Rates of Convergence of Spectral Methods for Graphon Estimation

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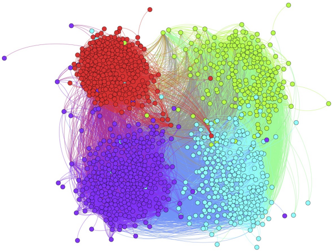
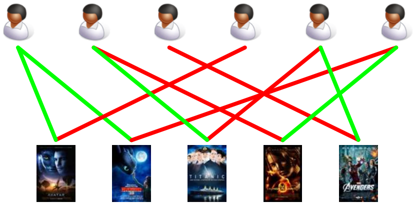
Statistical network analysis

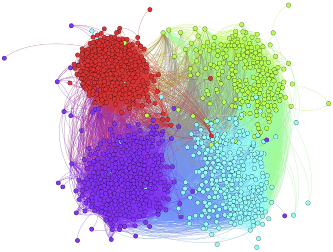


Simmons college friendship network



Moive recommender systems

Protein-protein interaction networks, DNA sequencing,...



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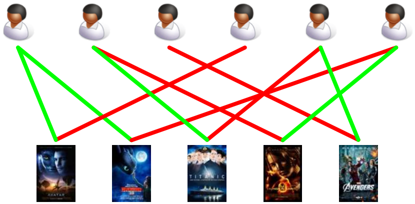

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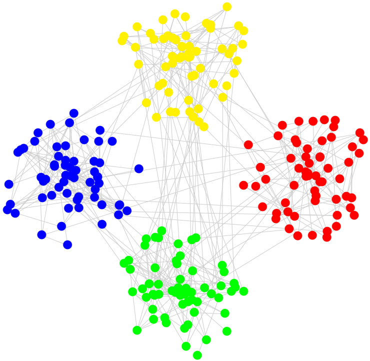
A key problem

Learn the network generating mechanism – how edges are formed



Communities

$$x : \{1, 2, \dots, n\} \to \{1, 2, \dots k\}$$

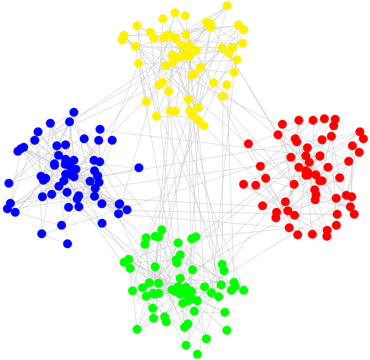
Connection probability

$$B = (b_{ab}) \in [0, 1]^{k \times k}$$
$$p_{ij} = b_{x(i)x(j)}$$

Observation of adjacency matrix

$$A_{ij} \mid x_i, x_j \sim \mathrm{Bern}\,(p_{ij})$$

Stochastic block models [Holland-Laskey-Leinhardt '83]



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Limitation of SBM

Real networks may not display the clustered structure

Graphon models

Latent features

$$x_i \overset{\text{i.i.d.}}{\sim} \mu \quad \forall i \in [n]$$

Connection probability

$$p_{ij} = f\left(x_i, x_j\right) \in [0, 1]$$

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- If $x_i \in \{1, \ldots, k\}$, reduces to SBM
- Originally proposed to study graph limits [Lovász '12]
- Popular framework for studying network generating mechanism estimation – estimate f or $P = \left(p_{ij}\right)$ from observation of A

Sparse graphon estimation

Partial observation of adjacency matrix

$$A_{ij} \mid x_i, x_j \sim \mathrm{Bern}\left(\rho \times p_{ij}\right),$$

$\rho = \rho_n$ may converge to 0 as $n \to \infty$

Sparse graphon estimation

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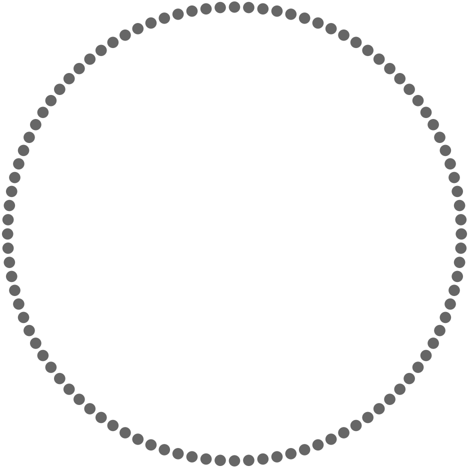
$\rho = \rho_n$ may converge to 0 as $n \to \infty$

Focus of this talk: estimating $P = (p_{ij})$

$$\mathsf{MSE}(\widehat{P}) = \frac{1}{n^2}\mathbb{E}\left[\|P - \widehat{P}\|_{\mathrm{F}}^2\right]$$

An Illustrating example



$n = 100$ and $f(x, y) = 1 - |x - y|$ and $\rho = 0.1$

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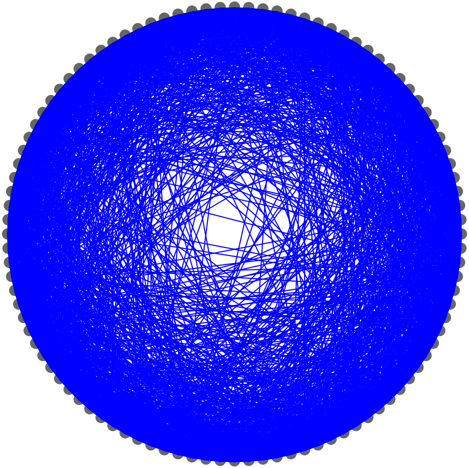

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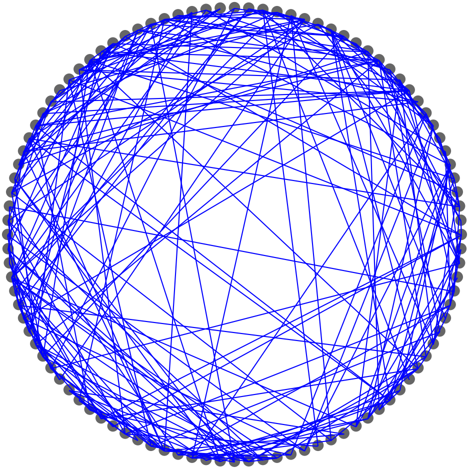

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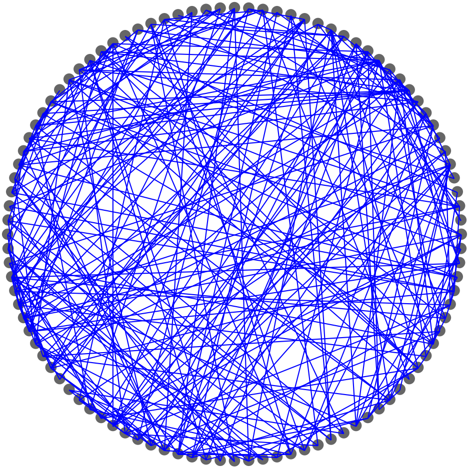

$n = 100$ and $f(x, y) = 1 - |x - y|$ and $\rho = 0.1$

Universal singular value thresholding (USVT) [Chatterjee '15]

1: Take singular value decomposition of A:

$$A = \sum_{i=1}^{n} s_i u_i v_i^\top$$

2: Threshold singular values:

$$\widehat{A} = \sum_{i=1}^{n} \mathbf{1}_{\{s_i \geq \tau\}} s_i u_i v_i^\top$$

3: Output

$$\widehat{P} = \frac{\widehat{A}}{\rho}$$

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Remarks: Threshold at $\tau = (1+\delta)\|A - \mathbb{E}[A]\|_2$ to suppress noise

Rates of Convergence of USVT for α-smooth graphons

Theorem (X. 17')

Suppose $n\rho = \Omega(\log n)$. *If* $f : [0,1]^d \times [0,1]^d \to [0,1]$ *is* α*-Hölder or Sobolev smooth, then*

$$\mathsf{MSE}(\widehat{P}) \lesssim (n\rho)^{-\frac{2\alpha}{2\alpha+d}}$$

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- Need $n\rho = \Omega(\log n)$ to ensure $\| A - \mathbb{E}[A] \|_2 \lesssim \sqrt{n\rho}$
- α-smooth $\approx (\alpha - 1)$-derivatives of f are well-behaved
- Suboptimal comparing to minimax rate $(d = 1)$ [Gao-Lu-Zhou '15, Klopp-Tsybakov-Verzelen '15, Gao-Lu-Ma-Zhou '16]

$$\inf_{\widehat{P}} \sup_{P \in \mathcal{P}} \mathsf{MSE}(\widehat{P}) \asymp \begin{cases} \frac{\log(n\rho)}{n\rho} & n\rho \le n^{\alpha} \log^{\alpha+1}(n) \\ (n^2\rho)^{-\frac{\alpha}{\alpha+1}} & n\rho \ge n^{\alpha} \log^{\alpha+1}(n) \end{cases}$$

- Gap to minimax rate shrinks as α increases

Rates of Convergence of USVT for analytic graphons

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Suppose $n\rho = \Omega(\log n)$. *If* $f : [0,1]^d \times [0,1]^d \to [0,1]$ *is analytic, then*

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- Analytic ≈ infinitely differentiable with locally converging Taylor series
- When $d = 1$, matches the minimax rate for α-smooth graphons with $\alpha \geq 1$

Proof ideas

1. To show [Shah-Balakrishnan-Guntuboyina-Wainwright '16, Koltchinskii-Lounici-Tsybakov '11, Klopp'11]

$$\mathsf{MSE}(\widehat{P}) \lesssim \min_{0 \le r \le n} \left(\frac{r}{n\rho} + \frac{1}{n^2} \sum_{i \ge r+1} \mathbb{E}\left[\lambda_i^2(P)\right] \right)$$

$\frac{r}{n\rho}$: estimation error of rank-r matrix

$\sum_{i \ge r+1} \lambda_i^2(P)$: approximation error of rank-r matrix

2. To show

$$\frac{1}{n^2} \sum_{i \ge r+1} \mathbb{E}\left[\lambda_i^2(P)\right] \lesssim \begin{cases} r^{-2\alpha/d} & \text{if } f \text{ is } \alpha\text{-smooth} \\ e^{-r^{1/d}} & \text{if } f \text{ is analytic} \end{cases}$$

using piecewise polynomial approximation of f [Birman-Solomyak '67 '77]

Summary and concluding remarks

When $n\rho = \Omega(\log n)$: the rates of convergence of USVT

$$\mathsf{MSE}(\widehat{P}) \lesssim \begin{cases} (n\rho)^{-\frac{2\alpha}{2\alpha+d}} & \text{if } f \text{ is } \alpha\text{-smooth} \\ \frac{\log^d(n\rho)}{n\rho} & \text{if } f \text{ is analytic} \end{cases}$$

When $n\rho = \Omega(1)$: Trim high-degree vertices or use convex programs

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Future work

- Minimax optimal rate for high-dimensional feature space $d > 1$
- Applications of graphon estimation in real data