On the Accuracy of the Wyner Model in Cellular Networks

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***Abstract*—The Wyner model has been widely used to model and analyze cellular networks due to its simplicity and analytical tractability. Its key aspects include fixed user locations and the deterministic and homogeneous interference intensity. While clearly a significant simplification of a real cellular system, which has random user locations and interference levels that vary by several orders of magnitude over a cell, a common presumption by theorists is that the Wyner model nevertheless captures the essential aspects of cellular interactions. But is this true? To answer this question, we compare the Wyner model to a model that includes random user locations and fading. We consider both uplink and downlink transmissions and both outage-based and average-based metrics. For the uplink, for both metrics, we conclude that the Wyner model is in fact quite accurate for systems with a sufficient number of simultaneous users, e.g., a CDMA system. Conversely, it is broadly inaccurate otherwise. Turning to the downlink, the Wyner model becomes inaccurate even for systems with a large number of simultaneous users. In addition, we derive an approximation for the main parameter in the Wyner model – the interference intensity term, which depends on the path loss exponent.**

***Index Terms*—Cellular IT models, TDMA, CDMA, multicell processing.**

I. Introduction

INTERCELL interference is a key bottleneck of modern cellular networks. As a consequence, multicell processing (MCP), which can efficiently suppress intercell interference, is of great current interest to both researchers and cellular standard bodies [1]–[9]. Surprisingly, analytical techniques for modeling such interference are in short supply. Grid-based models of multicell networks are too complex to handle analytically so researchers, particularly in the pursuit of information theoretic results, often resort to a simple multicellular model originally proposed by Wyner [10]. Compared to real cellular networks, the most popular linear version of the Wyner model (see Fig. 1) makes three major simplifications: (i) only interference from two adjacent cells is considered; (ii) random user locations and therefore path loss variations are ignored; (iii) the interference intensity from each neighboring base station (BS) is characterized by a single fixed parameter α $(0 \leq \alpha \leq 1)$.

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A. Background and Related Work

Although this model was explicitly introduced in [10] to derive the capacity of uplink cellular networks with multicell processing (MCP), it was also used in well-known CDMA cellular analysis papers that predate it [11], [12], although naturally not called the "Wyner model". In [10], Wyner used this deterministic model to "prove" that intracell TDMA is optimal and achieves capacity, which was later generalized by Somekh and Shamai [13] to account for flat fading, by which they showed that wideband transmission is advantageous versus intracell TDMA and that fading increases capacity when the number of users is sufficiently large. In [14] and [15], the Wyner model is further used to analyze the throughput of cellular networks under single-cell processing (SCP) and two-cell-site processing (TCSP). Later, scaling results for the sum capacity were derived under the Wyner model with multiple-input-multiple-output (MIMO) links in [16]. Recently, the Wyner model was extended to incorporate shadowing in [17].

Turning to the downlink, the Wyner model was used to analyze the average throughput with MCP in [18], where it is shown that a linear preprocessing and encoding scheme eliminates the intercell interference and dramatically improves downlink cellular performance compared to the conventional SCP approach. Thereafter, it was generalized to derive the downlink sum capacity [19] and sum throughput of various precoding schemes such as zero-forcing (ZF), beamforming (BF), minimum mean square error (MMSE) [20] and ZFBF [21], [22]. More recently, the Wyner model and its modifications have been widely used to evaluate different constrained coordination strategies in cellular networks, e.g., [23]–[26].

Despite the fairly large amount of literature based on the Wyner model, to our knowledge no serious effort has ever been made to validate this simple model, or to understand when it might be a reasonable approximation. Namely, how much is lost from these three simplifications? What might be an appropriate value of the key parameter α? It is not at all clear when conclusions based on the Wyner model broadly hold for more realistic scenarios. In this paper, we attempt to provide detailed analysis and simulations to determine when the Wyner model is accurate, and when it is not. "Accurate" is a necessarily subjective term here which we use to mean that major trends are captured within a "small" constant, for example a metric can be tracked within a factor of some small constant. "Inaccurate" means that key trends are in our opinion totally missed and the error cannot in general be bounded within a small constant, and results may be off by an order of magnitude or more. Different audiences may care to draw their own conclusions on its relative accuracy: this paper intends to offer a mathematics-based starting point for such a judgment, and to provide initial conclusions as to when it *may* be sufficiently accurate and also when it is almost *certainly inaccurate*.

B. Contributions

We compare a 1-D Wyner model to a 1-D grid-based network with random user locations and Rayleigh fading. In the first part of this paper, we study the uplink channel. Two important performance metrics in cellular systems are outage/coverage probability and average/sum throughput. From an outage and average throughput point of view, we show that if there are many simultaneous users, the Wyner parameter α can be accurately tuned to capture the key trends, since the stochastic intercell interference ultimately exhibits very little randomness due to the law of large numbers. On the other hand, the Wyner model is broadly inaccurate for systems with a small number of simultaneous users because the law of large numbers does not hold anymore and the statistics cannot be captured by the single deterministic parameter α. Even for the mean feature like average throughput, the Wyner model is inaccurate due to the looseness of Jensen's inequality.

In the second part of this paper, we turn to the downlink channel, and find conclusions quite different from the uplink case. With single-cell processing, we show that the Wyner model becomes inaccurate even with a large number of simultaneous users, since interference only comes from a few nearby BSs and still appears random. In the case of multicell processing with a suboptimal power allocation, it is shown that the Wyner model is only accurate to provide a lower bound to sum throughput in CDMA with a properly selected α.

The unifying conclusion is that the Wyner model, which itself models an "average" interference condition, can be adapted to handle mean-based metrics like sum or average throughput but is subject to inaccuracy due to Jensen's inequality. It cannot handle metrics that depend on the whole statistics of the intercell interference, like outage probability, with one exception being the CDMA uplink with a sufficient number of simultaneous users, in which case the law of large numbers holds and thus a single appropriately tuned α, which we derived, can capture outage as well.

The rest of this paper is organized as follows. Section II introduces the system model and key definitions. Section III and Section IV are devoted to the analysis of the uplink channel with single-cell processing and multicell processing respectively, while Section V and Section VI present the corresponding analysis of the downlink channel. Section VII extends the results to OFDMA systems. Section VIII ends the paper with a summary and concluding remarks. Proofs are deferred to the appendices.

II. System Model and Definition

The system model and key definitions in this paper are described in this section.

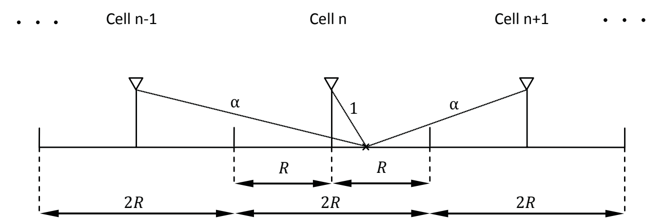

Fig. 1. The 1-D Wyner model.

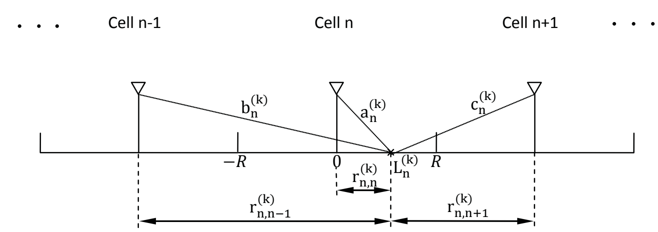

Fig. 2. The 1-D grid-based model used for comparison, which has random user locations and Rayleigh fading.

A. The System Model

Consider the 1-D linear Wyner model, depicted in Fig. 1, where there are N cells located on a line, indexed by n ($1 \leq n \leq N$), each covering a segment of length $2R$. BSs are located at the center of each cell and there are K active users per cell. Intercell interference only comes from two neighboring cells. The channel gain between BS and its home user is 1 and the intercell interference intensity is characterized by a deterministic and homogeneous parameter α. Therefore, the locations of users are implicitly fixed.

In this paper, random user locations are considered and the network is redrawn in Fig. 2. In contrast to fixed user locations and deterministic and homogeneous intercell interference intensity, we assume all the users are randomly, independently, and uniformly distributed on the line in each cell. Thus, the intercell interference intensity is a random variable depending on user locations. The channel in this paper is assumed to experience pathloss and Rayleigh fading. For simplicity, lognormal shadowing is neglected.

Under these assumptions, for the uplink, the received signal at BS n at a given time is given by (1), where $X_n^{(k)}$ is the transmit signal from user k in cell n with $\mathbb{E}[|X_n^{(k)}|^2] = 1$, $\{Z_n\}$ is mutually independent zero-mean additive white Gaussian noise with variance σ^2, $r_{n,m}^{(k)}$ is the distance from user k in cell n to the BS m, $P_n^{(k)}$ is the transmit power of user k in cell n, β is the pathloss exponent, $h_{n,m}^{(k)} \sim \mathcal{CN}(0,1)$ [1] is the Rayleigh fading channel from user k in cell n to the BS m and is independent across n and k. Every user is assumed to achieve perfect channel inversion with respect to user location in the uplink, i.e., the received signal power of each user at its home BS with fading averaging out is P:

$$P_n^{(k)}(r_{n,n}^{(k)})^{-\beta} = P. \tag{2}$$

[1] $X \sim \mathcal{CN}(0,1)$ means that X is a complex Gaussian random variable with zero mean and unit variance.

$$Y_n = \underbrace{\sum_{k=1}^{K} \sqrt{P_n^{(k)} (r_{n,n}^{(k)})^{-\beta}} h_{n,n}^{(k)} X_n^{(k)}}_{\text{home user signals}} + \underbrace{\sum_{i=1}^{K} \sqrt{P_{n-1}^{(i)} (r_{n-1,n}^{(i)})^{-\beta}} h_{n-1,n}^{(i)} X_{n-1}^{(i)}}_{\text{interference from cell } n-1} + \underbrace{\sum_{j=1}^{K} \sqrt{P_{n+1}^{(j)} (r_{n+1,n}^{(j)})^{-\beta}} h_{n+1,n}^{(j)} X_{n+1}^{(j)}}_{\text{interference from cell } n+1} + Z_n. \quad (1)$$

For the downlink, the received signal of user k in cell n at a given time is given by

$$Y_n^{(k)} = a_n^{(k)} h_{n,n}^{(k)} X_n + b_n^{(k)} h_{n,n-1}^{(k)} X_{n-1} + c_n^{(k)} h_{n,n+1}^{(k)} X_{n+1} + Z_n^{(k)}, \quad (3)$$

where X_n is the transmit signal of BS n, $\{Z_n^{(k)}\}$ is the mutually independent zero-mean additive white Gaussian noise with variance σ^2. Path loss from BS n, $n-1$, $n+1$ to user k in cell n are denoted by $\{a_n^{(k)}, b_n^{(k)}, c_n^{(k)}\}$ respectively, which are given by

$$a_n^{(k)} = \left(\frac{r_0}{r_{n,n}^{(k)}}\right)^{\frac{\beta}{2}}, b_n^{(k)} = \left(\frac{r_0}{r_{n,n-1}^{(k)}}\right)^{\frac{\beta}{2}}, c_n^{(k)} = \left(\frac{r_0}{r_{n,n+1}^{(k)}}\right)^{\frac{\beta}{2}},$$

where r_0 is the received power reference distance.

Let $L_n^{(k)}$ denote the location of user k in cell n. Since users are assumed to be independently and uniformly distributed over cells, $\{L_n^{(k)}\}$ are i.i.d. random variables over n and k with distribution $\mathbb{U}\,[-R, R]$, where $\mathbb{U}$ denotes the uniform distribution. As depicted in Fig. 2, $r_{n,n}^{(k)}, r_{n,n-1}^{(k)}, r_{n,n+1}^{(k)}$ can be written as a function of $L_n^{(k)}$ and R, which are

$$r_{n,n}^{(k)} = |L_n^{(k)}|, \quad r_{n,n\mp1}^{(k)} = 2R \pm L_n^{(k)}.$$

Also, it follows that $\{r_{n,m}^{(k)}\}$ are independent over n and k.

B. Terminology

In this paper, we consider various different system settings, which are explained as follows.

1) **Intracell TDMA**: one user per cell is allowed to transmit at any time instant while users in different cells can transmit simultaneously.
2) **CDMA**: all users in every cell are allowed to transmit simultaneously over the whole bandwidth. For the uplink, asynchronous CDMA is often used with pseudonoise sequences as signatures, so there exists intra-cell interference; while for the downlink, synchronous CDMA is often employed with orthogonal spreading codes and synchronous reception, so there is no intra-cell interference.
3) **OFDMA**: OFDM is employed at the physical layer and all users in every cell are allowed to transmit simultaneously but every user in each cell is assigned a distinct subset of subcarriers, which eliminates the intracell interference.
4) **Single-cell Processing (SCP)**: for the uplink, BSs only process transmit signals from their own cells and treat intercell interference as Gaussian noise; while for the downlink, BSs transmit signals with information only intended for their home users.
5) **Multicell Processing (MCP)**: for the uplink, a joint receiver has access to all the received signals and an optimal decoder decodes all the transmit signals jointly; while for the downlink, the transmit signal from each BS contains information for all users.

The relevant performance metrics used in this paper are as follows.

1) **Outage Probability (OP)** : the probability that the received signal-interference-ratio (SIR) is smaller than a threshold.
2) **Average Throughput (AvgTh)**: the expected user throughput where the expectation is taken over all possible user locations and fading realizations.
3) **Sum Throughput (SumTh)**: the maximum sum rate of all users.

III. Uplink with SCP

In this section, we will study the accuracy of the Wyner model for describing the uplink channel in cellular networks with SCP. Two common multiaccess schemes (intracell TDMA and CDMA) are considered in this section and two key performance metrics, namely outage probability and average throughput, are investigated separately. Since the interference-limited case is of interest in SCP, the Gaussian noise Z_n is ignored in this section.

A. Outage Probability

Consider intracell TDMA first. BS n receives interference from two concurrent users in neighboring cells. It follows that the signal-interference-ratio (SIR) of user k in cell n is a random variable given by

$$\text{SIR}_n^{(k)} = \frac{P_n^{(k)} |h_{n,n}^{(k)}|^2 (r_{n,n}^{(k)})^{-\beta}}{\frac{P_{n-1}^{(i)} |h_{n-1,n}^{(i)}|^2}{(r_{n-1,n}^{(i)})^{\beta}} + \frac{P_{n+1}^{(j)} |h_{n+1,n}^{(j)}|^2}{(r_{n+1,n}^{(j)})^{\beta}}}.$$

Substituting (2), the SIR can be simplified to

$$\begin{aligned}\text{SIR}_n^{(k)} &= \frac{|h_{n,n}^{(k)}|^2}{|h_{n-1,n}^{(i)}|^2 \left(\frac{r_{n-1,n-1}^{(i)}}{r_{n-1,n}^{(i)}}\right)^{\beta} + |h_{n+1,n}^{(j)}|^2 \left(\frac{r_{n+1,n+1}^{(j)}}{r_{n+1,n}^{(j)}}\right)^{\beta}} \\ &= \frac{|h_{n,n}^{(k)}|^2}{|h_{n-1,n}^{(i)}|^2 U_{n-1}^{(i)} + |h_{n+1,n}^{(j)}|^2 V_{n+1}^{(j)}}, \end{aligned} \quad (4)$$

where $U_{n-1}^{(i)} = \left(\frac{r_{n-1,n-1}^{(i)}}{r_{n-1,n}^{(i)}}\right)^{\beta}$ and $V_{n+1}^{(j)} = \left(\frac{r_{n+1,n+1}^{(j)}}{r_{n+1,n}^{(j)}}\right)^{\beta}$, and note that $\{U_n^{(i)}\}$ and $\{V_n^{(j)}\}$ are i.i.d. random variables over n and i, j respectively. Also, $U_n^{(i)}$ and $V_m^{(j)}$ are i.i.d. for any i, j, and $n \neq m$. For ease of notation, define auxiliary random variables $A_{n-1}^{(i)} = |h_{n-1,n}^{(i)}|^2 U_{n-1}^{(i)}$ and $B_{n+1}^{(j)} = |h_{n+1,n}^{(j)}|^2 V_{n+1}^{(j)}$.

To decode the received signal reliably, the SIR is constrained to be larger than a threshold θ. If not, there arises outage and the outage probability q is given by

$$q = \mathbb{P}[\mathrm{SIR}_n^{(k)} < \theta] = \mathbb{P}\left[A_{n-1}^{(i)} + B_{n+1}^{(j)} > \frac{|h_{n,n}^{(k)}|^2}{\theta}\right].$$

The outage probability can be derived as

$$\begin{aligned} q &\overset{(a)}{=} 1 - \mathbb{E}\left[e^{-\theta\left(A_{n-1}^{(i)}+B_{n+1}^{(j)}\right)}\right] \\ &\overset{(b)}{=} 1 - \mathbb{E}\left[e^{-\theta|h_{n-1,n}^{(i)}|^2 U_{n-1}^{(i)}}\right]\mathbb{E}\left[e^{-\theta|h_{n+1,n}^{(i)}|^2 V_{n+1}^{(j)}}\right] \\ &\overset{(c)}{=} 1 - \mathbb{E}\left[\frac{1}{1+U_{n-1}^{(i)}\theta}\right]\mathbb{E}\left[\frac{1}{1+V_{n+1}^{(j)}\theta}\right] \\ &\overset{(d)}{=} 1 - \mathbb{E}^2\left[\frac{1}{1+U_{n-1}^{(i)}\theta}\right] \\ &\overset{(e)}{=} 1 - \left[\int_0^{\frac{1}{2}}\left(\frac{1}{1+\theta(\frac{x}{1-x})^\beta} + \frac{1}{1+\theta(\frac{x}{1+x})^\beta}\right)dx\right]^2 \quad (5) \\ &\overset{(f)}{\le} 1 - \left(\frac{1}{1+\mathbb{E}[U_{n-1}^{(i)}]\theta}\right)^2, \quad (6) \end{aligned}$$

where (a) follows from the distribution of exponential random variable $|h_{n,n}^{(k)}|^2$; (b) follows from the independence property; (c) follows from the Laplace transform of the exponential random variables $|h_{n-1,n}^{(i)}|^2$ and $|h_{n+1,n}^{(i)}|^2$; (d) follows from the identical distribution of $U_{n-1}^{(i)}$ and $V_{n+1}^{(j)}$; (e) follows from the definition of $U_{n-1}^{(i)}$; (f) follows from the Jensen's inequality.

In the 1-D Wyner model, SIR $\equiv \frac{1}{2\alpha^2}$ and α is a deterministic parameter. Therefore, there is no notion of outage. However, with random user locations and Rayleigh fading, intercell interference and thus SIR become random. The Wyner model is certainly unable to capture the whole statistics of the SIR by a single deterministic parameter α in TDMA. However, surprisingly, we will see next that the Wyner model is able to do so in CDMA systems thanks to the law of large numbers.

In the 1-D Wyner model with fading, SIR $= \frac{|h_{n,n}^{(k)}|^2}{\alpha^2(|h_{n-1,n}^{(i)}|^2+|h_{n+1,n}^{(i)}|^2)}$ and the outage probability $q = 1 - \frac{1}{(1+\alpha^2\theta)^2}$. Define the average intercell interference intensity with random user locations and fading $\bar{\alpha}_{\mathrm{ul}}^2$ as

$$\begin{aligned} \bar{\alpha}_{\mathrm{ul}}^2 &= \mathbb{E}[U_n^{(k)}] = \mathbb{E}[V_n^{(k)}] \\ &= \int_0^{\frac{1}{2}}\left[\left(\frac{x}{1-x}\right)^\beta + \left(\frac{x}{1+x}\right)^\beta\right]dx \\ &\approx \frac{1}{(\beta+1)2^\beta}, \end{aligned} \quad (7)$$

where the approximation is explained in Appendix A. With $\alpha = \bar{\alpha}_{\mathrm{ul}}$, the outage probability q in the Wyner model with fading is the same as equation (6). Therefore, even including fading, the Wyner model is still inaccurate and only provides an upper bound to the outage probability.

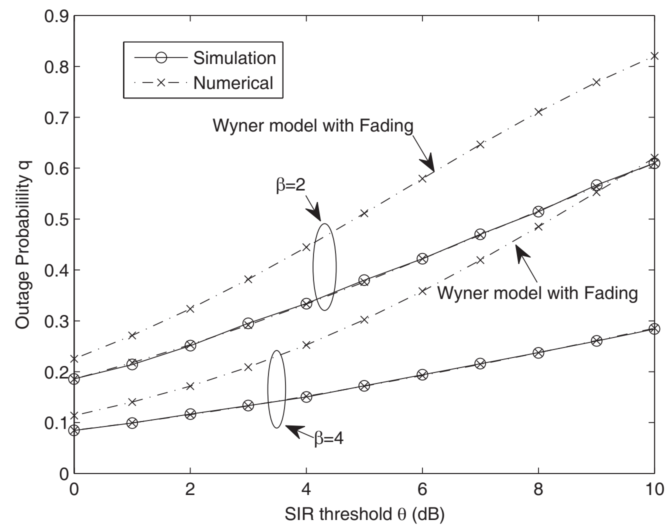

Fig. 3. Outage probability in intracell TDMA for the uplink with varying SIR threshold θ according to (5).

Simulation and numerical results according to (5) are shown in Fig. 3. It can be seen that the outage probability changes gradually with the SIR threshold and the Wyner model with fading provides a loose upper bound to the the outage probability.

Let us consider the asynchronous CDMA with full frequency diversity where BSs will receive intercell interference from all user transmissions in two neighboring cells, and the fading of intended signal is averaged out. It follows then the SIR of user k at BS n in CDMA can be derived as

$$\mathrm{SIR}_n = \frac{G}{\sum_{i=1,i\neq k}^K |h_{n,n}^{(i)}|^2 + \sum_{i=1}^K (A_{n-1}^{(i)} + B_{n+1}^{(i)})}, \quad (8)$$

where G is the processing gain in CDMA, the first term of the denominator is the intracell interference, and the second term is the intercell interference. When $K \gg 1$, since $h_{n,n}^{(i)}$, $h_{n-1,n}^{(i)}$, $h_{n+1,n}^{(i)}$, $U_{n-1}^{(i)}$ and $V_{n+1}^{(i)}$ are i.i.d., by the central limit theorem, $\sum_{i=1,i\neq k}^K |h_{n,n}^{(i)}|^2 + \sum_{i=1}^K (A_{n-1}^{(i)} + B_{n+1}^{(i)})$ can be approximated by a Gaussian random variable with mean μ and variance σ_u^2, which can be easily derived as

$$\begin{aligned} \mu &= K - 1 + 2K\mathbb{E}[U_n^{(i)}] \approx K - 1 + \frac{2K}{(\beta+1)2^\beta}, \\ \sigma_u^2 &= K - 1 + 2K\left(\mathbb{E}[(U_n^{(i)})^2] - \mathbb{E}^2[U_n^{(i)}]\right) \\ &\approx K - 1 + \frac{2K}{2^{2\beta}}\left[\frac{1}{2\beta+1} - \frac{1}{(\beta+1)^2}\right], \end{aligned} \quad (9)$$

where the approximation is explained in Appendix A. Therefore, the outage probability can be derived as

$$\begin{aligned} q &= \mathbb{P}\left[\sum_{i=1,i\neq k}^K |h_{n,n}^{(i)}|^2 + \sum_{i=1}^K (A_{n-1}^{(i)} + B_{n+1}^{(i)}) > \frac{G}{\theta}\right] \\ &\approx \mathcal{Q}\left(\frac{G/\theta - \mu}{\sigma_u}\right), \end{aligned} \quad (10)$$

where $\mathcal{Q}(x)$ is the complementary cumulative distribution function (CCDF) of the standard Gaussian random variable.

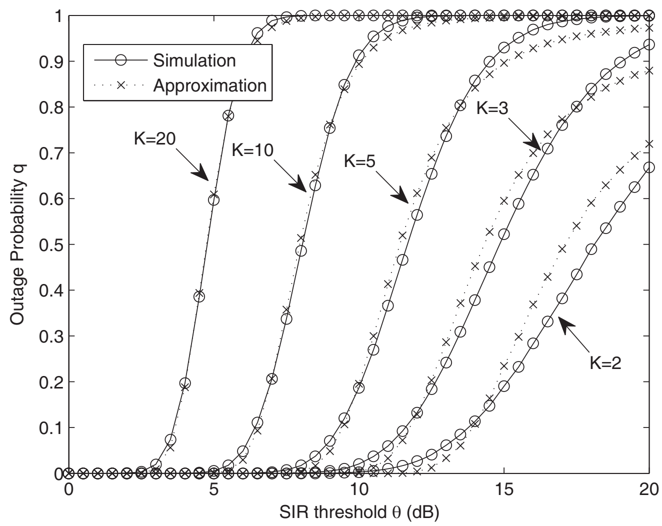

Fig. 4. Outage probability in CDMA for the uplink with varying number of users K per cell according to (10), pathloss exponent $\beta = 4$, $G = 64$.

When $K \to \infty$, due to the law of large numbers:

$$\lim_{K\to\infty} \frac{1}{K}\sum_{i=1}^{K} A_{n-1}^{(i)} = \lim_{K\to\infty} \frac{1}{K}\sum_{i=1}^{K} B_{n+1}^{(i)} = \bar{\alpha}_{\text{ul}}^2,$$

and

$$\lim_{K\to\infty} \frac{1}{K-1}\sum_{i=1, i\neq k}^{K} |h_{n,n}^{(i)}|^2 = 1,$$

Then SIR becomes deterministic and homogeneous, which is given by

$$\text{SIR}_n = \frac{G}{K-1+2\bar{\alpha}_{\text{ul}}^2 K}. \tag{11}$$

Thus, the outage probability becomes either 0 or 1.

The equation (11) is the same as that derived under the Wyner model, if $\alpha = \bar{\alpha}_{\text{ul}}$. In other words, when K is sufficiently large, the Wyner model is accurate and the parameter α can be tuned to characterize intercell interference intensity quite well. If users are uniformly distributed in each cell, α can be approximately determined by $\alpha^2 = \mathbb{E}[U_n^{(k)}] \approx \frac{1}{(\beta+1)2^\beta}$. From this expression, we can see that when the path loss exponent β increases, α^2 decreases exponentially fast. In particular, if $\beta = 4$, $\alpha \approx 0.25$ by numerical computation according to (7). This implies that to model the cellular system with uniform user distribution, a reasonable value of α is typically small.

Simulation and numerical results according to equation (10) are shown in Fig. 4. It shows that the Gaussian approximation of random interference is accurate and the outage probability increases sharply from 0 to 1 when $K = 20$. This implies that when there are more than 20 active users per cell, the Wyner model does characterize the outage probability in the CDMA system very accurately as shown by the analysis.

B. Average Throughput

In this subsection, to make a fair comparison with the available results obtained from the Wyner model, optimal multiuser detection (minimum mean square error detector plus successive interference cancelation) and perfect synchronization are assumed for the CDMA uplink with $G = K$, in which case the intracell interference is suppressed in effect, but the intercell interference still exists and is treated as Gaussian noise. Therefore, the average throughput [2] in intracell TDMA and CDMA under the Wyner model are the same and given by [14]

$$\bar{R}_{\text{TDMA}}^\star = \bar{R}_{\text{CDMA}}^\star = \frac{1}{2K}\log\left(1+\frac{1}{2\alpha^2}\right). \tag{12}$$

After considering random user locations and Rayleigh fading, the average throughput in intracell TDMA and CDMA become

$$R_{\text{TDMA}}^\star = \frac{1}{2K}\mathbb{E}\left[\log\left(1+\frac{|h_{n,n}^{(k)}|^2}{A_{n-1}^{(i)}+B_{n+1}^{(j)}}\right)\right],$$
$$R_{\text{CDMA}}^\star = \frac{1}{2K}\mathbb{E}\left[\log\left(1+\frac{\sum_{i=1}^{K}|h_{n,n}^{(i)}|^2}{\sum_{i=1}^{K}(A_{n-1}^{(i)}+B_{n+1}^{(i)})}\right)\right]. \tag{13}$$

When $K \to \infty$ and the parameter α in the Wyner model is set to be $\bar{\alpha}_{\text{ul}}$, by the law of large numbers, it gives

$$\lim_{K\to\infty} \frac{R_{\text{CDMA}}^\star}{\bar{R}_{\text{CDMA}}^\star} = 1, \tag{14}$$

which implies that the Wyner model accurately characterizes the average throughput in CDMA systems with a large number of users. However, it is unable to do so in TDMA systems, as shown by the simulation results in Fig. 5. The upper plot corresponds to the finite SNR regime with SNR $= 10$ dB; while the bottom plot corresponds to the infinite SNR case. It can be seen that over the entire range of K and β, the Wyner model only gives a loose lower bound to the average throughput in TDMA systems, e.g., when SNR $= 10$ dB, $\beta = 4$ and $K = 20$, the throughput in the Wyner model is off 50%. It can also be deduced that with low SNR, the Wyner model becomes more accurate in TDMA systems, since in the low SNR regime, the noise power becomes comparable to the intercell interference and thus weakens the influence of the random user location and fading. In conclusion, the Wyner model is quite accurate to characterize the average throughput in CDMA when K is sufficiently large, but inaccurate for TDMA especially in the high SNR regime.

What causes the inaccuracy of the Wyner model in TDMA systems? Consider the Wyner model with Rayleigh fading, in which case the average throughput is given by

$$R_{\text{TDMA-F}}^\star = \frac{1}{2K}\mathbb{E}\left[\log\left(1+\frac{|h_{n,n}^{(k)}|^2}{\left(|h_{n-1,n}^{(i)}|^2+|h_{n+1,n}^{(j)}|^2\right)\bar{\alpha}_{\text{ul}}^2}\right)\right],$$
$$R_{\text{CDMA-F}}^\star = \frac{1}{2K}\mathbb{E}\left[\log\left(1+\frac{\sum_{i=1}^{K}|h_{n,n}^{(i)}|^2}{\sum_{i=1}^{K}(|h_{n-1,n}^{(i)}|^2+|h_{n+1,n}^{(i)}|^2)\bar{\alpha}_{\text{ul}}^2}\right)\right]. \tag{15}$$

[2]The average throughput here refers to the average rate per user per *real dimension*.

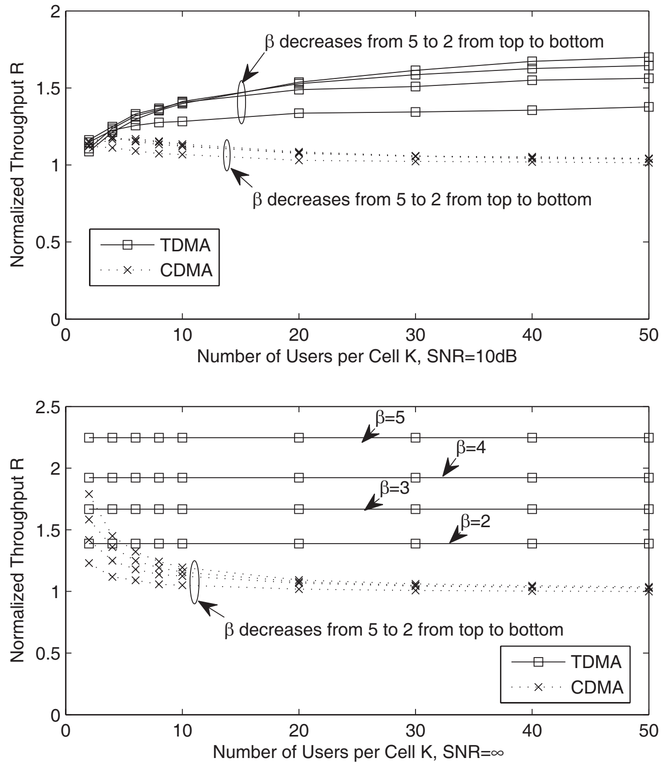

Fig. 5. The uplink average throughput according to (13) normalized by that in the Wyner model according to (12) with varying number of users K per cell, $G = K$ in CDMA. The top plot is for SNR of 10dB and the bottom plot has no noise.

Simulation results are shown in Fig. 6. It can be seen that Rayleigh fading only has negligible effects on the average throughput, i.e.,

$$R^{\star}_{\text{TDMA-F}} \approx R^{\star}_{\text{CDMA-F}} \approx \bar{R}^{\star}_{\text{TDMA}} = \bar{R}^{\star}_{\text{CDMA}}. \tag{16}$$

Moreover, by Jensen's inequality and the fact that $\log(1 + \rho x^{-1})$ is a convex function of $x > 0$ (for any $\rho \geq 0$), we have

$$\begin{aligned} &R^{\star}_{\text{TDMA}} \\ &\geq \frac{1}{2K}\mathbb{E}\left[\log\left(1 + \frac{|h^{(k)}_{n,n}|^2}{\left(|h^{(i)}_{n-1,n}|^2 + |h^{(j)}_{n+1,n}|^2\right)\bar{\alpha}^2_{\text{ul}}}\right)\right] \\ &= R^{\star}_{\text{TDMA-F}} \approx \bar{R}^{\star}_{\text{TDMA}}. \end{aligned} \tag{17}$$

This implies that even if it includes Rayleigh fading, the Wyner model only provides a loose lower bound to the average throughput in TDMA due to the looseness resulting from Jensen's inequality and random user locations. This substantiates the importance of considering the user locations when analyzing average throughput.

IV. Uplink with MCP

In this section, the average throughput in intracell TDMA and CDMA with MCP is investigated. The average throughput in intracell TDMA can be derived as

$$\begin{aligned} C^{\star}_{\text{TDMA}} &= \lim_{N\to\infty}\frac{1}{KN}\mathbb{E}[I(\{Y_n\};\{X_n\})|\{L_n\},\{h_{n,m}\}] \\ &= \lim_{N\to\infty}\frac{1}{KN}\mathbb{E}\left[\log(\det(\mathbf{\Lambda_N}))\right], \end{aligned} \tag{18}$$

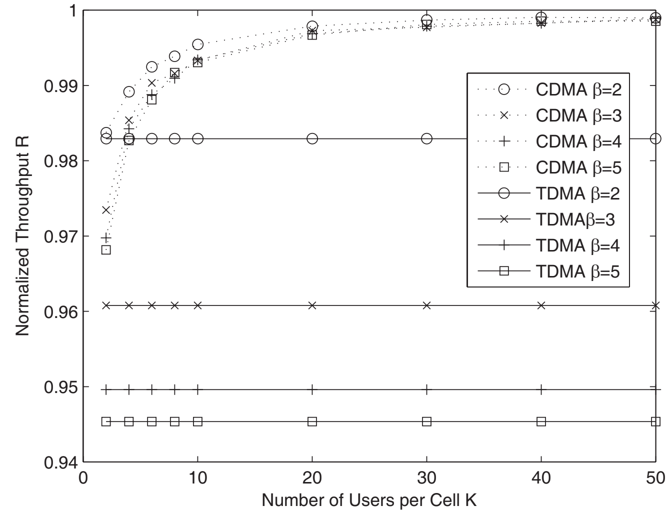

Fig. 6. The uplink average throughput in the Wyner model with Rayleigh fading according to (15) normalized by that in the Wyner model according to (12) with varying number of users K per cell, $G = K$ in CDMA.

where N is the number of cells and the expectation is taken over N i.i.d. random user locations $\{L_n\}$ and $3N$ i.i.d. Rayleigh fading $\{h_{n,m}\}$. The $N \times N$ matrix $\sigma^2\mathbf{\Lambda_N}$ is the covariance matrix of the conditioned Gaussian output vector $\{Y_n\}$ and $\mathbf{\Lambda_N}$ is given by (19), where $KS = \frac{KP}{\sigma^2}$ is the SNR of each user in intracell TDMA and P is the received power defined in (2).

Similarly, the average throughput in CDMA is given by

$$\begin{aligned} C^{\star}_{\text{CDMA}} &= \lim_{N\to\infty}\frac{1}{KN}\mathbb{E}[I(\{Y_n\};\{X^{(k)}_n\}|\{L^{(k)}_n, h^{(k)}_{n,m}\})] \\ &= \lim_{N\to\infty}\frac{1}{KN}\mathbb{E}\left[\log(\det(\mathbf{\Lambda_N}))\right], \end{aligned}$$

where the expectation is over NK i.i.d. random locations of users $\{L^{(k)}_n\}$ and $3NK$ i.i.d. Rayleigh fading $h^{(k)}_{n,m}$. The $N \times N$ matrix $\sigma^2\mathbf{\Lambda_N}$ is the covariance matrix of the conditioned Gaussian output vector $\{Y_n\}$ and $\mathbf{\Lambda_N}$ is given by (20), where $S = \frac{P}{\sigma^2}$ is the SNR of each user. As expected, by setting $K = 1$, the covariance matrix is the same in both systems.

In the following theorem, we prove that CDMA is advantageous to intracell TDMA.

Theorem 1. *With uniform user distribution and Rayleigh fading, the average throughput of the CDMA system is larger than that of TDMA, i.e., $C^{\star}_{CDMA} \geq C^{\star}_{TDMA}$. Equality is achieved when the user locations are fixed and their distances from the home BSs are the same.*

Proof: See Appendix B. ■

Remark 1. *This theorem was proved in [13] with flat fading but fixed user location. In [10], it was shown that intracell TDMA is optimal and achieves the capacity under the Wyner model. However, when taking random user locations and Rayleigh fading into consideration, intracell TDMA becomes suboptimal and the Wyner model leads to an inaccurate conclusion. But, it is able to characterize the average throughput in CDMA with large number of users accurately, which is shown by the following corollary.*

$$[\mathbf{\Lambda_N}]_{m,n} = \begin{cases} 1 + KS\left(|h_{n,n}^{(k)}|^2 + |h_{n-1,n}^{(i)}|^2 U_{n-1}^{(i)} + |h_{n+1,n}^{(j)}|^2 V_{n+1}^{(j)}\right) & (n,n) \\ KS\left(h_{n,n}^{(k)}(h_{n,n+1}^{(k)})^*(U_n^{(k)})^{\frac{1}{2}} + h_{n+1,n}^{(j)}(h_{n+1,n+1}^{(j)})^*(V_{n+1}^{(j)})^{\frac{1}{2}}\right) & (n,n+1) \\ KS\left(h_{n-1,n-1}^{(i)}(h_{n-1,n}^{(i)})^*(U_{n-1}^{(i)})^{\frac{1}{2}} + h_{n,n}^{(k)}(h_{n,n-1}^{(k)})^*(V_n^{(k)})^{\frac{1}{2}}\right) & (n,n-1) \\ KS\left(h_{n+1,n}^{(j)}(h_{n+1,n+2}^{(j)})^*(U_{n+1}^{(j)})^{\frac{1}{2}}(V_{n+1}^{(j)})^{\frac{1}{2}}\right) & (n,n+2) \\ KS\left(h_{n-1,n}^{(i)}(h_{n-1,n-2}^{(i)})^*(U_{n-1}^{(i)})^{\frac{1}{2}}(V_{n-1}^{(i)})^{\frac{1}{2}}\right) & (n,n-2) \\ 0 & \text{otherwise} \end{cases}. \tag{19}$$

$$[\mathbf{\Lambda_N}]_{m,n} = \begin{cases} 1 + S\sum_{k=1}^K\left(|h_{n,n}^{(k)}|^2 + |h_{n-1,n}^{(k)}|^2 U_{n-1}^{(k)} + |h_{n+1,n}^{(k)}|^2 V_{n+1}^{(k)}\right) & (n,n) \\ S\sum_{k=1}^K\left(h_{n,n}^{(k)}(h_{n,n+1}^{(k)})^*(U_n^{(k)})^{\frac{1}{2}} + h_{n+1,n}^{(k)}(h_{n+1,n+1}^{(k)})^*(V_{n+1}^{(k)})^{\frac{1}{2}}\right) & (n,n+1) \\ S\sum_{k=1}^K\left(h_{n-1,n-1}^{(k)}(h_{n-1,n}^{(k)})^*(U_{n-1}^{(k)})^{\frac{1}{2}} + h_{n,n}^{(k)}(h_{n,n-1}^{(k)})^*(V_n^{(k)})^{\frac{1}{2}}\right) & (n,n-1) \\ S\sum_{k=1}^K\left(h_{n+1,n}^{(k)}(h_{n+1,n+2}^{(k)})^*(U_{n+1}^{(k)})^{\frac{1}{2}}(V_{n+1}^{(k)})^{\frac{1}{2}}\right) & (n,n+2) \\ S\sum_{k=1}^K\left(h_{n-1,n}^{(k)}(h_{n-1,n-2}^{(k)})^*(U_{n-1}^{(k)})^{\frac{1}{2}}(V_{n-1}^{(k)})^{\frac{1}{2}}\right) & (n,n-2) \\ 0 & \text{otherwise} \end{cases}. \tag{20}$$

Corollary 1. *The average throughput of the CDMA system with large number of concurrent users* $(K \gg 1)$ *is given by*

$$C^\star_{\text{CDMA-LK}} = \frac{1}{K}\log\left(1 + KS(1 + 2\bar{\alpha}_{ul}^2)\right). \tag{21}$$

This corollary is proved in [13] with flat fading only. The conclusion does not change when considering random user locations.

Recall that the average throughput in the Wyner model is derived as [10]

$$C^\star_W = \frac{1}{K}\int_0^1 \log\left(1 + KS(1 + 2\alpha\cos 2\pi\theta)^2\right)d\theta. \tag{22}$$

Set the parameter α to be $\bar{\alpha}_{\text{ul}}$ and recall that $\bar{\alpha}_{\text{ul}}$ is typically small. In particular, with $\beta = 4$, $\bar{\alpha}_{\text{ul}} \approx 0.25$. Therefore, we have the asymptotic expansion for $C^\star_W$ for small α proved in [10]:

$$C^\star_W = \frac{1}{K}\log(1 + KS) - \frac{\bar{\alpha}_{\text{ul}}^2(KS-1)S}{(KS+1)^2}. \tag{23}$$

When the number of users K and SNR S is large, i.e., $KS \gg 1$, the average throughput can be further simplified as

$$C^\star_{\text{CDMA-LK}} \overset{(a)}{=} \frac{1}{K}\log(KS) + 2\frac{\bar{\alpha}_{\text{ul}}^2}{K} + o(\bar{\alpha}_{\text{ul}}^2),$$
$$C^\star_W = \frac{1}{K}\log(KS) - \frac{\bar{\alpha}_{\text{ul}}^2}{K} + o(\bar{\alpha}_{\text{ul}}^2), \tag{24}$$

where (a) follows from the fact that $\log(1+x) = x + o(x)$ when x is small. Therefore, we have

$$\frac{C^\star_{\text{CDMA-LK}} - C^\star_W}{C^\star_W} = \frac{3\bar{\alpha}_{\text{ul}}^2}{\log(KS)} \ll 1. \tag{25}$$

This suggests that the Wyner model is accurate to characterize the average throughput in CDMA systems with large number of users and relatively high SNR, which is also validated by simulation in Fig. 7.

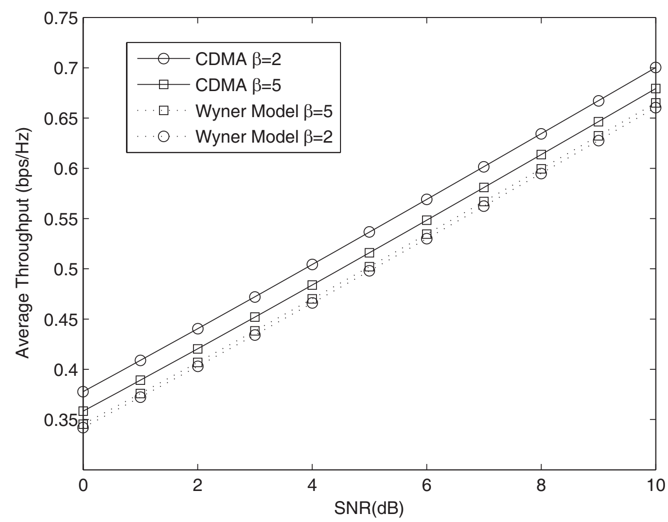

Fig. 7. Average throughput for the uplink with MCP with varying SNR and pathloss exponent β according to (21) and (22) .

V. Downlink with SCP

In this section, we will study the accuracy of the Wyner model in the downlink with SCP by analyzing the coverage probability and average throughput in intracell TDMA and CDMA systems. In the downlink, it is assumed that BSs are always transmitting at the fixed maximum power P and simple equal transmit power per user policy is used. Since the interference-limited case is of interest in SCP, the Gaussian noise Z_n is ignored in this section.

A. Coverage Probability

Consider intracell TDMA first. In the downlink, a typical user receives intended signal from its home BS and intercell interference from two neighboring BSs. It follows that the SIR of user k in cell n in intracell TDMA is given by

$$\mathrm{SIR} = \frac{(a_n^{(k)})^2|h_{n,n}^{(k)}|^2 P}{(b_n^{(k)})^2|h_{n,n-1}^{(k)}|^2 P + (c_n^{(k)})^2|h_{n,n+1}^{(k)}|^2 P} = \frac{|h_{n,n}^{(k)}|^2}{|h_{n,n+1}^{(k)}|^2 U_n^{(k)} + |h_{n,n-1}^{(k)}|^2 V_n^{(k)}},$$

where $U_n^{(k)}$ and $V_n^{(k)}$ are the same as defined in the uplink. Note that the difference from the uplink is that the intercell interference comes from the neighboring BSs and has nothing to do with the users in neighboring cells.

The coverage probability is defined as

$$q_c = \mathbb{P}[\mathrm{SIR} > \theta] = \mathbb{P}\left[\frac{|h_{n,n}^{(k)}|^2}{A_n^{(k)} + B_n^{(k)}} > \theta\right]. \quad (26)$$

It can be further simplified as

$$\begin{aligned} q_c &= \mathbb{E}\left[e^{-\theta\left(A_n^{(k)}+B_n^{(k)}\right)}\right] \\ &\overset{(a)}{=} \mathbb{E}\left[\mathbb{E}\left[e^{-\theta\left(A_n^{(k)}+B_n^{(k)}\right)} \mid U_n^{(k)}, V_n^{(k)}\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[e^{-\theta A_n^{(k)}} \mid U_n^{(k)}\right]\mathbb{E}\left[e^{-\theta B_{n+1}^{(k)}} \mid V_n^{(k)}\right]\right] \\ &\overset{(b)}{=} \mathbb{E}\left[\frac{1}{(1+U_n^{(k)}\theta)(1+V_n^{(k)}\theta)}\right] \\ &= 2\int_0^{\frac{1}{2}} \frac{1}{\left(1+\theta(\frac{x}{1-x})^\beta\right)\left(1+\theta(\frac{x}{1+x})^\beta\right)}dt, \end{aligned} \quad (27)$$

where (a) follows from the definition of conditional expectation; (b) follows from the Laplace transform of exponential random variable.

Consider synchronous CDMA downlink channel next. With orthogonal spreading codes and synchronous reception, there is no intra-cell interference and the transmit power for each user is $\frac{P}{K}$. It follows that the SIR of user k in cell n in given by

$$\mathrm{SIR} = \frac{G(a_n^{(k)})^2|h_{n,n}^{(k)}|^2 \frac{P}{K}}{(b_n^{(k)})^2|h_{n,n-1}^{(k)}|^2 P + (c_n^{(k)})^2|h_{n,n+1}^{(k)}|^2 P} = \frac{G|h_{n,n}^{(k)}|^2}{K\left(A_n^{(k)} + B_n^{(k)}\right)},$$

where G is the processing gain. With $G = K$, the SIR in CDMA is the same as that in TDMA. That implies that unlike the uplink CDMA, in the downlink CDMA the Wyner model is still unable to capture the whole statistics of the SIR. The key underlying reason is that in the downlink, the intercell interference, instead of coming from randomly distributed users in neighboring cells, actually comes from two neighboring BSs which have fixed transmit power and locations. Thus, the law of large numbers does not hold anymore.

With 1-D Wyner model with fading, $\mathrm{SIR} = \frac{|h_{n,n}^{(k)}|^2}{\alpha^2(|h_{n-1,n}^{(k)}|^2+|h_{n+1,n}^{(k)}|^2)}$ and the coverage probability $q_c = \frac{1}{(1+\alpha^2\theta)^2}$. With $\alpha = \bar{\alpha}_{\mathrm{ul}}$, the coverage probability q_c can also be rewritten as $\frac{1}{(1+\mathbb{E}[U_{n-1}^{(i)}]\theta)^2}$. Simulation and numerical results according to (27) are shown in Fig. 8. It can be seen that the outage probability changes gradually with the SIR threshold and the Wyner model with fading provides a loose lower bound to the outage probability.

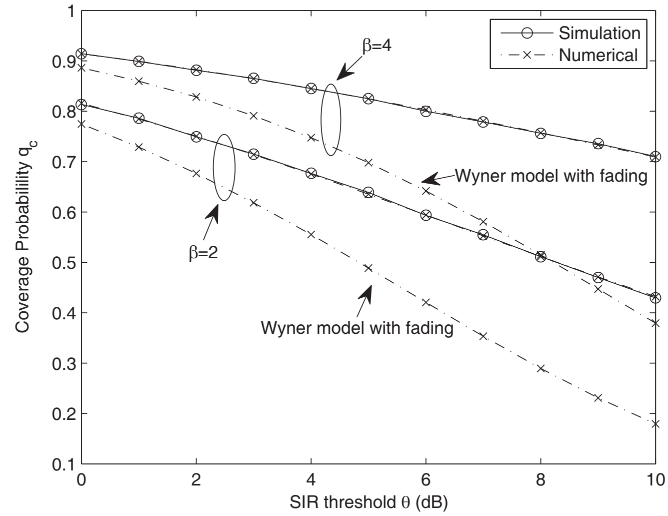

Fig. 8. Coverage probability in intracell TDMA and CDMA with $G = K$ for the downlink with varying SIR threshold θ according to (27).

B. Average Throughput

In this subsection, it is assumed that $G = K$ in CDMA to make a fair comparison. With random user location and fading, the average throughput in intracell TDMA and synchronous CDMA are the same and given by

$$\bar{R}_{\mathrm{TDMA}} = \bar{R}_{\mathrm{CDMA}} = \frac{1}{2K}\mathbb{E}\left[\log\left(1 + \frac{|h_{n,n}^{(k)}|^2}{A_n^{(k)} + B_n^{(k)}}\right)\right], \quad (28)$$

and the average throughput in the Wyner model is given as (12). Similar to the TDMA uplink channel, the Wyner model can only provide a loose lower bound to the average throughput by tuning the parameter α to be $\bar{\alpha}_{\mathrm{ul}}$, as shown by simulation results in Fig. 9.

Also, consider the Wyner model with Rayleigh fading and $\alpha = \bar{\alpha}_{\mathrm{ul}}$. The average throughput is given by

$$\bar{R}_{\text{TDMA-F}} = \bar{R}_{\text{CDMA-F}} = \frac{1}{2K}\mathbb{E}\left[\log\left(1 + \frac{|h_{n,n}^{(k)}|^2}{\left(|h_{n,n+1}^{(k)}|^2 + |h_{n,n-1}^{(k)}|^2\right)\bar{\alpha}_{\mathrm{ul}}^2}\right)\right],$$

As shown in Fig. 9, similar to the uplink, Rayleigh fading has only negligible effects on the average throughput. Moreover, by Jensen's inequality, we have

$$\begin{aligned} \bar{R}_{\mathrm{TDMA}} &= \bar{R}_{\mathrm{CDMA}} \\ &\geq \frac{1}{2K}\mathbb{E}\left[\log\left(1 + \frac{|h_{n,n}^{(k)}|^2}{\left(|h_{n,n+1}^{(k)}|^2 + |h_{n,n-1}^{(k)}|^2\right)\bar{\alpha}_{\mathrm{ul}}^2}\right)\right] \\ &= \bar{R}_{\text{TDMA-F}} = \bar{R}_{\text{CDMA-F}} \approx \bar{R}_{\text{TDMA-F}} = \bar{R}_{\text{CDMA-F}}. \end{aligned} \quad (29)$$

It implies that even including the Rayleigh fading, the Wyner model only provides a loose lower bound to the average

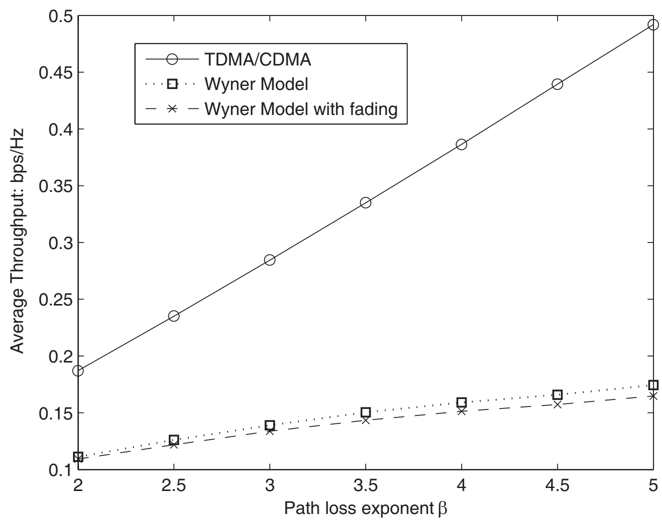

Fig. 9. Average throughput for the downlink with varying pathloss exponent β according to (28), $G = K = 10$ in CDMA.

throughput in TDMA and CDMA downlink due to Jensen's inequality and random user location.

VI. Downlink with MCP

In this section, the sum throughput with MCP in CDMA and intracell TDMA is investigated. Rewrite (3) in a compact matrix form as

$$\mathbf{Y} = \mathbf{H}^*\mathbf{X} + \mathbf{Z}, \tag{30}$$

where $\mathbf{H}^*$ is the $NK \times N$ location-dependent channel matrix. The downlink per-cell power constraint is given by

$$(\mathbb{E}[\mathbf{X}\mathbf{X}^*])_{ii} \le P, \qquad \text{for } i = 1, \ldots, N. \tag{31}$$

The key tool used in the analysis of sum throughput is the minimax uplink-downlink duality theorem established in [27], restated here for clarity.

Theorem 2. *(Minimax uplink-downlink duality [27]) For a given channel matrix $\mathbf{H}$, the sum capacity of the downlink channel (30) under per-cell power constraint (31) is the same as the sum capacity of the dual uplink channel affected by a diagonal "uncertain" noise under the sum power constraint :*

$$C^{sum}(\mathbf{H}, S) = \min_{\mathbf{A}} \max_{\mathbf{D}} \log\left(\frac{\det(\mathbf{H}\mathbf{D}\mathbf{H}^T + \mathbf{A})}{\det(\mathbf{A})}\right),$$

where $S = \frac{P}{\sigma^2}$, $\mathbf{A}$ and $\mathbf{D}$ are N-dim and NK-dim nonnegative diagonal matrices such that $Tr(\mathbf{A}) \le 1/S$ and $Tr(\mathbf{D}) \le 1$.

Consider CDMA first. Applying Theorem 2, the sum throughput with random user location and fading in CDMA is derived in the following corollary.

Corollary 2. *The downlink per-cell sum throughput with random user locations in CDMA is*

$$C^{sum}_{CDMA} = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\min_{\mathbf{A}} \max_{\mathbf{D}} \log\left(\frac{\det(\mathbf{H}\mathbf{D}\mathbf{H}^T + \mathbf{A})}{\det(\mathbf{A})}\right)\right],$$

where the expectation is taken over all possible $NK \times N$ channel matrices $\mathbf{H}$.

With random user location and fading, the channel matrices are random and thus asymmetric. Therefore, the above optimization problem has no simple and explicit solution. To obtain a lower bound to the optimal solution, equal transmit power per user in the dual uplink channel is used, i.e. $\mathbf{D} = \frac{1}{KN}\mathbf{I}$, which gives:

$$C^{l}_{\text{CDMA}} = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\min_{\mathbf{A}} \log\left(\frac{\det\left(\frac{1}{KN}\mathbf{H}\mathbf{H}^T + \mathbf{A}\right)}{\det(\mathbf{A})}\right)\right]. \tag{32}$$

For arbitrary K, it is still hard to solve the above minimization problem, but if $K \to \infty$, the following lemma holds, which simplifies the minimization problem.

Lemma 1. *When $K \to \infty$, $\frac{1}{K}\mathbf{H}\mathbf{H}^T$ converges to a deterministic and symmetric matrix, that is*

$$\frac{1}{K}\mathbf{H}\mathbf{H}^T \overset{K\to\infty}{\longrightarrow} \left(1 + 2\bar{\alpha}_{dl}^2\right)\mathbf{I}, \tag{33}$$

where $\bar{\alpha}_{dl}^2$ is defined as $\bar{\alpha}_{dl}^2 = \mathbb{E}[(b_n^{(k)})^2]$ with $\mathbb{E}[(a_n^{(k)})^2] = 1$.

Proof: See Appendix C. The proof essentially based on the law of large numbers. ■

Lemma 1 enables us to derive the main theorem in this section.

Theorem 3. *When $K \to \infty$, the lower bound to the per-cell sum throughput in CDMA with uniform user distribution and Rayleigh fading converges to*

$$C^{l}_{CDMA} \overset{K\to\infty}{\longrightarrow} \log(1 + S(1 + 2\bar{\alpha}_{dl}^2)). \tag{34}$$

Proof: By lemma 1, $\frac{1}{K}\mathbf{H}\mathbf{H}^T$ converges to a deterministic and symmetric matrix. Therefore, the minimum in (32) is achieved by substituting $\mathbf{A} = \frac{1}{NS}\mathbf{I}$. Then, the theorem follows by simple calculations. ■

As shown in [19], the per cell sum throughput given by the Wyner model is $KC^{\star}_{W}$. Therefore, similar to the uplink with MCP case, by setting $\alpha = \bar{\alpha}_{\text{dl}}$, we have that with high SNR S,

$$\frac{C^{l}_{\text{CDMA}} - KC^{\star}_{W}}{KC^{\star}_{W}} = \frac{3\bar{\alpha}_{\text{dl}}^2}{\log S} \ll 1. \tag{35}$$

This implies that the Wyner model accurately characterizes the lower bound to the average throughput in the CDMA downlink with a large number of users and high SNR. But it generally cannot accurately characterize the true average throughput, since the opportunistic scheduling based on random user location and fading can be used to improve the average throughput, which is not captured by the Wyner model.

In the following, let us move to the intracell TDMA. Assume equal transmit power per user in the dual uplink channel, i.e., $\mathbf{D} = \frac{1}{N}\mathbf{I}$, the lower bound to the sum throughput in intracell TDMA can be derived as

$$C^{l}_{\text{TDMA}} = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\min_{\mathbf{A}} \log\left(\frac{\det\left(\frac{1}{N}\tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\mathbf{T}} + \mathbf{A}\right)}{\det(\mathbf{A})}\right)\right],$$

where the expectation is taken over all possible $N \times N$ channel matrix $\tilde{\mathbf{H}}^{\mathbf{T}}$. Similar to the uplink case, we can derive the following theorem.

Theorem 4. *With uniform user distribution and Rayleigh fading, the lower bound to the per cell sum throughput in CDMA is larger than that of TDMA, i.e.,* $C^l_{CDMA} \geq C^l_{TDMA}$. *Equality is achieved when the user locations are fixed and their distances from the home BSs are the same.*

Proof: The proof is similar to that of Theorem 1. It follows by Jensen's inequality and the fact that $\min_{\mathbf{A}} \log\left(\frac{\det\left((1/N)\tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\mathbf{T}}+\mathbf{A}\right)}{\det(\mathbf{A})}\right)$ is a concave function of $\tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\mathbf{T}}$. ∎

Remark 2. *This theorem implies that with equal transmit power in the dual uplink channel, intracell TDMA which is originally capacity-achieving under the Wyner model becomes suboptimal after considering random user locations and fading. Also, in contrast to CDMA, even when* $K \to \infty$, *the random channel matrix* $\tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\mathbf{T}}$ *does not converge to a deterministic limit. Therefore, we cannot properly tune the parameter* α *to make the Wyner model characterize the lower bound of the sum throughput in intracell TDMA accurately.*

VII. Extension to OFDMA

In this section, OFDMA is considered and we find that in terms of the accuracy of the Wyner model, OFDMA lies between TDMA and CDMA. For concreteness, we explain this fact just for the uplink, but the downlink is similar.

OFDMA systems allocate users distinct time-frequency slices (*resource block*) consisting of N_f subcarriers in frequency and N_t consecutive OFDM symbols in time. Therefore, there is no intracell interference within a cell. However, with universal frequency reuse, there is intercell interference. Note that in a resource block, the allocation of subcarriers changes over OFDM symbols (due to frequency hopping).

The codeword of each user is modulated onto its resource block. Thus, the interference seen over transmitting a codeword may be attributed to transmissions from a total of $M(1 \leq M \leq K)$ users in the neighboring cell. Note that in intracell TDMA, $M = 1$; while in CDMA, $M = K$.

From the above analysis, by assuming the frequency diversity averages the fading of intended signal out, the SIR can be derived as

$$\text{SIR}_n = \frac{M}{\sum_{i=1}^{M}(A^{(i)}_{n-1} + B^{(i)}_{n+1})}.$$

Following the same derivation in the CDMA case, when $M \gg 1$, the outage probability can be derived as

$$\begin{aligned} q &= \mathbb{P}[\text{SIR}_n < \theta] \\ &= \mathbb{P}\left[\sum_{i=1}^{M}(A^{(i)}_{n-1} + B^{(i)}_{n+1}) > \frac{M}{\theta}\right] \\ &\approx \mathbb{Q}\left(\frac{M/\theta - \mu}{\sigma_u}\right), \end{aligned}$$

where μ and σ_u^2 are defined in (9) with replacement of K by M. Also, when $M \to \infty$, SIR $\to \frac{1}{2\alpha^2}$, which is exactly the same as that given by the Wyner model.

Therefore, if $M \gg 1$, OFDMA resembles the CDMA model and the Wyner model is accurate to characterize the SIR distribution. From the simulation results in Fig. 4, it can be deduced that for such a conclusion to hold, M may need to be larger than 20 in the uplink. In practice, the actual value of M in OFDMA systems depends on the specific resource allocation, frequency hopping, and ARQ (automatic repeat request) scheme design. For instance, it was argued that $M \approx K$ in Flash-OFDM system by properly designing the resource blocks for each BS (not the same among all the BSs) and assuming perfect symbol synchronization among transmissions of neighboring BSs (see Section 4.4 in [28]). However, for the LTE cellular standards [9], M is typically much closer to 1 than to K, since the resource blocks are fixed among all the BSs and the protocols are designed to allow for orthogonality amongst users in a cell, as well as potentially among two neighboring cells. Therefore in most implementations of LTE, we would expect the conclusions of the TDMA model to hold quite accurately. In general though, OFDMA systems will lie somewhere between TDMA and CDMA with the specific system design determining which they are more like.

VIII. Conclusion and Future Work

In this paper, we studied the accuracy of the 1-D Wyner model as compared to a 1-D model with random user locations and Rayleigh fading. Future extensions could consider the 2-D Wyner model as compared to a 2-D grid. Our results, summarized in Table I, suggest that the more averaging there is – for example due to spread spectrum, diversity, and/or mean-based metrics – the more accurate the Wyner model is.

Appendix

A. Approximation of μ and σ_u^2 in section III-A

We have $\mathbb{E}[U^{(k)}_n] = \mathbb{E}\left[\left(\frac{r^{(k)}_{n,n}}{r^{(k)}_{n,n+1}}\right)^{\beta}\right] = \mathbb{E}\left[\left(\frac{|L^{(k)}_n|}{2R - L^{(k)}_n}\right)^{\beta}\right]$. Consider the following approximation:

$$\begin{aligned} \mathbb{E}\left[\left(\frac{|L^{(k)}_n|}{2R - L^{(k)}_n}\right)^{\beta}\right] &\approx \frac{E[|L^{(k)}_n|]^{\beta}}{(2R)^{\beta}} \\ &= \frac{2}{(2R)^{\beta+1}}\int_0^R r^{\beta}\mathrm{d}r = \frac{1}{(\beta+1)2^{\beta}}. \end{aligned}$$

The reason behind this coarse approximation is that the numerator accounts more than the denominator for the value.

B. Proof of Theorem 2

Rewrite the covariance matrix $\mathbf{\Lambda_N}$ for CDMA as a summation of K covariance matrices $\mathbf{\Lambda^k_N}$ as $\mathbf{\Lambda_N} = \frac{1}{K}\sum_{k=1}^{K}\mathbf{\Lambda^k_N}$, where each $\mathbf{\Lambda^k_N}$ corresponds to a covariance matrix given by

TABLE I
THE ACCURACY OF THE WYNER MODEL IN CELLULAR NETWORKS

	Uplink			Downlink		
	Intracell TDMA	CDMA	OFDMA	Intracell TDMA	CDMA	OFDMA
$P_{\rm out}$ with SCP	Low	High	Medium	Low	Low	Low
Avg Rate with SCP	Low	High	Medium	Low	Low	Low
Avg or Sum Rate with MCP	Low	High	Medium	Low	Medium	Medium

(19) in intra-cell TDMA. Then it follows that

$$\begin{aligned} C^\star_{\rm CDMA} &= \lim_{N\to\infty} \mathbb{E}\left[\log(\det(\mathbf{\Lambda_N}))\right] \\ &= \lim_{N\to\infty} \mathbb{E}\left[\log(\det(\frac{1}{K}\sum_{k=1}^{K}\mathbf{\Lambda_N^k}))\right] \\ &\overset{(a)}{\geq} \frac{1}{K}\sum_{k=1}^{K}\lim_{N\to\infty}\mathbb{E}\log(\det(\mathbf{\Lambda_N^k})) \\ &= \frac{1}{K}\sum_{k=1}^{K} C^\star_{\rm TDMA} = C^\star_{\rm TDMA}, \end{aligned}$$

where (a) follows from Jensen's inequality and the fact that $\log(\det \mathbf{\Lambda})$ is concave for the covariance matrix $\mathbf{\Lambda}$.

C. Proof of Lemma 1

The (n,n)th entry of $\frac{1}{K}\mathbf{HH}^\star$ is

$$\left[\frac{1}{K}\mathbf{HH}^*\right]_{n,n} = \frac{1}{K}\sum_{k=1}^{K}\Big((a_n^{(k)})^2|h_{n,n}^{(k)}|^2 + (b_{n+1}^{(k)})^2|h_{n+1,n}^{(k)}|^2 + (c_{n-1}^{(k)})^2|h_{n-1,n}^{(k)}|^2\Big).$$

Applying the law of large numbers and i.i.d. property, it follows that

$$\left[\frac{1}{K}\mathbf{HH}^*\right]_{n,n} \overset{K\to\infty}{\longrightarrow} \mathbb{E}[(a_n^{(k)})^2] + \mathbb{E}[(b_n^{(k)})^2] + \mathbb{E}[(c_n^{(k)})^2].$$

Similarly, the $(n,n+1)$th entry is

$$\begin{aligned} \left[\frac{1}{K}\mathbf{HH}^*\right]_{n,n+1} &= \frac{1}{K}\sum_{k=1}^{K}\Big(a_n^{(k)}c_n^{(k)}h_{n,n}^{(k)}h_{n,n+1}^{(k)} \\ &\quad + a_{n+1}^{(k)}b_{n+1}^{(k)}h_{n+1,n}^{(k)}h_{n+1,n+1}^{(k)}\Big) \\ &\overset{K\to\infty}{\longrightarrow} 0, \end{aligned}$$

and the $(n,n+2)$th entry is

$$\begin{aligned} \left[\frac{1}{K}\mathbf{HH}^*\right]_{n,n+2} &= \frac{1}{K}\sum_{k=1}^{K}\Big(b_{n+1}^{(k)}c_{n+1}^{(k)}h_{n+1,n}^{(k)}h_{n+1,n+2}^{(k)}\Big) \\ &\overset{K\to\infty}{\longrightarrow} 0, \end{aligned}$$

Following the same derivation, it can be found that all other entries except (n,n)th are approaching 0. Moreover, for fair comparison with results obtained under the Wyner model, it is assumed that $\mathbb{E}[(a_n^{(k)})^2] = 1$ and define the parameter $\bar{\alpha}^2_{\rm dl}$ as $\bar{\alpha}^2_{\rm dl} = \mathbb{E}[(b_n^{(k)})^2]$. Then it follows that $\frac{1}{K}\mathbf{HH}^* \overset{K\to\infty}{\longrightarrow} \left(1+2\bar{\alpha}^2_{\rm dl}\right)\mathbf{I}$.

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