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On properties of discrete \((r, q)\) and \((s, T)\) inventory systems

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We consider single-item \((r, q)\) and \((s, T)\) inventory systems with integer-valued demand processes. While most of the inventory literature studies continuous approximations of these models and establishes joint convexity properties of the policy parameters in the continuous space, we show that these properties no longer hold in the discrete space, in the sense of linear interpolation extension and \(L^2\)-convexity. This non-convexity can lead to failure of optimization techniques based on local optimality to obtain the optimal inventory policies. It can also make certain comparative properties established previously using continuous variables invalid. We revise these properties in the discrete space.

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1. Introduction

This paper is concerned with two basic single-item inventory systems with integer-valued demand processes. We first investigate the properties of the key performance measures, including the recently developed discrete convexity concepts (see, e.g., Murota, 2003). We then revisit several other properties previously established assuming demand is continuous.

The primary focus of the paper (Sections 2, 3, and 4) is on the continuous-review \((r, q)\) inventory system where \(r\) is the reorder point and \(q\) is the order quantity. The demand process is a Poisson process with rate \(\lambda\). Unsatisfied demand is backlogged. The replenishment leadtimes are exogenous and sequential (see Zipkin, 2000, Chapter 7 for detail), having identical distribution as the generic random variable \(L\). This leadtime model includes the constant leadtime, \(L\), for short – and therefore a global optimum is guaranteed to be locally optimal – a property we call Property LC for short – and therefore a global optimum can be found by descent algorithms. However, while the notion of convexity has long been established for functions on the real vector space, it was not so for functions defined on the integer vector space (termed discrete convexity) until more recently (see, e.g., Murota, 2003). This partly explains why in the inventory literature most studies on the properties of \((r, q)\) systems assume continuous demands (Hariga and Ben-Daya, 1999; Hill, 1996; Hill et al., 2007; Jassen et al., 1998). It was also argued that the cost function for continuous demands can be an adequate approximation of the cost function for discrete demands only when the order quantity \(q\) is large enough (Zheng, 1992).

and the average on-hand inventory equals

\[ l(r, q) = E[IN^-] = r + (q + 1)/2 - E[D] + B(r, q). \]

(2)

To avoid the trivial case and without loss of generality, we assume that \(P_r(D = 0) < 1\). There are linear inventory-holding and backorder-penalty costs at unit rates \(h > 0\) and \(p > 0\), respectively. Let \(K\) be the fixed ordering cost for each replenishment order. Then, the expected long-run average system cost is

\[ C(r, q) = \frac{h}{q} + h(l(r, q) + pB(r, q) = \frac{h}{q} + \frac{iK + \sum_{y=r+1}^{\infty} G(y)}{q}, \]

(3)

where

\[ G(y) = hE[(y - D)^+] + pE[(D - y)^+]. \]

(4)

To find the optimal \((r^*, q^*)\) that minimizes \(C(r, q)\), a natural and traditional approach is to establish the joint convexity of \(C\) in \((r, q)\), so that a local optimum is guaranteed to be globally optimal – a property we call Property LC for short – and therefore a global optimum can be found by descent algorithms. However, while the notion of convexity has long been established for functions on the real vector space, it was not so for functions defined on the integer vector space (termed discrete convexity) until more recently (see, e.g., Murota, 2003). This partly explains why in the inventory literature most studies on the properties of \((r, q)\) systems assume continuous demands (Hariga and Ben-Daya, 1999; Hill, 1996; Hill et al., 2007; Jassen et al., 1998). It was also argued that the cost function for continuous demands can be an adequate approximation of the cost function for discrete demands only when the order quantity \(q\) is large enough (Zheng, 1992).

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For the continuous-demand model, it was shown that, as long as the cumulative demand is a nondecreasing stochastic process with stationary increments and continuous sample paths, \(IP\) is uniformly distributed on the interval \([r, r + q]\), and \(D\) and \(IP\) are independent of each other (Serfozo and Stidham, 1978; Zipkin, 1986). Under these conditions, Zipkin (1986) proves that \(B(r, q)\) and \(I(r, q)\) are jointly convex in \((r, q)\). His proof is given by explicitly expressing \(B(r, q)\) in terms of integrals involving the probability distribution function of \(D\), taking second partial derivatives and then showing the Hessian matrix is nonnegative definite. Zhang (1992) develops a similar proof for the discrete \((r, q)\) system and compares the exact optimal solution with the approximate solution obtained from using the EQQ formula. He shows that the EQQ heuristic is a lower bound on the optimal order quantity and the relative cost error by employing this heuristic is bounded by \(1/8 = 0.125\). This error bound has subsequently been improved by Axšeter (1996) to \((\sqrt{5} - 2)/2 \approx 0.118\) and by Gallego (1998) to 0.0607 for a variant of the EQQ heuristic. Rao (2003) to 0.0607 for a variant of the EOQ heuristic. For the continuous-demand model, it was shown that, as long as \((x, y), (x, y) + (\delta, \delta)\) are jointly convex in \((x, y)\), the expected long-run average cost of the continuous \((r, q)\) system is

\[
\bar{C}(r, q) = \frac{\int K + \int_q^\infty G(y)dy}{q},
\]

where \(G(.)\) is given by (4).

### 2. Nonconvexity of discrete \((r, q)\) systems

In this section, we examine two notions on discrete convexity that assures Property LG, i.e., guarantees local optimality to be global optimality. Before introducing these notions, we first define some notations and present some preliminaries.

Throughout the paper, denote \(\mathcal{R}\) to be the real line, \(\mathbb{R}^+\) the nonnegative real line, \(\mathcal{Z}\) the set of all integers and \(\mathcal{Z}^+\) the set of nonnegative integers. Let \(e_1 = (1,0), e_2 = (0,1), d_1 = (1, -1), d_2 = (-1, 1), \) and \(F = \{-e_1, d_1, e_2\}\). For any given function \(f\) defined on \(S = \mathcal{Z} \times \mathcal{Z}^+\) define

\[
\Delta_1 f(x, y) = f(x - 1, y) - f(x, y),
\]

\[
\Delta_2 f(x, y) = f(x + 1, y) - f(x, y),
\]

\[
\Delta_3 f(x, y) = f(x, y + 1) - f(x, y),
\]

\[
\Delta_4 f(x, y) = \Delta_6 f(x, y) = \Delta_5 f(x, y) - \Delta_3 f(x, y).
\]

For any given \((r, q)\) policy, define

\[
V(r, u) = E[D - r - u] = \bar{V}(r + u).
\]

The following properties are straightforward to obtain but useful for later development.

**Lemma 1.** For any \(x, u \in \mathcal{Z}\) and \(q \in \mathcal{Z}\) with \(q \geq 1:\)

(i) \(V(x)\) is convex and \(AV(x) = V(x + 1) - V(x) = -Pr(D \geq x + 1);\)

(ii) \(V(r + u) = V(r, u + 1);\)

(iii) \(V(r, u) - V(r, w + 1) = \sum_{i \geq r}^n Pr(D > r + i), \ w \geq u;\)

(iv) \(B(r, q) = \frac{1}{q} \sum_{i \geq r}^n V(r, u);\)

(v) \(\Delta_6 B(r, q) = (1/q)(q + 1) \sum_{i \geq r}^n (V(q + 1) - V(r, u));\)

(vi) \(\Delta_6 B(r, q) = (1/q)(V(r, q + 1) - V(r, 1)).\)

#### 2.1. Linear interpolation

**Definition 1.** For any given function \(f\) defined on \(S\), a function \(\bar{f}\) defined on \(\mathcal{R} \times \mathcal{R}^+\) is called its extension if \(\bar{f}\) agrees with \(f\) on \(S\). Let \(\bar{f}\) be the linear interpolation extension of \(f\) on \(\mathcal{R} \times \mathbb{R}^+\). The function \(f\) is called convex in the sense of linear interpolation extension if its linear interpolation extension \(\bar{f}\) is convex (see Altman et al., 2000 or Murota, 2005).

The following results are due to Altman et al. (2000), Hajek (1985), and Glasserman and Yao (1994).

**Lemma 2.**

(i) The real-valued function \(f\) defined on \(S\) is convex in the sense of linear interpolation extension if and only if it is multimodular, i.e.,

\[
\Delta_4 \Delta_6 f(x, y) \leq 0,
\]

when \((x, y), (x, y) + (\delta_x, \delta_y)\) and \((x, y) + (\delta_i, \delta_j)\) are all elements in \(S\) for \(\delta_i, \delta_j \in F\).
Proposition 1. The functions \( B(r, q), I(r, q) \) and \( C(r, q) \) are not jointly convex in the sense of linear interpolation extension, or, equivalently, not multimodal.

Proof. In order to show from Lemma 2 that \( B(r, q) \) is not jointly convex in the sense of linear interpolation extension, it is sufficient to prove that there exists \( (r_0, q_0) \in S \) such that
\[
\Delta_2 B(r_0, q_0) = \Delta_2 B(r_0 + 1, q_0) - \Delta_2 B(r_0, q_0 + 1) > 0.
\]
(7)

By Lemma 1 (v),
\[
\begin{align*}
\Delta_2 B(r + 1, q) & - \Delta_2 B(r, q + 1) \\
& = \sum_{q=1}^{q+1} (V(r, q + 2) - V(r, q)) \\
& = \sum_{q=1}^{q+1} (V(r, 1) - V(r, q + 1)) \\
& = \sum_{q=1}^{q+1} (V(r, 1) - V(r, 1 + 1)) \\
& = \frac{q(q + 1)(q + 2)}{q(q + 1)(q + 2)}.
\end{align*}
\]
(8)

Note that from Lemma 1 (iii),
\[
(V(r, 1) - V(r, u + 1)) - (V(r, u + 1) - V(r, q + 2))
\]
\[
= \sum_{i=1}^{u} Pr(D > i + \ell) - \sum_{i=1}^{q+1} Pr(D > i + \ell).
\]

For \( q = 1 \), we have
\[
\begin{align*}
\sum_{i=0}^{q} (V(r, 1) - V(r, u + 1)) & - \sum_{i=0}^{q} (V(r, u + 1) - V(r, q + 2)) \\
& = Pr(D > r + 1) - Pr(D > r + 2) = Pr(D = r + 2).
\end{align*}
\]
(9)

But we can always find \( r \) such that \( Pr(D = r + 2) > 0 \). Thus, (8) and (9) imply (7).

Note that, from (2), \( \Delta_2 I(r, q) = \Delta_2 B(r, q) + 1/2 \). Thus, the nonconvexity of \( B(r, q) \) implies the nonconvexity of \( I(r, q) \).

Finally, from (3),
\[
\Delta_2 C(r + 1, q) - \Delta_2 C(r, q + 1) = \left( \frac{2j}{q + 1} - \frac{jK}{q} \right) (p + h) \\
\times (\Delta_2 B(r + 1, q) - \Delta_2 B(r, q + 1)).
\]

Take \( K = 1, \ell = 1, Pr(L = 1) = 1, p = 10, h = 1, r_0 = 0 \) and \( q_0 = 10 \), then
\[
\Delta_2 C(r_0 + 1, q_0) - \Delta_2 C(r_0, q_0 + 1) = (35.5601)/(10 \cdot 11 \cdot 12) > 0.
\]

This implies that \( C(r, q) \) is not jointly convex. □

2.2. \( L^1 \)-convexity

In this subsection, we look at another notion of discrete convexity that assures Property \( LG - L^1 \)-convexity (see Murota, 2003). Among several equivalent characterizations of \( L^1 \)-convexity, perhaps the following midpoint convexity is most geometrically appealing:

Definition 2. A function \( f \) defined on \( S \) is \( L^1 \)-convex if the following midpoint property holds for all \( (x_1, y_1), (x_2, y_2) \in S \):
\[
f(x_1, y_1) + f(x_2, y_2) \geq f\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\
+ f\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right),
\]
with \( \max\{x_1 - x_2, y_1 - y_2\} \leq 2 \).

A related property is submodularity:

Definition 3. A function \( f \) defined on \( S \) is supermodular if
\[
f((x_1, y_1) \land (x_2, y_2)) + f((x_1, y_1) \lor (x_2, y_2)) \geq f(x_1, y_1) + f(x_2, y_2),
\]
where \( (x_1, y_1), (x_2, y_2) \in S \). Similarly, a function \( f \) defined on \( S \) is submodular if \(-f\) is supermodular.

The following properties are due to Hajek (1985) and Murota (2003).

Lemma 3

(i) A function \( f \) defined on \( S \) is supermodular if and only if for \( (x, y) \in S \), \( \Delta_2 f(x, y) - \Delta_2 f(x + 1, y) \leq 0 \).

(ii) \( L^1 \)-convexity assures Property \( LG \) and implies submodularity.

It has been shown in both theory and practice that, sometimes although a function is not \( L^1 \)-convex in its original coordinates, it is \( L^1 \)-convex after a coordinate transformation. For this reason, we are also interested in studying natural linear transformations of the policy variables in the \( (r, q) \) system. Let \( s = r + q \) be the order-up-to level and make a variable change \( q = s - r \), the performance measures can be expressed as
\[
B(r, s) = B(r, s - r), \quad I(r, s) = I(r, s - r), \quad C(r, s) = C(r, s - r).
\]

Alternatively, expressing the performance measures as functions of \( (q, s) \), we have
\[
B(q, s) = B(q - s, q), \quad I(q, s) = I(q - s, q), \quad C(q, s) = C(q, s - q).
\]

Before proceeding to examine \( L^1 \)-convexity of the discrete system, to facilitate understanding of the concept of \( L^1 \)-convexity, we first consider its extension to the real vector space; see, for example, Murota (2003), Murota and ShiouY (2004), and Zipkin (2008).

Lemma 4. For a twice differentiable real function \( f(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R} \), \( L^1 \)-convexity means the combination of the following three properties:

(i) Joint convexity:
\[
\frac{\partial f}{\partial x} \geq 0, \quad \frac{\partial f}{\partial y} \geq 0, \quad \mathbf{and} \quad \frac{\partial^2 f}{\partial x \partial y} - \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \geq 0;
\]

(ii) Submodularity: \( \frac{\partial^2 f}{\partial x \partial y} \leq 0 \).

(iii) Diagonal dominance of the Hessian matrix:
\[
\frac{\partial^2 f}{\partial x^2} \geq \frac{\partial^2 f}{\partial y^2} \geq \frac{\partial^2 f}{\partial x \partial y}.
\]

Thus, \( L^1 \)-convexity is much stronger than the usual joint convexity.

For the continuous \( (r, q) \) policy, we know that \( \tilde{B}(r, q), \tilde{I}(r, q) \) and \( \tilde{C}(r, q) \) are all jointly convex, so Lemma 4 (i) holds. Checking through the partial derivatives as in Zipkin (1986), we can show the following:

Proposition 2. For the continuous-demand model:

(i) \( \tilde{B}(r, q), \tilde{I}(r, q) \) and \( \tilde{C}(r, q) \) are supermodular. Their Hessian matrices are Leontief\(^\dagger\) and diagonally dominant for the variable \( r \) but not for the variable \( q \). They are not \( L^1 \)-convex;

\(^\dagger\) A matrix \( A \) is called Leontief if it has exactly one positive element in each column and there is a nonnegative vector \( x \) such that \( Ax > 0 \). See Veinott (1969) and Koehler and Winston (1980) for applications of Leontief matrices in production and inventory systems.
(ii) $\mathbb{E}(q,s), \mathbb{I}(q,s)$ and $\mathcal{C}(q,s)$ are jointly convex and submodular. Their Hessian matrices are diagonally dominant for the variable $s$ but not for the variable $q$. They are not $L$-convex.

Thus, the average cost of the continuous $(r,q)$ model is not $L$-convex, even though it is jointly convex. This implies that while joint convexity in the continuous model is sufficient for Property LG, we need several more properties of the cost function when we optimize in the discrete space and therefore it is much harder to achieve. The proposition also reveals exactly which properties of the $L$-convexity break down in the $(r,q)$ model. In addition, note that for continuous demand, Zipkin (2008) shows that, in the single-item lost-sales model, while the original single-stage lost-sales systems do not necessarily possess $L$-convexity, certain linear transformation of the state variables creates the property. 

Huh and Janakiraman (2010) extend this finding to the serial lost-sales systems, and Pang et al. (2012) show similar properties in the inventory-pricing problem. Unfortunately, our results above show that the natural linear transformations of the policy variables in the continuous $(r,q)$ system also fail to possess $L$-convexity.

We now examine the properties of the discrete system. We have

**Proposition 3.** For the discrete-demand model:

(i) $B(r,q), I(r,q)$ and $C(r,q)$ are (ia) componentwise convex, (ib) supermodular, and (ic) not $L$-convex;

(ii) $B(r,s), I(r,s)$, and $C(r,s)$ are (ia) componentwise convex, (ib) supermodular, and (ic) not $L$-convex;

(iii) $\mathbb{E}(q,s), \mathbb{I}(q,s)$, and $\mathcal{C}(q,s)$ are (iaa) componentwise convex, (ibb) submodular, and (iic) not $L$-convex.

The proof is given in Appendix A. Result (ia) implies that we can easily optimize one policy parameter with the other parameter fixed. Denote $r(q) = \arg\min_r C(r,q), \quad q(r) = \arg\min_q C(r,q)$.

Result (ib) implies that $r(q)$ is nonincreasing in $q$ and $q(r)$ is nonincreasing in $r$. Similar implications apply to parts (ii) and (iii). In addition, by the definition of $L$-convexity, the parts (ic) and (iic) are immediate from (iib) and (iib), respectively. Thus, in the discrete space too, both the original $(r,q)$ system and the natural linear transformations do not possess $L$-convexity.

3. Comparison with the EOQ model

Due to its closed-form solution and robustness, the EOQ formula is often used as a heuristic order quantity in $(r,q)$ systems even though it assumes deterministic demand. So, it is of interest to study how the EOQ model compares to the stochastic model.

When demand is deterministic, the long-run average cost is given by (3) with $G(y)$ replaced by

$$G(y) = p/ED - y + h(y - ED).$$

Denote by $(r^*, q^*)$ the resulting optimal policy, where the super-script “0” indicates “zero” demand variability.

For any fixed order quantity $q$, let $r(q) = \arg\min_r C(r,q)$ to be the corresponding optimal reorder point. The relative cost error (i.e., optimality loss) of using the policy $(r(q), q)$, as opposed to using the optimal policy $(r^*, q^*)$, is

$$\text{err}(q) = \frac{C(r(q), q) - C(r^*, q^*)}{C(r^*, q^*)}.$$

Let $(r^*, q^*)$ denote the optimal policy for the continuous-demand model, i.e., $\mathcal{C}(r^*, q^*) = \min_r C(r,q)$. When demand is continuous and deterministic, we obtain the EOQ model, and the long-run average cost is given by (5) with $G(y)$ replaced by (10). The corresponding optimal policy parameters $(r^*, q^*)$ have the following closed-form expressions:

$$q^* = \sqrt{\frac{2K}{ho}}, \quad r^* = ED - (1 - \omega)q^*,$$

where $\omega = p/(h + p)$. The resulting system cost is $\bar{C}(\bar{r}(q^*), q^*)$, where $\bar{r}(q) = \arg\min_r C(r,q)$. Denote the relative cost error of this heuristic by

$$\text{err}(q^*) = \frac{\bar{C}(\bar{r}(q^*), q^*) - C(r^*, q^*)}{C(r^*, q^*)}.$$

Let $q = \sqrt{(q^*)^2 + (1/\bar{p})\bar{C}(\bar{r}(q^*), q^*)}$. Proposition in Axsäter (1996) and Theorem 1 in Gallego (1998) show that

**Lemma 5.** For the continuous $(r,q)$ system,

(i) $q^* \leq q \leq q^*$;

(ii) $\text{err}(q^*) \leq \sqrt{5} - 2 / 2 \approx 0.118$.

For discrete demand, the continuous-demand cost function (5) is an adequate approximation of (3) only when the order quantity $q$ is not too small. In this case, it is reasonable to apply Lemma 5. Otherwise, this assertion may not be reliable. Indeed, we found counterexamples, which are presented in the next section.

3.1. Counterexamples

Note that the EOQ order quantity $q^*$ may not be an integer, so the first question is how to adapt the EOQ-based heuristic to the discrete model. Natural choices are $\{q^*\}$ and $\{q^*\}$. (Gallego, 1998 has a similar discussion and suggestion.) The following two examples show Lemma 5 fail to hold for both choices.

**Example 1.** Counterexample to Lemma 5 (i).

Let $D = 12$ with probability one, $p = 2.5, h = 1$ and $K = 157.5$. We have $q^* = q = 21$ and $q^* = 20.2$. Thus $q^* < q^* = [q^*] = q^*$.

Let $L = 1$ with probability one, $\varepsilon = 0.2, p = 9$ and $h = 2, K = 2$. We have $q^* = 0.6992$ and $q^* = 2.4$. This consequently implies $q^* > q = 3.1540$.

**Example 2.** Counterexample to Lemma 5 (ii). Let $L = 1$ with probability one, $\varepsilon = 1, p = 3.2, h = 5.4$ and $K = 1$. Then

$$r^* q^* r^* r(q^*), q^* q^* C(r(q^*)), C(r^*, q^*), C(r^*, q^*), C(r^*, q^*) \text{err}(q^*)$$

$$-1 \ 2 \ 0 \ 0.374 \ 3.682 \ 4.164 \ 0.131$$

Recall that in the continuous model, $q^*$ is the optimal order quantity when demand is deterministic. So, Lemma 5 compares the result of the stochastic system with its deterministic counterpart. Thus, it is interesting to see whether such a comparison carries through in the discrete model, i.e., whether Lemma 5 holds if we replace $q^*$ by $q^*$ and $q^*$ by $q^*$. The following examples show that, unfortunately, the answer is No.

**Example 3.** Counterexample to Lemma 5 (i). Letting $p = 1, h = 10, \varepsilon = 2, K = 28$, and Pr$(L = 1) = 1$, we have

$$r^* q^* r^* q^* r^* q^* q^* r^* q^* q^* q^* q^* q^*$$

$$-9 \ 11 \ -9 \ 12 \ -8.8280 \ 11.5844 \ -8.0905 \ 11.0995$$

Thus, in the discrete system, we can have $q^* < q^*$, i.e., the order quantity under stochastic demand can be less than that under deterministic demand.
Example 4. Counterexample to Lemma 5 (ii). Letting $p = 9$, $h = 3$, $z = 4$, $K = 4.5$, and $Pr(L = 1) = 1$, we have

\[ r^* \quad q^* \quad r^0 \quad q^0 \quad C(r(q^p), q^p) \quad C(r^*, q^*) \quad err(q^*) \]
\[ 2 \quad 6 \quad 3 \quad 3 \quad 16.1591 \quad 13.2072 \quad 0.2235 \]

Thus, the upper bound $\left(\sqrt{5} - 2\right)/2$ in Lemma 5 (ii) no longer holds in the discrete-demand case. For this continuous system with this set of parameters, we have

\[ r^* = q^* = r^0 = q^0 = 3, \quad C(r(q^p), q^p) = C(r^*, q^*) = 16.1591, \quad err(q^*) = 0.2235 \]

3.2. Bounds for discrete system

We now aim to revise the properties in Lemma 5 for the discrete system. To see how the properties are related to determine $(r^*, q^*)$ and $(r^0, q^0)$, we use the method provided by Federgren and Zheng (1992), which is based on the fact that $-G(y)$ is unimodal. Let $y^1$ be the smallest integer that minimizes $G(.)$ over all integers. Assuming that $(y^1, \ldots, y^n)$ have been generated for some $k \geq 1$, define $L(k) = \min\{y^1, \ldots, y^k\}$ and $R(k) = \max\{y^1, \ldots, y^k\}$. Let

\[ q^* = \min\{y^1, \ldots, y^k\} \quad \text{such that} \quad q^* \leq y^* \quad \text{and} \quad \text{the optimal reorder point is} \quad r^* = \frac{q^*}{C3} - 1. \]

A key observation is

\[ q < q^* \quad \text{if and only if} \quad \frac{1}{q} \left( r^* K + \sum_{k=1}^{q} G(y^k) \right) > G(y^q) \quad \text{for any} \quad 1 \leq q \leq q^* + 1. \]

When demand is deterministic, because $-G(y)$ is also unimodal, $(r^0, q^0)$ can be computed using the same procedure with $G(y)$ replacing $G(y')$. In particular, similarly to (13), we can generate $(z^1, \ldots, z^n)$ to be contiguous, and $G'(z^1)$ is the smallest $G(.)$ value, $k = 1, \ldots, q$. Define $L(k) = \min\{z^1, \ldots, z^k\}$ and $R(k) = \max\{z^1, \ldots, z^k\}$. Then

\[ q^0 \quad \text{is the smallest integer} \quad q \quad \text{such that} \quad \frac{1}{q} \left( r^0 K + \sum_{k=1}^{q} G'(z^k) \right) > \frac{G'(z^q)}{q}. \]

Let $H'(y) = G'(y)$ and $\tilde{H}(y) = \frac{G'(y)}{q}$. Then $\tilde{H}'(y) = \frac{G''(y)}{q(q-1)}$. By checking the signs of the derivatives of these functions, it is straightforward to show

\[ f_1(x) = \frac{1}{4} \left( 1 + \frac{1}{x} \right) ^2 \frac{x}{x-1}, \quad f_2(x) = \frac{1}{8} \left( 1 + \frac{3}{x} \right) ^2 \frac{x}{x-1}. \]

By checking the signs of the derivatives of these functions, it is straightforward to show

\[ f_1(x) = \frac{1}{4} \left( 1 + \frac{1}{x} \right) ^2 \frac{x}{x-1}, \quad f_2(x) = \frac{1}{8} \left( 1 + \frac{3}{x} \right) ^2 \frac{x}{x-1}. \]

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Recall that \( |q''| - 1 \leq q^* + 1 \) by Proposition 4 (ii). Using \( |q''| - 1 \) as an EOQ-based heuristic order quantity for the discrete system, we can view Proposition 4 (ii) and Proposition 5 (ii) as the discrete version of Lemma 5 in terms of assessing the effectiveness of the EOQ-based heuristic. Similarly, Proposition 4 (i) and Proposition 5 (i) are the discrete counterparts of Lemma 5 in terms of assessing the impact of ignoring demand variability.

4. Nonconvexity of discrete \((s, T)\) systems

We now examine the discrete periodic-review \((s, T)\) inventory system. Here, the inventory position \(IP(t)\) is reviewed every \(T\) periods. If, upon review, \(IP(t)\) is below \(s\), then order enough to bring \(IP(t)\) back to \(s\); otherwise, do nothing. The demand process is a Poisson process with per-period demand rate \(\lambda\). That is, let \(D(1)\) denote the demand for one period, then \(D(1)\) has a Poisson distribution with parameter \(\lambda\). The leadtime for each replenishment order is assumed to be a positive integer \(T\). There is a fixed order cost \(K\), a per-period unit holding cost \(h\), and a per-period unit backorder cost \(p\). Note that the cumulative demand in \(T\)-periods, denoted by \(D(t)\), has a Poisson distribution with parameter \(\lambda T\).

Let \(p_i(n, t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}\), \(P_i(n, t) = \sum_{k=0}^{n} p_i(k, t)\).

Following a similar argument as in Rao (2003) (see also Liu and Song 2012), the long-run average inventory and backorders under the \((s, T)\) policy are respectively

\[
I(s, t) = \frac{1}{T} \sum_{t=1}^{T} E[|s - D(t)|^+] = \frac{1}{T} \sum_{t=1}^{T} \sum_{n=0}^{\infty} (s - n)p_i(n, t),
\]

\[
B(s, t) = \frac{1}{T} \sum_{t=1}^{T} E[|D(t) - s|^+] = \frac{1}{T} \sum_{t=1}^{T} \sum_{n=0}^{\infty} (\lambda T - s) + I(s, t).
\]

Therefore, the long-run average cost under the \((s, T)\) policy is

\[
C(s, T) = \frac{K}{T} [1 - e^{-\lambda T}] + hI(s, T) + pB(s, T).
\]

Parallel to the discrete \((r, q)\) system, in the following, we carry out the analysis for the discrete \((s, T)\) system. Before proceeding, we first present some properties which are straightforward from (27) and (28).

Lemma 9

(i) \(\Delta_s I(s, T) = \Delta_s I(s, T) - 1\);

(ii) \(\Delta_s I(s - 1, T) = (1/T) \sum_{t=1}^{T} P_i(s - 1, t)\);

(iii) \(\Delta_t B(s, T) = \lambda / 2 + \Delta_t I(s, T)\);

(iv) For any function \(f(\cdot)\) defined on \(S\), we have \(\Delta_s f(x - 1, y) - \Delta_s f(x, y) = \Delta_t f(x - 1, y) - \Delta_t f(x, y + 1)\).

It follows from Lemma 9 that

Proposition 6. The function \(C(s, T)\) for the discrete system is not convex in the sense of linear interpolation extension nor in the sense of \(L^1\)-convexity.

The proof of the proposition is similar to the proof of Proposition 1, and is given in Appendix A. Furthermore, we have the following result which is the discrete version of that in Liu and Song (2012).

Proposition 7. For the discrete \((s, T)\) system, \(C(s, T)\) is componentwise convex and submodular. Hence, \(s(T) = \arg \min_s C(s, T)\) is nondecreasing in \(T\), and \(T(s) = \arg \min_t C(t, s)\) is nondecreasing in \(s\).

Proof. For any fixed \(T\), the convexity of \(B(s, T)\) in \(s\) directly follows from Lemma 9 (ii). In view of Lemma 9 (i), we have the convexity of \(C(s, T)\) in \(s\).

We now argue that for any fixed \(s, C(s, T)\) is convex in \(T\). Note that

\[
I(s, T) = \frac{1}{T} \sum_{t=1}^{T} (s - \lambda t) + B(s, T)
\]

and

\[
\frac{1}{T} \sum_{t=1}^{T} (s - \lambda t) = s - \frac{T + 1 + 2\lambda}{2}.
\]

By the convexity of \(K[1 - e^{-\lambda T}] + \frac{1}{T} \sum_{t=1}^{T} (s - \lambda t)\), it suffices to show that the convexity of \(B(s, T)\) in \(T\).

Let \(U(T)\) be a uniform distribution on \([1, \ldots, T]\) and independent of the demand process \(D(t)\). Then from (28),

\[
B(s, T) = E[\sum_{t=1}^{T} (D(t) + U(T)) - s^+].
\]

By Example 8.A.2 of Shaked and Shanthikumar (2007), we know that the Poisson process \(D(t)\) is stochastically increasing and linear \((SIL)\). Noting that for any fixed \(s\), the function \((t - s^-)^+\) is an increasing convex in \(t\), thus, by Theorem 8.A.17 of Shaked and Shanthikumar (2007) and (30), to get the convexity of \(B(s, T)\) in \(T\), it suffices to show that the uniform distribution \(U(T)\) is also \((SIL)\). To the end, for any increasing convex (concave) function \(g(\cdot)\) defined on \([0, 1, 2, \ldots]\), we need to show that

\[
Eg(U(T)) = \frac{1}{T + 1} \sum_{t=1}^{T} g(t + 2) + E(U(T)) \geq 2Eg(U(T + 1)).
\]

As \(U(T + 1)\) is stochastically larger than \(U(T)\), it is straightforward to see the monotonicity of \(Eg(U(T))\). Here we only consider the convexity of \(Eg(U(T))\) when \(g(\cdot)\) is convex. The concavity for the case in which \(g(\cdot)\) is concave can be proved similarly. It is sufficient to show that

\[
Eg(U(T + 2)) + Eg(U(T)) - 2Eg(U(T + 1)) \geq 0.
\]

Note that

\[
= \frac{1}{T + 1} \sum_{t=1}^{T} g(t + 2) - g(t)
\]

\[
\geq \frac{1}{T + 1} \sum_{t=1}^{T} g(t + 1) - g(t)
\]

\[
\geq \frac{1}{T + 1} \sum_{t=1}^{T} g(T + 1) - g(T)
\]

Hence, we have (31). Thus the convexity of \(B(s, T)\) in \(T\) is obtained.

Next we prove submodularity. Using Lemma 9 (ii) and (iv), we have

\[
\Delta_s I(s, T + 1) - \Delta_s I(s, T) = \frac{1}{T + 1} \sum_{t=1}^{T + 1} P_i(s, t) - \frac{1}{T} \sum_{t=1}^{T} P_i(s, t)
\]

\[
= \frac{1}{T + 1} \sum_{t=1}^{T + 1} (P_i(s, t) + P_i(s, T + 1) - P_i(s, t)).
\]

Note that the first term is the probability that the demand is less than or equal to \(s\) in the interval \([0, (l + T + 1)]\) and the second term is the probability that the demand is less than or equal to \(s\) in the interval \([0, t]\) by a Poisson process with parameter \(\lambda\). When \(t \leq l + T\), we know that

\[
\Delta_s I(s, T + 1) - \Delta_s I(s, T) \leq 0.
\]

Hence, Lemma 3 implies the submodularity of \(I(s, T)\). Consequently, the submodularity of \(C(s, T)\) follows from (29) and Lemma 9 (i).
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Appendix A. Proofs of Section 4.2 and Proposition 6 in Section 4

Proof of Proposition 3. First, we prove (ia). For any fixed q, the convexity in r can be found in Chapter 6 of Zipkin (2000). We now argue that for any fixed r, these functions are convex in q. Here we only give the proof for B(r, q), from which the results for I(r, q) and C(r, q) follow directly due to (2) and (3). For any fixed r, from Lemma 1 (iii),

\[ V(r, u) - V(r, q + 1) = \sum_{i=r}^{q} P_D(r > r + i) \]

\[ \geq (q - u + 1)P_D(r > r + q + 1). \]

By Lemma 1 (v) and (32),

\[ B(q + 2) + B(q) - 2B(q + 1) \]

\[ = \Delta_2B(q + 1) - \Delta_0B(q) \]

\[ \leq \frac{1}{q + 1} \left( \sum_{u=0}^{q-1} [V(q, r + 1) - V(q + 1, r + 1)] \right) \]

\[ \geq \frac{q(q + 1)Pr(D > r + q + 1) - 2\sum_{u=0}^{q-1} [V(q - 1, r + 1) - V(q + 1, r + 1)]}{q(q + 1)(q + 2)} \]

\[ \geq \frac{-q(q + 1)Pr(D > r + q + 1) + 2\left( \frac{q(q + 1)}{2} \right)Pr(D > r + q + 1)}{q(q + 1)(q + 2)} = 0, \]

which implies \( B \) is convex in q.

Next, we prove (ib). Similar to the above, it is sufficient to give a proof for B(r, q). Using (32) with \( u = 1 \) yields

\[ q \times \left( V(r, q + 1) - V(r, q + 2) \right) + (V(r, q + 1) - V(r, 1)) \leq 0. \]

Hence,

\[ \Delta_2B(r, q) - \Delta_0B(r + 1, q) \]

\[ \leq q \times \left( V(r, q + 1) - V(r, q + 2) \right) + (V(r, q + 1) - V(r, 1)) \)

\[ \geq 0. \]

The result thus follows by Lemma 3 (ic), by the definition of \( L^1 \)-convexity, directly follows from (ib).

For part (iiia), because linear transformation preserves convexity, all these measures are componentwise convex. For (iiib), similar to the above, we only consider I(r, s). The result holds because

\[ \Delta_2I(r, s) - \Delta_0I(r + 1, s) = \Delta_2\delta(r, s + 1) - \Delta_2\delta(r + 1, s - r + 1) \leq 0, \]

where the inequality follows from (8) with \( q = s - r - 1 \) and (9). The part (iic) is immediate from (ib).

Similarly, part (iiia) is immediate, and we only need to show (iiib) and (iic) for \( \Omega(r, s) \). The result for (iiib) holds because, similar to (32), we have

\[ V(s - q, 1) - V(s - q, q + 1) \leq q \times \left( V(s - q, 0) - V(s - q, 1) \right). \]

From this and Lemma 1 (vi), we obtain

\[ \Delta_2B(q, s) - \Delta_0B(q + 1, s) \]

\[ = \Delta_2B(s - q, q) - \Delta_0B(s - q - 1, q + 1) \]

\[ = V(s - q, q + 1) + q \cdot \left( V(s - q, 0) - (q + 1) \cdot V(s - q, 1) \right) \]

\[ \frac{q(q + 1)}{q(q + 1)(q + 2)} \]

\[ \geq 0. \]

To show (iiic), it suffices to find a pair \( (q, s) \) such that

\[ B(q + 2, s + 1) + B(q, s) < B(q + 1, s + 1) + B(q + 1, s). \]  

\[ G(q + 2, s + 1) + B(q, s) - B(q + 1, s + 1) - B(q + 1, s) \]

\[ = B(s - q - 1, q + 2) + B(s - q, q) - B(s - q + 1) \]

\[ - B(s - q - 1, q + 1) \]

\[ = \Delta_2B(s - q - 1, q + 1) - \Delta_0B(s - q, q). \]

In view of (8) with \( r = s - q - 1 \) and the proof of Proposition 1, we know there exists a pair \( (q, s) \) such that (35) holds. Hence the proposition is proved. "

Proof of Lemma 6. We first prove part (i). Let \( \Delta \) be the difference operator, i.e., for any discrete function \( f(.) \), \( \Delta f(x) = f(x + 1) - f(x) \). Note that for any integer \( y \),

\[ \Delta G(y) = -p + (p + h)Pr(D \leq y), \]

which implies that

\[ -p \leq \Delta G(y) \leq h. \]

On the other hand,

if \( y \gg ED \), then \( \Delta G(y) = h; \)

if \( y + 1 \ll ED \), then \( \Delta G(y) = -p. \)

Here we give a proof of the lemma for the case \( p \gg h \). The other case \( (p < h) \) can be proved in a similar way. If \( y' \gg y \), then by (36),

\[ G(y' + 1) - G(y') \leq G(y' + 1) - G(y') \leq G(y' + 1) - G(R(i) + 1) - G(R(i)) \]

which gives the lemma. If \( y' \ll y \),

\[ G(y' + 1) - G(y') \leq G(R(i) + 1) - G(y') \leq G(R(i) + 1) - G(R(i)) = h, \]

which also gives (i). For part (ii), we use induction. For \( q = 1 \), we have

\[ G(y') - G(y') \leq \min(p, h) \text{ (by part (i))} \]

\[ \leq [G(y^2) - G(y^2)] + [G(z^2) - G(z^2)] \]

(by (37) and (38)),

which implies (ii) for \( q = 1 \). Now suppose that the result holds for \( q = n \), that is,

\[ \sum_{i=1}^{n} G(y^{n+1} - G(y')) \leq \sum_{i=1}^{n} G(z^{n+2} - G(z)). \]

Then for \( q = n + 1 \),

\[ \sum_{i=1}^{n+1} G(y^{n+2} - G(y')) = \sum_{i=1}^{n} G(y^{n+1} - G(y')) + (n + 1)[G(y^{n+2} - G(y^{n+1})]

\[ \leq \sum_{i=1}^{n} G(z^{n+2} - G(z)) + (n + 1)[G(y^{n+2} - G(y^{n+1})]. \]
If
\[ z^{n+3} = z^{n+2} + 1 \text{ or } z^{n+3} = z^{n+2} - 1, \] (40)
then, by part (i), we have
\[
\sum_{i=1}^{n+1} [G(y^{n+2}) - G(y')] \leq \sum_{i=1}^{n+1} [G(z^{n+2}) - \tilde{G}(z')] + (n+1)[G(y^{n+2}) - G(y^{n+1})]
\]
\[
\leq \sum_{i=1}^{n+1} \tilde{G}(z^{n+2}) - \tilde{G}(z')\] + (n+1) \cdot \min(p,h)
\]
\[
= \sum_{i=1}^{n+1} [G(z^{n+2}) - \tilde{G}(z')] + (n+1)[\tilde{G}(z^{n+3}) - \tilde{G}(z^{n+2})]
\]
\[
\leq \sum_{i=1}^{n+2} [G(z^{n+3}) - \tilde{G}(z')],
\]
which gives (ii).

Now we prove that if (40) does not hold, then the result still holds with \( q = n + 1 \). The remainder of the proof is divided into four cases:

(a) \( y^{n+2} > y^1 \) and \( z^{n+2} < z^1 \);
(b) \( y^{n+2} > y^1 \) and \( z^{n+2} > z^1 \);
(c) \( y^{n+2} < y^1 \) and \( z^{n+2} < z^1 \);
(d) \( y^{n+2} < y^1 \) and \( z^{n+2} > z^1 \).

Here we only consider Case (a), the other cases can be treated similarly. If (40) does not hold, and Case (a) holds, we have
\[ y^{n+2} > y^1 \text{ and } z^{n+2} < z^1. \] (41)
The set \( \{y^1, \ldots, y^{m}\} \) is partitioned into two parts, \( \{y'_1, \ldots, y'_q\} \) and \( \{y^1, \ldots, y^{m-2-k}\} \) such that
\[ y^1 < \cdots < y'_1 = y^1, \quad y^{n+2-k} > \cdots > y'_1 > y^1. \] (42)
Similarly, the set \( \{z^1, \ldots, z^m\} \) is partitioned into two parts, \( \{z'_1, \ldots, z'_q\} \) and \( \{z^1, \ldots, z^{m-2-k}\} \) such that
\[ z^1 < \cdots < z'_1 = z^1, \quad z^{n+2-m} > \cdots > z'_1 > z^1. \] (43)
Without loss of generality, we assume that \( z^1 \not< ED \). There are two subcases, namely,

(a1) \( k = m; \) (a2) \( k \neq m \).

First, we look at subcase (a1). In view of (37) and (38),
\[ \Delta G(y) \geq -p = \Delta \tilde{G}(z), \quad y \in \{y'_1, \ldots, y'_q\}, \quad z \in \{z^1, \ldots, z_q\}, \] (44)
\[ \Delta G(y) \leq h = \Delta \tilde{G}(z), \quad y \in \{y'_1, \ldots, y^{n+2-k}\}, \quad z \in \{z^1, \ldots, z^{n+2-m}\}. \] (45)
Let \( \bar{n} = \min\{q; y^i < \mathcal{L}(n+2)\} \). Hence, if \( k = m \), then by (13),
\[
\sum_{i=1}^{n+1} [G(y^{n+2}) - G(y')] = \sum_{i=1}^{k} [G(y^{n+2}) - G(y')] + \sum_{i=k+1}^{n+2-k} [G(y^{n+2}) - G(y')] \\
\leq \sum_{i=1}^{k} [G(y^1) - G(y')] + \sum_{i=k+1}^{n+2-k} [G(y^{n+2}) - G(y')] \\
= \sum_{i=1}^{k} [G(y^1) - G(y')] + \sum_{i=k+1}^{m} [G(y^{n+2}) - G(y')] \\
= \sum_{i=1}^{m} [G(y^1) - G(y')] + \sum_{i=m+1}^{m+2-k} [G(y^{n+2}) - G(y')] \\
\leq \sum_{i=1}^{m} [G(z^{n+3}) - \tilde{G}(z')] + \sum_{i=m+1}^{m+n-2-m} [G(z^{n+3}) - \tilde{G}(z')] \\
\leq \sum_{i=1}^{m} [G(z^{n+3}) - \tilde{G}(z')],
\] (46)
which implies (ii).

To complete Subcase (a2), therefore, we consider \( k < m \) (\( k > m \) can be proved similarly).
\[
\sum_{i=1}^{n+1} [G(y^{n+2}) - G(y')] = \sum_{i=1}^{k} [G(y^{n+2}) - G(y')] + \sum_{i=k+1}^{m} [G(y^{n+2}) - G(y')] \\
+ \sum_{i=m+1}^{n+2-k} [G(y^{n+2}) - G(y')] \\
\leq \sum_{i=1}^{m} [G(y^1) - G(y')] + \sum_{i=m+1}^{m+n-2-m} [G(y^{n+2}) - G(y')] \\
\leq \sum_{i=1}^{m} [G(z^{n+3}) - \tilde{G}(z')] + \sum_{i=m+1}^{m+n-2-m} [G(z^{n+3}) - \tilde{G}(z')] \\
\leq \sum_{i=1}^{m} [G(z^{n+3}) - \tilde{G}(z')],
\]
which also implies (ii). \( \square \)

**Proof of Proposition 4.** First, (i) directly follows from Lemma 6 (ii). (i) We now prove part (ii). Let \( q \) be an integer that satisfies
\[ \{z^1, \ldots, z^q\} \subseteq \{r^1, r^2 + q^2\} \] (47)
\[ \{z^1, \ldots, z^q\} \cup \{z^{q+1}\} \subseteq \{r^1, r^2 + q^2\}. \] (48)
Thus
If either \( r^2 \) or \( q^2 \) is not integer, then \( \lfloor q^2 \rfloor \leq q \leq \lfloor q^2 \rfloor \).
If both \( r^2 \) and \( q^2 \) are integers, then \( q = q^2 + 1 \).

Let\[ z_i < z_{i-1} < \cdots < z_1 \leq ED < z_i < \cdots < z_{i-k} \] (49)
\[ \{z^1, \ldots, z_i\} \cup \{z^{i+1}\} \subseteq \{r^1, r^2 \} \]
\[ \{z^1, \ldots, z^i\} \subseteq \{r^1, r^2 \}. \]
Here we only prove (ii) for the case (47). The other case can be handled similarly.
First, we prove \( |q^2| - 1 \leq q^2 \). Let \( n = |q^2| - 2 \). In view of (17), we only need to prove that
\[ \frac{iK}{q} + \sum_{i=1}^{n+1} [G(z') - \tilde{G}(z')] > \tilde{G}(z^{n+1}). \] (49)
Let \( z^m = \min\{z^i : z^i \in \{z^1, \ldots, z_q\} \setminus \{z^{q+1}, z^{q+1-1}\}\} \),
\[ z^{p-m} = \max\{z^i : z^i \in \{z^1, \ldots, z^{q+1}\} \setminus \{z^{q+1}, z^{q+1-1}\}\}. \] By \( iK = \int_{z^{i-m}}^{z^{i-m+q}} [G(r^1) - \tilde{G}(z')] \ dy \) and **Lemma 6**, we have
\[
iK = \int_{z^{i-m}}^{z^{i-m+q}} [G(r^1) - \tilde{G}(z')] \ dy + \int_{z^{i-m}}^{z^{i-m+q}} [G(r^1) - \tilde{G}(z')] \ dy + \int_{z^{i-m}}^{z^{i-m+q}} [G(r^1) - \tilde{G}(z')] \ dy \\
> \int_{z^{i-m}}^{z^{i-m+q}} [G(z^{n+1}) - \tilde{G}(z')] + n \cdot \min(p,h) + \sum_{i=1}^{m} [\tilde{G}(z^{n+1}) - \tilde{G}(z')] \\
\geq \sum_{i=1}^{m} [\tilde{G}(z^{n+1}) - \tilde{G}(z')],
\]
which implies (49).
Let \( n = |q^2| + 1 \). In view of (17), we only need to prove that
\[ J_K + \frac{m^q}{n^{q+1}} \tilde{G}(z') \leq \tilde{G}(z'^{q+1}). \]  

(50)

Similarly, we have
\begin{align*}
J_K &= \left( \int_{r_1}^{x_1} [\tilde{G}(r') - \tilde{G}(y)] \, dy + \sum_{i=2}^{n} \int_{r_i}^{x_i} [\tilde{G}(r') - \tilde{G}(y)] \, dy \right) \\
& \quad + \int_{r_{n+1}}^{x_{n+1}} [\tilde{G}(r') - \tilde{G}(y)] \, dy \\
& \quad + \left( \sum_{i=2}^{n} \int_{r_i}^{x_i} [\tilde{G}(r') - \tilde{G}(y)] \, dy + \int_{r_{n+1}}^{x_{n+1}} \tilde{G}(r') \, dy \right) \\
& \quad \leq \sum_{i=1}^{n} \left( \tilde{G}(z'^{q+1}) - \tilde{G}(z') \right) + \int_{r_{n+1}}^{x_{n+1}} \tilde{G}(r') \, dy \\
& \quad + \sum_{i=1}^{n} \left( \tilde{G}(z'^{q+1}) - \tilde{G}(z') \right) \leq \sum_{i=1}^{n} \left( \tilde{G}(z'^{q+1}) - \tilde{G}(z') \right),
\end{align*}

which implies (50). Therefore, we have (ii). \[ \square \]

**Proof of Lemma 7.** First, we consider
\[ H(q) \geq \tilde{H}(q). \]  

(51)

Note that
\[ G(y^1) = pE(D - y^1)^+ + hE(y^1 - D)^+ \]
\[ \geq pE[(D - y^1)^+] + h(y^1 - D)^+ \] (by Jensen’s inequality)
\[ \geq pE[(D - z^1)^+] + h(z^1 - D)^+ \] (by the definition of \( z^1 \))
\[ = \tilde{G}(z^1), \]

which implies that (51) holds with \( q = 1 \).

We now show that (51) holds for any \( q \geq 1 \). Suppose that it holds for \( n \). Then there are three cases, namely,

(a) \( \{y^1, \ldots, y^n\} = \{z^1, \ldots, z^n\} \); (b) \( \mathcal{L}(n) < \tilde{\mathcal{L}}(n) \); (c) \( \mathcal{L}(n) > \tilde{\mathcal{L}}(n) \).

We first look at Case (a). Suppose that \( y^{n+1} = \mathcal{L}(n) - 1 \).

Then,
\[ H(n + 1) = G(\mathcal{L}(n) - 1) \]
\[ = pE[D - (\mathcal{L}(n) - 1)]^+ + hE((\mathcal{L}(n) - 1) - D)^+ \]
\[ \geq pE[(D - \mathcal{L}(n))^+] + h[|\mathcal{L}(n) - 1| - D]^+] \]
\[ \geq pE[(D - \tilde{\mathcal{L}}(n))^+] + h[|\tilde{\mathcal{L}}(n) - 1| - D]^+] \]
\[ \geq \min\{pE[(D - \tilde{\mathcal{L}}(n))^+] + h[|\tilde{\mathcal{L}}(n) - 1| - D]^+] \} \]
\[ \geq \tilde{G}(z^n - 1) \leq \tilde{G}(\mathcal{L}(n) - 1) + 1.
\]

Similarly, we can prove that if \( y^{n+1} = \mathcal{L}(n) + 1 \), then (51) holds for \( n + 1 \).

Next, we consider Case (b). Suppose that \( y^{n+1} = \mathcal{L}(n) + 1 \). Then \( \mathcal{L}(n) \leq \tilde{\mathcal{L}}(n) - 1 \), which implies
\[ pE[D - \mathcal{L}(n)]^+ + h|\mathcal{L}(n) - D|^+ \]
\[ \geq pE[(D - \tilde{\mathcal{L}}(n))^+] + h[|\tilde{\mathcal{L}}(n) - 1| - D]^+ \].

Hence,
\[ H(n + 1) = G(\mathcal{L}(n) + 1) = G(\mathcal{R}(n) + 1) \]
\[ = pE[D - (\mathcal{R}(n) + 1)]^+ + hE((\mathcal{R}(n) + 1) - D)^+ \]
\[ \geq pE[D - \mathcal{L}(n)]^+ + h|\mathcal{L}(n) - D|^+ \]
\[ \geq pE[(D - \tilde{\mathcal{L}}(n))^+] + h[|\tilde{\mathcal{L}}(n) - 1| - D]^+] \]
\[ \geq \min\{pE[(D - \tilde{\mathcal{L}}(n))^+] + h[|\tilde{\mathcal{L}}(n) - 1| - D]^+] \} \]
\[ \geq \tilde{G}(z^n + 1) \leq \tilde{G}(\mathcal{L}(n) + 1) + 1.
\]

Thus, (51) holds. Similarly we can prove it if \( y^{n+1} = \mathcal{L}(n) - 1 \).

Case (c) can be treated as Case (b). Therefore, (51) holds, proving the first inequality of part (i) of the theorem.

Now, we prove the second inequality of part (i). From the definition of \( G(y) \) (see (10)), for any positive integer \( q \),
\[ \tilde{G}(ED - (1 - \omega)(q - 1)) = \tilde{G}(ED + \omega(q - 1)) = h\omega(q - 1), \]

(53)

\[ \tilde{G}(y) \geq \tilde{G}(\tilde{y}), \]
\[ y \in (-\infty, ED - (1 - \omega)(q - 1)) \cup (ED + \omega(q - 1), +\infty), \]
\[ \tilde{y} \in (ED - (1 - \omega)(q - 1), ED + \omega(q - 1)). \]

(54)

Note that the length of the interval \([ED - (1 - \omega)(q - 1), ED + \omega(q - 1)]\) is \( q - 1 \). Thus, this interval contains at most \( q \) integers, and at least \( q - 1 \) integers. By (54), we know that
\[ \{z^1, \ldots, z^n\} \subseteq \{ED - (1 - \omega)(q - 1), ED + \omega(q - 1)\}. \]

If \([ED - (1 - \omega)(q - 1), ED + \omega(q - 1)]\) contains \( q \) integers, then \( ED - (1 - \omega)(q - 1) \) and \( ED + \omega(q - 1) \) must be integers, and in view of (54),
\[ \tilde{G}(z^n) = \tilde{G}(ED - (1 - \omega)(q - 1)) = \tilde{G}(ED + \omega(q - 1)) = h\omega(q - 1), \]

(55)

If \([ED - (1 - \omega)(q - 1), ED + \omega(q - 1)]\) contains only \( q - 1 \) integers, then again in view of (54),
\[ \tilde{G}(z^n) > \tilde{G}(ED - (1 - \omega)(q - 1)) = \tilde{G}(ED + \omega(q - 1)) = h\omega(q - 1). \]

(56)

Therefore, the second inequality of (i) directly follows from (55) and (56).

Finally, we prove part (ii). To this end, we show that
\[ \sum_{i=1}^{n+1} (\tilde{G}(z^{i+2}) - \tilde{G}(z^{i+1})) \leq h\omega(q + 1)^2. \]  

(57)

Here we only consider \( p \geq h \), the other case can be proved similarly. When \( q = 1 \),
\[ \tilde{G}(z^n - 1) \leq h\omega \times 2^2, \]

which implies that (57) holds for \( q = 1 \). Now suppose (57) holds for \( q = n \). Then for \( q = n + 1 \),
\[ \sum_{i=1}^{n+1} (\tilde{G}(z^{i+2}) - \tilde{G}(z^{i+1})) \leq \sum_{i=1}^{n+1} (\tilde{G}(z^{i+2}) - \tilde{G}(z^{i+1})) + \sum_{i=1}^{n+1} (\tilde{G}(z^{i+1}) - \tilde{G}(z^{i})) \]
\[ \leq \sum_{i=1}^{n+1} (\tilde{G}(z^{i+2}) - \tilde{G}(z^{i+1})) + h\omega(n + 1)^2 \]
\[ \leq (n + 1)h + h\omega(n + 1)^2 \leq h\omega(n + 2)^2, \]

which implies that (57) holds for \( q = n + 1 \). Therefore, by induction, (57) holds. Using (17) and (57), we have \( \tilde{K} \leq \sum_{i=1}^{m} (\tilde{G}(z^{i+1}) - \tilde{G}(z^{i})) \]
\[ \leq h\omega(q^2 + 1)^2, \]

which implies (ii). \[ \square \]

**Proof of Proposition 5.** We first look at (i). For any positive integers \( t \) and \( m \) with \( t < m \),
\[
\begin{align*}
\sum_{i=1}^{\ell} G(y_i') - \ell \sum_{i=1}^{m} G(y_i) &= (m - \ell) \sum_{i=1}^{\ell} G(y_i') - \ell \sum_{i=1}^{m} G(y_i') \\
&= \sum_{i=1}^{\ell} \left[ (m - \ell) G(y_i') - \sum_{i=1}^{m} G(y_i') \right] 
\end{align*}
\]

Therefore, if \( q^2 \leq q^* \), then

\[
\frac{C(r(q^2), q^2) - C(r^*, q^*)}{C(r^*, q^*)} = \frac{1}{C(r^*, q^*)} \left( \frac{\ell K + \sum_{i=1}^{\ell} G(y_i') - \ell K + \sum_{i=1}^{\ell} G(y_i')} {q^2} \right)
\]

\[
= \frac{1}{C(r^*, q^*)} \left[ \frac{\ell K \left( \frac{1}{q^*} - \frac{1}{q^2} \right)} {q^2} \left( q^2 \sum_{i=1}^{\ell} G(y_i') - q^2 \sum_{i=1}^{\ell} G(y_i') \right) \right]
\]

\[
\leq \frac{\ell K}{C(r^*, q^*)} \left( \frac{1}{q^2} - \frac{1}{q^*} \right).
\]

(58)

Noting that for any positive integer \( q \), \( \frac{\ell K + \sum_{i=1}^{\ell} G(y_i')} {q + 1} > G(y^{q+1}) \)

if and only if \( \frac{\ell K + \sum_{i=1}^{\ell} G(y_i')} {q} = G(y^{q+1}) \),

in view of (16) and Lemma 7 (i),

\[
C(r^*, q^*) \geq G(y^{q^2}) = H(q^*) \geq H_0(q^* - 1).
\]

(59)

Combining (58) and (59), if \( q^2 \leq q^* \), then

\[
\frac{C(r(q^2), q^2) - C(r^*, q^*)}{C(r^*, q^*)} \leq \frac{1}{4} \left( 1 + \frac{1}{q^2} \right) \frac{q^2}{q^2 - 1} \leq f_1(q^2).
\]

(60)

Finally, we consider the scenario \( q^2 > q^* \). Note that, by (16) and Proposition 4 (i) and (ii),

\[
\sum_{i=1}^{q_2} G(y_i') - \sum_{i=1}^{q^2} G(y_i') = \frac{1}{q^2 - 1} \left( q^2 \sum_{i=1}^{q^2} G(y_i') - q^2 \sum_{i=1}^{q_2} G(y_i') \right)
\]

\[
= \frac{1}{q^2 - 1} \left( q^2 \sum_{i=1}^{q^2} G(y_i') - q^2 \sum_{i=1}^{q_2} G(y_i') \right)
\]

\[
+ \left( q^2 - q_2 \right) \sum_{i=q_2}^{q^2} G(y_i') = \frac{1}{q^2 - 1} \left( q^2 \sum_{i=1}^{q^2} G(y_i') - q^2 \sum_{i=1}^{q_2} G(y_i') \right)
\]

\[
\leq \frac{1}{q^2 - 1} \left( q^2 \sum_{i=1}^{q^2} G(y_i') - q^2 \sum_{i=1}^{q_2} G(y_i') \right) + \ell K (q^2 - q^* - 1)
\]

\[
= \frac{1}{q^2 - 1} \left( \left( q^2 - 1 \right) \left( G(y^{q^2}) - G(y^{q_2}) \right) \right) + \ell K (q^2 - q^* - 1)
\]

\[
= \frac{1}{q^2 - 1} \left( \left( q^2 - 1 \right) \left( G(y^{q^2}) - G(y^{q_2}) \right) \right) + \ell K (q^2 - q^* - 1).
\]

(61)

Hence,

\[
\frac{C(r(q^2), q^2) - C(r^*, q^*)}{C(r^*, q^*)} \leq \frac{1}{q^2 - 1} \left( \left( q^2 - 1 \right) \left( G(y^{q^2}) - G(y^{q_2}) \right) \right) + \ell K (q^2 - q^* - 1)
\]

\[
= \frac{1}{q^2} \left( G(y^{q^2}) - G(y^{q_2}) \right) = \frac{\min \{ h, p \} } {q^2}.
\]

(62)

Using (59) and (61), in view of Proposition 4 (i) and (ii), if \( q^2 > q^* \),

\[
\frac{C(r(q^2), q^2) - C(r^*, q^*)}{C(r^*, q^*)} \leq \frac{\min \{ h, p \} } {q^2} \frac{1}{h_0(q^2 - 1)} \leq \frac{2}{q^2 \left( q^2 - 2 \right)}.
\]

(63)

(64)

Finally, we look at part (ii). Let \( x = \left[ q^2 \right] - 1 \), then from (58) we have that if \( x \leq q^* \),

\[
\frac{C(r(x), x) - C(r^*, q^*)}{C(r^*, q^*)} \leq \frac{\ell K}{C(r^*, q^*)} \left( \frac{1}{x} - \frac{1}{q^*} \right).
\]

Recall that

\[
q^* = \sqrt{\frac{2 \ell K}{h_0}}.
\]

By (59), if \( x \leq q^* \), then

\[
\frac{C(r(x), x) - C(r^*, q^*)}{C(r^*, q^*)} \leq \frac{2}{x(x - 2)}.
\]

(65)

Also, if \( x \geq 3 \),

\[
\frac{2}{x(x - 2)} < f_2(x).
\]

This proves (ii) by (63) and (64) and the fact that \( f_2(x) \) is decreasing on \([3, \infty)\) and \( f_2(3) = 3/4 \).

\[\square\]

**Proof of Proposition 6.** We first analyze the linear interpolation extension. Using Lemma 9 (i), we have

\[
\Delta_q \cdot B(s - 1, T) - \Delta_q \cdot B(s, T) = \Delta_q I(s - 1, T) - \Delta_q I(s, T).
\]

Hence, (29) gives

\[
\Delta_q C(s - 1, T) - \Delta_q C(s, T) = (h + p) \Delta_q I(s - 1, T) - \Delta_q I(s, T).
\]

Thus, it is sufficient to prove that \( I(s, T) \) is not convex in the sense of linear interpolation extension. Using Lemma 9 (ii) and (iv), we have

\[
\Delta_q I(s - 1, T) - \Delta_q I(s, T) = \Delta_q I(s - 1, T) - \Delta_q I(s - 1, T + 1)
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} P_s(s - 1, t) - \frac{1}{T} \sum_{t=1}^{T} P_s(s - 1, t + 1)
\]

\[
= \frac{1}{T + 1} \sum_{t=1}^{T} P_s(s - 1, t + 1)
\]

\[
+ \frac{1}{T} \sum_{t=1}^{T} P_s(s - 1, t - 1).
\]

(66)

Taking \( s = 1 \) and \( T = 1 \) in (66), then

\[
\Delta_q I(s - 1, T) - \Delta_q I(s, T) = \frac{1}{2} e^{-\ell(t+1)}(1 - e^{-\ell}) > 0.
\]

which, by (65), implies that \( C(s, T) \) is not convex in the sense of linear interpolation extension.

We next examine \( L^1\)-convexity. Taking \( T = 1 \) in (65) and (66), we have

\[
\Delta_q C(s - 1, T) - \Delta_q C(s, T) = \frac{p + h}{2}(-P_s(s - 1, \ell + 1) + P_s(s - 1, \ell + 1))
\]

Taking \( \ell = 0.1 \), \( \ell = 56 \), \( T = 1 \) and \( s = 36 \), we get that

\[
\Delta_q C(s - 1, T) - \Delta_q C(s, T) = -2.84 \times 10^{-13} < 0.
\]
This implies that \( c(s,T) \) is not \( L^1 \)-convex. Using similar techniques, we can conclude the same results for the submodularity property.

References


