \tau)$ for conditions (1) and (2). When (3) holds, the optimal upgrade offer for the seller is to offer $\tau$ units for the price $p_\tau$. Each buyer then finds it optimal to accept the upgrade offer, given that all other buyers also accept. We then have

**Lemma 2** Suppose the sequence of buyer utilities $u_\tau$ satisfies (3). Then there exists an efficient equilibrium with a buyer payoff of $u_1$.

The proof that (3) is sufficient for the existence of an equilibrium outcome with payoff $u_1$ is by construction. Taking a given $u_1$, the rest of the utility sequence is specified in the next section. For this sequence, we must show that it is not profitable for the seller to deviate by offering multiple upgrade options (as well as options with non-contiguous units); note that (3) only rules out seller deviations involving a single upgrade offer. This requires that we specify buyer strategies in response to any offer from the seller. In addition, we must specify strategies for continuation equilibria in the event that buyer holdings are distributed asymmetrically across units even though such events are off the

\(^8\)In the benchmark case of a single good, where a buyer exits the market after a purchase, (3) reduces to $0 \geq u_1$ and all surplus accrues as profit.
equilibrium path. In all cases, the support returns to the equilibrium path after 1 period. See Appendix B for the support construction.

2.3 Existence and Payoffs

The sequence of utilities that we construct to satisfy the sufficient condition (3) has two phases. When the quality gap is $T$ or smaller, the support makes the seller indifferent between a cash-in and delay; for larger quality gaps the support keeps buyer utility constant at $u_T$. Thus, for $T \geq 2$ we define a $T$-stage support utility sequence by

$$u_\tau = \begin{cases} v\tau + \delta u_{\tau+1} & \text{for } \tau = 1, \ldots, T - 1 \\ u_T & \text{for } \tau \geq T. \end{cases}$$

(4)

For any given $u_1$ and length $T$, the sequence $(u_2, \ldots, u_T)$ is determined.

A direct consequence of a $T$-stage support is that we only need to satisfy the support constraints, (3), over the range $\tau = 1, \ldots, T$; see Lemma A1 in Appendix B. This is because, when (3) holds at $\tau = T$, then it necessarily holds at all larger $\tau$ whenever utility remains constant and the seller is the residual claimant of surplus growth. Thus, an advantage of a $T$-stage support is that we only have to check a finite set of conditions. We now turn to finding the appropriate length for the $T$-stage support.

Consider any given $u_1 \in [0, \delta S_1]$, and then define $T$ by $(1 - \delta^{T-1})S_1 < u_1 \leq (1 - \delta^T)S_1$. For each $u_1$, this condition defines a unique $T \geq 1$, since $\delta \leq 1 - \delta^T$ holds for $T$ sufficiently large. Thus, higher buyer payoffs require a larger $T$. We see from (4) that for $\tau \leq T$

$$u_\tau = \sum_{s=\tau}^{T-1} vs\delta^{s-\tau} + \delta^{T-\tau}u_T.$$ 

Setting $\tau = 1$, the given $u_1$ together with the specified $T$ yield $u_T$, as required. From $u_T$ and $T$, the other utility values follow directly by taking $\tau = 2, \ldots, T - 1$.

Figure 1 then illustrates the relationship between $u_1$ and $T$ for the range of $\delta \in [1/2, 1]$. Define a set of critical $\delta$ cutoffs where $\delta_\tau$ is the root of $\delta^\tau + \delta = 1$. For example, when $1/2 \leq \delta < \delta_2$, we use a 1-stage support for the area A, where $u_1 < (1 - \delta)S_1$, and then a 2-stage support for larger $u_1$ in the area B. Because the maximal buyer payoff, $\delta S_1$, lies below $(1 - \delta^2)S_1$ for this range of $\delta$ we have covered all possible buyer payoffs. In the next range, $\delta \in (\delta_2, \delta_3)$, after areas C and D, we must also use a 3-stage support to cover the highest buyer payoffs (in the area E). In Figure 1, for each $\delta$ range, to cover higher
buyer payoffs we rise vertically with a 1-stage, then a 2-stage, and so on up to the value for $T$ at the maximal buyer payoff of $u_1 = \delta S_1$.

To build intuition for how the $T$-stage support works, we begin with the equilibrium path. Suppose that the buyer payoff is small, $u_1 < \frac{v}{1-\delta}$. By the defining condition for $T$, we then have $T = 1$ and buyer utility is constant at $u_\tau = u_1$ for all $\tau$. In Figure 1, this case corresponds to the region below the $(1 - \delta S_1)$ curve. The two deviation options for the seller in state 1 are to delay or to raise price. With utility constant (and small), the seller has a strict incentive not to delay. For the second option, buyers must refuse a price increase. Refusing with other buyers yields $\delta u_2$. An individual who accepts will not purchase next period’s cash-in offer because the continuation utility of $u_2$ from purchasing with others is below $\frac{v}{1-\delta}$. Thus, an individual buyer who accepts has a payoff of $v_1 + u_2$.

For buyers to refuse any $p > p_1$, the rejection condition is $\delta u_2 \geq \frac{v}{1-\delta} - p_1$. By equilibrium construction, the r.h.s. is just $(1 - \delta) u_1$. With utility constant, the rejection condition reduces to $\delta u_1 \geq (1 - \delta) u_1$, which holds when $\delta \geq 1/2$. Thus, (3) holds at state 1.

Consider the equilibrium when utility is large ($u_1 > \frac{v}{1-\delta}$). We now have $T > 1$, and we are in an upper region of Figure 1. Because buyers are initially the residual claimants of surplus growth, we have $\pi_1 = \delta \pi_2$ and there is no incentive for the seller to delay. If the seller attempts a price increase, buyers who reject expect $u_2$. Suppose an individual accepts. With $T > 1$, we have $u_2 > \frac{v}{1-\delta}$, since buyer utility rises by $S_2 - S_1$. Thus, the individual buyer will purchase again in state 2 and the payoff to accepting is $v + p + \delta u_2$. Buyers must refuse any price above $p_1$ and the rejection condition of $\delta u_2 \geq v - p_1 + \delta u_2$, reduces to flow value, $p_1 \geq v$. Thus, (3) is again satisfied in state 1.

More generally, (3) must hold for all states $\tau$ in addition to the equilibrium path. By setting $T$ according to the definition and then generating $u_2, ..., u_T$ from $u_1$ by capitalizing surplus growth, we accomplish two things: (i) $u_{\tau+1} > v_\tau/(1 - \delta)$ for $\tau = 1, ..., T - 1$ and (ii) $v_\tau/(1 - \delta) \geq u_T$ for $\tau \geq T$. When $\tau \geq T$, the situation is fully analogous to the case of a small buyer payoff on the equilibrium path. That is, once $u_\tau$ is constant at $u_T$ and relatively small ($u_T \leq \frac{v}{1-\delta}$), the seller has a strict incentive not to delay since he is the residual claimant of surplus growth. When $\tau < T$, we have $u_\tau > \frac{v_{\tau-1}}{1-\delta}$ and an individual buyer who is ahead of the market would choose to purchase with other buyers in state $\tau$ and the market returns to the equilibrium path with a gap of 1. As a result, the rejection condition for a price increase reduces to flow value, $p_\tau \geq v_\tau$. We then have

**Proposition 1** Let $\delta > 1/2$. Then, for every $u_1 \in [0, \delta S_1]$, there exists an efficient equilibrium with buyer payoff $u_1$. 


Significantly, the minimum possible equilibrium payoff for the seller is the flow dominance lower bound from Lemma 1. Thus, there is an equilibrium in which the seller’s market power is reduced to the static flow value of each unit with all of the future surplus from a unit accruing to buyers.

Consider the range of equilibrium payoffs as the rate of innovation increases. The discount factor is given by $\delta = e^{-r\Delta}$, where $r$ is the interest rate and $\Delta$ is the period length (time between innovations). The flow value of buyer surplus is given by $v = \int_0^\Delta \lambda e^{-r\tau} d\tau = \lambda(1 - e^{-r\Delta})/r$, where $\lambda$ is the instantaneous flow value of quality to buyers, and limiting outcomes can be calculated directly. The flow dominance lower bound of $v/(1 - \delta)$ on the seller payoff reduces to $\lambda/r$. In the limit, flow dominance corresponds to the seller collecting a payment of $\lambda$ at each instant. This is, however, a vanishingly small fraction of total surplus, since $v/(1 - \delta)^2$ grows without bound.

**Corollary 1** In the limit, as upgrades become increasingly frequent ($\Delta \to 0$), the seller’s minimum share of the surplus goes to zero and the buyers’ maximum share goes to one.

In the limit, while surplus grows without bound, flow dominance only ensures a finite profit for the seller. Thus, there is no guarantee of market power, as measured by profit
as a share of the total surplus, when innovations arrive very frequently: a seller may earn a high level of absolute profit while capturing only a small share of the market surplus.\footnote{In our model, $\delta$ reflects the time between innovations as well the commitment period of the seller (time between offers). We can allow the seller to make offers more frequently and follow the same principles for constructing the support utilities. In contrast to Coasian settings, these offers will be off-the-equilibrium path. It is straightforward to check that an efficient equilibrium with a $T - Stage$ support is robust to allowing the seller to make an interim offer at any time between innovations. Specifically, in the constant utility part of the $T - Stage$ support we can specify a cash-in at the same utility level for the interim offer. When utility is rising it is simplest to specify a delay outcome for the support. While allowing interim offers in the Coasian setting with heterogeneous buyers and a single good will speed up sales, we find that our original equilibrium path does not change with the commitment period of the seller.}

\section{Delay and Generational Equilibria}

We now consider inefficient equilibria. First, we show that equilibria must have a simple cyclical structure and, second, that innovation needs to be sufficiently frequent for delay to occur. We then consider seller and buyer incentives in the delay states, derive sufficient conditions, and show existence.

\subsection{Cyclical Equilibria and Upgrade Frequency}

In a $t - cycle$ equilibrium the only sale is for $t$ units every $t$ periods. Thus, 1 through $t - 1$ are delay states and the quality gap falls back to 1 every $t$ periods.

\begin{proposition}
Every equilibrium path follows a $t - cycle$: the buyers purchase the bundle of units $\{1, ..., t\}$ from the seller in state $t$, all payments to the seller occur in state $t$, and the maximal buyer quality is zero until state $t$.
\end{proposition}

What makes this argument work is flow dominance and the fact that the seller can profitably deviate by speeding up a cycle that does not have buyers moving to the state of the art in state $t$. If the sale only involves $\tau < t$ units, the seller can feasibly offer these units in state $t - 1$. By pricing these units at $\hat{p} = v\tau + \delta p - \varepsilon$, where $p$ is the price for $\tau$ units in state $t$, the seller payoff rises if all buyers accept since

\[ \hat{p} + \delta \pi(t, \tau) = (v\tau + \delta p - \varepsilon) + \delta^2 \pi(t + 1, \tau) > \delta p + \delta^2 \pi(t + 1, \tau) \]

and, upon substituting for $\hat{p}$ and noting that $(t, \tau)$ is a delay state, this reduces to $v\tau > \varepsilon$.

Can buyers reject this offer? If other buyers reject, an individual will always find it optimal to purchase the deviation offer since, by accepting, an individual buyer receives
\( \delta u(t,0) + \varepsilon \). To see this, note that the deviating buyer does not change the state, so \( \tau \) units will be offered next period. Since the buyer already has these units, the purchase in state \( t \) can be skipped and the buyer will have the same holdings as all other buyers as of \( t + 1 \). Thus, her payoff is improved relative to waiting whenever \( \varepsilon > 0 \). Hence, all buyers rejecting the offer is not an equilibrium continuation. But when all buyers accept, the seller can profit by making the deviation offer. Thus, an equilibrium with sales of \( \tau \) less than \( t \) cannot be supported. By contrast, this speed up argument does not apply to a \( t - cycle \) equilibrium. Suppose the seller offers \( \tau < t \) units. By acquiring \( \tau \) units in \( t - 1 \) when no other buyers accept, an individual buyer can no longer safely skip all purchases in state \( t \), since other buyers will be acquiring units 1 through \( t \).

We refer to a \( t - cycle \) equilibrium with \( t \geq 2 \) as a generational equilibrium, since buyers upgrade to the quality frontier. Payoffs in a \( t - cycle \) equilibrium are then \( \pi_t = p_t/(1 - \delta^t) \) for the seller, as the revenue flow of \( p_t \) is received once every \( t \) periods, and \( u_t = \frac{1}{1-\delta} \left( \frac{vt}{1-\delta} - p_t \right) \) for the buyers, as a purchase of \( t \) units at a price \( p_t \) is made once every \( t \) periods. From the above seller and buyer payoffs, joint surplus is

\[
\Psi_t \equiv \pi_t + u_t = \frac{vt}{(1 - \delta)(1 - \delta^t)}.
\]

Due to delay, the realized joint surplus in a \( t - cycle \) equilibrium is less than the maximal surplus \( S_1 \). These are short run efficiency losses since each cycle resolution ends with buyers holding all feasible units as of the sale date.

When innovations are infrequent, \( \delta < 1/2 \), the value of one unit is larger than the discounted value of all future units: \( v/(1 - \delta) > \delta v/(1 - \delta)^2 = \delta S_1 \). Flow dominance now plays a stronger role: it rules out equilibrium delay. Further, buyers are unable to credibly reject prices below \( v/(1 - \delta) \) and all surplus is captured by the seller.

**Proposition 3** If \( \delta < 1/2 \), then there is no equilibrium with delay: the unique equilibrium outcome follows the efficient path and the seller captures all surplus.

When \( \delta < 1/2 \) the seller and the buyers both value current flows more heavily than future ones. Intuitively, if upgrade innovations are sufficiently infrequent, then a speed-up deviation to avoid delay is mutually beneficial. An individual buyer with \( t - 1 \) units on hand would not purchase the \( t \) bundle in state \( t \) and this makes a speed-up offer for \( t - 1 \) units in state \( t - 1 \) attractive to an individual buyer. When all buyers accept, current revenue dominates the payoff from waiting to sell next period.

18
3.2 Delay Conditions and Existence

Equilibrium delay requires that the seller can find no offer that is acceptable to buyers and also profitable relative to waiting to sell in state $t$. Thus, we will specify cut-off prices for buyer strategies such that the seller does not find it profitable to make a sale before the quality gap reaches $t$ units.

Satisfying the buyer and seller delay incentives reduces to finding a $u_t$ that satisfies

$$
(1 - \delta^\tau)(\Psi_t - u_t) \geq \frac{v\tau(\delta^{\tau-t} - 1)}{(1 - \delta)} + \max \left[ \frac{v\tau}{(1 - \delta)}, u_t \right] - u_t
$$

(5)

for $\tau = 1, ..., t - 1$. As shown in Appendix C, when $u_t$ satisfies (5), there exist cut-off prices that support delay. The analysis of (5) is involved because the buyer and seller delay incentives can change significantly as the quality gap rises. To begin, note that (5) implies a lower as well as an upper bound on $u_t$. As $u_t$ approaches 0, (5) fails at all $\tau$, since the incentive for an individual buyer to jump ahead is very strong and the seller can profitably attract buyers as early as $\tau = 1$. If the buyers get too much of the available surplus ($u_t$ approaches $\Psi_t$), then (5) necessarily fails at all $\tau$: the seller can exploit flow dominance to sell early and profitably attract buyers by offering $\tau$ units at a price of $v\tau$.

An extreme payoff on either the buyer or the seller side thus allows the seller to profitably induce a speed up. As a consequence, both buyer and seller payoffs are compressed relative to the range of payoffs for efficient equilibria.

Lemma 3 The set of payoffs for all generational equilibria is a strict subset of the set of payoffs for efficient equilibria.

In order to be willing to wait until period state $t$, a buyer must decline seller deviation offers. This requires a strictly positive payoff for buyers. By contrast, there is an efficient equilibrium in which buyers receive no surplus. Similarly, the seller must be willing to delay and this includes foregoing the option to sell units prematurely at flow value. Thus, the seller necessarily earns more than the flow dominance lower bound. The upper bounds then follow from the lower bound on the other side of the market. Thus, while delay generates less surplus, both sides of the market necessarily receive a larger payoff relative to the minimum subgame perfect equilibrium payoff.

Satisfying the delay conditions for a $t$ – cycle equilibrium reduces to finding a buyer utility for the sale date, $u_t$, that satisfies (5) for a given $t$ and $\delta$. As the sale date gets
closer, buyer incentives change significantly when \( u_t \) lies between \( \frac{v}{1-\delta} \) and \( \frac{v(t-1)}{1-\delta} \). Initially, for small \( \tau \), a deviating buyer who acquired \( \tau \) units would be willing to purchase again at the sale date \( t \), owing to a sufficiently large quality increase between \( \tau \) and \( t \). However, for \( \tau \) closer to \( t \), a deviating buyer would choose to fall behind the market since the incremental surplus from \( t - \tau \) units is insufficient.

Because of this change in buyer deviation incentives, the delay conditions may bind at an interior \( \tau \). While a complete characterization of (5) is quite involved, it turns out that many of the complications only arise at relatively low discount factors.\(^{10}\) In Appendix C, we develop a sufficient condition on \( \delta^t \) such that if \( \delta^t \) is above a threshold, \( d^* \), then the delay conditions (5) are satisfied for an interval, \((u^A, \pi^A)\), of utility levels. We then have

**Lemma 4** If \( \delta^t > d^* \), then there exist bounds \( u^A \) and \( \pi^A \) such that the delay conditions (5) are satisfied for any \( u_t \in (u^A, \pi^A) \) in a \( t \) - cycle equilibrium.

Lemma 4 provides a lower and upper bound on the buyers’ payoffs. The utility bounds, \( u^A \) and \( \pi^A \), depend on \( \delta \) and \( t \) and are derived in Appendix C. At \( d^* \), \( u^A = \pi^A \), and for all \( t \) and \( \delta \) pairs where \( \delta^t > d^* \), we have \( u^A < \pi^A \). See Figure 2. Numerically, the threshold, \( d^* \), is about .439. For example, if \( t = 2 \), then \( \delta \) must be at least \( \sqrt{.439} = 0.663 \). One can interpret the \( t \) - cycle as having two stages, the delay phase and the sale date, where the discount factor between stages is \( \delta^t \). Hence, the longer delay in equilibrium, the higher must be \( \delta \) so that the seller will not find a profitable deviation.

We must also specify the cash-in (off equilibrium) support conditions for \( t \) - cycle equilibria, in addition to the delay conditions. The buyer cut-off rules and seller profits must satisfy the analog of (3). The main difference relative to the efficient case is that the efficient surplus \( S_\tau \) is replaced by \( \Psi_\tau \). Furthermore, the set of support utilities for off the equilibrium path states where the quality gap exceeds \( t \) needs to be modified for delay equilibria (see Appendix C). We then combine the delay and cash-in support conditions to establish the existence of delay equilibria.

**Proposition 4** Let \( t \geq 2 \) and suppose that \( \delta^t > d^* \). Then every \( u_t \in (u^A, \pi^A) \) can be supported in a \( t \) - cycle inefficient equilibrium.

Thus, the delay incentives govern the range of equilibrium buyer utilities. Intuitively, as \( \delta \) rises we can employ longer support lengths to support higher buyer payoffs.

---

\(^{10}\) This is because, in general, when condition (5) holds at \( \tau = 1 \) and at \( \tau = t - 1 \) it does not necessarily follow that (5) holds at \( 1 < \tau < t - 1 \). This can be seen by explicitly solving the cases of \( t = 2 \) and \( t = 3 \).
Figure 2: Delay Conditions \( v = 1 \) and \( t = 2 \)

Consider how delay equilibria relate to the division of surplus. The length of time between sales in a \( t \)-cycle equilibrium is \( D = \Delta t \). As \( \Delta \) declines, so that innovation is more rapid, any given \( t \)-cycle will continue to exist, since \( \delta^t = e^{-r\Delta t} > d^* \) still holds, but the delay length \( D \) will go to zero. The utility bounds relative to realized equilibrium surplus, \( \bar{s}_b \equiv \bar{u}^A/\bar{\Psi}_t \) and \( s_b \equiv u^A/\Psi_t \), index the range of equilibrium payoffs. Simplifying reveals an invariance property as the bounds depend only on the length of delay, \( D \). Defining \( D_{\text{max}} \equiv (-\ln d^*)/r \), straightforward calculations yield:

**Corollary 2** As equilibrium delay vanishes, the bounds on buyer utility converge to those for efficient equilibria, \( \bar{s}_b \to 1 \) and \( s_b \to 0 \) as \( D \to 0 \). As equilibrium delay approaches \( D_{\text{max}} \), the bounds on buyer utility converge to each other at a strictly positive value.

As with the limit in Corollary 1, flow dominance does not guarantee market power. At the other extreme, even a maximal delay and, correspondingly, a maximal loss in efficiency does not allow the seller to capture fully the available surplus.

### 4 Discussion of Upgrade Structure

We begin with four benchmark settings in which the seller captures all surplus, Next, we relate our endogenous bundling framework to observed practice in upgrade markets.
We then discuss our results relative to assumptions in the upgrade literature.

4.1 Benchmarks

First, consider a finite horizon. Once the final period arrives, buyers have no prospect of acquiring upgrades in the future. Subgame perfection then implies that each buyer will necessarily accept any offer that provides a positive payoff. The unique equilibrium outcome in the final period, given any current units held by the buyers, is that the seller offers an upgrade to the state of art at a price equal to the full value of the upgrade to the buyers. By backward induction, this holds for all prior periods since buyers never expect a positive payoff in the future. Hence, the equilibrium follows the efficient path and the seller captures the full social surplus.\footnote{Further details on this and the other benchmarks are provided in the working paper version.}

Second, suppose that there is no surplus growth: the seller has one unit to offer and trade may take place at any time over an infinite horizon. With no heterogeneity among buyers, we have a special case of the standard durable goods model (gap case). The result of Fudenberg, Levine, and Tirole (1985) implies that there is never delay and the full surplus is always extracted from buyers in any subgame perfect equilibrium.

Suppose that there is only one buyer in an infinite horizon model with growth. A variation on the speed-up argument of Fudenberg, Levine, and Tirole implies that the seller can always profitably tempt the buyer to purchase all available units immediately. The choice of a single buyer necessarily changes the state, in contrast to the case with a continuum. When responding to a current offer, a refusal is optimal only if the discounted continuation payoff exceeds the payoff from accepting the offer. Then a positive buyer payoff, $u_1 > 0$, requires a supporting utility path that rises indefinitely at an exponential rate. Because this exceeds the available surplus in finite time, a credible threat for refusing a price increase, relative to $u_1$, unravels. The only equilibrium has $u_1 = 0$.

Finally, with a complete absence of complementarity across quality levels, units $1, 2, \ldots$ are independent goods. Then, as one might expect, the seller regains the ability to extract buyers due to the lack a credible threat. The essential difference is that a buyer can accept a current offer when others do not, skip the subsequent cash-in offer from the seller, and then resume purchasing. Because the goods are independent, there is no payoff consequence due to complementarity from any missing units. Thus, we necessarily have $u_1 = 0$. Intuitively, complementarity is essential for a credible threat to refuse price increases as a deviating individual buyer faces the extra cost of having to acquire the...
missing unit. This is why, in our upgrade structure, it matters to an individual buyer whether or not others are expected to purchase the seller’s offer.

Thus, as all four benchmarks lead to strong monopoly power, each element is essential to a credible threat for buyers in an upgrade market.

4.2 Bundling in Practice

In practice, the upgrade process varies greatly with respect to how buyers move to higher quality levels. Contract contingencies, especially with respect to a buyer’s current holdings, are frequently observed. MacKichan, for example, offers the technical word processor Scientific Word 5.5 in a number of versions differentiated by features and each version has an upgrade price for prior users (serial number required) as well as a (higher) price for new users. Airliners typically offer seat upgrades, club memberships, and other amenities at prices that vary with frequent flier status, a result of past purchases.

A new version with a price contingent on a buyer’s current version is very close and often will be equivalent to an upgrade offer. Consider a buyer who holds units \( \{1, \ldots, \sigma\} \) and two offers. One is an upgrade bundle \( \{\sigma + 1, \ldots, \sigma + k\} \) for price \( p \). The other is a new version \( \{1, \ldots, \sigma, \ldots, \sigma + k\} \) at price \( p \) that is only available to buyers who hold units \( \{1, \ldots, \sigma\} \). The direct value to the buyer is the same with either bundle. Now, consider the same offers, but suppose that the buyer does not hold any units. By downward complementarity, the upgrade bundle has no direct value (non-contiguous units) while the buyer does not qualify for the other offer. Further, observe that if the pricing contingency is stated as a minimum requirement then buyers who hold at least the minimum will place a common value on the two bundles. Such a minimum requirement is common in practice. Microsoft allows any 2000-2007 Office program or suite to qualify a buyer for Office Professional at the upgrade (discount) price.

Thus, our equilibria will be robust to allowing price contingencies if we can demonstrate that it is still optimal for the seller to make the same offers (either in the upgrade form or the appropriate version form with a holding contingency). Consider the possibility that, by conditioning offers on current holdings, a seller may be able to curtail the credible threat of buyers to reject offers with high prices and thus eliminate equilibria with low seller payoffs. In this regard, our equilibria are robust. First, recall that buyers all have the same holdings on any equilibrium path and a contractual contingency in this regard has no force. With respect to the support for the equilibrium path, the same observation applies to the cash-in support. Finally, when buyers have asymmetric holdings, we constructed
continuation equilibria in which the seller captured the available surplus and, as a result, this support is robust to the addition.

At a more intuitive level, recall that the credible threat to reject seller offers with high prices is based on the expectation of a sufficiently high future surplus (when all buyers reject the offer). Consider, for example, the support condition (3) for an efficient equilibrium and the impact of allowing contract contingencies on buyer holdings. An individual buyer who fails to purchase when others do will fall behind the equilibrium path, but such a buyer is already extracted in our analysis. On the other hand, a buyer who purchases when others do not will jump ahead of the market and, in our analysis, such a buyer does have a strict preference for purchasing the subsequent equilibrium support cash-in offer from the seller. The seller could then employ a contractual contingency to isolate such a buyer and eliminate the (valuable) option to accept an offer designed for buyers with fewer units and “rejoin” the equilibrium path. But if the contingency is used to make an offer that extracts the deviating buyer then this will only serve to reduce further the deviation payoff to purchasing when others do not (refer to the max condition for $g(\tau, \tau)$ in (3)). Thus, the support condition (3) continues to be satisfied by our $T$-stage utility path and buyers retain a credible threat.

Thus, whether we regard the offers as upgrade bundles or versions with price contingencies, as in Fudenberg and Tirole (1998) or Ellison and Fudenberg (2000), does not affect the equilibrium structure.\footnote{In both Fudenberg and Tirole and Ellison and Fudenberg the second period offers can distinguish between buyers who purchased in period 1 and those who did not. These models also feature buyer heterogeneity.} In terms of information structures, our original offer set corresponds to the semi-anonymous regime of Fudenberg and Tirole (1998), where harsher terms cannot be imposed on buyers who hold more units than others. This establishes that our results do not depend on the precise form of the seller’s offers.\footnote{In our working paper we also address network effects, compatibility issues, and adoption costs. We argue that our equilibrium results are robust to these forces as they all reinforce the incentive for a buyer to keep up with the market while reducing the payoff of a buyer who jumps ahead of the market. See the related policy discussion in Anton and Biglaiser (2010a).} Bundles can be presented as upgrades or as versions with a contingency on past purchases. Our formulation of the offer space is simpler as it avoids the complications of contingencies.

4.3 Unbreakable version offers

Suppose that, as an exogenous condition, all bundles must be versions and no contingencies on purchase are allowed. That is, any offer for the current quality increment, unit
τ, is necessarily also an offer for units \( \{1, \ldots, \tau\} \) and similarly for any lower quality level. This might reflect a necessary property of the production technology, where a quality increment cannot be ‘broken out’ for separate sale. This offer structure is examined in Waldman (1996) for a two-period model and in Fishman and Rob (2000) for an infinite horizon model, both of whom focus on innovation incentives.

In our setting, where buyers have identical preferences, this “unbreakable” upgrade structure necessarily limits the market power of the seller. The reason is that a buyer always has the option of passing on a current offer and waiting to purchase a later offer. As long as the seller eventually offers a higher quality, the cost of waiting (relative to purchasing when other buyers do) is the lost flow value. Any subsequent higher-quality offer that attracts prior buyers will necessarily provide the buyer who delays with a strictly larger surplus than that received by prior buyers. As a result, there is no equilibrium in which a monopoly seller captures the full surplus when upgrades are unbreakable.

In a two-period version of our model and unbreakable goods, it is straightforward to derive the equilibrium. For large discount factors, the seller delays until period 2 and then offers units \( \{1, 2\} \) at the extraction price for the remaining surplus. For low discount factors, however, unit 1 is sold in period 1 for \( v \) and then period 2 has extraction pricing for units \( \{1, 2\} \). The seller is never able to capture the full surplus, in contrast to the finite horizon equilibrium with upgrade offers or generation offers with price contingencies.

In the case of an infinite horizon with non-contingent pricing, ongoing quality growth necessarily imposes a more severe limit on the seller. Fishman and Rob (2000) point out that the option to wait implies that the seller can charge no more than the flow value.\(^\text{14}\) In their rational expectations equilibrium, the low price leads to a rate of innovation that is inefficiently low. In our analysis, where bundling is endogenous, the option to wait (fall behind the market) is inconsequential. The basis of a credible threat for buyers resides with the extent to which the seller can tempt a buyer to jump ahead of others.

\(^{14}\)Full extraction of total surplus by the seller requires implementing the efficient path and, in the unbreakable version of our model, the seller would be limited to prices that reflect only the flow value and not the present discounted value to buyers from quality increments. Formally, consider the efficient path and let \( p(\tau, \tau-1) \) be an equilibrium price for version \( \tau \) when all buyers hold \( \{1, \ldots, \tau-1\} \). By rejecting an offer and resuming purchases next period, an individual buyer obtains \( v(\tau-1) + \delta u(\tau+1, \tau) \). Purchasing today yields \( v\tau - p(\tau, \tau-1) + \delta u(\tau+1, \tau) \). Combining, we must have \( p(\tau, \tau-1) \leq v \).
5 Directions for Future Work

Buyers are homogeneous in our model. This was assumed to focus on the structure of credible threats for buyers in a dynamic upgrade model in what one would expect to be the ideal situation for a seller to capture the full (efficient) joint surplus. Allowing for buyer heterogeneity is an important direction for subsequent work. In practice, it is common for sellers in upgrade markets to offer simultaneously different versions or quality levels of their products. This is typically taken to be a form of price discrimination. As noted before, several papers examine a finite horizon model, but there has been very little theoretical work on infinite horizon models in which buyers are always in the market and quality improvements are ongoing. One could introduce buyer heterogeneity in our model where buyers are privately informed of their valuation. This allows for an endogenous determination of pricing and whether the buyer segments remain distinct over time or whether the seller chooses to price over a cycle that periodically brings high and low types together at a common quality level (a generational cycle). An interesting feature of equilibrium price discrimination in this dynamic context is that incentive constraints can bind in both directions (with low-value buyer types choosing to mimic high-value buyer purchases as well the standard downward incentive constraint).

We also assumed an exogenous rate for the increase in quality. Of course, a model that addresses the question of how rewards for a given quality innovation are determined is a necessary step toward an endogenous determination of quality change. We are currently studying innovation and pricing incentives in a model where innovations can be generated not only by an incumbent but also by potential entrants. In this setting, property rights for innovation in relation to imitation incentives are crucial for buyer decisions regarding adopting the products of an incumbent or an entrant and, in turn, for assessing public policy choices and welfare in upgrade markets.

References


6 Appendix

6.1 Appendix A - Formal Structure

We present the formal structure of the model. First, we define bundles and offers. Next, we define strategies and histories, present payoffs and formally define equilibrium.

(i) The Bundle Offer Structure.

Consider the feasible offer set for the seller in period $\tau$. Let $\mathcal{P}_\tau \equiv \mathcal{P}(\{1, 2, ..., \tau\})$ denote the power set for the first $\tau$ integers. Any set $z \in \mathcal{P}_\tau$ is called a bundle. An offer is a collection of bundles and associated (non-negative) prices, $(z, p_z)_{z \in Z}$ for some $Z \in \mathcal{P}(\mathcal{P}_\tau)$. Define the offer set $\Omega_\tau$ by

$$\Omega_\tau \equiv \{\omega \in \mathcal{P}(\mathcal{P}_\tau \times R_+) \mid (i) \ (\emptyset, 0) \in \omega, (ii) \text{ if } (z, p) \in \omega \text{ and } (z, p'}) \in \omega, \text{ then } p = p'\}. $$

By (i), we are including the null bundle in every offer by the seller. This is for two reasons: first, the seller can make no offer by choosing only the null bundle and, second, it streamlines the buyer choice formalism, as a buyer chooses to make no purchase by selecting the null bundle. By (ii), every offered bundle has a unique price. Clearly, if two prices were offered for the same bundle, no buyer would choose the higher price (buyers act as price takers due to the zero measure property for strategies across histories). An acceptance by a buyer is an element of the set $\mathcal{P}(\mathcal{P}_\tau)$. 


Consider the maximal contiguous quality. For any \( z \in \mathcal{P}_\tau \), define \( M : \mathcal{P}_\tau \to \{0, 1, \ldots, \tau\} \) by finding the unique \( m \in \{0, \ldots, \tau\} \) such that \( m' \in z \forall m' \leq m \) and \( m + 1 \notin z \), and set \( M(z) = m \). Clearly, \( M(z) \) is the maximal contiguous quality held by a buyer and \( M(z) \) exists for any bundle \( z \). For an arbitrary sequence of holdings \( z_\tau \), define \( q_\tau = M(z_\tau) \).

(ii) Strategies and Histories.

A (pure) strategy for the seller is a sequence of offers, \( \mathcal{O} = (\mathcal{O}_\tau) \). Each offer is a map from the history of play up through period \( \tau - 1 \) into the offer set \( \Omega_\tau \). A history is the sequence of previous offers by the seller and acceptances by the buyers. Letting \( \mathcal{H}_\tau \) denote the space of all histories up through period \( \tau - 1 \), we have

\[
\mathcal{O}_\tau : \mathcal{H}_\tau \to \Omega_\tau.
\]

Given an observed history, \( h_\tau \in \mathcal{H}_\tau \), the seller’s strategy specifies an offer \( \omega_\tau = \mathcal{O}_\tau(h_\tau) \).

A buyer (pure) strategy profile is a sequence of acceptance decisions, \( \mathcal{A} = (\mathcal{A}_\tau) \). Given a history \( h_\tau \) and a seller offer \( \omega_\tau \), each buyer \( x \in [0, 1] \) needs to choose which bundles in \( \omega_\tau \) to accept. Thus, we have acceptance strategies for each buyer

\[
\mathcal{A}_\tau^x : \mathcal{H}_\tau \times \Omega_\tau \to \mathcal{P}(\mathcal{P}_\tau).
\]

Hence, for observed history \( h_\tau \in \mathcal{H}_\tau \) and in response to a seller offer of \( \omega_\tau \in \Omega_\tau \), buyer \( x \) chooses to accept the set of bundles \( \mathcal{A}_\tau^x(h_\tau, \omega_\tau) \subseteq \mathcal{P}(\mathcal{P}_\tau) \). Of course, any accepted bundle, \( z \in \mathcal{A}_\tau^x(h_\tau, \omega_\tau) \), must have been offered by the seller, \( (z, p) \in \mathcal{O}_\tau(h_\tau) \) for some \( p \). This is a feasibility restriction. Note that a buyer is free to accept one or more of the bundles (i.e., any subset) included in an offer \( \omega_\tau \). For example, by “accepting” only the null bundle, a buyer makes no purchase in period \( \tau \). Finally, we use \( \mathcal{A}_\tau \) for the strategy profile across buyers.

We need to specify the history space \( \mathcal{H}_\tau \). First, define \( \Omega^\tau \equiv \Omega_1 \times \Omega_2 \times \ldots \times \Omega_\tau \); this product space contains each feasible sequence of previous offers. Second, we need to calculate acceptance sets from buyer bundle purchases and this entails a measurability assumption on buyer strategies.

Let \( \mathcal{F}_\tau \) denote the set of Borel measurable functions for \( [0, 1] \to \mathcal{P}(\mathcal{P}_\tau) \). By definition, \( f_\tau : [0, 1] \to \mathcal{P}(\mathcal{P}_\tau) \) is Borel measurable (that is, \( f_\tau \in \mathcal{F}_\tau \)) if for any \( z \in \mathcal{P}_\tau \) we have \( \mathcal{X}_\tau(z) \in \mathcal{B} \) (the Borel sets of \( [0, 1] \)), where \( \mathcal{X}_\tau(z) = \{ x \in [0, 1] \mid z \in f_\tau(x) \} \). Thus, the set of buyers who chose bundle \( z \) is a Borel set and we can calculate market share and revenues by using standard Lebesgue measure. Define the product space \( \mathcal{F}^\tau \equiv \mathcal{F}_1 \times \mathcal{F}_2 \times \ldots \times \mathcal{F}_\tau \).
Then the history space is specified by \( H_1 = \emptyset \) and for \( \tau > 1 \),

\[
H_\tau = \Omega^{\tau-1} \times F^{\tau-1}.
\]

Note that the bundles and prices offered by the seller are recorded in \( \Omega^{\tau-1} \) while the bundles accepted by each buyer are recorded in \( F^{\tau-1} \). Thus, we know the price a buyer paid for a bundle from the history. We assume that for each \( h_\tau \in H_\tau \), and \( \omega_\tau \in \Omega_\tau \), we have \( A_\tau \in F_\tau \), i.e. \( A_\tau^x(h_\tau, \omega_\tau) \) is a Borel measurable function on \( x \in [0,1] \). An equivalent, but less convenient, formulation would be to assign an index to each element in the finite set \( P(\mathcal{P}_\tau) \) and define measurability in the standard way for a real valued function.

(iii) Payoffs and Equilibrium.

Turning to the calculation of player payoffs, we begin with the buyers. First, for each \( h_{\tau+1} \), calculate the units acquired by buyer \( x \) in each period \( k = 1, \ldots, \tau \). These units are given by \( Z_k(x) = \{ i \in \{1, \ldots, k\} \mid i \in z \text{ for some } z \in A_k^x(h_k, \omega_k) \} \), the bundles accepted by buyer \( x \). Thus, the set of units that buyer \( x \) has accumulated through the end of period \( \tau \) is given by

\[
Z^\tau(x) \equiv \bigcup_{k=1}^{\tau} Z_k(x) \subseteq \mathcal{P}_\tau.
\]

Recalling that \( M(z) \) is the maximal contiguous quality for any subset \( z \) of \( \{1, \ldots, \tau\} \), we see that the maximal contiguous quality unit held by buyer \( x \) is given by \( m_\tau(x) \equiv M(Z^\tau(x)) \).

Next, the total expenditure of buyer \( x \) in period \( \tau \) is given by \( p_\tau(x) \equiv \sum_{z \in A_\tau^x(h_\tau, \omega_\tau)} p_z \), which is the sum of the payments for each bundle that the buyer accepted. Thus, the payoff to buyer \( x \) from strategy \( A^x \) when other buyers follow \( A^{\sim x} \) and the seller follows \( O \) is the present discounted value of surplus from the maximal unit held less expenditures in each period:

\[
U(O, A^x, A^{\sim x}) = \sum_{\tau=1}^{\infty} \delta^{\tau-1} [vm_\tau(x) - p_\tau(x)].
\]

The infinite sum is always well defined, since (i) the sequence of maximal holdings \( m_\tau \) is non-decreasing in \( \tau \), (ii) \( m_\tau \leq \tau \), and (iii) \( \sum_{\tau=1}^{\infty} \delta^{\tau-1} \tau = 1/(1-\delta)^2 \).

We now compute the seller payoff. Given a history and an offer by the seller, \( A_\tau(z) \) as defined above is the set of buyers for whom \( z \in A_\tau^x(h_\tau, \omega_\tau) \). The Lebesgue measure of

30
such buyers is \( \alpha_r(z) = \int \alpha_r(\mathcal{X}(z)) \, dx \). Thus, the revenue of the seller in period \( \tau \) is

\[
q_\tau = \sum_{z \in \mathcal{P}_\tau} \alpha_r(z)q_z.
\]

\( \alpha_r(z) = 0 \) must hold if the seller did not offer bundle \( z \) or if no buyer purchased \( z \).

The seller payoff under strategies \( (\mathcal{O}, \mathcal{A}) \) is then

\[
\Pi(\mathcal{O}, \mathcal{A}) = \sum_{\tau=1}^{\infty} \delta^{\tau-1}q_\tau.
\]

The definitions for Nash and subgame perfect equilibrium are standard. The strategies \( (\mathcal{O}, \mathcal{A}) \) form a Nash equilibrium if

\[
\Pi(\mathcal{O}, \mathcal{A}) \geq \Pi(\tilde{\mathcal{O}}, \mathcal{A}) \quad \text{for all } \tilde{\mathcal{O}},
\]

\[
U(\mathcal{O}, \mathcal{A}^x, \mathcal{A}^{-x}) \geq U(\mathcal{O}, \tilde{\mathcal{A}}^x, \tilde{\mathcal{A}}^{-x}) \quad \text{for all } \tilde{\mathcal{A}}^x.
\]

A subgame perfect equilibrium requires that \( (\mathcal{O}, \mathcal{A}) \) form a Nash equilibrium at any given \( h_\tau \), where the seller makes an offer, and at any given \( h_\tau \) and \( \omega_\tau \), where the buyers respond to the offer.

When buyers are distributed across maximal holdings, the state is given by \( (\tau, (Q_\tau^m)_{m=0,\ldots,\tau-1}) \), where \( Q_\tau^m \) is the set of buyers with maximal contiguous quality \( m \). More generally, when buyers are distributed as \( (Q_\tau^m)_{m=0,1,\ldots,\tau} \), then the translated state is given by \( (\tau + 1 - m_\tau, (Q_\tau^m)_{m=m_\tau,\ldots,\tau}) \) where \( m_\tau \) is the smallest index of \( Q_\tau^m \) with a non-zero measure.

### 6.2 Appendix B - Efficient Equilibria

We first prove Lemmas 1 and 2 and then state several lemmas dealing with properties of the \( T - \text{stage} \) support. This is followed by the proof of Proposition 1.

(i) Flow Dominance and Sufficient Conditions for Equilibrium

**Proof of Lemma 1.** Depending on the history, buyers may also hold a subset, possibly null, of units \( \{Q + 2, \ldots, \tau - 1\} \). Without unit \( Q + 1 \), a buyer who rejects the upgrade offer will receive a flow payoff of \( vQ \) and have the same quality holdings in period \( \tau + 1 \). A buyer who accepts will receive a flow payoff of \( v\tau \) in period \( \tau \) and hold \( \{1, \ldots, \tau\} \) next period. We will show that accepting yields a strictly higher payoff than rejecting, for any strategy choices of other buyers and the seller following the upgrade offer.

Obviously, accepting yields a higher flow payoff in period \( \tau \) since \( v\tau - p > vQ \). Consider
the continuation payoff. An accepting buyer (i) begins with more units than a non-accepting buyer and (ii) has the option of mimicking the strategy of any non-accepting buyer (any seller offer may purchased by any buyer). By the upgrade payoff structure, the continuation payoff of an accepting buyer is therefore at least as large as that of a non-accepting buyer. This is because the mimicking option implies the same payments, but the accepting buyer will never hold fewer units.

It is now clear that every buyer will choose to accept the upgrade offer. Given any strategy choices of other buyers and the seller following the upgrade offer, a buyer who accepts always has a weakly larger payoff from \( \tau + 1 \) onward, with respect to the continuation sequence of offers implied by the strategies, and a strictly larger flow payoff in period \( \tau \). By the zero measure property, the continuation sequence of offers does not depend on the choice of a specific individual buyer to accept or reject the upgrade offer. ■

**Proof of Lemma 2.** To establish existence of an equilibrium we need to show that (i) our candidate upgrade offer is optimal for the seller with respect to the offer set \( \Omega_\tau \), and (ii) construct a continuation equilibrium for states where buyers are distributed asymmetrically providing, for any such state, both an optimal offer for the seller and optimal buyer responses for any given seller offer.

Now, define an upgrade offer \( B \) as any offer in \( \Omega_\tau \), with the property that if \( (z,p_z) \in B \), then \( M(z) = \sup \{ i \mid i \in z \} \). By construction, every bundle in \( B \) is an upgrade bundle, since the maximal contiguous quality in \( z \) coincides with the largest quality unit in \( z \). Thus, we can denote any \( (z,p_z) \in B \) by \( (b,p_b) \), where \( b = \sup \{ i \mid i \in z \} \) is the upgrade level for \( z \) and \( p_b \equiv p_z \) is the price. An upgrade offer need not include all of the feasible upgrade bundle levels \( 1, \ldots, \tau \). A buyer will optimally choose at most one bundle in \( B \).

The restriction to upgrade offers can be shown to be without loss of generality. This is because we construct continuation equilibria in which only upgrade offers are made by the seller. Thus, even if a period \( \tau \) offer includes non-upgrade bundles, all players expect that every possible period \( \tau + 1 \) continuation state will involve only upgrade offers. Furthermore, in every possible period \( \tau + 1 \) continuation state, every buyer will move to a quality holding of at least \( \tau \) in the continuation equilibrium outcome. Consequently, a buyer will value bundles in an offer only to the extent that the bundle allows the buyer to move to a higher quality level in period \( \tau \). If a buyer’s purchases in \( \tau \) result in the acquisition of non-contiguous quality levels, these non-contiguous units have no current or future payoff effect due to the structure of the continuation equilibria.

More generally, if buyers are asymmetrically distributed across maximal quality levels
and if buyers hold non-contiguous quality units (above a buyer’s maximal level), we can still work with upgrade offers without loss of generality. This is because any offer in conjunction with a buyer’s current maximal quality and non-contiguous holdings can always be reduced to an implied set of payments for achievable (higher) maximal quality levels. With the continuation equilibria noted above, any resulting non-contiguous quality units will have no payoff impact.

To demonstrate (i), it is sufficient to show that the seller cannot profitably deviate in state \((\tau, 0)\) to some other upgrade offer (with multiple upgrade bundles). First, define a buyer preference relation, \(\succeq_B\), for any two upgrade bundles by

\[
(b, p_b) \succeq_B (i, p_i) \iff vb - p_b + \delta u(\tau + 1, b) \geq vi - p_i + \delta u(\tau + 1, b | i),
\]

where \(u(\tau + 1, b | i)\) is equal to \(\frac{vi}{1-\delta}\) if \(i < b\) and equal to \(\max\left\{\frac{vi}{1-\delta}, u(\tau + 1, b)\right\}\) if \(i \geq b\). Note that the \(\succeq_B\) relation reflects implicit coordination in that an individual buyer has no incentive to choose \((i, p_i)\) if all other buyers choose \((b, p_b)\). Now, define an upgrade \((b, p_b) \in B\) to be a buyer continuation equilibrium (BCE) in state \(\tau\) for offer \(B\) if \((b, p_b) \succeq_B (i, p_i) \forall (i, p_i) \in B\). We must show that for any offer \(B\) there exists a BCE such that the seller cannot gain in state \(\tau\) by deviating to offer \(B\) instead of making the cash-in offer of \(p_\tau\). The proof is lengthy so we only provide a sketch. First, one shows that any two upgrade offers are comparable under \(\succeq_B\). Next, one can show that \(\arg \max_{(b, p_b) \in B} \left\{ \frac{vb}{1-\delta} - p_b + \delta u_{\tau+1-b} \right\}\) is a BCE (existence is trivial as \(B\) has a finite number of bundles; if it is not unique then select the arg max with largest upgrade level). Essentially, this follows because the argmax is the highest possible coordinated payoff for buyers and because utility differences across a \(T\)-stage support satisfy the bound \(u_\sigma - u_{\sigma'} \leq \frac{v(\sigma-\sigma')}{1-\delta}\).

We then have two cases for the offer \(B\). If every \((b, p_b) \in B\) satisfies \(p_b \geq G_b \equiv g(b, u_{\tau+1})\), then it is easy to show that \((0, 0)\), where all buyers refuse to purchase, is a BCE for \(B\). This is equivalent to a delay outcome and we know from (3) that the seller prefers to make the cash-in offer \(p_\tau\). If \(p_b < G_b\) for some \((b, p_b) \in B\) then we first find the \(\arg \max\) specified above for the subset of all such bundles in \(B\), call it \((b^*, p_{b^*})\). We then show that \((b^*, p_{b^*}) \succeq_B (i, p_i) \forall (i, p_i) \) where \(p_i \geq G_i\). Then, \((b^*, p_{b^*})\) is a BCE for \(B\) and the seller payoff of \(p_{b^*} + \delta u_{\tau+1-b^*}\) is, by (3), not profitable relative to \(p_\tau\).

Consider (ii) and any state in which buyers are asymmetrically distributed across
quality levels. Since all buyers with the same maximal contiguous quality level are treated identically in the continuation it is sufficient to keep track only of market shares. Thus, let us denote such a state by \((\tau, \alpha)\) where \(\alpha = (\alpha_0, \ldots, \alpha_{\tau-1})\) specifies for each \(\sigma = 0, \ldots, \tau - 1\) the fraction \(\alpha_\sigma \in [0, 1]\) of buyers entering period \(\tau\) with maximal quality level of \(\sigma\). By hypothesis, \(1 > \alpha_0 > 0\) and \(\sum_{\sigma=0}^{\tau-1} \alpha_\sigma = 1\). We specify a continuation equilibrium for \((\tau, \alpha)\) as follows. The seller makes an upgrade offer \(\{(b_\sigma, p_\sigma)\}_{\sigma=0,\ldots,\tau-1}\) where each \((b_\sigma, p_\sigma)\) is an upgrade from \(\sigma\) to \(\tau\), that is the bundle \(\{\sigma + 1, \ldots, \tau\}\), for price 
\[
    p_\sigma = \frac{v(\tau - \sigma)}{1 - \delta} + \delta u_1.
\]
For buyer strategies, we specify that a buyer with \(\sigma\) units chooses to accept \((b_\sigma, p_\sigma)\). It is straightforward to verify that, when all other buyers follow this strategy, it is optimal for an individual buyer with \(\sigma\) to do so as well. Since these upgrade offers leave each buyer with a payoff of \(v \sigma / (1 - \delta)\), the payoff to the seller is equal to the continuation surplus of \(S_\tau\) less these individual-rationality payoffs aggregated across buyers according to the distribution \(\alpha\). By feasibility, this bounds the seller’s payoff in any continuation.

Finally, to complete the argument that the above upgrade offer is an optimal choice for the seller, we need to specify a \(BCE\) if the seller makes some other upgrade offer. Allowing for partial upgrades, denote such an offer by \(\mathcal{B} = \{(b, \sigma; p_{b,\sigma}) \mid \sigma \leq b \leq \tau, 0 \leq p_{b,\sigma}\}\) in \((\tau, \alpha)\), where each \((b, \sigma; p_{b,\sigma})\) denotes an upgrade bundle for units \(\{\sigma + 1, \ldots, b\}\) at price \(p_b\). Since the offer \(\mathcal{B}\) has upgrades that begin at different levels and a buyer is free to purchase multiple bundles, we construct from \(\mathcal{B}\) for each possible buyer quality level \(\sigma = 0, \ldots, \tau - 1\), the set \(\mathcal{B}_\sigma\) of all upgrade bundles that move \(\sigma\) to a higher quality level; note \(\mathcal{B}_\sigma\) might contain only the refusal option for some \(\sigma\). We then have each buyer with quality level \(\sigma\) choose to accept the (largest index)
\[
    \arg \max_{\mathcal{B}_\sigma} \left\{ \frac{v b}{1 - \delta} - p_{b,\sigma'} \right\}.
\]
Then, these choices can be shown to form a \(BCE\) for \(\mathcal{B}\). The proof is trivial if the buyer choices result in a nondegenerate distribution across quality levels since the buyers are then held to their individual-rationality payoffs in the continuation state. If not, then all buyers move to some common quality level, say \(\hat{b}\), and we must use the continuation payoff \(u_{\tau+1-b}\) from the \(T\)-stage support. ■

(ii) Properties of the \(T\) – stage Support
Lemma A1 Consider a $T$-stage support, where $u_\tau = u_T \equiv \bar{u}$ for $\tau \geq T$. If the support condition (3) holds at $\tau = T$ for $\sigma = 0, \ldots, T$, then (3) holds at $\tau > T$ for $\sigma = 0, \ldots, \tau$.

Lemma A2 Consider a $T$-stage support. Then for any $T$, we have

$$u_\tau = \frac{1}{\delta^{\tau-1}} \left[ u_1 - \frac{v}{1-\delta} \left( \frac{1-\delta^{\tau-1}}{1-\delta} - (\tau - 1)\delta^{\tau-1} \right) \right].$$

Lemma A3 The $T$-stage support sequence $(u_1, \ldots, u_T)$ defined by $u_1 = (1 - \delta^{T-1})S_1$ satisfies (i) $u_\tau = (1 - \delta^{T-\tau})S_1 + \frac{v(\tau-1)}{1-\delta}$, (ii) $u_\tau \geq \frac{vT}{1-\delta}$ if and only if $\delta \geq \delta^{T-\tau}$ and (iii) $u_{T-1} \equiv \frac{v(T-1)}{1-\delta}$ and $u_{T-2} \equiv \frac{v(T-1)}{1-\delta} - v$.

Lemma A4 Consider a $T$-stage support with (i) $u_\tau \geq \frac{vT}{1-\delta}$ for $\tau = 1, \ldots, T-1$ and (ii) $\frac{v(T-1)}{1-\delta} < u_T < \frac{vT}{1-\delta}$. If the support condition (3) holds at $(\tau, \tau)$ for $\tau = 1, \ldots, T$, then the $T$-stage support satisfies (3) for all $(\sigma, \tau)$, where $0 \leq \sigma \leq \tau$ and $\tau \geq 1$.

(iii) Existence of Equilibrium

Proof of Proposition 1. First, note that conditions (i) and (ii) of Lemma A4 are valid when $(1 - \delta^{T-1}) S_1 \leq u_1 \leq (1 - \delta^T) S_1$. This follows by applying Lemma A3 to the reference sequences, $u_\tau$, for $T$ and for $T+1$. Then, by Lemma A4, it is sufficient to verify condition (3) at $(\tau, \tau)$ for $\tau = 1, \ldots, T$.

It is immediate that (3) at $\tau = 1$ requires $\delta S_1 \geq u_1$. Hence, we are done if $T = 1$. Now, consider $T \geq 2$ and note that the same observation implies that (3) holds at $(1, 1)$.

Now, consider (3) at $(\tau, \tau)$ for $\tau \leq T-1$. Then $u_{\tau+1} > \frac{vT}{(1-\delta)}$ and we have $g(\tau, u_{\tau+1}) = v\tau$. Thus, the equilibrium support condition (3) becomes

$$\frac{\delta v\tau}{1-\delta} + \delta u_1 \geq u_\tau.$$ 

We claim that condition $(\tau, \tau)$ implies condition $(\tau+1, \tau+1)$ for $\tau \leq T-2$. In other words, we claim that $\frac{\delta v\tau}{1-\delta} + \delta u_1 \geq u_\tau$ implies $\frac{\delta v(\tau+1)}{1-\delta} + \delta u_1 \geq u_{\tau+1}$. Recall that $u_{\tau+1} = \frac{1}{\delta} (u_\tau - v\tau)$. So, condition $(\tau+1, \tau+1)$ can be written as

$$\frac{\delta^2 v(\tau+1)}{1-\delta} + \delta^2 u_1 + v\tau \geq u_\tau.$$ 

Thus, it is sufficient to show that $\frac{\delta^2 v(\tau+1)}{1-\delta} + \delta^2 u_1 + v\tau > \frac{\delta v\tau}{1-\delta} + \delta u_1$. But, this holds if and only if $\delta S_1 + \frac{vT}{\delta} > u_1$, which is always the case for $\tau \geq 1$. Thus, (3) holds at $(1, 1)$ and this implies (3) holds at $(\tau, \tau)$ for $\tau = 2, \ldots, T-1$. 

35
We are then left with the \((T,T)\) condition, which reduces to

\[ \delta u_1 \geq (1 - \delta) u_T, \]

since \(g(T, u_{T+1}) = \frac{v_T}{1 - \delta} - \delta u_T\) by Lemma A3. We know that the condition holds at \((T-1, T-1)\) and we have

\[
\frac{\delta v(T-1)}{1 - \delta} + \delta u_1 \geq u_{T-1} = v(T-1) + \delta u_T \iff \\
\frac{1}{\delta} \left[ \frac{\delta v(T-1)}{1 - \delta} - (T-1)v + \delta u_1 \right] \geq u_T.
\]

Thus, it is sufficient for \((T,T)\) to show that

\[
\frac{\delta}{1 - \delta} u_1 \geq \frac{1}{\delta} \left[ \frac{\delta v(T-1)}{1 - \delta} - (T-1)v + \delta u_1 \right]. \tag{6}
\]

Simplifying and noting that \(\delta \geq 1/2\), condition (6) holds if and only if \(\delta u_1 \geq v(T-1)\).

From \(u_1 \geq (1 - \delta^{T-1}) S_1\), it is sufficient to show that \(\delta(1 - \delta^{T-1}) S_1 \geq v(T-1)\). At \(T = 2\) this reduces to \(\delta \geq 1/2\). Now, we carry out an induction: assume it holds for \(T\) if \(\delta + \delta^{T-1} > 1\) and show that it holds for \(T+1\) if \(\delta + \delta^T > 1\). So, we must show that \(\delta(1 - \delta^T) S_1 \geq vT\) or, equivalently, that

\[
\delta(1 + \ldots + \delta^{T-1}) > T(1 - \delta).
\]

The condition at \(T\) is \(\delta(1 - \delta^{T-1}) S_1 \geq v(T-1)\), which holds if and only if

\[
(1 - \delta) + \delta(1 + \ldots + \delta^{T-2}) > T(1 - \delta).
\]

\[
\delta(1 + \ldots + \delta^{T-1}) > (1 - \delta) + \delta(1 + \ldots + \delta^{T-2}) \iff \delta + \delta^T > 1
\]

then establishes the induction. Thus, the \((T,T)\) condition holds, and we have therefore shown that the \(T\)–stage support satisfies (3) for all \((\sigma, \tau)\), where \(0 \leq \sigma \leq \tau\) and \(\tau \geq 1\).

To see that every buyer payoff, \(u_1 \in [0, \delta S_1]\) can be supported in this way, simply note that each \(\delta \in [1/2, 1]\) lies in exactly one of the \(\delta_\tau\) cutoff sequence intervals. Note that the cut-off sequence \(\delta_\tau\) is strictly increasing in \(\tau\), from \(\delta_1 = 1/2\) to \(\lim_{\tau \to \infty} \delta_\tau = 1\), and satisfies \(\delta_\tau^{-1} < 1 - \delta < \delta_\tau\) for \(\delta \in (\delta_{\tau-1}, \delta_\tau)\). With \(\delta \in [\delta_\tau, \delta_{\tau+1}]\), we then see that every \(u_1 \in [0, \delta S_1]\) lies in exactly one of the \([ (1 - \delta^{T-1}) S_1 ; (1 - \delta^T) S_1 ]\) intervals, where
$T$ ranges from 1 up to the index on the $\delta_T$ root.

6.3 Appendix C - Delay and Generational Equilibria

First we prove Propositions 2 and 3. Next, we examine the delay conditions and then provide the analysis of the off equilibrium support. Finally, we prove Proposition 4.

(i) Proofs of Propositions 2 and 3.

Proof of Proposition 2. By Lemma 1 we have $\pi_1 > 0$ in any equilibrium and therefore there is a first date, say $t$, at which a sale involving unit 1 takes place. If $t = 1$, we are done as equilibrium implies we have a 1-cycle equilibrium. So, consider $t > 1$. By construction, the maximal quality held by buyers before period $t$ is zero, so the state is $t$ and $q_t > 0$ results from sales in period $t$. A potential complication is that state $t$ corresponds to histories in which buyers acquired no quality units as well as histories in which they acquired some subset of $\{2, ..., t - 1\}$. By definition, however, Markovian behavior requires that the seller offer in state $t$ and buyer acceptance choice(s) are the same across these histories since strategies only depend on the state.

Suppose that the sale does not result in $q_t = t$ or, in other words, buyers do not acquire the full feasible set of units $\{1, 2, ..., t\}$. This implies that, for some $\tau$ where $1 \leq \tau < t$, buyers hold units $\{1, ..., \tau\}$ but not unit $\tau + 1$ as they enter period $t + 1$. Also, let $p$ denote the total payment of a buyer for all bundles purchased in state $t$. Finally, note that whether or not any of the units in $\{\tau + 2, ..., t\}$ are held by buyers before period $t$ or acquired in $t$, the state in period $t + 1$ will be $(t + 1, \tau)$.

By construction, the equilibrium buyer continuation payoff from state $t$ is

$$u_t = v\tau - p + \delta u(t + 1, \tau)$$

as the quality flow utility is $v\tau$ and the payment is $p$ in state $t$ and next period’s state is $(t + 1, \tau)$. We will show that the seller has a profitable deviation in period $t - 1$: offer the feasible upgrade bundle of units $1, ..., \tau$ for some price $\hat{p}$.

Before proceeding with the main argument, we need to develop two properties of buyer payoffs. First, the equilibrium path will follow a cycle, since state $(t + 1, \tau)$ has the same quality gap as state $(t + 1 - \tau, 0)$. Thus, the maximal buyer quality remains at $\tau$ until period $t + \tau$, when the state reaches $(t + \tau, \tau)$, at which time the maximal buyer quality rises to $2\tau$ and the cycle begins again. Equilibrium also implies that a buyer only needs to make purchases in the states where maximal quality rises in order to achieve
the equilibrium buyer payoff. As noted above, the history of play only matters to the extent that it impacts maximal buyer quality. Thus, the bundle(s) offered by the seller in any state of the form \((t + k\tau, k\tau)\), where \(k = 1, 2, \ldots\), must, at a minimum, always include the next \(\tau\) units of quality. In particular, this is true for the history where buyers hold exactly the first \(k\tau\) quality units (and no other units), since the maximal quality for this history is \(k\tau\). Thus, an individual buyer never needs to hold more than these units in order to be able to reach the next equilibrium path level of maximal quality via purchases in state \((t + k\tau, k\tau)\). Furthermore, such a buyer can always choose from the same offered bundle(s) and price as any other buyer. It follows directly that the continuation payoff of a buyer only depends on holding the current maximal quality and it is independent of whether the buyer holds higher but non-contiguous units. This is the first property of buyer payoffs that we will need.

The second property is that, in equilibrium, the seller only receives revenues in states of the form \((t + k\tau, k\tau)\), where \(k = 1, 2, \ldots\). In \((t + k\tau, k\tau)\), in equilibrium, the seller offer must include units \(\{k\tau + 1, \ldots, k\tau + \tau\}\) and all buyers must acquire these units. Thus, no buyer ever pays a positive price for any bundle in states \(1, 2, \ldots, \tau\), since only units in \(\{2, \ldots, \tau\}\) can be offered by the seller in equilibrium and these units will necessarily be acquired via the upgrade bundle in state \(t\) when buyers also acquire unit 1. The same logic then applies for the next \(\tau\) units, and so on.

We now proceed with the main deviation argument. To keep things simple, let us first consider the case where the history for state \(t\) has buyers holding no quality units. For the seller deviation in \(t - 1\), choose \(\hat{p}\) for the bundle of units \(\{1, \ldots, \tau\}\) so that

\[
\hat{u} \equiv v\tau - \hat{p} + \delta v\tau + \delta^2 u (t + 1, \tau) = \delta u_t + \epsilon,
\]

for a small \(\epsilon > 0\). Combining this with the earlier expression for \(u_t\), we find that \(\hat{p} = v\tau + \delta p - \epsilon\). We claim that in any equilibrium continuation after this offer, all buyers will accept. By symmetry of strategies, in response to this offer in state \(t - 1\), all buyers must either accept or reject. Suppose the buyer strategy calls for a rejection and consider the decision of an individual buyer. Because no other buyer accepts, the continuation state will be \(t\). By accepting, the payoff for an individual buyer is \(\delta u_t + \epsilon\), since the individual buyer receives a flow of \(v\tau - \hat{p} = -\delta p + \epsilon\) in \(t - 1\) and a flow of \(v\tau\) in \(t\).

Finally, consider the continuation state \((t + 1, \tau)\). A complication is that, in addition to the first \(\tau\) units, the outcome in state \(t\) may involve buyers acquiring units in \(\{\tau + 2, \ldots, t\}\). By making no purchases in \(t\), the deviating buyer will lack these units in the future while
other buyers possess them. This is of no consequence in equilibrium: a buyer holding exactly $\tau$ units obtains the same continuation payoff of $u(t+1, \tau)$.

Now, adding the terms in $t-1$, $t$, and $t+1$ for a deviating buyer, we arrive at $\hat{u}$ as in the above equation. Thus, accepting the seller’s deviation offer in $t-1$ for $\tau$ units results in a higher payoff than rejecting and waiting whenever $\varepsilon > 0$. Thus, all buyers rejecting the offer is never an equilibrium continuation. In a symmetric equilibrium, it must be that all buyers accept the offer in $t-1$.

Now, to see that the deviation is profitable for the seller, note that the payoff to the deviation offer in $t-1$ (where all buyers accept) is

$$\hat{\pi} = \hat{p} + \delta \pi(t, \tau) = v\tau + \delta p - \varepsilon + \delta^2 \pi(t+1, \tau) = v\tau - \varepsilon + \delta \pi_t > \delta \pi_t,$$

as follows from the definition of $\hat{p}$ and the equilibrium hypothesis for state $t$, which implies $\pi_{t-1} = \delta \pi_t$ and $\pi_t = p + \delta \pi(t+1, \tau)$. Thus, we cannot have $\tau < t$ in equilibrium.

Finally, we must verify that the same deviation will work for the seller when the history for state $t$ has buyers holding quality units (but not unit 1). By equilibrium, seller’s payoffs $\pi(t, \tau)$ and $\pi(t+1, \tau)$ are independent of these holdings. The only remaining possible complication is that the deviation offer in $t-1$ sacrifices revenues that would otherwise have been received by the seller from an offer of units in $\{2, \ldots, t-1\}$. Equilibrium, however, rules out any such revenues for the seller as we showed above.

**Proof of Proposition 3.** We first show that $\delta < 1/2$ implies $u_\tau = 0 \forall \tau \geq 1$. To begin, we establish that the utility sequence in any equilibrium satisfies the bound $u_\tau \leq \frac{e}{1-\delta} u_{\tau+1}$ $\forall \tau \geq 1$. The proof is by induction. For $\tau = 1$, consider state 1. If there is delay, then $u_1 = \delta u_2$ and hence $u_1 \leq \frac{e}{1-\delta} u_2$ follows directly since $\delta < 1$. If there is a sale in 1 at price $p$, the continuation state is $(2, 1)$ and buyer utility satisfies $(1-\delta)u_1 = v/(1-\delta) - p$. In equilibrium, buyers must reject any price above $p$, which leads to state 2, and the continuation payoff must satisfy the lower bound of $\delta u_2 \geq \frac{v}{1-\delta} - p = (1-\delta)u_1$, since a buyer who accepts always has the option of making no further purchases. The bound then follows directly. Now suppose the bound is valid for 1 through $\tau-1$. We must show this implies $u_\tau \leq \frac{e}{1-\delta} u_{\tau+1}$. If there is delay in $\tau$, then $u_\tau = \delta u_{\tau+1}$ and the bound follows directly from $\delta < 1$. If there is a sale in $\tau$, let $q \geq 1$ denote the number of units sold and let $p$ denote the price. Buyer utility is then $u_\tau = \frac{vq}{1-\delta} - p + \delta u_{t+1-q}$, since a sale of $q$ implies that $(\tau+1, q)$ is the continuation state. Equilibrium requires that buyers reject
any higher price, so we must have
\[ \delta u_{t+1} \geq \frac{vq}{1-\delta} - p = u_{\tau} - \delta u_{t+1-q}. \]

As before, this follows from the buyer’s option to make no subsequent purchases. If \( q = 1 \), this reduces to \( \delta u_{t+1} = (1-\delta)u_{\tau} \) and we are done. If \( q > 1 \), then the gap \( \tau + 1 - q \) is less than \( \tau \) and, by the induction hypothesis, \( u_{\tau+1-q} \leq \left( \frac{\delta}{1-\delta} \right)^{q-1} u_{\tau} \). For \( \delta < 1/2 \), we have \( \frac{\delta}{1-\delta} < 1 \) and, therefore, \( u_{\tau+1-q} \leq u_{\tau} \). Hence, \( \delta u_{t+1} \geq u_{\tau} - \delta u_{t+1-q} \geq (1-\delta)u_{\tau} \) and the bound is valid.

The next step is to apply the bound and show that utility is forced to 0. Fix any \( \sigma \) and consider \( \sigma > \tau \). The bound yields \( u_{\tau} \leq \left( \frac{\delta}{1-\delta} \right)^{\sigma-\tau} u_{\sigma} \). Recalling that utility is always bounded by the feasible total surplus, \( u_{\sigma} < S_{\sigma} \), we then have
\[ u_{\tau} \leq \left( \frac{\delta}{1-\delta} \right)^{\sigma-\tau} \left[ \frac{v\sigma}{1-\delta} + \frac{\delta v}{(1-\delta)^2} \right]. \]

Since \( \sigma \delta^\sigma \) goes to zero as \( \sigma \to \infty \), we must have \( u_{\tau} = 0 \) for any \( \tau \geq 1 \) in any equilibrium.

Finally, we show that all equilibria must be efficient for \( \delta < 1/2 \). From Proposition 2, every equilibrium follows a \( t - cycle \) for some \( t \geq 1 \) with joint surplus \( \Psi_t \). For \( t = 1 \), we have \( u_1 = 0 \) from above. Thus, the efficient equilibrium in which the seller captures the full surplus of \( S_1 \) is the only possible equilibrium for \( t = 1 \). Existence is straightforward to verify, since utility at zero satisfies the sufficient condition (3), even for \( \delta < 1/2 \). Consider \( t \geq 2 \). From Lemma 1, we have the flow dominance bound of \( \pi_1 \geq v/(1-\delta) \). A candidate \( t - cycle \) equilibrium is ruled out if \( v/(1-\delta) > \delta^{t-1} \Psi_t \). Comparing,
\[ \frac{v}{1-\delta} > \frac{\delta^{t-1} \Psi_t}{(1-\delta)(1-\delta^t)} \]
\[ \iff \frac{\delta(1-\delta^t)}{t \delta^t} > 1. \]

The numerator is strictly increasing in \( t \). The denominator is strictly decreasing in \( t \) for \( \delta < 1/2 \). It is readily verified that the inequality is valid at \( t = 3 \) for all \( \delta < 1/2 \). Thus, there is no \( t - cycle \) equilibrium for \( t \geq 3 \). To rule out \( t = 2 \) (the above inequality only rules out \( t = 2 \) for \( \delta \) below \( \sqrt{2} - 1 \approx .414 \)), suppose the seller offers 1 unit in state 1 for price \( p \in (v, v/(1-\delta)) \). Supporting strategies for a \( 2 - cycle \) equilibrium require that the buyers either accept or reject such an offer. Clearly, it is not an equilibrium for the buyers to reject this offer: by accepting, an individual buyer earns a strictly positive
payoff whereas rejecting yields zero. But if accepting is an equilibrium, the seller has a profitable deviation. For \( p \) sufficiently close to \( v/(1-\delta) \) we have \( p+\delta \pi_1 = p+\delta^2 \Psi_2 > \delta \Psi_2 \). Thus, there are no equilibria with delay when \( \delta < 1/2 \). ■

(ii) Delay Conditions.

The cut-off rules for buyers in states \( \tau < t \) are to reject any (upgrade) offer for \( \sigma \leq \tau \) units at a price greater than \( p(\sigma, \tau) \), where \( p(\sigma, \tau) \) satisfies

\[
\frac{v\sigma (1-\delta^{t-\tau})}{(1-\delta)} + \delta^{t-\tau} \max \left[ \frac{v\sigma}{(1-\delta)}, u_t \right] - \delta^{t-\tau} u_t \leq p(\sigma, \tau) \leq \frac{v\sigma}{(1-\delta)} + \delta^{t-(\tau-\sigma)} u_t. \tag{7}
\]

Note that we have used the continuation properties \( \delta^{t-\tau} u_t = u(\tau+1,0) \) and \( \delta^{t-(\tau-\sigma)} u_t = u(\tau+1-\sigma,0) \) as the first sale on the equilibrium path occurs in state \( t \). The left-hand-side of (7) provides the lower bound on the cut-off price; otherwise, an individual buyer would be better off accepting when other buyers reject. This bound reflects the difference in gross surplus for an individual buyer between buying and rejecting, since other buyers are expected to reject and, hence, the continuation state would be \( \tau+1 \). The first term is the buyer’s interim flow payoff, from \( \tau \) until \( t \), generated by \( \sigma \) units while the second terms correspond to the option of buying (or not) with the other buyers once the state reaches \( t \). The right-hand-side provides an upper bound on the cut-off price; if it failed, an individual buyer would be better off rejecting when others accept. Given that all other buyers are expected to buy the package and the state will be \( (t+1, \sigma) \), the bound reflects the payoff difference for an individual buyer between buying and not buying the offer for \( \sigma \) units. Clearly, there exist \( p(\sigma, \tau) \) that satisfy (7) for \( \sigma \leq \tau \) and \( \tau = 1, \ldots, t-1 \). As with efficient equilibria, this is due to the implicit coordination on cut-off prices among buyers.

Given the cut-off prices, delay must be optimal for the seller. Thus, in state \( \tau \) for \( \tau < t \), the seller prefers the equilibrium path payoff of \( \pi_\tau = \delta^{t-\tau} \pi_t \) to selling \( \sigma \) units in state \( \tau \) at a price of \( p = p(\sigma, \tau) \) and receiving a payoff of \( p(\sigma, \tau) + \delta^{t-(\tau-\sigma)} \pi_t \). Hence,

\[
\delta^{t-\tau} (1-\delta^\sigma) \pi_t \geq p(\sigma, \tau) \tag{8}
\]

for \( \sigma = 1, \ldots, \tau \) and \( \tau = 1, \ldots, t-1 \). We then have

Lemma A5. If the buyer and seller delay conditions, (7) and (8), hold for \( \sigma = \tau \), at each \( \tau = 1, \ldots, t-1 \), then the conditions hold for all feasible pairs \((\sigma, \tau)\).

As a result, we need only find \( t-1 \) distinct prices, \( p(1,1), \ldots, p(t-1, t-1) \) and it is sufficient to deter the seller from selling the maximum feasible number of units, “cashing-in,” in each delay state. Intuitively, if it is not profitable to sell \( \tau \) units in state \( \tau \), the
first time it is possible to do so, it will not be profitable to sell \( \tau \) units in a later delay state. For example, if the seller does not offer one unit in state 1, then there will be no temptation to sell one unit at a later date when the additional unsold units will create a longer delay in the continuation state.

Combining (7) and (8), supporting prices exist if and only if

\[
\frac{v\tau(1 - \delta^{t - \tau})}{(1 - \delta)} + \delta^{t - \tau} \max \left[ \frac{v\tau}{(1 - \delta)}, u_t \right] - \delta^{t - \tau} u_t \leq p(\tau, \tau) \leq \delta^{t - \tau}(1 - \delta^\tau) \pi_t \tag{9}
\]

for \( \tau = 1, \ldots, t - 1 \). Condition (5) in the text follows from (9) upon simplifying and substituting with \( \pi_t + u_t = \Psi_t \).

**Proof of Lemma 4.** In (5), \( \tau \) assumes integer values \( 1, \ldots, t - 1 \). Let us replace \( \tau \) with a continuous variable, \( x \), that assumes values in the interval \([0, t]\). This greatly simplifies the derivation of the sufficiency condition. It is useful to define three functions:

\[
A(x, u, \delta, t) \equiv (\delta^{t - x} - \delta^t) \left[ \frac{v\tau}{(1 - \delta)} - u \right]
\]

\[
B(x, \delta, t) \equiv \frac{v\tau}{1 - \delta}(1 - \delta^{t - x})
\]

\[
C(x, u, \delta, t) \equiv \frac{v\tau}{1 - \delta} - \delta^{t - x} u,
\]

where \( u \equiv u_t \). For \((\delta, t)\), consider \( u \in [0, \frac{vt}{1 - \delta}] \); we treat the case of \( u > \frac{vt}{1 - \delta} \) later in the proof. In terms of \( x \), condition (5) becomes

\[
A(x, u, \delta, t) \geq B(x, \delta, t) \quad \text{for} \quad 0 \leq x \leq \frac{(1 - \delta)u}{v},
\]

\[
A(x, u, \delta, t) \geq C(x, u, \delta, t) \quad \text{for} \quad \frac{(1 - \delta)u}{v} < x \leq t.
\]

First, consider when \( A(x, u, \delta, t) \geq B(x, \delta, t) \) for all \( x \) in the interval \( [0, \frac{(1 - \delta)u}{v}] \). Suppressing arguments, note that \( A \) is increasing and convex in \( x \), and equals 0 when \( x = 0 \), while \( B \) is strictly concave in \( x \) and equals 0 when \( x = 0 \). Thus, if we have \( \frac{\partial A}{\partial x} \geq \frac{\partial B}{\partial x} \) at \( x = 0 \), then \( A \geq B \) must hold for all positive \( x \). Calculating the partial derivatives yields

\[
\bar{u}^A \equiv \frac{v}{(1 - \delta)} \left[ \frac{t}{1 - \delta^t} - \frac{1 - \delta^t}{(- \ln \delta) \delta^t} \right] \geq u. \tag{10}
\]
Next, $A(x, u, \delta, t) \geq C(x, u, \delta, t)$ for all $x \in \left[\frac{(1-\delta)u}{v}, t\right]$ if
\[
\frac{\delta^t(1-\delta)u}{v} \geq x - t(\delta^t - x - \delta^t) \equiv h(x, t, \delta).
\] (11)

It is easy to show $h(x, t, \delta)$ is strictly concave and equals 0 at $x = 0$ and $x = t$. Thus, $h(x, t, \delta)$ has a unique interior maximum at some $x^*(\delta, t)$ which is implicitly defined by
\[
\delta^t(1-\delta) = x^* - t(\delta^t - x^* - \delta^t) \equiv h(x, t, \delta).
\]

Refer to Figure 3. Now, define $u^A$ by
\[
u^A = \frac{v}{\delta^t(1-\delta)} h(x^*, t, \delta) = \Psi_t + \frac{v(1-a(\delta^t))}{\delta^t(1-\delta) \ln \delta^t},
\]
where $a(\delta^t)$ is defined below. Then any $u \geq u^A$ will satisfy (11).

To find when (10) and (11) hold at a candidate $u$, we have $\bar{u}^A \geq u^A \Leftrightarrow \delta^t \geq a(\delta^t)$, where we define $a(d) = -\ln \left[-\frac{d \ln(d)}{1-d}\right]$ for $d \in (0, 1)$. It is easy to show that $a(d)$ is strictly decreasing for $d \in (0, 1)$ and, by L’Hospital’s Rule, that $\lim_{d \to 1} a(d) = 0$. Then, the equation $d = a(\delta^t)$ has a unique root, $d^*$ in $(0, 1)$. Thus, we have established that $\delta^t \geq d^*$ implies the interval $(u^A, \bar{u}^A)$ is non-empty.
The last step of the proof is to consider the range of values for \( u \) that can be supported. A straightforward argument establishes that \( u^A < \frac{vt}{1-\delta} \) holds for \( \delta^t \geq d^* \). The comparison of \( u^A \) with \( \frac{vt}{1-\delta} \) reveals that \( u^A \) crosses \( \frac{vt}{1-\delta} \) exactly one time, from below, at the root of the equation \(-\ln \delta^t = (\delta^{-t} - 1)^2\); numerically, the root is .572, which exceeds \( d^* \). Thus, for \( \delta^t \) below this root, we have \( \pi^A < \frac{vt}{1-\delta} \) and any \( u \in (u^A, \pi^A) \) satisfies the delay condition (5). For \( \delta^t \) above this root, we have \( u^A > \frac{vt}{1-\delta} > u^A \) and there are two cases. First, by the above analysis, any \( u \in (u^A, \pi^A) \) satisfies the delay condition (5). Second, for the case of \( u \in [\frac{vt}{1-\delta}, \pi^A] \), the delay condition (5) requires that \( A(x, u, \delta, t) \geq B(x, \delta, t) \) for all \( x \) in the interval \([0, t]\); note that since \( u \) is large, the case of \( \frac{(1-\delta)u}{v} < x \leq t \) never arises. Then \( A(x, u, \delta, t) \geq B(x, \delta, t) \) and utilities in this payoff range satisfy (5).

(iii) Off Equilibrium Support (Cash-in offers)

Now, we deal with off equilibrium states \((\tau, 0)\) where \( \tau > t \). Since we are using cash-in supports, the payoffs are \( \pi_\tau = p_\tau + \delta \pi_1 \) for the seller and \( u_\tau = v\tau - p_\tau + \delta u(\tau + 1, \tau) = \frac{vt}{1-\delta} - p_\tau + \delta u_1 \) for buyers. Note that from \((\tau, 0)\) the surplus on the continuation path is

\[
\Psi_\tau \equiv \frac{vt}{1-\delta} + \delta \Psi_1 = \pi_\tau + u_\tau \text{ for } \tau > t
\]

and that cash-in states contrast with delay states, \( \tau < t \), where we have \( \Psi_\tau \equiv \delta^{t-\tau} \Psi_t \).

Recall from the text that we must satisfy (1) and (2). Thus, we seek a utility sequence that satisfies the analog of (3), as given by

\[
\Psi_\tau - \delta \Psi_{\tau+1-\sigma} \geq u_\tau - \delta u_{\tau+1-\sigma} + g(\sigma, u_{\tau+1}) \tag{12}
\]

for \( \tau \geq t \) and \( \sigma = 0, \ldots, \tau \). Analogous to our support utility sequence for efficient equilibria, we define a \( T \)-stage support sequence, where \( T \geq t + 1 \), for inefficient equilibria by

\[
u_\tau = \begin{cases} 
\delta u_{\tau+1} + \Psi_\tau - \delta \Psi_{\tau+1} & \text{for } \tau = 1, \ldots, T - 1 \\
u_T & \text{for } \tau \geq T.
\end{cases}
\]

For the special case of \( T = t \), we specify a constant sequence \( u_\tau = u_t \) for all \( \tau \). As before, the seller is indifferent with respect to delay and the cash-in up to state \( T \), and the seller prefers to cash-in in state \( T \) provided that the buyers’ utilities, \( u_T \), are not too large.

We can satisfy (12) by an appropriate choice of the support length \( T \) for a given \( \delta \) and buyer payoff. Recalling Figure 1 for efficient equilibria, we find a similar structure for the relationship between \( u_t, \delta \), and the length of a \( T \). For any given \( \delta \) we can support successively higher buyer payoffs by increasing the support horizon, \( T \). In particular as
u_t rises, then T must also rise until we achieve $u_t \in [\Psi_t - \delta^{T-t}S_1, \Psi_t - \delta^{T-t+1}S_1]$. Thus,

**Lemma A6** Let $T \geq t$ and suppose $\delta + \delta^T \geq 1$. Let buyer utilities follow a $T$-stage support, where $u_t$ satisfies (i) $0 \leq u_t \leq \Psi_t - \delta S_1$ for $T = t$ and (ii) $\Psi_t - \delta^{T-t}S_1 \leq u_t \leq \Psi_t - \delta^{T-t+1}S_1$ for $T > t$. Then (12) holds for every $\tau \geq t$ and $\sigma = 0, \ldots, \tau$.

(iv) Equilibrium Existence

The following lemma provides sufficient conditions for equilibrium existence.

**Lemma A7** Suppose a sequence of buyer utilities $u_{\tau}$ satisfies (5) and (12) for some $t \geq 2$ and for all $\sigma$ and $\tau$, such that $0 \leq \sigma \leq \tau$ for any $\tau \geq 1$. Then there exists an inefficient $t$-cycle equilibrium with a buyer payoff of $u_1$.

**Proof.** The delay conditions are satisfied by Lemma 4 for $u_t \in (u^A, \pi^A)$. With (12) satisfied as well, we know that no single-bundle deviation is profitable for the seller. As with the efficient case, we must now show that the seller cannot profitably deviate to an upgrade offer set in state $\tau$. While the logic is similar to that for efficient equilibria, the proof is more complicated and we only provide a brief sketch. We must distinguish between delay states, where $\tau < t$, and support states where $\tau \geq t$. We define a BCE based on the same buyer preference relation $\succeq_B$ but now $u(\tau + 1, b)$ and $u(\tau + 1, b | i)$ must account for whether there is a delay or a cash-in the continuation state. We can then show the same arg max construction yields a BCE. Next, note that any upgrade offer can be written in the form $B = I \cup J \cup K$ where an upgrade to $b$ is in $I$ if $p_b \leq G_b$, in $J$ if $p_b \geq H_b$, and in $K$ if $G_b < p_b < H_b$. The threshold prices, following the buyer and seller decisions in the text at (12), are specified by

\[
G_b = \begin{cases} 
\frac{v_b}{1 - \delta} (1 - \delta^{t-\tau}) + \delta^{t-\tau} \max \left\{ \frac{v_b}{1 - \delta}, u_t \right\} - \delta^{t-\tau} u_t & \text{if } \tau < t \\
v_b + \delta \max \left\{ \frac{v_b}{1 - \delta}, u_{\tau+1} \right\} - \delta u_{\tau+1} & \text{if } \tau \geq t
\end{cases}
\]

and

\[
H_b = \begin{cases} 
\delta^{t-\tau} (1 - \delta^b) (\psi_t - u_t) & \text{if } \tau < t \\
(\psi_{\tau} - u_\tau) - \delta (\psi_{\tau+1-b} - u_{\tau+1-b}) & \text{if } \tau \geq t
\end{cases}
\]

The proof then proceeds by considering the various cases for the decomposition $B = I \cup J \cup K$. For instance, if $J$ is empty then we are done as the argmax property can be applied to $I \cup K$ and we can specify that buyers choose this BCE, which is priced below $H_b$. The most difficult case arises when none of the component sets in $I \cup J \cup K$ is empty. We can show that there always exists a BCE priced below $H_b$ in this case.

The next step is to specify continuation equilibrium strategies at asymmetric buyer states, $(\tau, \alpha)$; see part (i) of Appendix B for the definition. These states are complicated
by the fact that joint surplus is not maximized in an inefficient \( t \)-cycle equilibrium. This contrasts with the efficient case, where the \((\tau, \alpha)\) offers fully extracted buyers relative to equilibrium payoffs for the continuation state of \((\tau + 1, \tau)\), since feasibility will no longer imply this is an optimal offer for the seller. Instead, we specify a continuation equilibrium that leaves buyers asymmetric. Thus, this part of the support neither returns to the equilibrium path nor to a cash-in support state.

Let \( \alpha_m \equiv \inf \{ \alpha_\sigma \mid \alpha_\sigma > 0, \} \) denote the smallest non-zero mass of buyers in \( \alpha \) and if the inf is not unique let \( m \) be the largest index (e.g., if \( \alpha_\sigma = 1/\tau \) for all \( \sigma \) then \( \alpha_m = 1/\tau \) and \( m = \tau - 1 \)). We specify that the seller offers a collection of upgrades \( \{(b_\sigma, p_\sigma)\}_{\sigma = 0, \ldots, \tau - 1} \) with prices of

\[
p_m = \frac{v(\tau - 1 - m)}{1 - \delta}, \quad \text{and} \quad p_\sigma = \frac{v(\tau - \sigma)}{1 - \delta} \quad \text{for} \ \sigma \neq m.
\]

It is straightforward to verify that all buyers optimally accept and that the continuation state is of the form \((\tau + 1, \alpha')\) where \( \alpha' = (0, \ldots, 0, \alpha_m, 1 - \alpha_m) \). Thus, in the state for period \( \tau + 1 \) the smallest mass group is at quality level \( \tau - 1 \) units, while all other buyers are at \( \tau \). We then apply the same logic to specify offers for \((\tau + 1, \alpha')\) and all resulting states so that the distribution of buyers is always concentrated in the same proportions \((\alpha_m, 1 - \alpha_m)\) at the two highest quality levels.

The argument that this is an optimal strategy for the seller is involved, so we only offer a brief sketch. For any arbitrary seller offer in \((\tau, \alpha)\) and pattern of buyer choices, one can bound the seller payoff via individual rationality of the buyer choices, using the hypothesis that the seller makes the offer in the continuation state in period \( \tau + 1 \) that induces a concentration of buyers at the two highest quality levels (if the buyer choices lead to a degenerate distribution, then payoffs must instead follow the \( t \)-cycle equilibrium path and cash-in support). Then, since the repeated concentration of buyers in proportions \((\alpha_m, 1 - \alpha_m)\) at the two highest quality levels allows the seller to extract all surplus generated on this path, the associated seller payoff can be shown to be at least as large as the bound derived from individual buyer rationality for any other seller offer. In this regard, note that every equilibrium with delay \((t \geq 2)\) has a buyer quality level that is strictly below the mean for the \((\alpha_m, 1 - \alpha_m)\) concentration.

Finally, finding a BCE for an arbitrary upgrade offer \( B \) in state \((\tau, \alpha)\) follows the same basic logic as that for an efficient equilibrium. The only difference is that we must take care to account for any possible delay if the continuation state is degenerate.
Proof of Proposition 4. Define $\Gamma(t, T, \delta) \equiv \Psi_t - \delta^{T-t}S_1$, the upper bound on payoffs that can be supported with a $T$-stage support. We claim $\Gamma(t, T, \delta) \geq \overline{w}^A(t, \delta)$ holds whenever $\delta$ satisfies the delay conditions. Comparing, $\Gamma(t, T, \delta) \geq \overline{w}^A(t, \delta) \Leftrightarrow$

\[ \frac{(1 - \delta^t)(1 - \delta)}{\delta^t} \geq -\ln \delta \]  

(13)

It is clear that both sides of (13) are positive, falling in $\delta$ and equal to 0 at $\delta = 1$. The l.h.s. of (13) is rising in $T$ and $t$. Differentiating both sides of (13) with respect to $\delta$ we find that the slope of the l.h.s. is greater/less than the slope of the r.h.s. if and only if

\[ \delta^T + \delta^{t+1} \geq T(1 - \delta) + \delta - (T - t)\delta^t(1 - \delta) \]  

(14)

In (14), the l.h.s. is positive and convex in $\delta$ and the r.h.s. is decreasing in $\delta$. Thus, the l.h.s. and the r.h.s. of (13) cross a single time in $\delta$. It easy to see that (13) is negative in a neighborhood of 1 and positive at $\delta = \sqrt[4]{d^t}$ for any $T \geq t$. So, there is a $\gamma(t, T) \in (\sqrt[4]{d^t}, 1)$ such that (13) holds whenever $\delta \leq \gamma(t, T)$. Clearly, $\gamma(t, T)$ is increasing in $T$.

Now, we will show that $\gamma(t, T) > \delta^t$ for all $t \geq 2$. At $\gamma = \gamma(t, T)$ we have

\[ \frac{1 - \gamma}{\gamma^t} = \frac{-\ln \gamma}{1 - \gamma^t} \]

The r.h.s. is falling in $\gamma$. Since for any $\gamma$ this is less than 1, $\gamma(t, T) > \delta^t$. ■