Inventory Planning with Forecast Updates: Approximate Solutions and Cost Error Bounds

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We consider a finite-horizon, periodic-review inventory model with demand forecasting updates following the martingale model of forecast evolution (MMFE). The optimal policy is a state-dependent base-stock policy, which, however, is computationally intractable to obtain. We develop tractable bounds on the optimal base-stock levels and use them to devise a general class of heuristic solutions. Through this analysis, we identify a necessary and sufficient condition for the myopic policy to be optimal. Finally, to assess the effectiveness of the heuristic policies, we develop upper bounds on their value loss relative to optimal cost. These solution bounds and cost error bounds also work for general dynamic inventory models with nonstationary and autocorrelated demands. Numerical results are presented to illustrate the results.

Subject classifications: inventory, forecasting, MMFE, approximation, error bounds.
Area of review: Manufacturing, Service, and Supply Chain Operations.
History: Received June 2003; revisions received August 2004, May 2005; accepted November 2005.

1. Introduction

Demand forecasting is essential for inventory planning, especially when the demand environment is highly dynamic and the procurement lead times are long. How to adjust inventory planning decisions according to demand forecasting updates is of great interest to managers and for decades has attracted many researchers.

It is well known that the optimal inventory policy for any dynamic forecasting inventory model is very complex, so both in practice and in the research literature considerable attention has been given to much simpler myopic policies. For example, several authors have either proposed using myopic policy as inventory policy (e.g., Graves 1999 and Aviv 2003) or established sufficient conditions under which a myopic policy can be optimal in specific demand-forecasting models (e.g., Johnson and Thompson 1975, Miller 1986). Also, due to tractability, some recent works in the supply chain management literature employ the myopic policy in a demand-forecasting environment to gain various insights such as the value of information sharing (e.g., Lee et al. 2000), collaborative planning, forecasting and replenishment (e.g., Aviv 2001, 2002), and quantifying bullwhip effect (e.g., Chen et al. 2000).

However, some basic questions remain. In particular, are the insights gained from the myopic policy still valid in a system under an optimal policy? This is equivalent to asking how good the myopic policy is in general dynamic demand-forecasting inventory models. Also, if the myopic policy is not good enough, are there any simple adjustments that can improve the performance significantly? More generally, how can we evaluate the performance of a myopic policy or any other heuristic policy in terms of their value loss relative to optimal policy? These are the questions we aim to address in this paper.

We consider a single-item, periodic-review inventory system with demand forecasting updates. The demand can be time-correlated and nonstationary over time or follow any demand-forecasting model. For simplicity, we assume that the forecast evolution follows the martingale model of forecast evolution (MMFE), developed independently by Graves et al. (1986) and Heath and Jackson (1994). The MMFE is quite straightforward, general, and flexible. It can represent nonstationary and time-correlated demands. It can also accommodate judgmental forecasts as well as commonly used time series models such as the autoregressive moving average (ARMA) model. Other features of the inventory model are standard, such as full backlogging; a constant replenishment lead time; and linear ordering, inventory-holding, and backorder-penalty costs.

Several authors have adopted the MMFE to study production-inventory planning issues. For instance, Güllü (1996) uses a two-period MMFE to assess the value gained from using a dynamic demand-forecasting model. Graves et al. (1998) address how to adjust the material requirement schedule when the safety-stock plans are modified from...
period to period due to the modification of the demand forecasts. Toktay and Wein (2001) focus on one type of forecast-corrected inventory policy in a system with finite capacity and obtain closed-form approximations. Gallego and Özer (2001) consider a model of advance demand information and characterize the form of optimal policy. Their demand model can be viewed as a special case of the MMFE.

Among the studies using MMFE, Iida and Zipkin (2006) is most closely related to ours. They show that a demand-forecast dependent base-stock policy is optimal and develop bounds on the optimal base-stock levels. They further develop a piecewise-linear approximation of the cost functions and a simulation-based technique to solve the problem approximately. Finally, they establish conditions under which the myopic policy is optimal.

In this paper, we make two major contributions to the literature. The first is the development of easier-to-compute bounds on optimal base-stock levels, using a different approach from that of Iida and Zipkin (2006). Our approach also allows us to provide a necessary and sufficient condition for the myopic policy to be optimal and gain deeper insights. The second contribution is the development of error bounds on the value loss of any heuristic policy relative to the optimal cost, a subject not studied by Iida and Zipkin. These cost-error bounds can also be used to evaluate heuristic policies in any dynamic demand-forecasting inventory systems.

Our main idea for developing bounds on the optimal base-stock levels is through a sample-path approach to approximate the first-order condition function in the dynamic-program formulation. We develop an explicit expression of the first-order condition to see clearly the trade-off between the marginal cost in the current period and the marginal cost for the future periods. We use both the probability of overstock for a number of periods (i.e., no orders would be placed for a number of periods because the pre-order inventory level is higher than the optimal base-stock level in these periods) and the magnitudes of these overstocks to estimate the marginal future cost.

The notion of obtaining bounds on the optimal base-stock levels by estimating the marginal future cost in dynamic inventory models is not new. See Morton (1978) and Morton and Pentico (1995) for models with nonstationary and independent demands. These works use the probability of overstock for a number of periods to develop lower bounds on the optimal base-stock levels. By employing more information than the probability of overstock, we obtain significantly tighter lower bounds; see, e.g., Example 7.

Another approach to obtaining bounds on the optimal base-stock levels in dynamic inventory models is to allow disposal of stock earlier than the end of the horizon. This approach transforms the original problem into a shorter planning horizon problem. Solving the shorter planning horizon problem with the upper or lower bound (on the marginal future cost) being terminal cost leads to the upper or lower bound on the optimal base-stock levels. Using lower bound zero and upper bound the maximum salvage cost in the terminal period, Morton (1978) and Iida and Zipkin (2006) develop bounds on the optimal base-stock levels for the independent demand model and MMFE, respectively. In fact, our approach is general enough to treat these methods as special cases; see Examples 5 and 6. However, solving a shorter planning horizon problem optimally is still computationally challenging even for small problem sizes under MMFE.

In the literature of dynamic inventory models, a commonly used method to evaluate the performance of a heuristic policy is to estimate the gap between the lower and upper bounds on the optimal base-stock levels. Morton (1978) and Iida and Zipkin (2006) show that, under certain conditions, the gap between the upper and lower bounds by solving a shorter planning horizon (say \( (k+1) \)-period) problem (with zero or maximum salvage value as the terminal cost) goes to zero as \( k \) goes to infinity. However, for the MMFE examined in this research, it is not practical to solve a \( (k+1) \)-period problem optimally even for small \( k \).

This motivates us to explore an alternative approach to evaluating the effectiveness of heuristic policies. Our approach is a sample-path worst-case approach. We develop upper bounds on the cost difference between any heuristic policy and the optimal policy. We also develop lower bounds on the optimal cost. This leads to upper bounds on the cost error of any heuristic relative to the optimal policy.

To our knowledge, Lovejoy’s (1990, 1992) are the only previous attempts in the dynamic inventory literature to establish cost error bounds on suboptimal policies. While Lovejoy focused on myopic and stopped myopic policies, we derive cost error bounds for any heuristic policy. Our techniques are also different from his. When applied to myopic policies, our bounds are significantly tighter. (Since the completion of our study, Levi et al. (2004) have examined a similar inventory model to ours and show that the cost of a dual-balancing policy is within 200% of the cost of optimal policy.)

Several other types of forecasting models have been studied in the literature. One is a Bayesian model of updating demand distribution from past history; see, e.g., Scarf (1959, 1960), Azoury (1985), and Lovejoy (1990). The second is a time series approach—the demand process is the ARMA process or the ARIMA (integrated ARMA) process; see, e.g., Johnson and Thompson (1975), Miller (1986), Reyman (1989), and Graves (1999). The third approach is to model the demand as a Markov-modulated stochastic process; see, e.g., Lovejoy (1992), Song and Zipkin (1993), and Treharne and Sox (2002). For further discussion of the literature, see Iida and Zipkin (2006).

The rest of this paper is organized as follows. Section 2 introduces the basic notation and the model formulation. Section 3 discusses the first-order condition and the myopic policy. Section 4 presents the solution bounds, while §5
develops the cost error bounds of any heuristic policy relative to optimal policy. Finally, §6 presents numerical examples, and §7 concludes the paper.

2. Model and Formulation

We consider a \( T \)-period periodic-review inventory system with stochastic demand and zero replenishment lead time. (The extension to systems with a fixed constant lead time can be done by following the standard argument.) Let \( D_t \) be the actual demand in period \( t \). The demand process \( \{ D_t, \ t = 1, 2, \ldots, T \} \) can be nonstationary and correlated over time. At any period \( t \), we generate forecasts of the demand for all future periods in the horizon. At the beginning of each period, an ordering decision is made based on the inventory status and the demand forecast. Then, the placed orders arrive. During the period, demand is realized and fulfilled as much as possible. Unsatisfied demand is fully backlogged. At the end of the period, inventory-holding and backorder-penalty costs are charged, and demand forecasts are updated. There are linear costs for ordering, inventory holding, and backlogging, respectively. We use \( \hat{c}_t \), \( \hat{h}_t \), and \( \hat{b}_t \) to represent the unit ordering, inventory holding, and backordering costs in period \( t \), respectively.

To update the demand forecasts, we can either follow a standard forecasting tool such as a time-series model, or use other techniques such as expert judgment, or do both. Let \( D_{t, t+i} \) be the forecast made at the end of period \( t \) for the demand in period \( t+i \), \( i = 0, 1, \ldots, T-t \). Because forecasts are made after the current demand information is revealed, \( D_{t, t+i} = D_{t+i} \). Let \( \mathbf{D}_t \) be demand forecast vector made at the end of period \( t \) = \( \{D_{t, t+1}, \ldots, D_{t, T}\} \), where \( \mathbf{D}_t \) is the initial forecast vector. We consider two types of forecast updates: additive and multiplicative.

For additive updates, define \( e_{t, t+i} = D_{t, t+i} - D_{t-1, t+i} \) as the forecast update made at the end of period \( t \) for demand in period \( t+i \). Denote \( \text{Var}[e_{t,s}] = \sigma_s^2 \) and let \( \mathbf{e}_t \) be demand forecast update vector made at the end of period \( t = \{e_{t,1}, e_{t,2}, \ldots, e_{t,T}\} \). We assume that the forecasts are unbiased, i.e., \( E[e_{t,s}] = 0, s \leq t \). We also assume that the forecast updates \( \{e_t, t = 1, 2, \ldots, T\} \) are independent over time. The forecast updates within a period, however, are not necessarily independent because they might rely on the same or related information.

For multiplicative updates, similarly define \( e_{t, t+i} = D_{t, t+i}/D_{t-1, t+i} \). Here, \( E[e_{t,s}] = 1 \). Again, let \( \mathbf{e}_t \) be the forecast update vector made at the end of period \( t \). We assume that both \( D_{t, t+i} \) and \( e_{t, t+i} \) are positive and the forecast updates are independent over time \( t \). As before, the forecast updates within a period are not necessarily independent.

For exposition simplicity, we mainly focus on the additive model throughout the paper (with the exception of §6.2). However, all the results hold true for the multiplicative model.

We further assume that the forecast updates have a continuous distribution, and thus the one-period cost has a unique minimum point. The forecast updates in different periods can have different distributions. This model is broader than the original MMFE (e.g., Heath and Jackson 1994), which assumes the multivariate normal distribution.

We now formulate the problem as a dynamic program. Let \( I_t \) (respectively, \( I_t^* \)) be the inventory level at the beginning of period \( t \) after (respectively, before) ordering. The state of the system at the beginning of period \( t \) is \( (I_t^*, \mathbf{D}_{t-1}) \). Because the lead time is zero, the system dynamics are

\[
\begin{align*}
I_{t+1} & = I_t - D_t - e_{t, t+1}, \\
D_{t, t+i} & = D_{t-1, t+i} + e_{t, t+i} \quad \text{for } 0 \leq i \leq T - t.
\end{align*}
\]

Let \( \hat{C}_t(I_t, \mathbf{D}_{t-1}) \) be the expected holding and backorder costs charged to period \( t \), given that \( I_t = y \) and the latest forecast for demand in period \( t \) is \( D_{t-1, t} \). Then,

\[
\hat{C}_t(I_t, \mathbf{D}_{t-1}) = \hat{h}_t E[(y - (D_{t-1, t} + e_{t, t}))^+] + \hat{b}_t E[(D_{t-1, t} + e_{t, t} - y)^+],
\]

where \( y^+ = \max\{y, 0\} \) and \( D_{t-1, t} + e_{t, t} = D_{t, t} \). Let \( \hat{V}_t(x, \mathbf{D}_{t-1}) \) be the optimal total expected costs from period \( t \) through \( T \), given that \( I_t = x \) and the latest forecast for the future demands is \( \mathbf{D}_{t-1} \). We have the following recursive functional equations:

\[
\begin{align*}
\hat{V}_{t+1}(x, \mathbf{D}_T) & = -\hat{c}_{t+1} x, \\
\hat{V}_t(x, \mathbf{D}_{t-1}) & = \min_{y \geq x} \{ \hat{c}_t (y - x) + \hat{C}_t(y, \mathbf{D}_{t-1}) \} \\
& \quad + E[\hat{V}_{t+1}(y - (D_{t-1, t} + e_{t, t}), \mathbf{D}_t)].
\end{align*}
\]

Next, we make a transformation to simplify the expression. Set \( \hat{h}_t = \hat{h}_t + (\hat{c}_t - \hat{c}_{t+1}) \) and \( \hat{b}_t = \hat{b}_t - (\hat{c}_t - \hat{c}_{t+1}) \). Let

\[
\begin{align*}
C_t(y, D_{t-1}) & = \hat{h}_t E[(y - (D_{t-1, t} + e_{t, t}))^+] + \hat{b}_t E[(D_{t-1, t} + e_{t, t} - y)^+] \\
& = (\hat{c}_t - \hat{c}_{t+1}) y + \hat{C}_t(y, D_{t-1}), \\
V_t(x, D_{t-1}) & = \hat{c}_t x + \hat{V}_t(x, D_{t-1}).
\end{align*}
\]

We obtain

\[
\begin{align*}
V_{t+1}(x, D_T) & = 0, \quad (1) \\
V_t(x, D_{t-1}) & = \min_{y \geq x} G_t(y, D_{t-1}), \quad 1 \leq t \leq T, \quad (2)
\end{align*}
\]

where

\[
G_t(y, D_{t-1}) \equiv C_t(y, D_{t-1}) + E[V_{t+1}(y - (D_{t-1, t} + e_{t, t}), D_t)]. \quad (3)
\]
From now on, we call the transformed functions $C_t(y, D_{t-1, t})$ and $V_t(x, D_{t-1})$ the one-period expected cost and the optimal total expected cost from period $t$ to period $T$, respectively.

It can be shown that $G_t(y, D_{t-1, t})$ is convex in $y$. Let $s^*_t(D_{t-1})$ be its minimizer. Then, the base-stock policy with time- and state-dependent base-stock levels $s^*_t(D_{t-1})$ is optimal (e.g., Iida and Zipkin 2006). However, the multidimensional Equations (1) and (2) are extremely difficult to compute, and so is $s^*_t(D_{t-1})$. In the rest of this paper, we develop tractable approximations of $s^*_t(D_{t-1})$ and provide error bounds on the cost of using the approximate policies.

For simplicity, we sometimes suppress the argument in the base-stock levels. For example, we write $s^*_t(y)$ with the understanding of $s^*_t(D_{t-1})$. Also, let $D_t(t, t + i)$ represent the cumulative demand in periods 1 through $t + i - 1$, i.e., $D_t(t, t + i) = \sum_{j=0}^{i-1} D_{t+j} = \sum_{j=0}^{i-1} D_{t+j, t+j}$.

Using this notation, (3) can be rewritten as

$$G_t(y, D_{t-1}) \equiv C_t(y, D_{t-1, t}) + E[V_{t+1}(y - D_t(t, t + 1), D_t)].$$

Throughout this paper, for any function $\theta(y, D)$, without any confusion, we use $\theta'(y, D)$ to denote the partial derivative $\partial \theta(y, D)/\partial y$. In addition, for any real numbers $u$ and $v$, we denote $u \vee v = \max \{u, v\}$ and $u \wedge v = \min \{u, v\}$.

3. The First-Order Condition and Myopic Policies

3.1. The First-Order Condition: A Sample-Path View

Because $s^*_t(D_{t-1})$ is the solution of the first-order condition $G'_t(y, D_{t-1}) = 0$, to develop approximations of $s^*_t(D_{t-1})$, we begin with examining the constitutions of $G'_t(y, D_{t-1})$.

Assume that the inventory level after ordering in period $t$ is $y$, i.e., $I_t = y$. By the dominated convergence theorem, it is straightforward to show that $(E[V_{t+1}(y - D_t(t, t + 1), D_t)]') = E[V_{t+1}(y - D_t(t, t + 1), D_t)]$. Therefore,

$$G'_t(y, D_{t-1}) = C'_t(y, D_{t-1, t}) + E[V_{t+1}(y - D_t(t, t + 1), D_t)].$$

(4)

Note that $V_{t+1}(x, D_t) = G_{t+1}(x \lor s^*_{t+1}(D_t), D_t)$, which implies

$$V_{t+1}(x, D_t) = \begin{cases} 0, & x \leq s^*_{t+1}(D_t), \\ G'_{t+1}(x, D_t), & x > s^*_{t+1}(D_t). \end{cases}$$

(5)

So, $V_{t+1} = 0$ if the optimal base-stock level in period $t + 1$, $s^*_{t+1}(D_t)$, is reachable (Veinott 1965), i.e., the pre-order inventory level, $I_{t+1} = y - D_t(t, t + 1)$, is no greater than the optimal base-stock level $s^*_{t+1}(D_t)$. Therefore, the decision $y$ at time $t$ has no effect on the future cost if $s^*_{t+1}(D_t)$ is reachable. In other words, $y$ affects (increases) the cost of periods $t + 1$ and beyond only if the optimal base-stock level in period $t + 1$ is not reachable, in which case $V_{t+1}$ is positive.

Similarly, for any future period $t + i$, $V'_{t+i}$ is nonzero (positive) only if none of $s^*_{t+1}, s^*_{t+2}, \ldots, s^*_{t+i}$ is reachable, i.e., $I_{t+j} = y - D_t(t, t + j) > s^*_{t+j}$, $j = 1, 2, \ldots, i$. This means that the events $A_{t+j}(y, D_{t-1})$ happen for all $j = 1, 2, \ldots, i$, where

$$A_{t+j}(y, D_{t-1}) = \{y - D_t(t, t + j) > s^*_{t+j}(D_{t+j-1})\}.$$ (6)

Without confusion, we suppress $(y, D_{t-1})$, the initial condition at time $t$, in this notation most of the time.

Let $I(A)$ be the indicator function of $A$ and $I(A') = 1 - I(A)$. Note that the decision $y$ at time $t$ affects period $t + k$ and beyond only if $A_{t+i}$ happens for all $i = 1, \ldots, k$.

This implies

$$E[I(A_{t+1} \cdots A_{t+k})]V'_{t+i}(y - D_t(t, t + i), D_{t+i-1}) > 0$$

and

$$E[I(A_{t+1} \cdots A_{t+k} A'_{t+i})]V'_{t+i}(y - D_t(t, t + i), D_{t+i-1}) = 0.$$ (7)

The following proposition provides a decomposition of $G_t$.

Proposition 1. For any time $t$,

$$G_t(y, D_{t-1}) = C_t(y, D_{t-1, t}) + \pi_t(y, D_{t-1}),$$

(8)

where

$$\pi_t(y, D_{t-1}) = E[V'_{t+1}(y - D_t(t, t + 1), D_t)] = \sum_{i=1}^{T-t} E[I(A_{t+1} \cdots A_{t+i})] C'_{t+i}(y - D_t(t, t + i), D_{t+i-1, t+i}).$$

(9)

Moreover, $\pi_t(y, D_{t-1}) \geq 0$.

Proof. According to (4), we need to work only on $E[V_{t+1}(y - D_t(t, t + 1), D_t)]$:

$$E[V_{t+1}(y - D_t(t, t + 1), D_t)] = E[I(A_{t+1}) V_{t+1}(y - D_t(t, t + 1), D_t)] + E[I(A_{t+1} A'_{t+1}) V_{t+1}(y - D_t(t, t + 1), D_t)]$$

$$= E[I(A_{t+1}) V_{t+1}(y - D_t(t, t + 1), D_t)] + E[I(A_{t+1}) C'_{t+1}(y - D_t(t, t + 1), D_{t+i+1})]$$

$$+ E[I(A_{t+1}) V'_{t+2}(y - D_t(t, t + 2), D_{t+i+1})]$$
\[ = E[I(A_{t+1})C_{t+1}^r(y - D[t, t + 1], D_{t+1})] \\
+ E[I(A_{t+1}A_{t+2})V_{t+2}^r(y - D[t, t + 2], D_{t+1})] \\
+ E[I(A_{t+1}A_{t+2}^r)C_{t+2}^r(y - D[t, t + 2], D_{t+1})] \\
= E[I(A_{t+1})C_{t+1}^r(y - D[t, t + 1], D_{t+1})] \\
+ E[I(A_{t+1}A_{t+2})V_{t+2}^r(y - D[t, t + 2], D_{t+1})] \\
= E[I(A_{t+1})C_{t+1}^r(y - D[t, t + 1], D_{t+1})] \\
+ \cdots + \sum_{i=1}^{T-t} E[I(A_{t+i} \cdots A_t)C_{t+i}^r(y - D[t, t+i], D_{t+i})] \\
= \pi_r(y, D_{t-1}). \]

The second and the fifth equalities are due to (7). Finally, \( \pi_r(y, D_{t-1}) \geq 0 \) follows from (5). \( \square \)

From Proposition 1, the optimal inventory decision is a trade-off between the marginal cost in the current period \( C_t^r(y, D_{t-1}) \) and the marginal future cost \( \pi_r(y, D_{t-1}) \). While the marginal current cost can be positive or negative, from (5) the marginal future cost is always nonnegative.

Applying (8), the first-order condition for optimality is
\[
G_t^r(s_t^*(D_{t-1}), D_{t-1}) = C_t^r(s_t^*(D_{t-1}), D_{t-1}) + \pi_t(s_t^*(D_{t-1}), D_{t-1}) = 0. \tag{10}
\]

Note that
\[
C_t^r(s_t^*(D_{t-1}), D_{t-1}) = (b_t + h_t) P(D_{t-1}, t + e_{i,t} \leq s_t^*(D_{t-1})) - b_t = (b_t + h_t) F_t(s_t^*(D_{t-1}) - D_{t-1}) - b_t,
\]
where \( F_t(\cdot) \) is the cumulative distribution function of forecast error term \( e_{i,t} \). Treating \( -\pi_t(s_t^*(D_{t-1}), D_{t-1}) \) as a known constant, from (10) we can express the optimal base-stock level as
\[
s_t^*(D_{t-1}) = D_{t-1} + F_t^{-1}\left(\frac{b_t - \pi_t(s_t^*(D_{t-1}), D_{t-1})}{b_t + h_t}\right). \tag{11}
\]

Because \( \pi_t(s_t^*(D_{t-1}), D_{t-1}) \) depends on the entire forecast evolution, it is very difficult to obtain.

### 3.2. Myopic Policies

One common approach to deal with the difficulty of obtaining \( \pi_t(s_t^*(D_{t-1}), D_{t-1}) \) is to ignore it and use only the first term, \( C_t^r(y, D_{t-1}) \), to approximate \( G_t^r(y, D_{t-1}) \). This results in the so-called myopic policy—a base-stock policy with the base-stock level solving \( C_t^r(y, D_{t-1}) = 0 \). Let \( s_t^{\text{my}}(D_{t-1}) \) be the myopic base-stock level at time \( t \) given that the forecast vector at the beginning of time \( t \) is \( D_{t-1} \). Then,
\[
s_t^{\text{my}}(D_{t-1}) = D_{t-1} + F_t^{-1}\left(\frac{b_t}{b_t + h_t}\right). \tag{12}
\]

If \( F_t \) is a normal distribution and we let \( \Phi \) be the standard normal distribution function (remember that \( \sigma_t \) is the standard deviation of the forecasting error term \( e_{i,t} \)), then
\[
s_t^{\text{my}}(D_{t-1}) = D_{t-1} + \sigma_t \Phi^{-1}\left(\frac{b_t}{b_t + h_t}\right). \tag{13}
\]

Here, \( \Phi^{-1}(b_t/(b_t + h_t)) \) is termed the safety factor.

The myopic policy simply uses a lower bound zero to approximate the term \( \pi_t(s_t^{\text{my}}(D_{t-1}), D_{t-1}) \) \((\geq 0)\) and thus is an upper bound on \( s_t^*(D_{t-1}) \), i.e.,
\[
s_t^*(D_{t-1}) \leq s_t^{\text{my}}(D_{t-1}). \tag{14}
\]

**Remark.** In the case of a constant lead time \( L \), (12) becomes
\[
s_t^{\text{my}}(D_{t-1}) = \sum_{j=0}^{L} D_{t-1+j} + [F_t(L+1)]^{-1}\left(\frac{b_t}{b_t + h_t}\right), \tag{14}
\]
where \( F_t(L+1)(\cdot) \) is the cumulative distribution function of \( \sum_{i=0}^{L} e_{i,t+1} \).

The simple expressions of (12) and (13) further explain why the myopic policy is popular and when it might cause suboptimality. At any time \( t \), the policy parameter is the sum of two terms: the latest forecast of the current period demand \( D_{t-1} \), and a safety stock that depends only on the distribution of forecast error \( e_{i,t} \). Note that in most forecasting models, it is reasonable to have \( \sigma_t = \sigma \). If, further, the cost parameters are stationary, then the safety stock is a constant. Under these settings, using the myopic policy, we would stock in the first period the constant safety stock plus the forecasted demand for the first period, \( D_{t-1} \). Then, in the subsequent periods, we need only to adjust the order quantity according to the realized demand in the previous period and the latest demand forecast for the current period. More specifically, if we have ordered in period \( t - 1 \), so the post-order inventory position in that period \( I_{t-1} = s_t^{\text{my}} \) and the pre-order inventory position \( I_t^s = s_t^{\text{my}} - D_{t-1} \), then
\[
q_t = s_t^{\text{my}} - I_t^s = D_{t-1} - D_{t-2} + D_{t-1} - D_{t-1} = D_{t-1} + e_{i,t-1}
\]
is the order quantity in period \( t \) if this quantity is nonnegative. If \( q_t \) is nonnegative for all sample paths for all \( t \), which means that the myopic base-stock level in each period is reachable, then the cost in each period is minimized, so the myopic policy is optimal. It turns out that \( q_t \geq 0 \) is both necessary and sufficient for the myopic policy to be optimal. In general, we have the following.

**Proposition 2.** The myopic policy is optimal if and only if
\[
P(s_t^{\text{my}}(D_{t-1}) - D[t, t + 1] > s_t^{\text{my}}(D_{t-1})) = 0 \quad \text{for all } t. \tag{15}
\]

If both the cost parameters and the forecast update process are stationary, (15) is equivalent to
\[
q_t = D_{t-1} + e_{i,t-1} \geq 0 \quad \text{for all } t. \tag{16}
\]

**Proof.** The sufficient condition can be shown easily by induction; we omit the details here. To show the necessary condition, suppose that \( P(s_t^{\text{my}} - D_{t-1} > s_t^{\text{my}}(D_{t-1})) > 0 \). This implies that \( E[I(s_t^{\text{my}} - D_{t-1}) > s_t^{\text{my}}(D_{t-1})] > 0 \).
Because $V_{t+2}(\cdot, \cdot) \geq 0$, we have $G'_s(s^m, D_{t-1}) > 0$, which means that the myopic policy cannot be optimal—a contradiction.

Now assume stationary costs and forecast updates. Condition (16) requires that the demand forecast for the current period (period $t$) be large enough to offset the negative deviation of the forecast error in the previous period. It is conceivable that this condition can be met easily if the demand process has a nondecreasing trend. This is consistent with the general understanding of when the myopic policy is expected to be optimal.

It is interesting to see that even when demand has a decreasing trend, the myopic policy can still be optimal. Indeed, Iida and Zipkin (2006) offers such an example (see their independent, nonstationary demand example), provided the following condition holds: For every sample path of forecast updates, demands are nonnegative, i.e.,

$$D_{t-1, t} = D_{t-1} + e_{t-1, t} \geq 0 \quad \text{for all } t.$$  

(17)

Because $e_{t-1, t-1}$ and $e_{t-1, t}$ have the same distribution and $e_{t, t}$ and $D_{t-1, t}$ are uncorrelated, (17) implies (16). Although the demand may have a decreasing trend, (17) ensures that the forecast errors will be bounded by the lowest possible demand. In other words, if the demand in one period is low, the forecast errors in all periods have to be sufficiently small.

Note that because $D_{t-1, t}$ can be correlated with $e_{t-1, t-1}$, (16) does not necessarily imply (17). Therefore, Iida and Zipkin’s condition is sufficient but not necessary. We now illustrate this by an example adapted from Güllü (1997). In this example, demands in two consecutive periods are negatively correlated. If the demand in the current period is high (respectively, low), then the demand in the next period is expected to be lower (respectively, higher). This type of demand pattern is common when certain marketing efforts are in place. For example, the demand right after a promotion period is expected to be lower because of forward buying during the promotion period.

Example 3. Let $\mu$ be the mean demand, and set

$$D_{0, t} = \mu,$$

$$e_{t, t} = \eta_t \sim N(0, \sigma^2),$$

$$e_{t, t+1} = \rho \eta_t,$$

$$e_{t, t} = 0 \quad \text{for all } t \geq t + 2,$$

$$D_{t, t} = D_{0, t} + e_{t-1, t} + e_{t, t} \quad \text{for all } t.$$  

(17)

Here, $N(0, \sigma^2)$ is the normal random variable $N(0, \sigma^2)$ truncated at $-25\sigma$ and $25\sigma$. (Obviously, given the large truncation limits, $N(0, \sigma^2)$ is basically $N(0, \sigma^2)$.) Assume that $\rho = -0.9$, $\mu = 100$, and $\sigma = 40$. Because $e_{t-1, t} = -0.9e_{t-1, t-1}$, $|\eta_t| \leq 25\sigma \leq 1,000$, we have $q_t = D_{t-1, t} + e_{t-1, t-1} + \mu + e_{t-1, t-1} + 100 + 0.1\eta_{t-1} \geq 0$. Hence, the myopic policy is optimal. However, $D_{t-1, t} + e_{t-1, t} + \mu + \eta_t - 0.9\eta_{t-1}$ can be negative, so (17) is not satisfied.

When would myopic policies behave poorly? From the necessary and sufficient condition (16) we see that, for stationary costs and forecast updates, the myopic policy is suboptimal if there exist sample paths in which $q_t < 0$ for some $t$. This could happen if there is a sudden drop in demand or there is a decreasing demand trend so that $D_{t-1, t}$ is too small to offset the negative deviation of $e_{t-1, t-1}$. A similar observation is made by Song and Zipkin (1996) using a Markov modulated stochastic process to model demand facing obsolescence.

4. Bounds on Optimal Base-Stock Levels

Recall that the myopic policy simply approximates the nonnegative term $\pi(s^*(D_{t-1}), D_{t-1})$ by zero and thus is an upper bound on the optimal base-stock level $s^*(D_{t-1})$. In this section, we develop a general class of upper (lower) bounds on $\pi(s^*(D_{t-1}), D_{t-1})$ (these imply bounds on $G'_s$), which, according to (11), yield lower (upper) bounds on the optimal base-stock level $s^*_l(D_{t-1})$. Observe from the expression of (9) that what makes $\pi$ difficult to compute is its dependence on the optimal base-stock levels from period $t + 1$ on through the events $A_{t+i}$ in (6), $i = 1, \ldots, t$. Our idea for approximation is to replace the optimal base-stock level $s^*_l(D_{t+1})$ in $A_{t+i}$ with simpler-to-compute bounds. We present a procedure to construct the bounds recursively. We later show through examples that our procedure includes many existing bounds in the literature as special cases.

4.1. Construction of the Bounds

Set $\pi^u_t(\cdot, D_{t-1}) = \pi^u_t(\cdot, D_{t-1}) = 0$ and $s^*_l(D_{t-1}) = s^*_l(D_{t-1}) = s^*_l(D_{t-1})$ (the myopic base-stock level in period $T$). For any time $t < T$, suppose that we have obtained upper and lower bounds on the optimal base-stock level in period $t + i$, $s^*_l(D_{t+i})$ and $s^*_u(D_{t+i})$, respectively, for all $i = 1, \ldots, T - t$. (For example, we may have $s^*_l(D_{t+i}) = s^*_l(D_{t+i})$ and $s^*_u(D_{t+i}) = 0$.) Assume that $I_i = y$, and define

$$A^u_{t+i}(y, D_{t+i}) = \{y - D(t, t + i) > s^*_u(D_{t+i})\},$$

$$A^l_{t+i}(y, D_{t+i}) = \{y - D(t, t + i) > s^*_u(D_{t+i})\},$$

$$A^u_{t+i}(y, D_{t+i}) = \{y - D(t, t + i) > s^*_u(D_{t+i})\},$$

$$A^l_{t+i}(y, D_{t+i}) = \{y - D(t, t + i) > s^*_u(D_{t+i})\},$$

$$\pi^u_t(y, D_{t-1}) = \sum_{i=1}^{T-1} E[I(A^u_{t+i} \cap A^m_{t+i} \cap C_{t+i}(y - D(t, t + i), D_{t+i+1} \cup D_{t+i+1}))],$$

$$\pi^l_t(y, D_{t-1}) = \sum_{i=1}^{T-1} E[I(A^l_{t+i} \cap A^m_{t+i} \cap C_{t+i}(y - D(t, t + i), D_{t+i+1} \cup D_{t+i+1}))].$$
and construction, we have

\[ s^t_i(D_{t-1}) = \text{zero point of } C'_t(y, D_{t-1}, D_{t-1}) + \pi'_t(y, D_{t-1}), \]

\[ S^t_i(D_{t-1}) = \text{zero point of } C'_t(y, D_{t-1}, D_{t-1}) + \pi'_t(y, D_{t-1}). \]

Note that \( \pi'_t(y, D_{t-1}), \pi''_t(y, D_{t-1}), S^t_i(D_{t-1}), \) and \( S^t_i(D_{t-1}) \) all depend on the known bounds \( s^t_i(D_{t+i}) \) and \( S^t_i(D_{t+i}) \), \( t = 1, \ldots, T - t, \) implicitly.

If all the events \( A^u_{t+i}(y, D_{t-1}), \ldots, A^m_{t+i}(y, D_{t-1}) \) happen, for example, the inventory level before ordering is greater than the myopic base-stock level in all periods \( t + 1 \) through \( t + i \) given that the inventory level after ordering in period \( t \) is \( y \). Therefore, no order will be placed in these periods under the myopic policy. Similar discussion applies to \( A^u_{t+i}(y, D_{t-1}) \) and \( A^m_{t+i}(y, D_{t-1}) \).

As we shall show below, using the right combination of \( A^u_{t+i}(y, D_{t-1}), A^l_{t+i}(y, D_{t-1}) \), and \( A^m_{t+i}(y, D_{t-1}) \) to replace \( A^u_{t+i}(y, D_{t-1}) \) in \( \pi'_t(y, D_{t-1}) \) can result in bounds on \( \pi'_t(y, D_{t-1}) \).

We have Theorem 4.

**Theorem 4 (Bounds on Optimal Base-Stock Levels).**

For any given \( t, t = 1, \ldots, T, \) under the above assumption and construction, we have

\[ \pi'_t(y, D_{t-1}) \leq \pi'_t(y, D_{t-1}) \leq \pi''_t(y, D_{t-1}). \]

This implies that \( G'_t(y, D_{t-1}) \) is bounded below and above by \( C'_t(y, D_{t-1}) + \pi'_t(y, D_{t-1}) \) and \( C'_t(y, D_{t-1}) + \pi''_t(y, D_{t-1}) \), respectively. Moreover, these bounding functions are increasing functions of \( y \), so their zero points \( S^t_i(D_{t-1}) \) and \( S^t_i(D_{t-1}) \) are unique and have the exact expression as in (11) with \( \pi'_t \) replaced by \( \pi'_t \) and \( \pi''_t \), respectively. Finally,

\[ S^t_i(D_{t-1}) \leq s^t_i(D_{t-1}) \leq S^t_i(D_{t-1}). \]

**Proof.** It is sufficient to show that (18) holds; other parts are straightforward.

Note that when \( t = T \), from the terminal condition (1), all the inequalities in (18) become equalities. Now, consider \( t < T \). First, note that from the construction, \( A^u_{t+i} \subseteq A^l_{t+i} \subseteq A^m_{t+i} \), and \( A^m_{t+i} \subseteq A^u_{t+i} \), implying \( I(A^u_{t+i}) \leq I(A^l_{t+i}) \leq I(A^m_{t+i}) \) and \( I(A^m_{t+i}) \leq I(A^u_{t+i}) \) for all \( i = 1, \ldots, T - t \). For every sample path of \( D[t, t + 1] \), we have

\[ C^u_{t+i}(y - D[t, t + 1], D_{t+i}) < 0 \]

for \( s^u_{t+i}(D_{t+i}) < y - D[t, t + 1] < s^m_{t+i}(D_{t+i}) \).

So,

\[ E[I(A^u_{t+i} - A^l_{t+i} + A^m_{t+i})] \leq 0. \]

Therefore,

\[ E[I(A^u_{t+i})] \leq 0. \]

\[ E[I(A^u_{t+i})] \leq 0. \]

Similarly,

\[ E[I(A^u_{t+i} - A^l_{t+i} + A^m_{t+i})] \leq 0. \]

The first inequality is due to the fact that \( I(A^u_{t+i} - A^l_{t+i} + A^m_{t+i}) \leq 0 \). The second inequality is from \( I(A^u_{t+i}) \). Combining (9), (19), and (20), we obtain \( \pi'_t(y, D_{t-1}) \leq \pi'_t(y, D_{t-1}) \).

On the other hand, recall that \( V'_{t+i}(y - D[t, t + 1], D_{t+i}) \) is nonnegative, so

\[ I(A^u_{t+i} - A^m_{t+i}) \leq 0. \]

Then,

\[ E[I(A^u_{t+i})] \leq 0. \]

All the inequalities are due to (21). Note that the second and fifth equalities are due to the fact that when \( A^u_{t+i} \) happens, we have \( y - D[t, t + 1] > s^u_{t+i}, \) so no order will be
placed in period \( t + i \) under the optimal policy. The third and sixth equalities are due to (7). From (9) and (22), we obtain 
\[
\pi_i(y, D_{t+i}) \geq \pi_i(y, D_{t-1}).
\]

To summarize, from Theorem 4, for any \( t \), based on any upper and lower bounds on the optimal base-stock levels from period \( t + 1 \) until \( T \), we can construct lower and upper bounds on the optimal base-stock level in period \( t \). In this way, we develop a general class of bounds on the optimal base-stock levels.

### 4.2. Connections with Existing Bounds

Clearly, the only difference between these new approximations and the myopic policy is the adjustments in the fractile that determines the magnitude of the safety stock. The myopic is a special case of our result in which we set \( \pi_i(y, D_{t-1}) = 0 \).

Below we provide several other examples to illustrate the connection of the bounds in Theorem 4 with the existing bounds in the literature.

The solution to the shorter-horizon, \((k + 1)\)-period problem with zero terminal cost is referred to as the \( k \)-period ahead solution. It has been shown in the literature that this solution is an upper bound on optimal base-stock level for the nonstationary independent demand model (Morton 1978) and for the MMFE (Iida and Zipkin 2006). In Example 5, we show that this result can be viewed as a special case of our procedure. For simplicity, we illustrate only the case of \( k = 1 \). The idea for general \( k \) is similar.

**Example 5 (One-Period Ahead Policy).** Setting \( s_{t+1}^{m} = s_{t+1}^{o} \) and \( s_{t+2}^{m} = \infty \) yields \( A_{t+2}^{m} = y - D[t, t + 2] \geq s_{t+2}^{o} = \infty \) so \( \pi_i(y, D_{t}) = 0 \). Therefore, 
\[
E[I(A_{t+1}^{m} \cdot A_{t+2}^{m})]_{C_{t+1}(y - D[t, t + 1], D_{t+i+1})} = 0
\]
for all \( i \geq 2 \) and \( \pi_i(y, D) = E[I(A_{t+1}^{m})]_{C_{t+1}(y - D[t, t + 1], D_{t+i+1})} \). According to Theorem 4, the upper bound to 
\[
C_i(y, D_{t-1},\ldots) + E[I(A_{t+1}^{m}) C_{t+1}(y - D[t, t + 1], D_{t+i+1})] = 0
\]
is an upper bound on \( s_{t+1}^{o} \).

In the following, we show that our procedure generalizes and tightens some lower bounds in the literature. For instance, Example 6(b) has been shown by Morton (1978) and Iida and Zipkin (2006) for independent and MMFE demand models, respectively. Example 6(c) provides tighter lower bounds on the optimal base-stock level than that of Example 6(b).

**Example 6.** Let \( s_{t+i}^{(k)}(D_{t+i}) \) be the optimal solution to a \((k + 1)\)-period problem (from period \( t \) until period \( t + k \)) for period \( t + i \) with terminal cost of \( \sum_{s-t}^{T-t} h_{t+i} \). Then, 

\( s_{t+i}^{(k)}(D_{t+i}) \) solves the following equations recursively:

\[
C_{t+i}(y, D_{t+i-1},\ldots) + \pi_{t+i}(y, D_{t+i-1}) = 0,
\]

where

\[
\pi_i(y, D_{t-1}) = E[I(y - D[t, t + 1] > s_{t+i}^{(k)}(D_{t+i})] \cdot C_{t+1}(y - D[t, t + 1], D_{t+i+1})] + \cdots + E[I(y - D[t, t + 1] > s_{t+i}^{(k)}(D_{t+i})] \cdot \cdots \cdot I(y - D[t, t + k + 1] > s_{t+i+k+1}^{(k)}(D_{t+i+k+1})) \cdot (\sum_{i=k+1}^{T-t} h_{t+i})].
\]

(b) \( s_{t+i}^{(k)}(D_{t+i}) \leq s_{t+i}^{*}(D_{t+i}), i = 0, \ldots, k \).

(c) Replacing \( \sum_{i=k+1}^{T-t} h_{t+i} \) in (24) with \( \sum_{i=k+1}^{T-t} h_{t+i} P(y - D[t, t + i] > s_{t+i}^{*}(D_{t+i})), \) solving (23) leads to a tighter lower bound for \( s_{t+i}^{*}(D_{t+i}) \).

**Proof.** By the same logic in developing Proposition 1, we obtain (a). Because the proof of (c) is similar to that of (b), below we show only (b).

We use induction. Because

\[
C_{t+i}(y, D_{t+i-1},\ldots) + \pi_{t+i}(y, D_{t+i-1}) \leq C_{t+i}(y, D_{t+i-1},\ldots) + \pi_{t+i}(y, D_{t+i-1})
\]

we know that \( s_{t+i}^{(k)}(D_{t+i}) \leq s_{t+i}^{*}(D_{t+i}). \)

Suppose that \( s_{t+j}^{(k)}(D_{t+j}) \leq s_{t+j}^{*}(D_{t+j}) \) holds for \( i \leq j \leq k \). We only need to show \( s_{t+i}^{(k)}(D_{t+i}) \leq s_{t+i}^{*}(D_{t+i}) \) for simplicity, we show the case of \( i = 1 \).

Applying a similar proof of (22) to the \((k + 1)\)-period problem, we know that \( s_{t+j}^{(k)}(D_{t+j}) \) leads to a lower bound on \( \pi_{t+i}(y, D_{t+i}) \). That is,

\[
C_{t+i}(y, D_{t+i-1},\ldots) + \pi_{t+i}(y, D_{t+i-1})
\]

is a lower bound on \( s_{t+i}^{*}(D_{t+i}) \).

Therefore \( s_{t+i}^{(k)}(D_{t+i}) \leq s_{t+i}^{*}(D_{t+i}) \) holds because \( s_{t+i}^{(k)}(D_{t+i}) \) solves \( C_{t+i}(y, D_{t+i-1},\ldots) + \pi_{t+i}(y, D_{t+i-1}) = 0 \) and \( s_{t+i}^{*}(D_{t+i}) \) solves \( C_{t+i}(y, D_{t+i-1},\ldots) + \pi_{t+i}(y, D_{t+i-1}) = 0 \).
Example 7. Set \( s_{t+1}^u(D_t) = s_{t+1}^m(D_t) \). According to Theorem 4, we have \( \pi_t^u(y, D_{t-1}) \leq \pi_t^m(y, D_{t-1}) \), where

\[
\pi_t^m(y, D_{t-1}) = \mathbb{E}[I(A_t^m)C_{t+1}(y - D[t, t+1], D_{t, t+1})] + \sum_{j=2}^{T-t} \mathbb{E}[I(A_{t+1} \cdots A_{t+j-1}^m)I(A_{t+j}^m)]C_{t+j}(y - D[t, t+j], D_{t+j-1, t+j}].
\]

Because

\[
C_{t+j}(y - D[t, t+j], D_{t+j-1, t+j}) = (b_{t+j} + h_{t+j})P(D_{t+j-1, t+j} + e_{t+j, t+j} \leq y - D[t, t+j]) - b_{t+j} \leq h_{t+j},
\]

we further have \( \pi_t^m(y, D_{t-1}) \leq \pi_t^u(y, D_{t-1}) \leq \pi_t^m(y, D_{t-1}) \), where

\[
\pi_t^u(y, D_{t-1}) = h_{t+1}P(A_{t+1}^u) + \sum_{j=2}^{T-t} h_{t+j}P(A_{t+1}^u \cdots A_{t+j-1}^u A_{t+j}^u),
\]

\[
\pi_t^m(y, D_{t-1}) = h_{t+1}P(A_{t+1}^m) + \sum_{j=2}^{T-t} h_{t+j}P(A_{t+1}^m \cdots A_{t+j-1}^m A_{t+j}^m).
\]

According to Theorem 4, the solution to each of the following three equations:

\[
C_t(y, D_{t-1}) + \pi_t^u(y, D_{t-1}) = 0, \quad i = 1, 2, 3,
\]

is a lower bound on the optimal base-stock level. Furthermore, because \( \pi_t^u(y, D_{t-1}) \) increases in \( y \), we have

\[
C_t(y, D_{t-1}) + \pi_t^u(y, D_{t-1}) \leq C_t(y, D_{t-1}) + \pi_t^u(s_t^u(D_{t-1}), D_{t-1}), \quad y \leq s_t^u(D_{t-1}).
\]

Suppose that \( S_t^u(D_{t-1}) \) solves \( C_t(y, D_{t-1}) + \pi_t^u(s_t^u(D_{t-1}), D_{t-1}) = 0 \). Then, \( S_t^u(D_{t-1}) \) is a newsveslver solution, which is easier to compute but is a looser lower bound (than the solution to (26)):

\[
s_t^u(D_{t-1}) \geq S_t^u(D_{t-1}) \geq S_t^u(D_{t-1}) \geq S_t^u(D_{t-1}).
\]

where

\[
S_t^u(D_{t-1}) = D_{t-1} + \Phi\left(\frac{b_t - \pi_t^u(s_t^u(D_{t-1}, D_{t-1}))}{b_t + h_t}\right), \quad i = 1, 2, 3.
\]

The lower bound \( S_t^u(D_{t-1}) \) is essentially the lower bound developed in Morton (1978) and Morton and Pentico (1995). They developed the lower bound on the optimal base-stock level for the case of independent demand inventory models and their result can be extended to the demand-forecasting inventory models. In addition to the probability of overstock they employ, we use more information such as the magnitude of overstock, \( y - D[t, t+j] - s_t^u(D_{t+j-1}) \) given \( I_t = y \), to estimate the marginal future cost. Observe from (25) that \( h_{t+j}P(A_{t+1}^u \cdots A_{t+j-1}^u A_{t+j}^u) \) can be much greater than \( E[I(A_{t+1}^u \cdots A_{t+j-1}^u A_{t+j}^u)C_{t+j}(\cdot, \cdot)] \), \( j = 1, 2, \ldots, T-t \). So, \( \pi_t^u(y, D_{t-1}) \) can be much smaller than \( \pi_t^m(y, D_{t-1}) \), which could lead to a significantly tighter lower bound on optimal base-stock levels. Indeed, for a two-period problem, one of our lower bounds, the solution to (26) for the case of \( i = 1 \), is the exact optimal solution, while their lower bound \( S_t^u(D_{t-1}) \) cannot be optimal. In fact, none of the bounds in (27) can be optimal.

Now we present some easier-to-implement bounds, which are used in the numerical studies in §6.

Example 8. Set \( s_t^{u+1}(D_t) = s_{t+1}^m(D_{t+1}) \), \( s_t^{u+2}(D_{t+1}) = s_{t+2}^m(D_{t+2}) \), and \( s_t^{u+3}(D_{t+2}) = \infty \). We obtain

\[
A_t^{u+3} = \{ y - D[t, t+3] > s_t^{u+3}(D_{t+1}) \}
\]

\[
= \{ y - D[t, t+3] > \infty \} = \emptyset,
\]

which implies

\[
E[I(A_{t+1}^u \cdots A_{t+j-1}^u)C_{t+j}(y - D[t, t+j], D_{t+j-1, t+j})] = 0 \quad \forall j \geq 3.
\]

According to Theorem 4, \( S_t^u(D_{t-1}) \), the solution to

\[
C_t(y, D_{t-1}) + E[I(A_{t+1}^u)C_{t+1}(y - D[t, t+1], D_{t, t+1})] + E[I(A_{t+1}^m A_{t+2}^m)C_{t+2}(y - D[t, t+2], D_{t+1, t+2})] = 0 \quad (28)
\]

is an upper bound for \( s_t^{u+1}(D_{t-1}) \).

Example 9. Setting \( s_t^{u+1}(D_t) = s_{t+1}^m(D_{t+1}) \) and \( s_t^{u+1}(D_{t+1}) = \infty \), \( 2 \leq i \leq T - t \), we have

\[
A_t^{i,i} = \{ y - D[t, t+i] > s_t^{u+i}(D_{t+i-1}) \}
\]

\[
= \{ y - D[t, t+i] > \infty \} = \Omega.
\]

Therefore,

\[
E[I(A_{t+1}^u \cdots A_{t+i-1}^u A_{t+i}^m)C_{t+i}(y - D[t, t+i], D_{t+i-1, t+i})] = E[I(A_{t+1}^u A_{t+i}^m)C_{t+i}(y - D[t, t+i], D_{t+i-1, t+i})] \quad \forall j \geq 3.
\]

Thus, \( S_t^u(D_{t-1}) \), the solution to

\[
C_t(y, D_{t-1}) + E[I(A_{t+1}^u)C_{t+1}(y - D[t, t+1], D_{t, t+1})] + \sum_{j=2}^{T-t} E[I(A_{t+1}^u A_{t+j}^m)C_{t+j}(y - D[t, t+j], D_{t+j-1, t+j})] = 0 \quad (29)
\]

is a lower bound on \( s_t^u(D_{t-1}) \) according to Theorem 4.
5. Cost-Error Bounds

In this section, we consider how to estimate the value loss of a heuristic relative to optimal cost. Let $V_t^H(x, D_{t-1})$ be the total expected cost of a given heuristic policy $H$ in periods $t$ through $T$, assuming that the pre-order inventory level in period $t$ is $x$ and the forecast vector made at the end of period $t-1$ is $D_{t-1}$. The cost error of $H$ relative to the optimal cost is defined by

$$\text{err} = \frac{V_t^H(x, D_{t_0}) - V_t(x, D_{t_0})}{V_t(x, D_{t_0})} \times 100\%.$$ 

Our approach is to develop an upper bound for $V_t^H(x, D_{t-1}) - V_t(x, D_{t-1})$ and a lower bound for $V_t(x, D_{t-1})$ because $V_t(x, D_{t-1})$ is usually computationally impossible to obtain.

Recall that $s^H_t$ is an upper bound for the optimal base-stock level; therefore, we assume that $s^H_t(D_{t-1}) \leq s^*_t(D_{t-1})$. 

5.1. Upper Bound on $V_t^H(x, D_{t-1}) - V_t(x, D_{t-1})$

Take any period $t$, and assume that $I_t = x$. There are two possible situations that require different treatments: $s^H_t(D_{t-1}) \leq s^*_t(D_{t-1})$ and $s^H_t(D_{t-1}) > s^*_t(D_{t-1})$. We study them separately and present the results in Lemmas 10 and 11. Based on these results, we develop upper bounds on $V_t^H(x, D_{t-1}) - V_t(x, D_{t-1})$, which is presented in Theorem 12.

**Lemma 10.** If $s^H_t(D_{t-1}) \leq s^*_t(D_{t-1})$, we have

$$V_t^H(x, D_{t-1}) - V_t(x, D_{t-1}) \leq \beta_1(t, D_{t-1}) + E[V_{t+1}^H(x \lor s^H_t(D_{t-1}) - D_t, D_t) - V_{t+1}(x \lor s^H_t(D_{t-1}) - D_t, D_t)],$$

where

$$\beta_1(t, D_{t-1}) = C_t(s^H_t(D_{t-1}), D_{t-1,i}) - C_t(s^*_t(D_{t-1}), D_{t-1,i}).$$

**Proof.** If $s^H_t(D_{t-1}) \leq s^*_t(D_{t-1})$, then $C_t(s^H_t(D_{t-1}), D_{t-1,i}) \geq C_t(s^*_t(D_{t-1}), D_{t-1,i})$. Furthermore, because $s^*_t$ minimizes $C_t(D_{t-1,i})$, we have

$$C_t(x \lor s^H_t(D_{t-1}), D_{t-1,i}) - C_t(x \lor s^*_t(D_{t-1}), D_{t-1,i}) \leq \beta_1(t, D_{t-1}).$$

Note that $V_t^H(x, D_{t-1}) = C_t(s^H_t(D_{t-1}) \lor x, D_{t-1,i}) + E[V_{t+1}^H(s^H_t(D_{t-1}) \lor x - D_t, D_t)]$ and $V_t(x, D_{t-1}) = C_t(s^*_t(D_{t-1}) \lor x, D_{t-1,i}) + E[V_{t+1}(s^*_t(D_{t-1}) \lor x - D_t, D_t)].$

We obtain

$$V_t^H(x, D_{t-1}) - V_t(x, D_{t-1}) = C_t(x \lor s^H_t(D_{t-1}), D_{t-1,i}) - C_t(x \lor s^*_t(D_{t-1}), D_{t-1,i}) + E[V_{t+1}^H(x \lor s^H_t(D_{t-1}) - D_t, D_t)] - E[V_{t+1}(x \lor s^*_t(D_{t-1}) - D_t, D_t)] \leq \beta_1(t, D_{t-1}) + E[V_{t+1}^H(x \lor s^H_t(D_{t-1}) - D_t, D_t)] - E[V_{t+1}(x \lor s^*_t(D_{t-1}) - D_t, D_t)] \leq \beta_1(t, D_{t-1}) + E[V_{t+1}^H(x \lor s^H_t(D_{t-1}) - D_t, D_t) - V_{t+1}(x \lor s^*_t(D_{t-1}) - D_t, D_t)].$$

Here, the first inequality is due to (32). The second inequality is because $x \lor s^H_t(D_{t-1}) - D_t \leq x \lor s^*_t(D_{t-1}) - D_t$ and $V_t(x, D_t)$ increases in $z$. □

**Lemma 11.** If $s^H_t(D_{t-1}) > s^*_t(D_{t-1})$, then

$$V_t^H(x, D_{t-1}) - V_t(x, D_{t-1}) \leq \beta_2(t, D_{t-1}) + E[V_{t+1}^H(x \lor s^H_t(D_{t-1}) - D_t, D_t) - V_{t+1}(x \lor s^*_t(D_{t-1}) - D_t, D_t)].$$

**Proof.** We first show that (33) holds. We have the following key observation:

$$V_t^H(x, D_{t-1}) - V_t(x, D_{t-1}) = V_{t+1}^H(s^H_t(D_{t-1}) \lor x, D_{t-1}) - G_t(s^*_t(D_{t-1}) \lor x, D_{t-1})$$

$$= G_t(x \lor s^H_t(D_{t-1}), D_{t-1}) - G_t(x \lor s^*_t(D_{t-1}), D_{t-1}) + V_t^H(x \lor s^H_t(D_{t-1}), D_{t-1}) - G_t(x \lor s^H_t(D_{t-1}), D_{t-1}).$$

We obtain

$$V_t^H(x, D_{t-1}) - V_t(x, D_{t-1}) = C_t(x \lor s^H_t(D_{t-1}), D_{t-1,i}) - C_t(x \lor s^*_t(D_{t-1}), D_{t-1,i}) + E[V_{t+1}^H(x \lor s^H_t(D_{t-1}) - D_t, D_t)] - E[V_{t+1}(x \lor s^*_t(D_{t-1}) - D_t, D_t)] \leq \beta_1(t, D_{t-1}) + E[V_{t+1}^H(x \lor s^H_t(D_{t-1}) - D_t, D_t) - V_{t+1}(x \lor s^*_t(D_{t-1}) - D_t, D_t)].$$

Here, the first inequality is due to (32). The second inequality is because $x \lor s^H_t(D_{t-1}) - D_t \leq x \lor s^*_t(D_{t-1}) - D_t$ and $V_t(x, D_t)$ increases in $z$. □
Therefore, The second inequality is due to (18). Combining (36) and 
Lu, Song, and Regan:

\[ G_t(x \in s^H_t(D_{t-1}), D_{t-1}) - G_t(x \in s^o_t(D_{t-1}), D_{t-1}) + C_t (x \in s^H_t(D_{t-1}), D_{t-1}) - C_t (x \in s^o_t(D_{t-1}), D_{t-1}) + E[V^H_{t+1}(x \in s^H_t(D_{t-1}), D_t, D_t)] - E[V^H_{t+1}(x \in s^o_t(D_{t-1}), D_t, D_t)] = G_t(x \in s^H_t(D_{t-1}), D_{t-1}) - G_t(x \in s^o_t(D_{t-1}), D_{t-1}) + E[V^H_{t+1}(x \in s^H_t(D_{t-1}), D_t, D_t)] - E[V^H_{t+1}(x \in s^o_t(D_{t-1}), D_t, D_t)]. \]

(36) Note that \( G_t(y, D_t)|_{s_H^0(\mathbf{d}) \leq y \leq s^o(\mathbf{d})} \geq 0 \). We also have \( G_t(x \in s^H_t(D_{t-1}), D_{t-1}) - G_t(x \in s^o_t(D_{t-1}), D_{t-1}) = (x \in s^H_t(D_{t-1}) - x \in s^o_t(D_{t-1})) \)

\[ = G_t(y, D_t)|_{s_H^0(\mathbf{d}) \leq y \leq s^o(\mathbf{d})} \leq |s^H_t(D_{t-1}) - s^o_t(D_{t-1})|^+ G_t(y, D_t)|_{y \leq s^o(\mathbf{d})} = |s^H_t(D_{t-1}) - s^o_t(D_{t-1})|^+ \{C_t(s^H_t(D_{t-1}), D_{t-1}) + \pi^\alpha_t(s^H_t(D_{t-1}), D_{t-1})\} \leq \pi^\alpha_t(s^H_t(D_{t-1}), D_{t-1}) + \pi^\alpha_t(s^o_t(D_{t-1}), D_{t-1}) \] (37) The second inequality is due to (18). Combining (36) and (37) yields (33).

We next show that (34) holds. If \( x \geq s^H_t(D_{t-1}) \), we have \( V^H_t(x, D_{t-1}) - V_t(x, D_{t-1}) = C_t(x, D_{t-1}) - C_t(x, D_{t-1}) + E[V^H_{t+1}(x - D_t, D_t)] - E[V^H_{t+1}(x - D_t, D_t)] = E[V^H_{t+1}(x - D_t, D_t)] - E[V^H_{t+1}(x - D_t, D_t)]. \) If \( x < s^H_t(D_{t-1}) \), we have \( V^H_t(x, D_{t-1}) = V^H_t(0, D_{t-1}) \). Because \( s^H_t(D_{t-1}) \geq s^o_t(D_{t-1}) \), we also have \( C_t(s^H_t(D_{t-1}), D_{t-1}) - C_t(s^o_t(D_{t-1}), D_{t-1}) \leq 0 \). Therefore, \( V^H_t(x, D_{t-1}) - V_t(x, D_{t-1}) = V^H_t(0, D_{t-1}) - V_t(x, D_{t-1}) \leq V^H_t(0, D_{t-1}) - V_t(0, D_{t-1}) = C_t(s^H_t(D_{t-1}), D_{t-1}) - C_t(s^o_t(D_{t-1}), D_{t-1}) + E[V^H_{t+1}(s^H_t(D_{t-1}) - D_t, D_t)] - E[V^H_{t+1}(s^o_t(D_{t-1}) - D_t, D_t)] \leq E[V^H_{t+1}(s^H_t(D_{t-1}) - D_t, D_t)] - E[V^H_{t+1}(s^o_t(D_{t-1}) - D_t, D_t)] \)

\[ = E[V^H_{t+1}(s^H_t(D_{t-1}) - D_t, D_t)] - E[V^H_{t+1}(s^o_t(D_{t-1}) - D_t, D_t)] + E[V^H_{t+1}(s^o_t(D_{t-1}) - D_t, D_t)] - E[V^H_{t+1}(s^o_t(D_{t-1}) - D_t, D_t)]. \]

(43) The first equality is due to (39). The second inequality is due to \( V_t(x, D_{t-1}) \) increases in \( x \), and the second inequality is due to (40).

We further show the following in the appendix:

\[ E[V^H_{t+1}(s^H_t(D_{t-1}) - D_t, D_t)] - E[V^H_{t+1}(s^o_t(D_{t-1}) - D_t, D_t)] \leq \beta_t(t, D_{t-1}). \] (44)

From (43) and (44), if \( x < s^H_t(D_{t-1}) \), we have that (34) holds. On the other hand, if \( x \geq s^H_t(D_{t-1}) \), from (38), (34) also holds. This completes the proof. □

Based on Lemmas 10 and 11 and noting that \( \beta_n(t, D_{t-1}) \), \( n = 1, 2, 3 \), do not depend on the initial inventory \( x \), we develop upper bounds on \( V^H_t(x, D_{t-1}) - V_t(x, D_{t-1}) \) as follows.

**Theorem 12 (Maximum Gap Between the Costs of Any Heuristic Policy and the Optimal Policy).** For any given heuristic policy \( H \) and any \( t \),

\[ V^H_t(x, D_{t-1}) - V_t(x, D_{t-1}) \leq \beta_1(t, D_{t-1}) \lor \beta_2(t, D_{t-1}) + E\left[ \sum_{k=t+1}^T \beta_1(k, D_{k-1}) \lor \beta_2(k, D_{k-1}) \right], \]

\( n = 2, 3 \). (45)

**Proof.** Suppose that \( n = 2 \). For \( t = T \), because we assume that \( s^H_T \leq s^o_T \equiv s^* \), we do not need to consider \( s^H_T > s^* \). For the case \( s^H_T \leq s^* \), from Lemma 10, we have that (45) holds. Suppose that (45) holds for \( V^H_{t+1}(x, D) - V_{t+1}(x, D), i = 1, \ldots, T - t \), for some \( t \). Also, recall that the upper bound in (45) is independent of \( x \). In the following, we show that (45) holds for period \( t \), the proof will then be completed by induction.

No matter whether \( s^H_t(D_{t-1}) \) is less than \( s^*_{t-1} \) or not, from (30) and (33), we have

\[ V^H_t(x, D_{t-1}) - V_t(x, D_{t-1}) \leq \beta_1(t, D_{t-1}) \lor \beta_2(t, D_{t-1}) + E\left[ V^H_{t+1}(x \lor s^H_t(D_{t-1}) - D_t, D_t) - V_{t+1}(x \lor s^H_t(D_{t-1}) - D_t, D_t) \right]. \]

By induction, (45) holds for \( n = 2 \). Similarly, based on (30) and (34), (45) also holds for \( n = 3 \). □

**Remarks.** (a) The maximum value loss of using the heuristic policy \( H \) is given by the right-hand side of (45) with \( t = 1 \).

(b) The right-hand side of (45) does not depend on the order inventory level \( x \), so what we have is a worst-case analysis. In other words, the right-hand side of (45) is an absolute upper bound on the cost difference between the heuristic and the optimal policy. We can modify \( \beta_1(t, D_{t-1}) \lor \beta_2(t, D_{t-1}) \) to develop a recursive upper bound for \( V^H_t(x, D_{t-1}) - V_t(x, D_{t-1}) \), which depends on \( x \). We illustrate this by modifying \( \beta_1(t, D_{t-1}), \beta_2(t, D_{t-1}) \) into...
\( \hat{\beta}_1(t, D_{t-1}) \) and \( \hat{\beta}_2(t, D_{t-1}) \) only. The other is much more complex.

\[
\hat{\beta}_1(t, D_{t-1}) = C(s^H_t(D_{t-1}) \lor x, D_{t-1,t})
\]

\[
- C(s^H_t(D_{t-1}) \land s^H_t(D_{t-1}) \lor x, D_{t-1,t})
\]

\[
\hat{\beta}_2(t, D_{t-1}) = \beta_2(t, D_{t-1}) = \tau_1[C(s^H_t(D_{t-1}), D_{t-1,t})
\]

\[
+ \pi^m_t(s^H_t(D_{t-1}), D_{t-1})
\]

\[
\tau_1 = [s^H_t(D_{t-1}) \lor x - s^*_t(D_{t-1}) \lor x]^+.
\]

(c) In the numerical study, we use the following fact:

\[
\beta_3(t, D_{t-1}) \leq \sum_{k=1}^{T-t} E[I(A^p_t A^m_{t+k}) \tau_2]
\]

\[
\cdot C_{t+k}(s^H_t(D_{t-1}) - D[t, t+k, D_{t+k-1,t+k}])
\]

\[
(46)
\]

Taking any upper bound on the optimal base-stock level that is no greater than the myopic base-stock level (such as the myopic solution) as a heuristic policy, we do not need to consider the case of \( s^H_t(D_{t-1}) \leq s^*_t(D_{t-1}) \). This yields the following corollary.

**Corollary 13.** The maximum gap between the costs of the optimal policy and any upper-bound policy \( s^B_t(D_{t-1}) \), \( t = 1, 2, \ldots, T \), that satisfies \( s^*_t(D_{t-1}) \leq s^m_t(D_{t-1}) \) for all \( t \) is

\[
V^H_t(x, D_0) - V_t(x, D_0)
\]

\[
\leq \beta_3(1, D_0) + \sum_{t=2}^{T} E[\beta_3(t, D_{t-1})]
\]

\[
\leq \beta_3(1, D_0) + \sum_{t=2}^{T} E[\beta_3(t, D_{t-1})]
\]

\[
(47)
\]

where

\[
\hat{\beta}_3(t, D_{t-1})
\]

\[
= \sum_{k=1}^{T-t} E[(s^*_t(D_{t-1}) - D_t - s^m_{t+k}(D_{t+k}))^+ I(A^m_{t+k})
\]

\[
\cdot (C_{t+k}(s^m_t(D_{t-1}) - D[t, t+k, D_{t+k-1,t+k}]))
\]

\[
\leq \left( \sum_{k=1}^{T-t} h_{t+k} \right) E[(s^*_t(D_{t-1}) - D_t - s^m_{t+k}(D_{t+k}))^+].
\]

\[
(49)
\]

In particular, the cost-error bounds for the myopic policy is given by replacing \( s^B_t(D_{t-1}) \) with \( s^H_t(D_{t-1}) \) in (47).

Note that, compared with (46), the bound (48) is much easier to evaluate but can be much looser (see the examples in §6).

Using a different approach, Lovejoy (1992) developed an upper bound for the cost difference between the optimal policy and the myopic policy. Because \( s^m_t(D_{t-1}) \geq s^*_t(D_{t-1}) \), instead of developing \( \hat{\beta}_3 \), Lovejoy estimates the cost of disposing of the overstock \( (s^m_t - D_t - s^m_{t+1})^+ \), which yields an upper bound on the extra cost due to the decision made in period \( t \). For example, according to Lovejoy (1992), the disposal cost can be set at \( \sum_{j=1}^{T-t} h_j \). Thus, Lovejoy’s bound is to replace \( \hat{\beta}_3(t, D_{t-1}) \) with \( (\sum_{j=1}^{T-t} h_j) E[(s^*_t(D_{t-1}) - D_t - s^m_{t+1}(D_{t+1}))^+] \). From (49), the error bound developed in this paper can be much tighter, as illustrated in the numerical examples in §6.

### 5.2. Lower Bound on \( V_t(x, D_{t-1}) \)

The derivation of a lower bound on \( V_t(x, D_{t-1}) \) is quite straightforward. We assume that an upper bound on optimal base-stock level \( s^*_t(D_{t-1}) \) is known (recall that \( s^*_t(D_{t-1}) \) is a special case of \( s^H_t(D_{t-1}) \)). We have the following.

**Proposition 14.**

\[
V_t(x, D_{t-1}) \geq C_t(x \lor s^H_t(D_{t-1}) \land s^m_t(D_{t-1})), D_{t-1,t-1},
\]

\[
+ \sum_{j=1}^{T-t} E[C_{t+j}(s^*_t(D_{t+j-1})
\]

\[
\land s^m_t(D_{t+j-1}), D_{t+j-1,t+j}].
\]

\[
(50)
\]

**Proof.** Because \( s^*_t(D_{t+j-1}) \leq s^m_t(D_{t+j-1}) \land s^m_t(D_{t+j-1}) \leq s^*_t(D_{t+j-1}) \) for all \( j \), we have

\[
V_t(x, D_{t-1}) \geq C_t(x \lor s^*_t(D_{t-1}), D_{t-1,t-1},
\]

\[
+ \sum_{j=1}^{T-t} E[C_{t+j}(s^*_t(D_{t+j-1})
\]

\[
\land s^m_t(D_{t+j-1}), D_{t+j-1,t+j}].
\]

This lower bound is similar to those in Lovejoy (1990, 1992) when \( s^*_t(D_{t-1}) \) is chosen to be \( s^m_t(D_{t-1}) \).

### 6. Numerical Study

#### 6.1. Result Illustration

In this subsection, we present a numerical study to illustrate the results developed in the previous sections. We also compare our cost-error bounds with those developed in Lovejoy (1990, 1992).

We use an AR(1) demand forecast model. That is,

\[
D_t = \mu_t + \rho(D_{t-1} - \mu_{t-1}) + \epsilon_t \quad \text{for all } t, \ |\rho| < 1,
\]

where \( E[D_t] = \mu_t \), the coefficient of correlation of demands in two successive periods is \( \rho \), and \( \epsilon_t \) are i.i.d. \( N(0, \sigma^2) \) random variables.

At any time period \( t \), after \( D_t \) is revealed, we generate a new forecast for the demand in period \( t+1 \) as

\[
D_{t+1} = \mu_{t+1} + \rho(D_t - \mu_t).
\]
Using the new forecast for period $t+1$, we obtain a new demand forecast for period $t+2$ as

$$D_{t,t+2} = \mu_{t+2} + \rho(D_{t,t+1} - \mu_{t+1}) = \mu_{t+2} + \rho^2(D_t - \mu_t).$$

Similarly, we obtain the new forecast for period $t+i$ as

$$D_{t,t+i} = \mu_{t+i} + \rho^i(D_t - \mu_t), \quad 0 \leq i \leq T - t.$$

Therefore, we have

$$D_t = (D_{t,t+1}, \ldots, D_{t,T}).$$

$$e_t = (e_{t,1}, \ldots, e_{t,T}) = (\epsilon_t, \rho \epsilon_t, \ldots, \rho^{T-t} \epsilon_t).$$

Note that $\epsilon_t = e_{t,i} = D_t - D_{t-i,1}$ is the one-step forecasting error.

The AR(1) model has been adopted by several authors in the recent supply chain management literature to study the value of information sharing and collaborative forecasting (see, e.g., Aviv 2001 and Lee et al. 2000). Due to tractability, these authors focus on myopic policies. A natural question is: Can the findings of these studies also apply to a system under an optimal policy? This is equivalent to asking whether the myopic policy is sufficiently good for systems with the AR(1) demand model. Previous research has established certain sufficient conditions under which the myopic policy is optimal when the demand follows AR(1) (see, e.g., Johnson and Thompson 1975 and Lida and Zipkin 2006). To shed light on the above issues, our numerical study focuses on parameters which do not satisfy these sufficient conditions.

More specifically, the time horizon $T = 10$ and there is a replenishment lead time $L = 3$. Two sets of cost parameters were chosen. One set is nonstationary: $h_2j+1 = 1, b_{2j+1} = 19, h_2 = 9, b_2 = 11$ ($j = 0, \ldots, 4$). The other set is stationary: $h_1 = 1, b_1 = 20, i \geq 1$.

**Initial Demand.** We choose several values of $D_0$: $D_{0,0} = \mu_0 + p\sigma_0$, where $\sigma_0 = \sigma/(1 - \rho^2)^{1/2}$, $p \in \{-3, -2, -1, 0, 1, 2, 3\}$. For any fixed $p$, $D_{0,t} = \mu_t + p\sigma_0\rho^t$ and $D_0 = (D_{0,1}, D_{0,2}, \ldots, D_{0,T})$. We set $\sigma_0/\mu_0 = 0.3$ or 0.25. With these parameters, the demand can demonstrate a wide range of variability. For example, suppose that $\sigma_0/\mu_0 = 0.3$, $\mu_{t+i} = \mu_t - 5$, $\mu_t = 100$, $\rho = 0.9$, and $p = -3$. We have $D_1 \sim N(19, (13.08)^2)$ and $D_2 \sim N(22.1, (17.59)^2)$, $D_3 \sim N(17.1, (34.13)^2)$.

**Demand Trend.** We first consider constant $\mu_t$ over time: $\mu_t = 100$ for all $t \geq 1$. We then consider $\mu_t$ with a decreasing trend: $\mu_{t+i} = \mu_{t+i-1} - 5$, $\mu_1 = 100$. We test different values of $p \in \{-0.9, -0.8, \ldots, 0.0, 0.9\}$.

**Heuristics Evaluated.** We evaluate the performances of three policies: the myopic policy $s_{0,t}$, the upper bound policy $S^{u}_{t}$ given in (28), and the heuristic policy $S^{H}_{t}$ defined by

$$S^{H}_{t}(D_{t-1}) = \gamma S^{u}_{t}(D_{t-1}) + (1 - \gamma) S^{l}_{t}(D_{t-1}), \quad 0 \leq \gamma \leq 1.$$

Here, $S^{l}_{t}(D_{t-1})$ is given by (29). The parameter $\gamma$ is chosen to minimize $\max \{\beta_{1}(t, D_{t-1}), \beta_{2}(t, D_{t-1})\}$ in period $t$ (i.e., to minimize the upper bound on the relative cost error). Also, we use (31) to compute $\beta_{1}(t, D_{t-1})$ and we use (35) and (46) to compute $\beta_{2}(t, D_{t-1})$.

To illustrate the heuristic $S^{H}_{t}(D_{t-1})$ under consideration, let us consider the first-period problem for the case of decreasing demand trend, nonstationary cost parameters, and $\sigma_0/\mu_0 = 0.3$. Suppose that $D_0 = (\mu_1, \mu_2, \ldots, \mu_{13}) = (100, 95, \ldots, 40)$. We have $S^{u}_{5}(D_0) = 486.55, S^{H}_{5}(D_0) = 448.28$, and $S^{H}_{5}(D_0) = 446.35$. We further have $S^{u}_{6}(D_0) = 447.67$ with the parameter $\gamma = 0.68$ (which means that the heuristic is closer to $S^{u}_{6}(D_0)$). In evaluating the heuristic $S^{H}_{6}(D_0)$, we take $\tau = 1.32, \beta_{1}(1, D_0) = 2.43, \beta_{3}(1, D_0) = 2.44$, and $\max \{\beta_{1}(1, D_0), \beta_{3}(1, D_0)\} = 2.44$.

**Upper Bounds on Relative Cost Error.** Tables 1–4 show $\text{err}$, the upper bound on relative cost error of the approximate policies, where $\text{err}$ is defined by

$$\text{err} = \frac{V^{H}(x, D_0) - V_{l}(x, D_0)}{V_{l}(x, D_0)} \times 100\% \leq \text{err} = \text{RHS of (45)} \times 100\%,$$

where RHS stands for right-hand side. “M-Lovejoy,” “M-easy,” and “Myopic” are upper bounds on relative cost error for myopic policy when we apply Lovejoy’s method defined in (49), the easier-to-compute upper bound defined in (48), and a general method defined in (47) to evaluate the myopic policy, respectively. Because Lovejoy’s method is not for evaluating a general heuristic policy, our method is used to evaluate $S^{H}_{t}(D_{t-1})$ given in (51).

Specifically, we use (31), (35), and (46) to compute the upper bound on $V^{H}(x, D_0) - V_{l}(x, D_0)$ as defined in (45) for $n = 3$ to evaluate the heuristic which referred to as $S^{H}(D_{t-1})$. Tables 1 and 3 report some special cases, while Tables 2 and 4 provide the averages. As mentioned before, we tested all cases of $p \in \{-0.9, -0.8, \ldots, 0.0, 0.9\}$ and $p \in \{-3, -2, -1, 0, 1, 2, 3\}$. For the cases we do not report, the results are usually better, i.e., the upper bounds on the relative cost errors are smaller for the same value of $p$.

**Software Used.** We used Matlab version 6.5 for the numerical study. We employed two routines for integration: the single integration function quad() and double integration function dblquad(). We also used the standard normal probability density and cumulative distribution functions: normpdf() and normcdf(). We set the maximum error to be 0.0001 for precision.

We observe that for nonstationary cost parameters, the myopic policy can be far from optimal; in one case the value loss is as high as 45%. On the other hand, with stationary cost parameters, the myopic policy is very close to optimal. The maximum value loss is around 2%.
The heuristic policy $S_t^H$ is very close to optimal whether or not the cost parameters are stationary. The maximum value loss is 1.3635% for nonstationary costs and 0.4533% for stationary costs.

This also indicates that our upper bound on the cost difference between the heuristic and optimal policy is very tight. This observation can be further verified by comparing the upper bound on the relative cost error on the myopic policy between our method and Lovejoy’s (1990, 1992) method. The examples show that our cost error bounds are usually 20% to 1% of Lovejoy’s (1992) bounds. The reason is, when the overstock (by ordering up to the myopic base-stock level) happens, the extra cost due to the overstock is usually much smaller than the total cost for holding the items from the current period until the end of the planning horizon—the basis for Lovejoy’s estimation.

The performance of $S_t^H$ is also very close to optimal; its maximum value loss is 2.26% in all the cases examined. Thus, this policy is recommended to be used if the myopic policy fails perform well.

Note that the two-period-ahead policy is a slightly tighter upper bound on the optimal base-stock level than $S_t^H$. So the good performance of $S_t^H$ implies good performance of the two-period-ahead policy. This is consistent with the finding by Treharne and Sox (2002), who show that the two-period-ahead policy is very close to optimal for the Markov modulated demand model.

It is interesting to observe that, as $\rho$ decreases, the performances of all three policies tend to be better (closer to optimal). We find this difficult to explain. For example, on one hand, if $\rho < 0$ and the demand in period $t$ is lower than $\mu_t$, the expected demand in period $t+1$ will be higher than $\mu_{t+1}$, so both the probability and the magnitude of overstock decrease, which favors the myopic policy. On the other hand, if the demand in period $t$ is higher than $\mu_t$, the expected demand in period $t+1$ will be lower than $\mu_{t+1}$, which seems to be against the myopic policy.

### Table 1. Upper bound on relative cost error for nonstationary cost parameters.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$p$</th>
<th>M-Easy $\sigma_0/\mu_1 = 0.3$</th>
<th>M-Lovejoy $\sigma_0/\mu_1 = 0.3$</th>
<th>Myopic $S_t^H$ $\sigma_0/\mu_1 = 0.3$</th>
<th>Trend</th>
<th>$\sigma_0/\mu_1 = 0.3$</th>
<th>M-Easy $\sigma_0/\mu_1 = 0.3$</th>
<th>M-Lovejoy $\sigma_0/\mu_1 = 0.3$</th>
<th>Myopic $S_t^H$ $\sigma_0/\mu_1 = 0.3$</th>
<th>Trend</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.9$</td>
<td>$-3$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0124</td>
<td>0.0315</td>
<td>0.0032</td>
<td>0.0000</td>
</tr>
<tr>
<td>$-0.6$</td>
<td>$-3$</td>
<td>0.0021</td>
<td>0.0292</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>4.2252</td>
<td>14.4251</td>
<td>2.116</td>
<td>0.0002</td>
</tr>
<tr>
<td>$-0.3$</td>
<td>$-3$</td>
<td>0.6085</td>
<td>6.2483</td>
<td>0.0929</td>
<td>0.0000</td>
<td>0.0000</td>
<td>14.2205</td>
<td>57.6793</td>
<td>8.0443</td>
<td>0.0039</td>
</tr>
<tr>
<td>0</td>
<td>$-3$</td>
<td>5.2664</td>
<td>39.0306</td>
<td>2.0305</td>
<td>0.0000</td>
<td>0.0000</td>
<td>26.666</td>
<td>108.6149</td>
<td>15.8132</td>
<td>0.0195</td>
</tr>
<tr>
<td>$0.3$</td>
<td>$-3$</td>
<td>15.2877</td>
<td>89.5447</td>
<td>7.5095</td>
<td>0.0000</td>
<td>0.0000</td>
<td>40.3978</td>
<td>153.7279</td>
<td>24.7017</td>
<td>0.1164</td>
</tr>
<tr>
<td>$0.6$</td>
<td>$-3$</td>
<td>28.0026</td>
<td>138.5369</td>
<td>14.4255</td>
<td>0.0000</td>
<td>0.0000</td>
<td>54.6722</td>
<td>190.039</td>
<td>33.4014</td>
<td>0.4347</td>
</tr>
<tr>
<td>$0.9$</td>
<td>$-3$</td>
<td>49.4354</td>
<td>205.2975</td>
<td>26.4638</td>
<td>0.1575</td>
<td>0.0000</td>
<td>78.7203</td>
<td>243.2094</td>
<td>45.1247</td>
<td>1.0117</td>
</tr>
</tbody>
</table>

### Table 2. Upper bound on relative cost error for nonstationary cost parameters.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$p$</th>
<th>M-Easy $\sigma_0/\mu_1 = 0.25$</th>
<th>M-Lovejoy $\sigma_0/\mu_1 = 0.25$</th>
<th>Myopic $S_t^H$ $\sigma_0/\mu_1 = 0.25$</th>
<th>Trend</th>
<th>$\sigma_0/\mu_1 = 0.25$</th>
<th>M-Easy $\sigma_0/\mu_1 = 0.25$</th>
<th>M-Lovejoy $\sigma_0/\mu_1 = 0.25$</th>
<th>Myopic $S_t^H$ $\sigma_0/\mu_1 = 0.25$</th>
<th>Trend</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.9$</td>
<td>$-3$</td>
<td>1.4581</td>
<td>12.9941</td>
<td>0.3318</td>
<td>0.0000</td>
<td>0.0000</td>
<td>17.5428</td>
<td>71.291</td>
<td>9.9122</td>
<td>0.0039</td>
</tr>
<tr>
<td>$-0.6$</td>
<td>$-3$</td>
<td>7.3877</td>
<td>49.3081</td>
<td>2.9722</td>
<td>0.0001</td>
<td>0.0000</td>
<td>29.5138</td>
<td>116.3659</td>
<td>17.2377</td>
<td>0.0428</td>
</tr>
<tr>
<td>$-0.3$</td>
<td>$-3$</td>
<td>17.6516</td>
<td>93.7289</td>
<td>8.1915</td>
<td>0.0004</td>
<td>0.0000</td>
<td>42.4281</td>
<td>154.9331</td>
<td>24.8103</td>
<td>0.2018</td>
</tr>
<tr>
<td>$0.3$</td>
<td>$-3$</td>
<td>31.587</td>
<td>147.6837</td>
<td>16.3204</td>
<td>0.0287</td>
<td>0.0000</td>
<td>67.5943</td>
<td>195.5852</td>
<td>41.31</td>
<td>1.3635</td>
</tr>
</tbody>
</table>
The following example, however, sheds some light on why the myopic policy may be near optimal for \( \rho \) near 1. Assume that cost parameters are stationary and \( \mu_t = 100 \) for all \( t \geq 1 \). Then, the safety stock, \( ss \), for the myopic policy is the same across different periods due to (14). More specifically, the forecast error of the lead time demand is
\[
(1 + \rho + \rho^2 + \rho^3)\epsilon + (1 + \rho + \rho^2)\epsilon + (1 + \rho)\epsilon + \epsilon + \epsilon + \epsilon + \epsilon.
\]
It has a normal distribution with mean zero and standard deviation
\[
\sigma \sqrt{1 + (1 + \rho)^2 + (1 + \rho + \rho^2)^2 + (1 + \rho + \rho^2 + \rho^3)^2},
\]
where \( \sigma \) is the standard deviation of \( \epsilon \). Thus, the myopic policy has safety stock
\[
ss = \sigma \sqrt{1 + (1 + \rho)^2 + (1 + \rho + \rho^2)^2 + (1 + \rho + \rho^2 + \rho^3)^2} \cdot \Phi^{-1}\left( \frac{b}{h} \right).
\]
In particular,
\[
s_t^{\text{m}} = 4\mu_1 + \rho(1 + \rho + \rho^2 + \rho^3)(D_{t-1} - \mu_{t-1}) + ss,
\]
\[
s_{t+1}^{\text{m}} = 4\mu_1 + \rho^2(1 + \rho + \rho^2 + \rho^3)(D_{t-1} - \mu_{t-1}) + \rho(1 + \rho + \rho^2 + \rho^3)\epsilon + ss.
\]

Now, consider the case \( \rho = -0.9 \) and \( \sigma_0/\mu_1 = 0.3 \). Because \( D_{t-1} - \mu_{t-1} \) has a normal distribution, by direct computation, the probability of overstock is \( \max_{-3 \leq \rho < 0} P((s_t^{\text{m}} - D_t) \geq s_{t+1}^{\text{m}} | (D_{t-1} - \mu_{t-1}) = \rho \sqrt{\text{Var}(D_{t-1} - \mu_{t-1})}) \leq 0.0023 \). With such a low (close to zero) probability of overstock under the myopic policy, the myopic level is likely to be always reachable and thus likely to be optimal.

### 6.2. Comparison with Iida and Zipkin (2006)

In this subsection, we compare our approximations with those in Iida and Zipkin (2006). For this purpose, we follow their choice of demand pattern and cost parameters. More specifically, demand follows a multiplicative model. The experiments include two patterns of initial forecasts: trends and cycles. At first, the initial forecast for period 1 is 250. We then adjust that value and set the initial forecasts for periods 2 to 16 as follows:

- **Trends:** Linear trends with three slopes: +10, 0, and −25, numbered 1 through 3 (represented by trend in the table).
- **Cycles:** Four types of cycle: none, 50 sin(\( \pi \)), 50 cos(\( \pi \)), and −50 cos(\( \pi \)); numbered 1 through 4 (represented by C in Tables 5 and 6).

#### Table 3. Upper bound on relative cost error for stationary cost parameters.

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( p )</th>
<th>( \sigma_0/\mu_1 = 0.3 )</th>
<th>( \text{Myopic} )</th>
<th>( S_0^I )</th>
<th>( \text{Trend} = 0 )</th>
<th>( \sigma_0/\mu_1 = 0.3 )</th>
<th>( \text{Myopic} )</th>
<th>( S_0^I )</th>
<th>( \text{Trend} = -5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0006</td>
<td>0.0092</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.7</td>
<td>−3</td>
<td>0.5662</td>
<td>3.9742</td>
<td>0.0171</td>
<td>0.0088</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.8</td>
<td>−3</td>
<td>1.0399</td>
<td>6.4845</td>
<td>0.0534</td>
<td>0.0049</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.9</td>
<td>−3</td>
<td>1.9345</td>
<td>10.2073</td>
<td>0.1707</td>
<td>0.0179</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

#### Table 4. Upper bound on relative cost error: Average case for stationary cost parameters.

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \sigma_0/\mu_1 = 0.3 )</th>
<th>( \text{Myopic} )</th>
<th>( S_0^I )</th>
<th>( \text{Trend} = 0 )</th>
<th>( \sigma_0/\mu_1 = 0.3 )</th>
<th>( \text{Myopic} )</th>
<th>( S_0^I )</th>
<th>( \text{Trend} = -5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0262</td>
<td>0.2742</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.6</td>
<td>0.2824</td>
<td>2.1451</td>
<td>0.0051</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.9</td>
<td>1.3592</td>
<td>6.0600</td>
<td>0.1060</td>
<td>0.0103</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0042</td>
<td>0.0515</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0964</td>
<td>0.8165</td>
<td>0.0007</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.9</td>
<td>0.5286</td>
<td>2.6658</td>
<td>0.0215</td>
<td>0.0018</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
These patterns can produce nonpositive initial forecasts (the trend term plus the cycle term); such values are reset to one.

The multiplicative forecast updates have multidimensional log-normal distributions. Let \( \xi_t, z = \log e_t, z \). The \( \xi_t, z \) then are joint-normally distributed with covariance matrix

\[
\begin{pmatrix}
0.08 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.02 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.005 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0008 \\
\end{pmatrix}
\]

(52)

and mean vector

\((-0.04, -0.01, -0.0025, -0.0004)\).

Other parameters are \( h_t = 2 \) and \( h_z = 10 \). The planning horizon \( T = 2, 4, 8, 12, 16 \). We also examine the case of demand uncertainty being revealed early. To represent early resolution of uncertainty, a new set of problems is constructed by reversing the order of the diagonal elements in the covariance matrix (52) as done in Iida and Zipkin.

With these data, we compare our heuristic solution \( S_t^h(D_{t-1}) \) defined in (51) with Iida and Zipkin’s (2006, §4.3) solutions obtained by using their two approximation techniques to solve the dynamic program.

The results are presented in Tables 5 and 6, respectively. Here, \( \delta_1 \) and \( \delta_2 \) are drawn from Tables 1 and 2 of an earlier version of their paper, dated December 20, 2004, and err is the relative cost error of our heuristic policy given by (51). Note that \( \delta_1 \) measures the error of functional approximations and \( \delta_2 \) measures the sampling error in Iida and Zipkin. In other words, \( \delta_1 \) is the “upper bound on the relative cost error” while \( \delta_2 \) is the “computational error.” See that earlier version of Iida and Zipkin for more detail.

For the problems tested, our heuristic policy is near optimal; its maximum value loss is less than 1% in all the cases examined. When the demand uncertainty is revealed early, our policy appears even better. The reason is that when the demand uncertainty is revealed earlier, the safety stock is reduced and thus the possibility of overstock is reduced.

As a consequence, the gap between the upper and lower bounds on the optimal base-stock level can be very small (or even zero), rendering our heuristic to be near optimal or even optimal.

Out of the total 120 cases reported in Tables 5 and 6, in 109 cases (more than 90%) our method is better than Iida and Zipkin’s (judging from the magnitudes of the errors). Our method is usually better when (1) the planning horizon is shorter, or (2) the demand pattern increases or stays the same, or (3) the demand uncertainty reveals early. Even in other cases, such as when the demand pattern decreases, our method can still be better.

7. Conclusions

We have examined a single-item, periodic-review inventory system with demand-forecast updates following the Martingale model of forecast evolution (MMFE). The optimal policy is a state-dependent base-stock policy that is computationally intractable to obtain. Using a sample-path approach, we developed a general class of tractable bounds on the optimal base-stock levels, which generalized and improved the existing bounds in the literature. We then used these bounds to construct near-optimal policies. Our numerical examples showed that our heuristics outperform the myopic policy significantly. The sample-path approach also allowed us to identify a necessary and sufficient condition for the myopic policy to be optimal, which sharpens our intuition on this policy. Furthermore, our sample-path

<table>
<thead>
<tr>
<th>Trend</th>
<th>C</th>
<th>( T = 2 )</th>
<th>( T = 4 )</th>
<th>( T = 8 )</th>
<th>( T = 12 )</th>
<th>( T = 16 )</th>
<th>C</th>
<th>( T = 2 )</th>
<th>( T = 4 )</th>
<th>( T = 8 )</th>
<th>( T = 12 )</th>
<th>( T = 16 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \delta_1 )</td>
<td>0.30</td>
<td>0.23</td>
<td>0.14</td>
<td>0.23</td>
<td>0.16</td>
<td>3</td>
<td>0.24</td>
<td>0.02</td>
<td>0.33</td>
<td>0.15</td>
<td>0.17</td>
</tr>
<tr>
<td></td>
<td>( \delta_2 )</td>
<td>0.81</td>
<td>0.72</td>
<td>0.34</td>
<td>0.33</td>
<td>0.23</td>
<td>0.72</td>
<td>0.44</td>
<td>0.32</td>
<td>0.15</td>
<td>0.17</td>
<td>0.08</td>
</tr>
<tr>
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<td>0.01</td>
<td>0.02</td>
<td>0.25</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
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<td>( \delta_1 )</td>
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<td>0.39</td>
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<td>0.25</td>
<td>0.4</td>
<td>0.29</td>
<td>0.03</td>
<td>0.41</td>
<td>0.30</td>
<td>0.19</td>
</tr>
<tr>
<td></td>
<td>( \delta_2 )</td>
<td>0.54</td>
<td>1.42</td>
<td>0.28</td>
<td>0.11</td>
<td>0.14</td>
<td>0.82</td>
<td>0.49</td>
<td>0.64</td>
<td>0.19</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \delta_3 )</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
<td>0.02</td>
<td>0.02</td>
<td>0.01</td>
</tr>
<tr>
<td>3</td>
<td>( \delta_1 )</td>
<td>0.09</td>
<td>0.26</td>
<td>0.25</td>
<td>0.37</td>
<td>0.30</td>
<td>3</td>
<td>0.16</td>
<td>0.03</td>
<td>0.11</td>
<td>0.28</td>
<td>0.09</td>
</tr>
<tr>
<td></td>
<td>( \delta_2 )</td>
<td>0.59</td>
<td>0.75</td>
<td>0.11</td>
<td>0.43</td>
<td>0.37</td>
<td>0.64</td>
<td>0.44</td>
<td>0.01</td>
<td>0.31</td>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \delta_3 )</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.02</td>
<td>0.02</td>
<td>0.03</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
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approach enabled us to perform worst-case analysis and derive upper bounds on the value loss of any heuristic policy (including the myopic policy). These appear to be the first set of cost-error bounds on the performance of any heuristic policies in dynamic inventory models. Numerical examples demonstrated that our error bounds improve the existing error bounds in the literature for evaluating the performance of myopic policies. Finally, both the solution bounds and the cost-error bounds developed in this paper can be easily adapted to general dynamic inventory models with non-stationary or autocorrelated demands.

**Appendix**

**Proof of (44).** Define

\[ I_{t+i} = \text{inventory level after ordering in period } t+i \text{ following the heuristic policy under consideration given } I_{t+i} = s_{t}^H(D_{t-1}) - D_t, \]

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We have

\[ \tilde{I}_{t+i} - \hat{I}_{t+i} \geq 0, \quad i = 1, 2, \ldots, T-t, \] (53)

\[ \tilde{I}_{t+i} - \hat{I}_{t+i} \leq (s_{t}^H(D_{t-1}) - D_t) \lor s_{t}^{H}(D_{t}) - (s_{t}^{H}(D_{t-1}) - D_t) \lor s_{t}^{H}(D_{t}) \]

\[ \leq [(s_{t}^H(D_{t-1}) - s_{t}^H(D_{t})) \land (s_{t}^H(D_{t-1}) - D_t - s_{t}^{H}(D_{t}))]^{+} \tau_2. \] (54)

The reason is, considering period \( t+1 \), if the heuristic orders for both \( I_{t+i} = s_{t}^H(D_{t-1}) - D_t \) and \( I_{t+i+1} = s_{t}^H(D_{t-1}) - D_t \), then \( I_{t+i} - I_{t+i+1} = 0 \); if the heuristic places an order for only one situation, because \( s_{t}^H(D_{t-1}) - D_t \geq s_{t}^{H}(D_{t-1}) - D_t \), we have \( \tilde{I}_{t+i} - \hat{I}_{t+i} = s_{t}^H(D_{t-1}) - D_t - s_{t}^{H}(D_{t}) \leq s_{t}^{H}(D_{t-1}) - s_{t}^{H}(D_{t}) \); if the heuristic does not place an order for both situations, we have \( \tilde{I}_{t+i} - \hat{I}_{t+i} = s_{t}^H(D_{t-1}) - s_{t}^{H}(D_{t}) \).

We focus on developing an upper bound for \( E[V_{t+1}^{H} \cdot (s_{t}^{H}(D_{t-1}) - D_t, D_t)] - E[V_{t+1}^{H}(s_{t}^{H}(D_{t-1}) - D_t, D_t)] \) by developing an upper bound for \( C_{t+2}(\tilde{I}_{t+i+1}, \cdot) - C_{t+2}(\hat{I}_{t+i+1}, \cdot) \), where \( \tilde{I}_{t+i} \leq \cdot \leq \hat{I}_{t+i+1} \), \( i = 1, \ldots, T-t \). Let

\[ i_0 = \min(j: s_{t}^H(D_{t-1}) - D(t, t+j) \leq s_{t}^{H}(D_{t+j})) \]

Then, period \( t+i_0 \) is the first period after \( t \) when the heuristic will place an order given \( I_{t+i} = s_{t}^{H}(D_{t-1}) - D_t \). Thus, we have \( \tilde{I}_{t+i_0} \geq s_{t}^{H}(D_{t+i_0}) \). Because \( I_{t+i} \geq s_{t}^{H}(D_{t+i}) = \tilde{I}_{t+i} \), from (53), we have \( \hat{I}_{t+i_0} \geq \hat{I}_{t+i} \). Therefore,

\[ C_{t+1}(\tilde{I}_{t+i}, \cdot) - C_{t+1}(\hat{I}_{t+i}, \cdot) = 0, \quad j = i_0, \ldots, T. \] (55)

The following events happen for all \( k < i_0 \):

\[ A_{t+k}^{H} = [s_{t}^{H}(D_{t-1}) - D(t, t+k) > s_{t}^{H}(D_{t+k})]. \]

In any of these periods, say period \( t+k, (k < i_0) \), if \( \tilde{I}_{t+k} \leq s_{t+k}^{H}(D_{t+k-1}) \), because \( \tilde{I}_{t+k} \leq \hat{I}_{t+k} \leq \tilde{I}_{t+k} \leq s_{t+k}^{H}(D_{t+k-1}) \), and one period expected cost \( C_{t+k}(z, \cdot) \) decreases in \( z \) when \( z < s_{t+k}^{H}(D_{t+k-1}) \), we have

\[ C_{t+k}(\tilde{I}_{t+k}, \cdot) - C_{t+k}(\hat{I}_{t+k}, \cdot) \leq 0. \] (56)

If \( \tilde{I}_{t+k} > s_{t+k}^{H}(D_{t+k-1}) + \tau_2 \), because \( \tilde{I}_{t+k} - \hat{I}_{t+k} \leq \tau_2 \) (from (54)) and \( C_{t+k}(z, \cdot) \) increases in \( z \) when \( z > s_{t+k}^{H}(D_{t+k-1}) \), we have

\[ C_{t+k}(\tilde{I}_{t+k}, \cdot) - C_{t+k}(\hat{I}_{t+k}, \cdot) \leq C_{t+k}(\tilde{I}_{t+k}, \cdot) - C_{t+k}(\hat{I}_{t+k}, \cdot) \] (57)
If \( s_{t+1}^m(\mathbf{D}_{t+k-1}) + \tau_2 \geq \bar{I}_{t+k} \), because
\( C_{t+k}(\bar{I}_{t+k}, \cdot) \leq C_{t+k}(s_{t+k}^m(\mathbf{D}_{t+k-1}), \cdot) \), we have
\[
C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k}, \cdot) \\
\leq C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(s_{t+k}^m(\mathbf{D}_{t+k-1}), \cdot). \tag{58}
\]
In summary, if \( I_{t+j} > s_{j+1}^m(\mathbf{D}_{t+j-1}), \) \( j = 1, \ldots, k, \) and \( \bar{I}_{t+k} > s_{t+k}^m(\mathbf{D}_{t+k-1}), \) from (57) and (58), we have
\[
C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k}, \cdot) \\
\leq C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k} - \tau_2) \vee s_{t+k}^m(\mathbf{D}_{t+k-1}), \cdot) \tag{59}.
\]

Considering all these possibilities, we have
\[
E[C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k}, \cdot)] \\
= E[I(A_{t+k}^h \cdots A_{t+1}^h - A_{t+k}^h)^y] [C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k}, \cdot)] \\
+ E[I(A_{t+k}^h \cdots A_{t+1}^h - A_{t+k}^h)[C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k}, \cdot)], \\
= E[I(A_{t+k}^h \cdots A_{t+1}^h - A_{t+k}^h)[C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k}, \cdot)] \\
+ E[I(A_{t+k}^h \cdots A_{t+1}^h - A_{t+k}^h)[C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k}, \cdot)] \\
+ E[I(A_{t+k}^h \cdots A_{t+1}^h - A_{t+k}^h)[C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k}, \cdot)]
\]
\[
\cdot [C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k} - \tau_2) \vee s_{t+k}^m(\mathbf{D}_{t+k-1}), \cdot)] \\
\leq E[I(A_{t+k}^h \cdots A_{t+1}^h - A_{t+k}^h)[C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k}, \cdot)] \\
\cdot [C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k} - \tau_2) \vee s_{t+k}^m(\mathbf{D}_{t+k-1}, \cdot)], \cdot]
\]
\[
= E[I(A_{t+k}^h \cdots A_{t+1}^h - A_{t+k}^h)[C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k}, \cdot)] \\
- D[t, t + k, D_{t+k-1}, \tau_2] - C_{t+k}(s_{t+k}^m(\mathbf{D}_{t+k-1})) \\
- D[t, t + k] - \tau_2] \vee s_{t+k}^m(\mathbf{D}_{t+k-1}, \cdot)]
\]
\[
\cdot [C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k} - \tau_2) \vee s_{t+k}^m(\mathbf{D}_{t+k-1}, \cdot)], \cdot]
\]
\[
\leq E[I(A_{t+k}^h \cdots A_{t+1}^h - A_{t+k}^h)[C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k}, \cdot)]
\cdot [C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k} - \tau_2) \vee s_{t+k}^m(\mathbf{D}_{t+k-1}, \cdot)].
\]

The second equality is due to (55), i.e., when \( (A_{t+k}^h \cdots A_{t+1}^h)^y \) happens, \( \bar{I}_{t+k} = \bar{I}_{t+k} \). The first inequality is due to (56), and the second inequality is due to (59). The last equality is due to the medium theory, and the last inequality is due to \( C_{t+k}(z, D_{t+k-1}, \tau_2) \) increases in \( z \) and \( C_{t+k}(z, D_{t+k-1, \tau_2}) = 0 \) when \( z \geq s_{t+k}^m(\mathbf{D}_{t+k-1}) \). Finally, we have
\[
E[V_{t+1}^m(s_{t+1}^m(\mathbf{D}_{t+1}) - D_{t+1}, D_{t+})] - E[V_{t+1}^m(s_{t+1}^m(\mathbf{D}_{t+1}) - D_{t}, D_{t+})]
\]
\[
= E \left[ \sum_{k=1}^{T} \left[ C_{t+k}(\bar{I}_{t+k}, \cdot) - C_{t+k}(\bar{I}_{t+k}, \cdot) \right] \right]
\]
\[
\leq \sum_{k=1}^{T} E[I(A_{t+k}^h \cdots A_{t+1}^h - A_{t+k}^h) \tau_2 C_{t+k}(s_{t+k}^m(\mathbf{D}_{t+k-1})) - D[t, t + k, D_{t+k-1}, t + \tau_2]] = \beta_3(t, D_{t+}). \]

\[ \square \]

Acknowledgments
The authors thank the associate editor and two anonymous referees for many helpful suggestions that improved the paper’s exposition. This research was supported in part by NSF grants DMI-0084922 and DMI-0335552 and the National Natural Science Foundation of China award no. 70328001.

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