Evaluation of Base-Stock Policies in Multiechelon Inventory Systems with State-Dependent Demands. Part II: State-Dependent Depot Policies

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We consider a two-echelon inventory system where the exogenous demands occur only at the retailer locations, and the demand rates are functions of an underlying continuous-time Markov chain. This underlying process may represent, for example, general economic conditions, the number of active users in the system, et cetera. Each retailer location follows a base-stock policy that is independent of the process. However, the warehouse (central depot) follows a state-dependent base-stock policy. We develop a procedure to compute the exact steady-state customer-delay distribution for both the warehouse and the retailer locations. © 1996 John Wiley & Sons, Inc.

1. INTRODUCTION

This is the second part of Song and Zipkin [1], where we analyze a multiechelon inventory system whose exogenous demands are functions of an underlying continuous-time Markov chain. More specifically, given the state of the underlying process, the demand process at each location at the lowest echelon forms a Poisson process with rate depending on this state. (Thus, the overall demand process forms what is called a Markov-modulated Poisson process.) This underlying process may represent, for example, the general economic situation, the number of active users in some system, et cetera. In that article we present a procedure to compute the steady-state performance measures, assuming the policies used at each location are all state independent.

This article analyzes a model of a two-echelon inventory system within the same demand environment. Each retailer location follows a base-stock policy that is again independent of the underlying Markov chain. The warehouse (central depot), however, follows a state-dependent base-stock policy.

Other components of the model are the same as in Part I. For instance, backorders are allowed at each location, and they are filled according to the FIFO rule. For each retailer there is a transit time from a release of a unit by the central depot until the unit arrives at the retailer. Also, we view the outside source as having ample stock, so it always releases a unit immediately in response to a demand from the central depot. However, there are also transit times between the source and the depot. We assume all the transit times have continuous phase-type distributions.

Suppose the system starts fully stocked; that is, each location's inventory equals its base-stock level. Because each of the retailers follows a one-for-one replenishment policy that is
independent of the state, the demand at the warehouse is also a Markov-modulated Poisson process.

Here is the notation we use throughout the article:

\[ A = \{ A(t), t \geq 0 \} \]
- underlying Markov chain, assumed to be ergodic,
\[ Q = [q_{ab}] = \text{infinitesimal generator of } A, \]
\[ \pi = \text{stationary probability density of } A, \]
\[ j = \text{index of retailer locations}, \]
\[ 0 = \text{index of the warehouse}, \]
\[ \lambda_j(a) = \text{exogenous demand rate at retailer } j \text{ in state } a, \]
\[ \lambda_0(a) = \text{demand rate at the warehouse} \]
\[ = \sum_j \lambda_j(a), \]
\[ \Lambda_j = \text{diag} \{ \lambda_j(a) \}_a, \]
\[ \Lambda_0 = \text{diag} \{ \lambda_0(a) \}_a, \]
\[ T_j = \text{transit time to retailer } j, \text{ the time from the release of a unit by the warehouse until the receipt of the unit at } j, \]
\[ T_0 = \text{lead time at the warehouse}, \]
\[ S_j = \text{base-stock level at retailer } j, \]
\[ S_0(a) = \text{base-stock level at the warehouse when } A = a, \]
\[ D_{0j} = \text{delay at the warehouse, the time from the order of a unit by retailer } j \]
\[ \text{until the release of that unit by the warehouse}, \]
\[ D_j = \text{delay at the retailer } j, \]
\[ L_j = \text{total lead time at retailer } j \]
\[ = D_{0j} + T_j, \]
\[ L_0 = T_0. \]

As in Part I, we assume that each \( T_j \) has a continuous phase-type distribution. Specifically, we model \( T_j \) with reference to a continuous-time, finite-state, absorbing Markov chain \( H_j = \{ H_j(t), t \geq 0 \}; T_j \) is the time until \( H_j \) is absorbed. When we talk about the states of \( H_j \) without qualification, we shall mean the transient states only. The row vector \( v_j \) specifies the initial probabilities of the (transient) states, and the matrix \( U_j \) is the generator of \( H_j \) (restricted to the transient states). We write \( T_j \sim \text{CPH}(v_j, U_j) \) to denote a continuous phase-type distribution with these parameters.

For any number \( x \), \( [x]^+ = \max \{ x, 0 \} \). We use \( e \) to denote a column vector of ones; its dimension will be clear from the context. Also, \( I \) is used to denote the identity matrix, again of appropriate dimension.

As we saw in Part I, the crucial work in analyzing this system is to compute the average customer delay at the warehouse, \( D_{0j} \). So, in the next section, we concentrate on the single-location model, where both the demand process and the replenishment policy are state dependent. The probability distribution of the customer delay is obtained at the conclusion of this section. In Section 3 we use this distribution to analyze the customer delays at the retailer locations.

### 2. ANALYSIS OF CUSTOMER DELAY AT THE WAREHOUSE

In this section we analyze the customer delay at the warehouse. Because this is a single-location model, we drop the node index 0.
To summarize the model, there is an underlying continuous-time Markov chain $A$, which is ergodic with stationary density $\pi$. Given $A(t) = a$, the demand process forms a Poisson process with rate $\lambda(a)$. We follow a state-dependent base-stock policy with base-stock level $S(a) \geq 0$ when $A(t) = a$. The initial state $A(0)$ has the steady-state distribution of $A$ just after a typical or equilibrium demand. Specifically, from Lemma 3.2 of Part I, we have

$$A(0) \sim \alpha = (\pi \Lambda e)^{-1} \pi \Lambda = \left( \frac{\pi(a)\lambda(a)}{\sum_b \pi(b)\lambda(b)} \right)_a.$$  

Orders are filled after a random transit time $T$; here, this is the full lead time; that is, $L = T$. We assume $T \sim \text{CPH}(v, U)$, and $ve = 1$; so $T > 0$ with probability 1. Also, we write $L \sim \text{CPH}(\gamma, \Gamma)$. Let $Z = \{Z(t), t \geq 0\}$ be the Markov chain whose time to absorption has the distribution of $L$. Because $L = T$, we have $\gamma = v$, $\Gamma = U$ and $Z = H$. We introduce this extra notation to simplify our discussion below of two-echelon models. We use $Z = \{Z(t), t \geq 0\}$ to denote the Markov-chain representation of the lead time $L$. Assume the process $A$ has $m$ states. We rearrange the states of $A$, if necessary, such that $S(b) \geq S(a)$ for all $b \geq a$.

Recall from Part I that the analysis of the customer delay is based on the key observation that the order initiated by the $n$th demand will be used to fill the $(n + S)$th demand, provided we start with $S$ (the base-stock level, constant over $A$) as the initial inventory. Let

$$\tau_s(n) = \text{time between demand } n \text{ and } n + S,$$

$$L(n) = \text{lead time initiated by the } n\text{th demand},$$

$$D(n + S) = \text{delay experienced by demand } n + S.$$  

We have

$$D(n + S) = [L(n) - \tau_s(n)]^+.$$  

Letting $\tau_s$ denote the steady-state time between $S$ demands, we have

$$D = [L - \tau_s]^+.$$  

So, $D$ is the remaining time until $Z$ is absorbed after the random time $\tau_s$.

By analogy, we want to find a similar relationship. Of course, things become more complex when the policy is state dependent: The order process is no longer identical to the demand process. A demand may or may not trigger an order; an order may or may not be due to a demand; and finally, the order size need not always equal one. To keep track of such information, at any time $t$, we need to consider together the current state $A(t)$ and the current inventory position $IP(t)$. Observe

(i) $IP(t) \geq S(A(t));$

(ii) when $IP(t) = S(A(t))$, each demand triggers an order;

(iii) when $IP(t) > S(A(t))$, an order is placed only if $A(t)$ changes, and then the
order size can be larger than one; a demand decreases $IP(t)$ by one but does not trigger any order.

It is clear that $(A, IP)$ is a continuous-time Markov chain with state space $\Omega = \{(a, x): x = S(a), S(a) + 1, \ldots, S(m); a = 1, 2, \ldots, m\}$. The nonzero transition rates of $(A, IP)$ are

$$
\begin{align*}
\tilde{q}((a, x), (a, x - 1)) &= \lambda(a), \quad S(a) < x \leq S(m) \\
\tilde{q}((a, x), (b, S(b))) &= q_{ab}, \quad S(a) \leq x \leq S(b), \quad b > a, \\
\tilde{q}((a, x), (b, x)) &= q_{ab}, \quad S(b) \leq x \leq S(m), \quad b > a, \\
\tilde{q}((b, x), (a, x)) &= q_{ba}, \quad S(b) \leq x \leq S(m), \quad b > a
\end{align*}
$$

and

$$
\tilde{q}((a, x), (a, x)) = \begin{cases} q_{aa} - \lambda(a), & S(a) < x \leq S(m), \\ q_{aa}, & x = S(a). \end{cases}
$$

Figure 1 presents the state-transition diagram of $(A, IP)$ for a simple case with $m = 2$. 
Here the numbers in parentheses indicate the order sizes. Figure 2 illustrates the three-state case \( m = 3 \), which, in fact, represents the basic structure for the general \( m \)-state case. Here, we have omitted most arcs representing transitions from \( A = 1 \) to \( A = 3 \), to reduce clutter.

From these transition rates, it is easy to compute the equilibrium probabilities of \( (A, IP) \), denoted by \( \{ P(a, x) : (a, x) \in \Omega \} \), and we assume this has been done.

Having described when and how an order is triggered under a state-dependent policy, we now turn to analyze the relationship between the customer delay and the equilibrium lead time. Recall that we assume full backlogging. That is, every demand will eventually be filled. Also, we assume demands are filled according to the FIFO rule. Suppose \( A(0) = a \) and we start with \( S(a) \) as the initial inventory. Then the initial inventory will fill the first \( S(a) \) demands. We can think of each order as composed of several order units, each of size one, and suppose we impose an arbitrary ordering on the units in each order. Then, we can
index all of the units in all the orders by \( n \). The index \( n \) determines the sequence in which order units are used to fill demands. Specifically, the \( n \)th order unit will be used to fill the \((n + S(a))\)th demand. Let

\[
\begin{align*}
Y(n) &= \text{arrival epoch of the } n\text{th demand,} \\
\Theta(n) &= \text{ordering epoch of the } n\text{th order unit,} \\
L(n) &= \text{lead time corresponding to the } n\text{th order unit.}
\end{align*}
\]

Thus, if order units \( n \) and \( n + 1 \) are ordered in the same batch (the order size is bigger than one), then \( \Theta(n) = \Theta(n + 1) \) and \( L(n) = L(n + 1) \). Define

\[
D(n + S(a)) = \text{delay experienced by the } (n + S(a))\text{th demand.}
\]

Then,

\[
D(n + S(a)) = [(\Theta(n) + L(n)) - Y(n + S(a))]^+ \\
= [L(n) - (Y(n + S(a)) - \Theta(n))]^+. \tag{4}
\]

Note that when the policy used is state independent, each demand triggers one order, so \( \Theta(n) = Y(n) \) and (4) is reduced to (2). Now, because the policy we use is state dependent, \( \Theta(n) \) is not necessarily equal to \( Y(n) \). Observe, however, that \( \Theta(n) \) and \( L(n) \) start at the same time. So, \( Y(n + S(a)) - \Theta(n) \) plays a role similar to that of \( \tau_S(n) \) in Part I. Thus, to analyze the equilibrium distribution of the customer delay, we need to study the equilibrium behavior of \( \Theta(n) \). Specifically, assume the \( n \)th unit ordered at time \( \Theta(n) \) is typical; then \( n + S(a) \) is a typical demand that receives this order unit. Therefore, \( D(n + S(a)) \) is the delay experienced by a typical demand.

Define

\[
N(t) = \text{cumulative demands up to time } t, \\
N(t_1, t_2) = N(t_2) - N(t_1) \\
= \text{number of demands during the interval } (t_1, t_2], \quad t_2 > t_1.
\]

Let

\[
\tilde{N}_d(n) = N(\Theta(n), Y(n + S(a))) = n + S(a) - N(\Theta(n)). \tag{5}
\]

Clearly, for fixed \( n \) this is a random variable. Later, we shall see that \( [\tilde{N}_d(n) | A(\Theta(n))] \) is independent of the initial state \( A(0) = a \). So, when \( A(\Theta(n)) \) is given, we shall drop the subscript \( a \) from the notation. Recall, \( L \sim CPH(\gamma, \Gamma) \), and \( Z = \{Z(t), t \geq 0\} \) is the continuous-time Markov chain whose time until absorption determines \( L \). Let \( \eta = (\eta(a))_a \) be the probability density of \( A \) just after the order for an arbitrary unit, that is, the density of \( A(\Theta(n)) \), and \( \tilde{A} \) the continuous-time Markov chain with same probability law as the process \( A \), but with initial condition \( A(0) \sim \eta \). Define \( \tau_{\tilde{A}} \) to be the time until \( \tilde{N}_d \) demands have occurred. [We have dropped the index \( n \) of \( \tilde{N}(n) \) to indicate equilibrium.] Then, (4) reveals that the average customer delay is the remaining time until \( Z \) is absorbed after the
random time $\tau_N$. Let $\rho(t)$ denote the density of $Z(t)$, and suppose we can determine $\rho(\tau_N)$. Then, as in Part I, $D \sim \text{CPH}(\rho(\tau_N), \Gamma)$. That is, $D$ results from restarting the process $Z$ with an initial density equal to the density $\rho(\tau_N)$. Also, $\Pr\{D = 0\}$ is precisely the defect of $\rho(\tau_N)$; that is, $1 - \rho(\tau_N)e$. Our problem, then, is to determine $\rho(\tau_N)$.

To this end, consider the discrete-time Markov chain $(X, W) = (X[n], W[n])$: $n \geq 0$, where $X[n]$ and $W[n]$ ($n \geq 1$) are just the processes $Z$ and $A$ observed at the $n$th demand epoch, respectively, and $W[0] \sim \gamma$, $X[0] \sim \eta$. Then, exactly as in Section 3 of Part I, $(X, W)$ has transition matrix

$$\Omega = (I \otimes \Lambda - \Gamma \otimes I - I \otimes Q)^{-1}(I \otimes \Lambda).$$

Let $\sigma[n]$ be the density of $(X[n], W[n])$ restricted to the transient states of $Z$. Then $\rho(\tau_N) = \sigma(\tilde{N})(I \otimes e)$. Denote by $e_a$ the ath unit row vector, and let

$$\xi_a(k) = \Pr\{\tilde{N} = k | \tilde{A}(0) = a\}.$$

We have

$$[(X[\tilde{N}], W[\tilde{N}])| W[0] = a] \sim H_a = (\gamma \otimes e_a) \sum_k \xi_a(k) \Omega^k,$$

and

$$\sigma(\tilde{N}) = \sum_a \eta(a) H_a$$

$$= \sum_k \left[ \gamma \otimes \sum_a \eta(a) \xi_a(k) e_a \right] \Omega^k$$

$$= \sum_k (\gamma \otimes \delta(k)) \Omega^k,$$

where $\delta(k) = (\eta(a) \xi_a(k))_a$.

In sum, we have

$$D \sim \text{CPH}(\rho(\tau_N), \Gamma) \quad \text{and} \quad E[D] = -\rho(\tau_N)\Gamma^{-1} e,$$

where

$$\rho(\tau_N) = \sigma(\tilde{N})(I \otimes e) = \sum_k (\gamma \otimes \delta(k)) \Omega^k (I \otimes e).$$

In the rest of the section, we shall first show that $\tilde{N}_a(n)$ defined by (5) is independent of $A(0) = a$, conditional on $A(\Theta(n))$. Then, we will calculate $\{\xi_a\}$ and $\eta$. All of this will be done first for the two-state case ($m = 2$), and the general results will be presented at the conclusion of the section.

According to (5), the key step in studying $\tilde{N}_a(n)$ is to analyze $N(\Theta(n))$ for a given $a$. In order to do so, we need to distinguish the order types, that is, to identify the situation under which the $n$th order unit is placed.

First, consider a simple case with $m = 2$ and refer to Figure 1. Observe that each arc with
a positive number in parentheses corresponds to one order. Specifically, arcs $(1, S(1)) \rightarrow (1, S(1))$ and $(2, S(2)) \rightarrow (2, S(2))$ represent orders due to demands, whereas arcs $(1, x) \rightarrow (2, S(2)) (S(1) \leq x \leq S(2) - 1)$ represent orders due to changes in $A$. In the following we shall identify the order types by their corresponding arcs in the transition-rate diagram.

Assume $A(0) = 2$. The following observation is crucial in calculating $N(\Theta(n))$:  

**PROPOSITION 1:** Suppose $A(0) = 2$. If $A(\Theta(n)) = 2$, that is, the $n$th order unit is of type corresponding to an arc headed into node $(2, S(2))$, then up to time $\Theta(n)$, the number of demands equals the number of order units. Otherwise, if $A(\Theta(n)) = 1$, then the number of demands up to time $\Theta(n)$ equals the number of order units plus $(S(2) - S(1))$.

**PROOF:** Suppose the $n$th order unit is of order type $(2, S(2)) \rightarrow (2, S(2))$. Then the assertion is obvious if there is no state transition in $A$ before $\Theta(n)$. Suppose there are some state transitions before $\Theta(n)$. Then the number of transitions must be even, because $A(0) = A(\Theta(n)) = 2$, so each transition from 2 to 1 must be offset by another from 1 to 2. Call the time required for each such pair of transitions from state 2 back to itself a cycle. That is, the time between two successive visits to node $(2, S(2))$, which also corresponds to a loop starting and ending with node $(2, S(2))$ (e.g., $(2, S(2)) \rightarrow (1, S(2)) \rightarrow (1, S(2) - 1) \rightarrow (2, S(2)))$. Now observe that within each cycle the number of order units equals the number of demands, proving the assertion for the order type $(2, S(2)) \rightarrow (2, S(2))$. Also, this same observation applies to the order types $(1, x) \rightarrow (2, S(2)) (S(1) \leq x \leq S(2) - 1)$.

Now, suppose the $n$th order unit is of type $(1, S(1)) \rightarrow (1, S(1))$. Because we have shown above that each time we arrive at node $(2, S(2))$, the number of demands equals the number of order units, each time we get to node $(1, S(1))$ from node $(2, S(2))$ there must be $S(2) - S(1)$ more demands than the order units. But, each order of type $(1, S(1)) \rightarrow (1, S(1))$ is triggered by a demand, so it will not affect the difference between the number of demands and the number of order units. This completes the proof for $A(\Theta(n)) = 1$. \hfill \square

Given the above facts, it is clear that $N(\Theta(n)) = n$ for order units of types $(2, S(2)) \rightarrow (2, S(2))$ and $(1, S(2) - 1) \rightarrow (2, S(2))$, because the order sizes are one. Thus, in this case, $N_2(n) = n + S(2) - n = S(2)$ [by (5)]. Likewise, if the $n$th order unit is of type $(1, S(1)) \rightarrow (1, S(1))$, then $N(\Theta(n)) = n + S(2) - S(1)$, and $N_2(n) = S(1)$.

We need a bit more effort, however, when the order size is bigger than one. For instance, suppose the $n$th order unit is of type $(1, S(2) - 2) \rightarrow (2, S(2))$. Because there are two units ordered at time $\Theta(n)$, we have either $\Theta(n) = \Theta(n - 1)$ or $\Theta(n) = \Theta(n + 1)$, and the two cases are equally likely. In the first case, we have $N(\Theta(n)) = n$, whereas in the second case, $N(\Theta(n)) = n + 1$. As a result, for this type of order, $N_2(n)$ equals $S(2)$ or $S(2) - 1$, each with probability $\frac{1}{2}$. Similarly, if $\Theta(n)$ corresponds to order type $(1, S(2) - S(1) - 3) \rightarrow (2, S(2))$, we have three equally likely cases: $\Theta(n - 2) = \Theta(n - 1)$, $\Theta(n) = \Theta(n - 1)$, or $\Theta(n) = \Theta(n + 1)$ and $\Theta(n) = \Theta(n + 1) = \Theta(n + 2)$. So, $N(\Theta(n))$ takes the values $n, n + 1$ and $n + 2$ each with probability $\frac{1}{3}$. This same argument can be generalized in obvious ways, and we summarize the results in Table 1, where $F\{c, d\}$ means the uniform distribution on the integers $\{c, c + 1, \ldots, d\}$ for $c \leq d$.

Now, suppose $A(0) = 1$; that is, we start with node $(1, S(1))$. Observe, again that each time we arrive at node $(1, S(1))$, the number of demands equals the number of order
Table 1. Conditional distribution of $N_t(n)$ for given order types.

<table>
<thead>
<tr>
<th>Order type</th>
<th>$N(\Theta(n))$</th>
<th>$N_t(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2, S(2)) \rightarrow (2, S(2))$</td>
<td>$n$</td>
<td>$S(2)$</td>
</tr>
<tr>
<td>$(1, x) \rightarrow (2, S(2))$</td>
<td>$F{n, n + S(2) - x - 1}$</td>
<td>$F{x + 1, S(2)}$</td>
</tr>
<tr>
<td>$S(1) \leq x &lt; S(2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(1, S(1)) \rightarrow (1, S(1))$</td>
<td>$n + S(2) - S(1)$</td>
<td>$S(1)$</td>
</tr>
</tbody>
</table>

units. And the two numbers are also equal between two successive visits to node $(2, S(2))$. Employing an analysis similar to that of Proposition 1, we have: If $A(\Theta(n)) = 1$, then the number of demands up to time $\Theta(n)$ equals that of order units. Otherwise, if $A(\Theta(n)) = 2$, then at time $\Theta(n)$, the number of demands equals the number of order units minus $S(2) - S(1)$. Analogous to the case $A(0) = 2$, we have the results of Table 2.

From Tables 1 and 2 we see that, given the order type of the $n$th order unit, the distribution of $N_t(n)$ is independent of $a$. Note, however, that the order type determines the state of $A$ right after the order. So, we conclude that $[N_t(n) | A(\Theta(n))]$ is independent of $a$.

Now, we replace $[N_t(n) | A(\Theta(n))]$ by the equivalent notation $[N | A(0)]$. Let

$$p_2 = \lambda(2)P(2, S(2)) + q_{12}\sum_{x=S(1)}^{S(2)} P(2, x) = \lambda(2)P(2, S(2)) + q_{12}\sum_{x=S(2)}^{S(2)-1} P(2, x).$$

(12)

Note that $p_2$ is the total order rate into node $(2, S(2))$. From either Table 1 or Table 2 we can obtain

$$\xi_1(S(1)) = \Pr[N = S(1) | A(0) = 1] = 1,$n

$$\xi_2(x) = \Pr\{N = x | A(0) = 2\} = \sum_{l=S(1)}^{x-1} P(1, l) p_2^{l-1}, \quad S(1) \leq x \leq S(2) - 1,$n

$$\xi_2(S(2)) = \Pr\{N = S(2) | A(0) = 2\} = \sum_{l=S(1)}^{S(2)-1} P(1, l) p_2^{l-1}.$$n

(13)

Next, we calculate the distribution of $A(0)$; that is, $\eta$. Observe, that the steady-state probability of $A$ being in state $a$ after a typical order can be obtained by calculating the proportion of order rates that flow into $(a, S(a))$ in the network of Figure 1. Specifically, letting

Table 2. Conditional distribution of $N_t(n)$ for given order types.

<table>
<thead>
<tr>
<th>Order type</th>
<th>$N(\Theta(n))$</th>
<th>$N_t(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2, S(2)) \rightarrow (2, S(2))$</td>
<td>$n + S(1) - S(2)$</td>
<td>$S(2)$</td>
</tr>
<tr>
<td>$(1, x) \rightarrow (2, S(2))$</td>
<td>$F{n, n + S(2) - x - 1}$</td>
<td>$F{x + 1, S(2)}$</td>
</tr>
<tr>
<td>$S(1) \leq x &lt; S(2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(1, S(1)) \rightarrow (1, S(1))$</td>
<td>$n$</td>
<td>$S(1)$</td>
</tr>
</tbody>
</table>
Table 3. Order rates in equilibrium.

<table>
<thead>
<tr>
<th>Order type</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>((a, S(a)) \rightarrow a, S(a))</td>
<td>(\lambda(a)P(a, S(a)))</td>
</tr>
<tr>
<td>((a, S(a)) \rightarrow (b, S(b)))</td>
<td>((S(b) - S(a))q_{ab}P(a, S(a)))</td>
</tr>
<tr>
<td>((a, x) \rightarrow (b, S(b)))</td>
<td>((S(b) - x)q_{ab}P(a, x))</td>
</tr>
<tr>
<td>(S(a) \geq x &lt; S(b))</td>
<td>(q_{ab})</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
    r_2 &= \lambda(1)P(1, S(1)) + \lambda(2)P(2, S(2)) + q_{12}\left(\sum_{x=1}^{S(2) - 1} (S(2) - x)P(1, x)\right) \\
    &= \lambda(1)P(1, S(1)) + p_2 \\
\end{align*}
\]

which is the total equilibrium order rate, we have

\[
\begin{align*}
    \eta(1) &= \Pr\{A = 1\} \\
    &= \left[\lambda(1)P(1, S(1))\right]r_2^{-1} \\
    \eta(2) &= \Pr\{A = 2\} \\
    &= \left[\lambda(2)P(2, S(2)) + q_{12}\left(\sum_{x=1}^{S(2) - 1} (S(2) - x)P(1, x)\right)\right]r_2^{-1} \\
    &= p_2r_2^{-1}.
\end{align*}
\]

Combining (6)–(15) gives us the distribution of the customer delay at the warehouse for the two-state model.

To generalize the above results to arbitrary \(m\), we refer the reader to Figure 2, where a three-state model is presented. The notation used here is consistent with that of Figure 1.

Observe that we have some new types of arcs in this network: the arcs \((1, x) \rightarrow (2, x)\) and \((2, x) \rightarrow (1, x)\), \(S(2) + 1 \leq x \leq S(3)\). These arcs reflect changes in the process \(A\). Because in each of the nodes adjacent to these arcs the inventory position is above the base-stock levels for both states 1 and 2, such state transitions do not cause orders. As a matter of fact, picking any pair of states, we get a subgraph that is quite similar to Figure 1, and the only difference is that we have some new arcs corresponding to no orders. Analogous to the two-state model, let a cycle correspond to the time between two successive visits to node \((a, S(a))\) for any given \(a\). (Again, each cycle constitutes a loop in the network in Figure 2.) Then every such cycle has the property that the total number of demands in the cycle equals that of order units. So, it is easy to see that the same kind of analysis as we used for the two-state model applies here. Hence, we shall only state the generalized results of Table 3 without proof.

Define

\[
r_m = \sum_{a=1}^{m} \lambda(a)P(a, S(a)) + \sum_{a=1}^{m} \sum_{b=a+1}^{m} q_{ab}\left(\sum_{x=S(a)}^{S(b) - 1} (S(b) - x)P(a, x)\right) \\
\]

Clearly, \(r_m\) is the total equilibrium order rate, a generalization of \(r_2\) above. Observe that
Table 4. Conditional distribution of $\tilde{N}_a(n)$ for given order types.

<table>
<thead>
<tr>
<th>Order type</th>
<th>$N(\Theta(n))$</th>
<th>$\tilde{N}_a(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(a, S(a)) \rightarrow (a, S(a))$</td>
<td>$n$</td>
<td>$S(a)$</td>
</tr>
<tr>
<td>$(a, x) \rightarrow (b, S(b))$</td>
<td>$F{n + S(a) - S(b), n + S(a) - x - 1}$</td>
<td>$F{x + 1, S(b)}$</td>
</tr>
<tr>
<td>$S(a) \leq x &lt; S(b)$</td>
<td>$n - (S(b) - S(a))$</td>
<td>$S(b)$</td>
</tr>
</tbody>
</table>

Table 4: Conditional distribution of $\tilde{N}_a(n)$ for given order types.

$$p_{mb} = \lambda(b)P(b, S(b)) + \sum_{a=1}^{b-1} q_{ab} \left( \sum_{x=S(a)}^{S(b)-1} (S(b) - x)P(a, x) \right), \quad b > 1,$$

is the total order rate into node $(b, S(b))$, analogous to $p_2$ above. Then, the steady state distribution of $A$ after a typical order is

$$\eta(1) = \lambda(1)P(1, S(1))r_m^{-1},$$

$$\eta(b) = p_{mb}r_m^{-1}, \quad b > 1.$$

Again, we can show that $[\tilde{N}_a(n) | \tilde{A}(\Theta(n))]$ is independent of $a$. So, without loss of generality, in Table 4, which is a generalization of Tables 1 and 2, we assume $A(0) = a$ for any $a$, and $b > a$.

Now we are ready to compute $\xi_a(k) = \Pr\{\tilde{N} = k | \tilde{A}(0) = a\}$, as defined in (7). Suppose $b > a \geq 1$. Then

$$\xi_1(S(1)) = 1,$$

$$\xi_b(x) = q_{ab} \left[ \sum_{l=S(a)}^{x-1} P(a, l) \right] p_{mb}^{-1}, \quad S(a) + 1 \leq x \leq S(b) - 1,$$

$$\xi_b(S(b)) = \left[ \lambda(b)P(b, S(b)) + \sum_{c=1}^{b-1} q_{cb} \sum_{x=S(c)}^{S(b)-1} P(c, x) \right] p_{mb}^{-1}.$$  

Thus, applying (6)–(11) and (16)–(19), we obtain the distribution of the customer delay at the warehouse for the general $m$-state model.

Suppose we want to evaluate a single policy, which means evaluating $E[D]$ in (10). As we can see, the complexity of the computations involved is dominated by that of the computation of (11). Note, in (11), $k$ ranges from $S(1)$ to $S(m)$. So we may first compute $\Omega^{S(1)}$, which can be done in $O(\log(S(1))m^3l^3)$ steps, with $l$ the number of transient states of the process $Z$ (see the analysis in Part I). We then compute $\Omega^k, k = S(1) + 1, \ldots, S(m)$, each requiring $O(m^3l^3)$ operations. Thus, the total complexity is

$$O(\log(S(1))m^3l^3) + O([S(m) - S(1)]m^3l^3).$$

3. ANALYSIS OF CUSTOMER DELAY AT THE RETAILER LOCATIONS

3.1. General Discussion

In this section, we analyze the delay for a typical customer at retailer $j$. First, recall we now follow a state-independent base-stock policy, and we assume all the retailers start with
full stock. So, each demand at retailer $j$ becomes one demand at the warehouse. Thus, the
lead time at retailer $j$ equals the sum of the delay at the warehouse and the transit time from
the warehouse to retailer $j$. Specifically, suppose $A(0) = a$. We want to characterize $D_j$, 
which we can identify with $D_j(n_j + S_j)$, viewing $n_j + S_j$ as a demand under equilibrium
conditions. So, we can also treat $n_j$ as an equilibrium demand. Now, demand $n_j$ at $j$ corre-
sponds to some demand $n_0 + S_0(a)$ at the warehouse (remember, we use 0 to index the
warehouse). Thus, we may write

$$L_j(n_j) = D_0(n_0 + S_0(a)) + T_j(n_j),$$

regarding $n_0$ as an equilibrium demand unit at the warehouse, conditional on demand $n_0 + S_0(a)$ coming from retailer $j$. Or, we may write simply

$$L_j = D_{0j} + T_j. \quad (21)$$

In Section 2 we have computed the delay for a typical customer at the warehouse, regardless
of which location it comes from. Does it really matter where this typical customer comes
from? In other words, do we have $D_{0j} = D_0$ for all $j$?

Examining the derivation of (10) in Section 2, the distribution of $D_0$, it is not hard to see
that the distribution $D_{0j}$ is still of the same type as (10). More specifically, we have

$$D_{0j} \sim \text{CPH}(\rho_{0j}(\tau_{\bar{N}}), \Gamma_0).$$

Here, $\rho_{0j}(\tau_{\bar{N}})$ is the density of $Z_0(\bar{N})$, conditional on demand $\bar{N}$ coming from retailer $j$.
That is, $D_{0j}$ results from restarting the chain $Z_0$ with an initial density equal to $\rho_{0j}(\tau_{\bar{N}})$.
We shall see later (Section 3.2) that, as in Part I, $\rho_{0j}(\tau_{\bar{N}})$, and therefore $D_{0j}$, is independent of $j$
if

$$\lambda_j(a) = \mu_j \kappa_a, \quad \text{for all } a. \quad (22)$$

We call (22) the proportional-rate case.

Note, however, that whether condition (22) holds or not, the convolution in (21) can be
represented as the time until absorption of a chain $Z_j$, with generator

$$\Gamma_j = \begin{bmatrix} \Gamma_0 & -\Gamma_0 \kappa \nu_j \\ 0 & U_j \end{bmatrix}.$$ 

As for the relationship between the delay and the lead time at retailer $j$, note that the
policy is state independent (with base-stock level $S_j$), so we have exactly the same relation
as in (2) and (3). Specifically,

- $n_j =$ index of demand units at retailer $j$,
- $\tau_{S_j}(n_j) =$ time between demand $n_j$ and $n_j + S_j$,
- $L_j(n_j) =$ lead time initiated by demand $n_j$,
- $D_j(n_j) =$ delay of demand $n_j + S_j$. 

Then

$$D_j(n_j + S_j) = [L_j(n_j) - \tau S_j(n_j)]^+.$$ 

In equilibrium,

$$D_j = [L_j - \tau S_j]^+. \quad (23)$$

So, $D_j$ is the remaining time until $Z_j$ is absorbed after $S_j$ demands. This, in turn, can be obtained by restarting the process $Z_j$ with an initial density equal to the density of $Z_j(\tau S_j)$, say, $\rho_j(\tau S_j)$. That is,

$$D_j \sim \text{CPH}(\rho_j(\tau S_j), \Gamma_j) \quad \text{and} \quad E[D_j] = -\rho_j(\tau S_j)\Gamma_j^{-1}e. \quad (24)$$

As in Part I, $\rho_j(\tau S_j)$ can be derived by observing the joint process $(X_j, W_j)$ for $S_j$ steps, where $X_j$ and $W_j$ are the processes $Z_j$ and $A$ embedded at demand epochs. Specifically, letting $\sigma_j[n]$ denote the density of this joint process at $n$, we have

$$\rho_j(\tau S_j) = \sigma_j[S_j](I \otimes e)$$

$$= \sigma_j[0]\Omega_j(I \otimes e), \quad (25)$$

where

$$\Omega_j = [I \otimes \Lambda_j - \Gamma_j \otimes I - I \otimes Q]^{-1} (I \otimes \Lambda_j)$$

is the one-step transition matrix of $(X_j, W_j)$ whose derivation can be found in Part I. Our task, then, is to determine $\sigma_j[0]$, the initial density of $(X_j, W_j)$ when $L_j$ starts.

In the following, we shall compute $\sigma_j[0]$ for the proportional-rate case and the non-proportional-rate case separately. Evidently, the first case is simpler.

### 3.2. Proportional-Rate Case

Now, assume (22) holds for all $j$. Then we need not distinguish where a demand at the warehouse comes from, and (21) becomes

$$L_j = D_0 + T_j. \quad (21')$$

Notice that there are two groups of states for the chain $X_j$, reflected in the partition of $\Gamma_j$, one group inherited from $X_0$, and a new group reflecting $T_j$. If $X_j[0]$ is in the first group, then $D_0 > 0$, and $L_j$ starts where $D_0$ starts. Recall, from Section 2, $D_0$ is obtained by observing the chain $(X_0, W_0)$, where

$$W_0[0] \sim \eta = \text{equilibrium distribution of } A \text{ at order epochs.}$$

Moreover, recalling from (8) and (9) in Section 2,
\[ \sigma_0[\tilde{N}] = \sum_k (\gamma_0 \otimes \delta(k))\Omega^k_0 \]  \hspace{1cm} (26)

gives the joint density of \((X_0, W_0)\) when \(D_0\) begins, and hence describes \((X_j[0], W_j[0])\) for this group of states. Here, \(\Omega_0\) is given by (6) with \(\Lambda_0\) and \(\Gamma_0\) replacing \(\Lambda\) and \(\Gamma\), respectively. If \(X_j[0]\) is in the second group, then \(D_0 = 0\), and \(L_j\) starts with \(T_j\), according to the probability vector \(v_j\). For these states (\(X_j[0], W_j[0]\)) has density \(v_j \otimes (\alpha_j - \sigma_0[\tilde{N}](e \otimes I))\). Here, \(\alpha_j\) is the equilibrium density of \(W_j\), which is identical to \(\sigma_0\) [by (1)] because of (22). In sum,

\[ \sigma_j[0] = (\sigma_0[\tilde{N}], v_j \otimes (\alpha_j - \sigma_0[\tilde{N}](e \otimes I))). \]  \hspace{1cm} (27)

### 3.3. Non-Proportional-Rate Case

Now, suppose we do not have (22). Then \(D_0\) depends on \(j\) in general. Let \(C_0[n]\) be the index of the retailer from which demand \(n\) comes. Following the same logic as in the previous subsection, one can easily see that all we need to do is to replace \(\sigma_0[\tilde{N}]\) in (27) by \(\sigma_0[j]\), the density of \(\{(X_0[\tilde{N}], W_0[\tilde{N}])|C_0[\tilde{N}] = j\}\). That is, in the general non-proportional-rate case,

\[ \sigma_j[0] = (\sigma_0[\tilde{N}], v_j \otimes (\alpha_j - \sigma_0[\tilde{N}](e \otimes I))). \]  \hspace{1cm} (28)

Thus, what remains is to compute \(\sigma_0[\tilde{N}]\).

Observe, for any \(k\), \(\{(X_0[\tilde{N}], W_0[\tilde{N}], C_0[\tilde{N}] = j)|W_0[0] = a, \tilde{N} = k\}\) has density equal to

\[ (\gamma_0 \otimes e_a)\Omega^{k-1}_0\Omega_0, \]

where

\[ \Omega_0 = (I \otimes \Lambda_0 - \Gamma_0 \otimes Q)^{-1}(I \otimes \Lambda_j) \]  \hspace{1cm} (29)

is the one-step transition matrix of \((X_0, W_0)\), assuming the state transition is due to a demand from retailer \(j\). (Refer to Section 4.2 of Part I for the derivation of \(\Omega_0\).) So, \(\{(X_0[\tilde{N}], W_0[\tilde{N}])|C_0[\tilde{N}] = j, W_0[0] = a\}\) has density

\[ \frac{1}{\Pr \{C_0[\tilde{N}] = j\}} (\gamma_0 \otimes e_a) \sum_k \xi_a(k)\Omega^{k-1}_0\Omega_0, \]

and

\[ \sigma_0[\tilde{N}] = \frac{1}{\Pr \{C_0[\tilde{N}] = j\}} \sum_a \eta(a)(\gamma_0 \otimes e_a) \sum_k \xi_a(k)\Omega^{k-1}_0\Omega_0 = \frac{1}{\Pr \{C_0[\tilde{N}] = j\}} \sum_k (\gamma_0 \otimes \delta(k))\Omega^{k-1}_0\Omega_0. \]  \hspace{1cm} (30)
Next, we compute \( \Pr \{ C_0[\hat{N}] = j \} \). First, from Lemma 3.2 of Part I we know that the chain \( W_0 \) has transition matrix

\[
\Phi_0 = (\Lambda_0 - Q)^{-1} \Lambda_0.
\]

Let \( \beta_0[n] \) denote the density of \( W_0[n] \). Then, the same logic as in the derivation of \( \sigma[\hat{N}] \) in Section 2 leads to

\[
\beta_0[n] = \sum_k \delta(k) \Phi_0^k.
\]

Hence,

\[
\Pr \{ C_0[\hat{N}] = j \} = \sum_a \Pr \{ W_0[\hat{N}] = a \} \Pr \{ C_0[\hat{N}] = j | W_0[\hat{N}] = a \}
\]

\[
= \sum_a \Pr \{ W_0[\hat{N}] = a \} \frac{\lambda(a)}{\lambda_0(a)}
\]

\[
= \beta_0[\hat{N}] \Lambda_0^{-1} \Lambda_j e
\]

\[
= \sum_k \delta(k) \Phi_0^k \Lambda_0^{-1} \Lambda_j e. \tag{31}
\]

Thus, combining (30) and (31), we get

\[
\sigma_0[\hat{N}] = \left[ \sum_k \delta(k) \Phi_0^k \Lambda_0^{-1} \Lambda_j e \right]^{-1} \sum_k (\gamma_k \otimes \delta(k)) \Omega_0^{-1} \Omega_0. \tag{32}
\]

In sum, (25), (28), and (32) together give the distribution of \( D_j \), the customer delay at retailer \( j \). It is easy to see that the total number of operations needed in computing \( E[D_j] \) is of the same order as that of \( D_0 \), as discussed at the end of Section 2.

Finally, we show that \( D_0 \) is independent of \( j \) when (22) holds. Clearly, what we need to show is that \( \sigma_0[\hat{N}] \) is independent of \( j \), which, in turn, can be done by proving that \( \sum_k \delta(k) \Phi_0^k \Lambda_0^{-1} \Lambda_j e \) is independent of \( j \). Let \( \hat{K} = \text{diag}(\kappa_0) \). Then, (22) means

\[
\Lambda_i = \mu_i \hat{K}, \quad i = 0, j.
\]

Notice that \( \Lambda_0^{-1} \Lambda_j = \mu_0^{-1} \mu_j I \) and \( \sum_k \delta(k) \Phi_0^k e = \beta_0[\hat{N}] e = 1 \). So,

\[
\left[ \sum_k \delta(k) \Phi_0^k \Lambda_0^{-1} \Lambda_j e \right]^{-1} \Lambda_j
\]

\[
= \mu_0 \mu_j \left[ \sum_k \delta(k) \Phi_0^k e \right]^{-1} \mu_j \hat{K}
\]

\[
= \mu_0 \hat{K}
\]

\[
= \Lambda_0, \tag{33}
\]
which is independent of $j$. Note that (33) also verifies that $\sigma_0[\tilde{N}]$ obtained in this subsection is identical to $\sigma_0[N]$ obtained in Section 3.2, provided (22) holds.

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