We consider a periodic-review inventory system in which multiple non-identical retailers replenish from a warehouse, which further replenishes from an outside ample source. Each facility faces Poisson customer demand and replenishes according to a base-stock policy in a fixed time interval. There are fixed costs incurred for placing an order. The warehouse fills the retailers’ orders in the same sequence as the original demand. The objective is to minimize the average system cost per period. This paper develops an evaluation scheme and provides a method to obtain the optimal base-stock levels and reorder intervals. Specifically, for fixed reorder intervals, we show that the optimal base-stock levels can be obtained by generalizing the result of Axskäler (1990). To find the optimal reorder intervals, we first decompose the total cost into each facility and then construct a lower bound to the allocated facility cost. These lower bound functions, each depending on the facility’s reorder interval, can be used to derive the solution bounds. One enabler to tighten the bounds is an effective heuristic cost. Thus, we propose a simple heuristic that combines a revised algorithm of solving the deterministic counterpart and the lower bound cost functions. The results of numerical studies suggest that the optimal policy tends to be an integer-ratio policy, that the power-of-two solution obtained from the deterministic counterpart may perform poorly, and that the suggested heuristic can generate an effective solution for large systems.
1 Introduction

We consider a one-warehouse-multi-retailer system (OWMR) in which each facility replenishes inventory in a fixed time interval. There are $N$ non-identical retailers, each facing independent Poisson demand with complete backlogging. The retailers replenish their inventory from the warehouse, which further replenishes from an outside supplier with unlimited stock. Let $j$ be the facility index, where $j = 0$ represents the warehouse and $j = 1, \ldots, N$ represent the retailers. Each facility implements a stationary echelon $(S, T)$ policy. Specifically, retailer $j$ reviews its inventory order position (inventory on order + inventory on hand - backorders) in every $T_j$ periods and orders up to a base-stock level $S_j$; the warehouse reviews its echelon inventory order position (inventory on order + inventory on hand + inventory at or in transit to the retailers - total backorders at the retailers) in every $T_0$ periods and orders up to an echelon base-stock level $S_0$. We call $T_j$ the reorder interval for facility $j$. We assume that the order schedules are synchronized, i.e., whenever the warehouse receives a shipment, the retailer places an order if it is in the retailer’s order period (see §2 for an example). We also assume that the warehouse applies the virtual allocation rule (e.g., Axsäter 1993, Graves 1996), under which the retailers’ orders are filled in the same sequence as the occurrence of the demand at the retailers. A linear echelon holding cost $h_j$ is incurred per unit and per period for location $j(\geq 0)$ and a backorder cost $b_j$ per unit and per period for retailer $j(\geq 1)$. In addition, a fixed order cost $K_j$ is incurred at facility $j(\geq 0)$ for placing an order. The objective is to determine the policy parameters $S_j$ and $T_j$ for all $j$ such that the total system cost per period is minimized.

This type of periodic replenishment scheme is commonly implemented in practice due to simplicity – it is just simpler for supply chain firms to plan labor, coordinate production, and deliver materials according to fixed schedules. Despite its common usage, it is not clear how to find the optimal reorder intervals. In the deterministic demand regime, although the optimal policy is not known, there exists a very simple integer-ratio policy that guarantees 94 %-effectiveness. (Let $\mathbb{N}$ denote the set of positive integers, and $T_b$ be some base period. An integer-ratio policy satisfies the following conditions: $T_j = n_jT_b$, $n_j \in \mathbb{N}$ for all $j$ and either $T_j/T_0$ or $T_0/T_j \in \mathbb{N}$ for $j = 1, \ldots, N$.) If the base period $T_b$ can be chosen as a non-integer value, there exists a power-of-two (POT) policy.

---

1 The virtual allocation rule is commonly seen in practice. For example, Wal-Mart’s distribution center assigns replenishment stocks to the demands as they occurred. The assigned stocks will be loaded onto a truck and shipped to the retail stores according to some fixed schedule (Chandran 2003). This rule is essentially the first-come-first-serve rule. We refer the authors to Graves (1996) for a detailed discussion on its applications.
that guarantees 98% effectiveness (Roundy 1985). (A POT policy is an integer-ratio policy where $T_j = 2^{k_j} T_b$, $k_j \in \mathbb{N}$ for all $j$.) It is viewed that the POT solution obtained from the corresponding deterministic model is an effective heuristic for the stochastic model. In fact, this approximation has been adopted for the stochastic inventory models in the literature (e.g., Chen and Zheng 1997, Rao 2003). However, there is no study to examine the effectiveness of the POT policy, or in general, the integer-ratio policy as it is not clear how to find the optimal solution.

This study aims to answer three fundamental questions. First, how to obtain the optimal $(S, T)$ policy for the stochastic OWMR problem? Liu and Song (2010) recently revisited the single-stage $(S, T)$ model in Rao (2003) and indicated that solving the single-stage problem is more complicated than the algorithm suggested by Rao due to non-convexity of the total cost function. With this recent finding, it is conceivable that solving the optimal policy for the OWMR model would be extremely difficult. Second, what properties do the optimal reorder intervals possess? Is the class of integer-ratio policies a good candidate for designing a heuristic policy? Third, how effective is the POT solution obtained from the corresponding deterministic model? Under what conditions does the POT solution perform poorly? Is there a simple heuristic that outperforms the POT policy in general?

We develop an approach to find the optimal $(S, T)$ policy. Since there is no ex ante knowledge that whether or not an integer-ratio policy is optimal, we do not impose any restrictions on the reorder intervals except that they are integers. We first characterize the dynamics of key inventory variables and provide a bottom-up recursion to evaluate the average system cost. While this evaluation scheme is new, we show that, with fixed reorder intervals, the optimal echelon base-stock levels can be obtained by generalizing the result of Axsäter (1990) (see Appendix A). Our main contribution in this paper is to provide a method for finding the optimal reorder intervals. This is achieved by the following steps. First, we decompose the original system and allocate the total system cost into each facility. Second, we construct a lower bound function to the allocated facility cost. These lower bounds are a function of each individual facility’s reorder interval, and thus independent of each other. Third, we propose a simple and effective heuristic. The resulting heuristic cost as well as the lower bound functions generate the bounds for the optimal reorder intervals. Consequently, the optimal reorder intervals can be found by enumerating all feasible solutions. To reduce the computational effort, we further prove a property that can reduce the number of enumerations: If the reorder intervals of the facilities satisfy integer-ratio relationships, the reorder interval of the warehouse must be no shorter than the minimum of those of the retailers.
This result corresponds to the “last-minute property” (the warehouse orders only when at least one retailer orders) established by Schwarz (1973) for the deterministic OWMR model.

A key element of our optimization algorithm is to identify an effective heuristic. Through an extensive numerical study, we find that almost all of the optimal solutions are an integer-ratio policy. This inspires us to construct an integer-ratio policy as a heuristic policy. It is natural to test whether the POT solution obtained from the corresponding deterministic model is effective. To our surprise, while the POT solution may perform well, it could perform poorly, especially when $K_0/(h_0\lambda_0)$ is significantly smaller than those of the retailers, where $\lambda_j$ is the demand rate for the retailer $j$, and $\lambda_0 = \sum_{i=1}^{N} \lambda_i$. Under this condition, the deterministic solution often yields a smaller reorder interval for the warehouse whereas the optimal solution suggests all facilities use the same reorder intervals. Two reasons may explain this observation. First, in the deterministic model, the demand variability is not considered so the resulting cost function does not accurately reflect the true cost function. Second, the deterministic model cannot reflect the benefit of risk pooling. That is, the warehouse can choose a larger reorder interval to consolidate the retailers’ orders to take an advantage of risk pooling. In light of these reasons, we propose a heuristic that mitigates these two effects. This innovative heuristic revises the deterministic POT clustering procedure (i.e., identifying facilities that use the same reorder interval) and utilizes the lower bound cost functions to incorporate the demand variability. A numerical study suggests that the heuristic is near-optimal: The average percentage error compared to the optimal cost is 0.34% for two-retailer cases, 0.52% for four-retailer cases, and 0.41% for eight-retailer cases. See §5 for a detailed summary of the effectiveness of the heuristic and the quality of the solution bounds.

We provide a literature review by focusing on the stochastic distribution system with reorder intervals. Graves (1996) considered a general distribution system in which the inventory is controlled by a local $(S, T)$ policy. Graves found that most of the safety stock should be held at the retailer sites and that the virtual allocation rule is near optimal. Axsäter (1993) studied a special case of Graves’s model in that the retailers have an identical reorder intervals and the order schedule is nested, i.e., $T_0/T_j \in \mathbb{N}$ for $j = 1, \ldots, J$. He demonstrated that the local $(S, T)$ policy under the virtual allocation rule in Graves (1996) is essentially the same as the echelon $(S, T)$ policy.

There are papers aiming to find optimal or near-optimal reorder intervals in a context of distribution systems. Çetinkaya and Lee (2000) considered a warehouse that replenishes its stock according to an $(s, S)$ policy and ships stocks to retailers in every fixed interval. Gürbüz et al. (2007) considered a replenishment policy in which retailers are replenished simultaneously in every
fixed interval. Chu and Shen (2010) considered the system with a pre-specified service level at each location. By assuming normal approximation on the demand and stationary safety stock policy at the warehouse, they developed a worst-case bound for the power-of-two solution. Marklund (2011) considered a continuous-review system in which the warehouse implements a local \((r, Q)\) policy and allocates the stock according to the virtual allocation rule. The retailers implement an \((S, T)\) policy. Marklund assumed that the batch size at the warehouse is fixed (i.e., no fixed cost consideration at the warehouse). He provided a method to evaluate the average system cost and proposed heuristics to generate the policy parameters.

There is a stream of research on the distribution model aiming to study the impact of order schedules on the bullwhip effect. Noteworthy examples include Lee et al. (1997), Cachon (1999), Chen and Samroengraja (2004), Cheung and Zhang (2008). Finally, our model is a generalization of the stochastic joint replenishment problem (JRP) studied by Atkins and Iyogun (1988), who showed that the periodic review \((S, T)\) policy outperforms the can-order policy (Silver 1981, Federgruen et al. 1984) when the major fixed cost is large. The optimization method developed in this paper can be used for solving the JRP in Atkins and Iyogun (1988).

The rest of the paper is organized as follows. §2 introduces the model and the notation. §3 characterizes the inventory variables and evaluates the total cost per period. §4 provides an approach for finding the optimal reorder intervals. §5 conducts a numerical study to examine the optimal solution and the effectiveness of the deterministic POT solution and the proposed heuristic. §6 concludes. Appendix A shows how to optimize the base-stock levels when the reorder intervals are fixed. Appendix B provides proofs. Appendix C presents the optimal solution as well as the solution bounds for the instances we tested in this paper.

## 2 The Model

We consider a periodic-review, two-echelon distribution system in which a single warehouse supplies \(N\) non-identical retailers. Time periods are indexed 0, 1, 2, \(\ldots\). Let \([t, t + \tau)\) and \([t, t + \tau]\) denote the time interval over periods \(t, t + 1, \ldots, t + \tau - 1\) and periods \(t, t + 1, \ldots, t + \tau\), respectively. Retailer \(j\) faces Poisson demand with stationary rate \(\lambda_j\). The demands are independent between retailers. Let \(D_j[t, t + \tau)\) and \(D_j[t, t + \tau]\) denote the cumulative demand over time periods in \([t, t + \tau)\) and \([t, t + \tau]\), respectively. There is a constant lead time \(L_j\) for facility \(j\), \(j = 0, \ldots, N\). Let \(L_{[0,j]} = L_0 + L_j\). Each facility implements a stationary echelon \((S, T)\) policy. More specifically, at the beginning of every
periods, retailer $j$ orders up to a base-stock level $S_j$ if its inventory order position (inventory on order + inventory on hand - backorders) is less than $S_j$. Similarly, at the beginning of every $T_0$ periods, the warehouse orders up to an echelon base-stock level $S_0$ if its echelon inventory order position (inventory on order + inventory on hand + inventory at or in transit to the retailers - total backorders at the retailers) is less than $S_0$. We call these $T_j$-th periods order periods and the moment of placing an order an order epoch. Let $h_j$ be the echelon holding cost rate for facility $j$, $j = 0, ..., N$, and the local holding cost rate $H_j = h_0 + h_j$ for $j = 1, ..., N$. Unmet demand is fully backlogged at each retailer. Let $b_j$ be the backorder cost rate for retailer $j$, $j = 1, ..., N$. Finally, there is a fixed cost $K_j$ associated with each order placed at facility $j$, $j = 0, ..., N$. The objective is to determine $(S_j, T_j)$, $j \geq 0$, such that the average total system cost per period is minimized.

We assume that the reorder intervals are integers. The ordering activities between the warehouse and the retailers are coordinated in a synchronized manner: Whenever the warehouse receives a shipment, the retailer places an order if it is in the retailer’s order period. For example, consider a one-warehouse-two-retailer system with $L_0 = 3$, $T_0 = 2$, $T_1 = 1$, and $T_2 = 3$. Suppose that the warehouse places an order at the beginning of period $t, t + 2, t + 4, ...$. Consider the order epoch $t$. This order placed at $t$ will arrive at the beginning of period $t + L_0 = t + 3$. This is the moment that both retailers place an order. Thus, the order periods for retailer 1 are $t + 3, t + 4, t + 5, ...$, and for retailer 2 are $t + 3, t + 6, t + 9, ...$. We term such an order coordination synchronized ordering. Define $lcm$ as an operator that generates the least common multiplier of two or more numbers, e.g., $lcm(2, 3, 4) = 12$. Let

$$M = lcm\{T_0, T_1, ..., T_N\}.$$ 

If we define $t$ and $t + L_0$ as the starting time of a cycle for the warehouse and the retailers, respectively, the next cycle will start in $M$ periods later under the synchronized ordering rule.

One concern for the synchronized ordering rule is that it may lead to a bigger bullwhip effect when the system is controlled by local policies (Lee et al. 1997). However, Cheung and Zhang (2008) pointed out that a high bullwhip effect may not necessarily lead to a high system cost. Moreover, the bullwhip effect has less impact in our model because demand is learned by the warehouse when it occurs (i.e., echelon control).

Under the echelon $(S, T)$ policy, the retailer fills the incoming demand as if it were a single-stage system. That is, when a unit of demand arrives, the retailer fills the demand and the retailer’s net inventory level (= inventory on hand minus backorders) is reduced by one unit. The retailer
does not place an order until its next order epoch. On the other hand, the warehouse immediately learns this arriving demand and assigns one unit (if available) to fill this demand. These assigned inventory units will be shipped at the retailer’s order epoch. In other words, one may view the retailers’ order epochs as the shipping times planned by the warehouse.

Because the centralized information is available, we assume that the warehouse commits its inventory, if available, to fill the incoming orders according to the sequence of the demands occurring at the retailer site. If the warehouse does not have an uncommitted unit to fill an arriving demand, the warehouse creates a backorder and adds this to the current list of outstanding orders. When inventory becomes available, the outstanding orders are filled in the sequence in which they are created. This is the so-called virtual allocation rule (Axsäter 1993, Graves 1996).

We end this section by listing the sequence of events in a period. For the warehouse, (1) an order, if any, is received from retailer $j$; (2) an order is placed with the outside supplier if the period is an order period; (3) a shipment, if any, is received; (4) a shipment is sent to retailer $j$ if the period is retailer $j$’s order period. For retailer $j$, order placement occurs at the beginning of retailer $j$’s order period, while customer demand arrives during the period. Costs are evaluated at the end of the period for all facilities.

3 Evaluation

Under the echelon $(S, T)$ policy, the system forms a regenerative process with a cycle of $M$ periods. Specifically, consider a warehouse order epoch $t$ at which the warehouse orders up to $S_0$. This order will arrive at the beginning of period $t + L_0$. We assume that this is the moment that each retailer $j$ will simultaneously place the first order up to $S_j$ in a regenerative cycle. Thus, the regenerative cycle that we consider in the subsequent analysis is $[t, t + M]$ for the warehouse and $[t + L_0, t + L_0 + M]$ for each retailer $j$. We call $t$ and $t + L_0$ regenerative epoch for the warehouse and the retailers, respectively. To evaluate the average total cost per period, we only need to characterize the distribution of the inventory variables in the above regenerative cycle. The long-run average total cost per period is equal to the sum of the expected total cost incurred at the warehouse in the cycle of $[t + L_0, t + L_0 + M]$ and at each retailer $j$ in the cycle of $[t + L_{[0,j]}, t + L_{[0,j]} + M]$ divided by the cycle length $M$.

We describe the inventory dynamics in the considered regenerative cycle. Let $r$ be the period
index in the regenerative cycle, i.e.,

\[ r = 0, 1, 2, ..., M - 1. \]

Also, define \([a]\) as the roundoff operator, which returns the greatest integer less than or equal to \(a\), a real number. Define

\[ r_j(r) = \left\lfloor \frac{r}{T_j} \right\rfloor T_j, \quad j = 0, 1, 2, ..., N. \]

Thus, for \(r = 0, 1, \ldots, M-1\),

\[ r_j(r) \in \left\{ 0, T_j, 2T_j, \ldots, \left\lfloor \frac{M-1}{T_j} \right\rfloor T_j \right\}. \]

Since \(t\) and \(t + L_0\) are regenerative epochs for the warehouse and the retailers, respectively, the warehouse’s order periods are \(t + r_0(r)\), and the retailer \(j\)’s order periods are \(t + L_0 + r_j(r)\) in a cycle of \(M\) periods. As we shall see, the order decision for the warehouse at the beginning of period \(t + r_0(r)\) will directly affect the inventory amount sent to the retailer \(j\) at the beginning of period \(t + L_0 + r_j(r)\).

We first define inventory variables for the warehouse.

\[
\begin{align*}
IOP_0(n) &= \text{echelon inventory order position at the beginning of period } n, \\
IL^-_0(n) &= \text{echelon inventory level at the beginning of period } n, \\
IL_0(n) &= \text{echelon inventory level at the end of period } n.
\end{align*}
\]

Let \(D_0[s, t] = \sum_{j=1}^{N} D_j[s, t]\) and \(D_0[s, t] = \sum_{j=1}^{N} D_j[s, t]\). The inventory dynamics for the warehouse are

\[
\begin{align*}
IOP_0(t + r_0(r)) &= S_0, \quad (1) \\
IOP_0(t + r) &= IOP_0(t + r_0(r)) - D_0[t + r_0(r), t + r) \\
&= S_0 - D_0[t + r_0(r), t + r), \quad (2) \\
IL^-_0(t + L_0 + r) &= IOP_0(t + r) - D_0[t + r, t + L_0 + r), \quad (3) \\
IL_0(t + L_0 + r) &= IOP_0(t + r) - D_0[t + r, t + L_0 + r]. \quad (4)
\end{align*}
\]

Equation (1) means that the warehouse’s echelon inventory order position after ordering is equal to \(S_0\). Equation (2) shows the warehouse’s echelon inventory order position at any given time \(t + r\) in
the regenerative cycle. Equations (3) and (4) specify the echelon inventory level of the warehouse at the beginning and the end of period $t + L_0 + r$, respectively.

Now, we consider retailer $j$. We define inventory variables for retailer $j$.

\[
\begin{align*}
IOP_j(n) &= \text{inventory order position at the beginning of period } n, \\
IL_j^{-}(n) &= \text{inventory level at the beginning of period } n, \\
IL_j^{+}(n) &= \text{inventory level at the end of period } n.
\end{align*}
\]

Notice that under the virtual allocation rule, $IOP_j(n)$ is always equal to $S_j$ because when a unit of demand arrives, this information is instantaneously transferred to the warehouse as if the retailer “virtually” orders whenever demand occurs. However, at retailer $j$’s order (shipping) epoch, the warehouse may not be able to fully fill retailer $j$’s order, and thus backorders occur. We define $IP_j$, $B_0$ and $B_{0j}$ to describe the warehouse backorders:

\[
\begin{align*}
IP_j(n) &= \text{inventory in-transit position at the beginning of period } n, \\
B_0(n) &= \text{total number of warehouse backorders at the beginning of period } n, \\
B_{0j}(n) &= \text{number of warehouse backorders that belong to retailer } j.
\end{align*}
\]

When $n$ is the order period of retailer $j$, the difference between $IOP_j(n)$ and $IP_j(n)$ is the unfilled demand (by the warehouse) coming from retailer $j$. That is, $B_{0j}(n) = IOP_j(n) - IP_j(n)$. Thus, the total number of warehouse backorders is $B_0(n) = \sum_{j=1}^{N} B_{0j}(n)$.

The corresponding regenerative cycle for retailer $j$ includes periods $t + L_0 + r$, $r = 0, 1, \ldots, M - 1$. Suppose that $IP_j(t + L_0 + r)$ is known, it will determine $IL_j(t + L_{[0,j]} + r)$ as follows:

\[
IL_j(t + L_{[0,j]} + r) = IP_j(t + L_0 + r) - D_j[t + L_0 + r, t + L_{[0,j]} + r].
\]  

(5)

After all $IL_j$ in each period of the regenerative cycle is obtained, the total cost of the system per period is

\[
C(S, T) = \sum_{j=0}^{N} K_j T_j + \frac{1}{M} \sum_{r=0}^{M-1} \mathbb{E}\left[h_0 \max[0, IL_0(t + L_0 + r)] + \sum_{j=1}^{N} h_j IL_j(t + L_{[0,j]} + r)\right] \\
+ (b_j + H_j)[IL_j(t + L_{[0,j]} + r)]^{-},
\]  

(6)

where $[x]^- = \max\{0, -x\}$. The first term in $C(S, T)$ represents the average total fixed cost per period\(^2\) and the rest represents the average inventory holding and backorder cost per period.

\(^2\)For simplicity, we assume that the facilities will place an order at each order epoch. If the fixed order cost is
Our remaining task is to characterize $IP_j(t + L_0 + r)$. First, for retailer $j$, notice that $IP_j(t + L_0 + r)$ depends on $IP_j(t + L_0 + r_j(r))$, i.e.,

$$IP_j(t + L_0 + r) = IP_j(t + L_0 + r_j(r)) - D_j[t + L_0 + r_j(r), t + L_0 + r).$$

(7)

Since retailer $j$ may not be able to receive the full quantity it ordered in period $t + L_0 + r_j(r)$, $IP_j(t + L_0 + r_j(r))$ is not necessary equal to $S_j$. More specifically,

$$IP_j(t + L_0 + r_j(r)) = IOP_j(t + L_0 + r_j(r)) - B_{0j}(t + L_0 + r_j(r)) = S_j - B_{0j}(t + L_0 + r_j(r)).$$

(8)

To characterize $B_{0j}(t + L_0 + r_j(r))$, we first have to characterize $B_0(t + L_0 + r_j(r))$. Define $IL_0^-(n)$ and $IOP_0^-(n)$ the local inventory level and inventory order position for the warehouse at the beginning of period $n$, respectively. By definition,

$$B_0(t + L_0 + r_j(r)) = [IL_0^-(t + L_0 + r_j(r))]^{-}$$

$$= [IOP_0^-(t + r_j(r)) - D_0[t + r_j(r), t + L_0 + r_j(r)])^{-}$$

$$= \left[ IOP_0(t + r_j(r)) - D_0[t + r_j(r), t + L_0 + r_j(r)] - \sum_{i=1}^{N} IOP_i(t + r_j(r)) \right]^{-}$$

$$= \left[ IOP_0(t + r_j(r)) - D_0[t + r_j(r), t + L_0 + r_j(r)] - \sum_{i=1}^{N} S_i \right]^{-}. \quad (9)$$

Furthermore, from (2), we can re-write (9) as follows:

$$B_0(t + L_0 + r_j(r))$$

$$= \left[ S_0 - D_0[t + r_j(r), t + r_j(r)] - D_0[t + r_j(r), t + L_0 + r_j(r)] - \sum_{j=1}^{N} S_j \right]^{-}$$

$$= \left[ S_0 - D_0[t + r_0(r_j(r), t + r_j(r)] - D_0[t + r_j(r), t + L_0 + r_j(r)] \right]^{-}$$

$$= \left[ S_0 - D_0[t + r_0(r_j(r), t + L_0 + r_j(r)] \right]^{-}. \quad (10)$$

where $s_0 = S_0 - \sum_{j=1}^{N} S_j$, the local base-stock level for the warehouse. Since $r_0(r_j(r)) \leq r_j(r)$, the time interval in $D_0$ in (10) must be at least $L_0$ periods. Specifically, $0 \leq r_j(r) - r_0(r_j(r)) < T_0$ where $r_j(r) - r_0(r_j(r)) = 0$ when $T_j$ is an integer multiple of $T_0$. 

incurred only when a facility places an order, the fixed cost term should be modified as $\sum_{j=1}^{N} (K_j Pr(D | T_j) > 0) / T_j$. This addition will not affect the algorithm of finding the optimal base-stock level $S$ for fixed $T$. Following Section 5 of Shang and Zhou (2010), it can be shown that our approach of finding the optimal reorder intervals $T$ in §4 can be carried over to this alternative cost expression.
Under the virtual allocation rule, we can apply binomial disaggregation on $B_0$ to obtain the distribution of $B_{0j}$ because demand arrival at each retailer follows an independent Poisson process. That is, for any period $n$, we have

$$P(B_{0j}(n) = k | B_0(n) = m) = \binom{m}{k} \left( \frac{\lambda_j}{\lambda_0} \right)^k \left( \frac{\lambda_0 - \lambda_j}{\lambda_0} \right)^{m-k}, \; k = 0, 1, ..., m, \quad (11)$$

where $\lambda_0 = \sum_{j=1}^{N} \lambda_j$. We refer the reader to Simon (1981) for a detailed description of the binomial disaggregation technique. Since this conditional probability is independent of time, we shall omit the period index $n$ in the further analysis. Consequently, the distribution of $B_{0j}$ is

$$P(B_{0j}(t + L_0 + r_j(r)) = k) = \sum_{m=k}^{\infty} P(B_0(t + L_0 + r_j(r)) = m) P(B_{0j} = k | B_0 = m)$$

$$= \sum_{m=k}^{\infty} P\left(D_0[L_0 + r_j(r) - r_0(r_j(r))] = s_0 + m\right) \binom{m}{k} \left( \frac{\lambda_j}{\lambda_0} \right)^k \left( \frac{\lambda_0 - \lambda_j}{\lambda_0} \right)^{m-k} \quad (12)$$

Note that the distribution of $B_{0j}$ is independent of time period $t$ but depends on $S$, $T_0$ and $T_j$. After $B_{0j}$ is characterized, we can use (7) and (8) to further obtain the distribution of $IP_j(t + L_0 + r)$ and $IP_j(t + L_0 + r_j(r))$, which also depend on $S$, $T_0$ and $T_j$. Finally, we apply (5) to obtain the distribution of $IL_j$ for each period in the considered regenerative cycle. The average total cost per period can be calculated from (6).

**A Bottom-up Evaluation Scheme**

We provide a bottom-up recursion to simplify the evaluation of $C(S, T)$ and to facilitate the subsequent analysis. This procedure is to first evaluate retailer $j$’s cost by assuming that the retailer has ample supply. Then, we evaluate the echelon cost of the warehouse, which is equivalent to the total system cost. More specifically, let $M_x(y)$ be an operator that returns the remainder of $y$ divided by $x$, where $x$ is a positive integer and $y$ is a nonnegative integer. Thus, $M_{T_j}(r)$ can be viewed as the number of periods between the order period $r_j(r)$ and the current period $r$ for facility $j$.

Consider a given warehouse regenerative epoch $t$ and retailer regenerative epoch $t + L_0$. For a given inventory position $IP_j(t + L_0 + r_j(r)) = y$ at the order epoch, $j = 1, ..., N$, let $G_j(y, T_j, r)$ denote the expected inventory holding and backorder cost at retailer $j$ in the $r$th period within a regenerative cycle. We have

$$G_j(y, T_j, r) = \mathbb{E}[h_j(IL_j(t + L_{[0,j]} + r) + (b_j + H_j)(IL_j(t + L_{[0,j]} + r))^{-}]$$
\[ \begin{align*}
= & \mathbb{E}[h_j(IP_j(t + L_0 + r_j(r)) - D_j[t + L_0 + r_j(r), t + L_{[0,j]} + r] ) \\
& + (b_j + H_j)(IP_j(t + L_0 + r_j(r)) - D_j[t + L_0 + r_j(r), t + L_{[0,j]} + r])] \\
= & \mathbb{E}[h_j(y - D_j[L_j + MT_j(r)]) + (b_j + H_j)(y - D_j[L_j + MT_j(r)])], \quad (13)
\end{align*} \]

where \( D_j[\tau] \) and \( D_j[\tau + 1] \) denote the total demand in \( \tau \) and \( \tau + 1 \) periods, respectively.

Similarly, we define \( G_0(S, T, r) \) the expected system-wide inventory cost in the \( r \)-th period within a regenerative cycle. That is,

\[ \begin{align*}
G_0(S, T, r) &= \mathbb{E} \left[ h_0(IL_0(t + L_0 + r)) + \sum_{j=1}^{N} G_j(IP_j(t + L_0 + r_j(r)), T_j,r) \right] \\
&= \mathbb{E} \left[ h_0(IP_0(r_0(r)) - D_0[t + r_0(r), t + L_0 + r]) + \sum_{j=1}^{N} G_j(IP_j(t + L_0 + r_j(r)), T_j, r) \right] \\
&= \mathbb{E} \left[ h_0(S_0 - D_0[L_0 + MT_0(r)]) + \sum_{j=1}^{N} G_j(S_j - B_0j[L_0 + r_j(r)], T_j, r) \right], \quad (14)
\end{align*} \]

where \( B_0j(L_0 + r_j(r)) \) can be found from (12).

Let

\[ G(S, T) \overset{def}{=} \frac{1}{M} \sum_{r=0}^{M-1} G_0(S, T, r). \]

**Proposition 1** For given echelon base-stock policies with parameters \((S, T)\), the average total cost per period is

\[ C(S, T) = \sum_{j=0}^{N} \frac{K_j}{T_j} + G(S, T), \]

where \( \sum_{j=0}^{N} (K_j/T_j) \) is the average total fixed order cost per period and \( G(S, T) \) is the average inventory holding and backorder cost per period.

Proposition 1 can be easily verified because the total cost obtained by using (13) and (14) is equivalent to the cost in (6).

### 4 Optimization

This section discusses how to obtain the optimal \((S, T)\) policy. When the reorder intervals are fixed, we show that Axsäter’s (1990) algorithm can be revised to obtain the optimal echelon base-stock
levels (see Appendix A). As a result, our focus is to show how to optimize the reorder intervals. We suggest a complete enumeration to find the optimal reorder intervals. To facilitate the search, we construct an upper and a lower bound for the optimal reorder interval $T^*_j$, $j = 0, \ldots, N$. The subsequent sections present the required results to obtain these solution bounds. In §4.1, we decompose the total system cost into each facility. In §4.2, we construct a lower bound function to the allocated facility cost. In §4.3, we derive the solution bounds by using these lower bound functions.

### 4.1 Decomposition of the System Cost

This section shows how to decompose the total system cost into each facility. Our starting point is the parameter-allocation scheme suggested by Chen and Zheng (1994), who studied the distribution system with the review period equal to one for all facilities. Since our model is more general, we have to extend their approach to construct the lower bound.

Following Chen and Zheng, imagine that the final product carried at retailer $j$, $j = 1, \ldots, N$ is composed of two components, 0 and $j$. The warehouse delivers component 0 to the retailer $j$, where component $j$ is added to component 0 to produce the final product. Figure 1 illustrates this idea. Here, the triangle represents component 0 and the rectangle represents component $j$. We allocate the holding cost and backorder cost of the final product to each component. More specifically, we use superscript to represent component index and subscript to represent the final product index. For $j = 1, \ldots, N$,

\[ h^0_j + h^j_j = h_j, \text{ and } b^0_j + b^j_j = b_j. \]

To facilitate the subsequent analysis, we define the cost functions of components 0 and $j$ at retailer $j$ as follows: For a given $IP_j(t + L_0 + r_j(r)) = y$,

\[ g^0_j(y, T_j, r) = E[h^0_j(y - D_j[L_j + M_{T_j}(r)]) + (b^0_j + h_0)(y - D_j[L_j + M_{T_j}(r)])], \]

\[ g^j_j(y, T_j, r) = E[h^j_j(y - D_j[L_j + M_{T_j}(r)]) + (b^j_j + h_j)(y - D_j[L_j + M_{T_j}(r)])]. \]

Clearly,

\[ g^0_j(y, T_j, r) + g^j_j(y, T_j, r) = G_j(y, T_j, r). \]

Thus, the expected inventory holding and backorder cost per period for any given set of the $(S, T)$ policies can be expressed as follows:

\[ G(S, T) = \frac{1}{M} \left( \sum_{r=0}^{M-1} E \left[ (h_0(IL_0(t + L_0 + r)) + \sum_{j=1}^{N} \left( g^0_j(IP_j(t + L_0 + r_j(r)), T_j, r) \right) \right] \right) \]
One can view the first two terms on the right-hand side of Equation (15) is the inventory holding and backorder cost for the distribution system of component 0. The last term $g_j$ function is the inventory holding and backorder cost for the single-stage system with component $j$, $j = 1, ..., N$.

In Chen and Zheng (1994), the authors can solve $N + 1$ separable systems (one distribution system and $N$ single-stage systems) after the above cost allocation scheme because the reorder intervals (review periods) are implicitly assumed to be one. However, our problem is more complicated because for each fixed $r$ in the regenerative cycle, the cost function of component 0 is a function of $T_0$, and the cost function of component $j$ is a function of $T_0$ and $T_j$. Thus, we cannot fully decouple the system as Chen and Zheng did.

Below we derive a new result that allocates the total system cost into each facility. Notice that

$$g_j^0(IP_j(t + L_0 + r_j(r)), T_j, r) = g_j^0(IP_j(t + L_0 + r), T_j, 0).$$

Thus, Equation (15) can be re-written as

$$G(S, T) = \frac{1}{M} \left\{ \sum_{r=0}^{M-1} E[h_0(IL_0(t + L_0 + r)) + \sum_{j=1}^{N} \left( g_j^0(IP_j(t + L_0 + r), T_j, 0) + g_j^0(IP_j(t + L_0 + r_j(r)), T_j, r) \right) \right\}.$$

From (7) and (8), recall that $IP_j(t + L_0 + r)$ is a function of $S$, $T_0$ and $T_j$. Let us further decompose the $g_j^0(\cdot, T_j, 0)$ function, i.e., the cost of component 0 incurred at retailer $j$. For notational simplicity,
without confusion, we omit the last two arguments in the \( g_j^0 \) function, i.e., \( g_j^0(\cdot, T_j, 0) = g_j^0(\cdot) \). We define the following functions to decouple the \( g_j^0(\cdot) \) function.

\[
g_j^0(y) = \mathbb{E}[h_j^0(y - D[L_j])] + (b_j^0 + h_j^0)(y - D[L_j]),
\]

\[
S_j = \arg \min_y g_j^0(y).
\]

We define

\[
g_{jj}^0(y) = \begin{cases} g_j^0(S_j), & \text{if } y \leq S_j, \\ g_j^0(y), & \text{otherwise}, \end{cases}
\]

and

\[
g_{j0}^0(y) = g_j^0(y) - g_{jj}^0(y).
\]

Thus, \( g_j^0(y) = g_{j0}^0(y) + g_{jj}^0(y) \) for all \( y \). Note that \( g_{jj}^0(y) \) and \( g_{j0}^0(y) \) are both convex functions: \( g_{jj}^0(y) \) is constant for \( y \leq S_j \) and convex increasing for \( y > S_j \) whereas \( g_{j0}^0(y) \) is convex decreasing for \( y \leq S_j \) and zero for \( y > S_j \). The \( g_{j0}^0(y) \) function is the so-called induced penalty function in the multi-echelon literature (Clark and Scarf 1960).

With this decomposition scheme, Equation (16) can be further expressed as

\[
G(S, T) = \frac{1}{M} \left\{ \sum_{r=0}^{M-1} \mathbb{E}[h_0(IL_0(t + L_0 + r))] + \sum_{j=1}^{N} (g_{j0}^0(IP_j(t + L_0 + r)) + g_{jj}^0(IP_j(t + L_0 + r))) + g_j^0(IP_j(t + L_0 + rj(r)), T_j, r)) \right\} \quad (18)
\]

\[
= g_0(S, T) + \sum_{j=1}^{N} g_j(S, T), \quad (19)
\]

where

\[
g_0(S, T) = \frac{1}{M} \left\{ \sum_{r=0}^{M-1} \mathbb{E}[h_0(IL_0(t + L_0 + r))] + \sum_{j=1}^{N} (g_{j0}^0(IP_j(t + L_0 + r))) \right\}, \quad (20)
\]

\[
g_j(S, T) = \frac{1}{M} \left\{ \sum_{r=0}^{M-1} \mathbb{E}[g_{jj}^0(IP_j(t + L_0 + r)) + g_j^0(IP_j(t + L_0 + rj(r)), T_j, r)] \right\}. \quad (21)
\]

We define \( g_0(S, T) \) as the allocated warehouse cost and \( g_j(S, T) \) as the allocated retailer \( j \)'s cost.

This completes our decomposition scheme.
4.2 Lower Bound on the Allocated Facility Cost

This section develops a cost lower bound to \( g_j(S,T), j = 0, 1, \ldots, N \). We first consider the allocated warehouse cost.

\[
g_0(S,T) = \frac{1}{M} \sum_{r=0}^{M-1} \mathbb{E} \left[ h_0 I L_0(t + L_0 + r) + \sum_{j=1}^{N} \left( g^0_{j0}(IP_j(t + L_0 + r)) \right) \right]
\]

\[
\geq \frac{1}{M} \sum_{r=0}^{M-1} \mathbb{E} \left[ h_0 I L_0(t + L_0 + r) + \min_{\sum_j y_j(r) \leq \sum_j IP_j(t + L_0 + r)} \sum_{j=1}^{N} g^0_{j0}(y_j(r)) \right]
\]

\[
\geq \frac{1}{M} \sum_{r=0}^{M-1} \mathbb{E} \left[ h_0 I L_0(t + L_0 + r) + \min_{\sum_j y_j(r) \leq g_0^j} \sum_{j=1}^{N} g^0_{j0}(y_j(r)) \right]
\]

\[
= \frac{1}{T_0} \sum_{r=0}^{T_0-1} \mathbb{E} \left[ h_0 I L_0(t + L_0 + r) + \min_{\sum_j y_j(r) \leq g_0^j} \sum_{j=1}^{N} g^0_{j0}(y_j(r)) \right]
\]

\[
= g_0(S_0, T_0).
\]

The first inequality holds because we re-optimize the inventory allocation between the retailers. (This corresponds to the so-called balance assumption in the literature; see, e.g., Clark and Scarf (1960).) The second inequality holds because \( IL_0(t + L_0 + r) \geq \sum_j IP_j(t + L_0 + r) \). The second equality holds because \( IL_0(t + L_0 + r) \) and \( IL_0(t + L_0 + r) \) are cyclic with cycle length \( T_0 \).

The following convexity result leads to the lower bound function.

**Proposition 2** For fixed \( T_0 \), \( g_0(S_0, T_0) \) is convex in \( S_0 \).

Let \( S_0(T_0) = \text{argmin}_{S_0} g_0(S_0, T_0) \), and \( g_0(T_0) = g_0(S_0(T_0), T_0) \). We have \( g_0(S,T) \geq g_0(T_0) \). The function \( g_0(T_0) \) is the lower bound to the allocated warehouse cost.

We next develop a lower bound to the allocated retailer cost.

**Proposition 3** For a given \( T_j \), \( \sum_{r=0}^{T_j-1} g^j_{j0}(y, T_j, r) \) is convex in \( y \).

Let \( S_j(T_j) = \text{arg min}_{y} (1/T_j) \sum_{r=0}^{T_j-1} g^j_{j0}(y, T_j, r) \). We have

\[
g_j(S,T) = \frac{1}{M} \left\{ \sum_{r=0}^{M-1} \mathbb{E} \left[ g^0_{j0}(IP_j(t + L_0 + r)) + g^j_{j0}(IP_j(t + L_0 + r_j(T_j)), T_j, r) \right] \right\}
\]

\[
\geq \frac{1}{T_j} \left\{ \sum_{r=0}^{T_j-1} \mathbb{E} \left[ g^0_{j0}(S_j) + g^j_{j0}(IP_j(t + L_0 + r_j(T_j)), T_j, r) \right] \right\}
\]

\[
\geq \frac{1}{T_j} \left\{ \sum_{r=0}^{T_j-1} \mathbb{E} \left[ g^0_{j0}(S_j) + g^j_{j0}(S_j(T_j), T_j, r) \right] \right\}
\]

\[
= g_j(T_j).
\]
The first inequality holds because the first term \( g_{0j}(S_j) \) is a constant, which is the smallest component 0 cost incurred at retailer \( j \) derived in (17) and because \( IP_j(t + L_0 + r_j(r)) \) is cyclic with cycle length \( T_j \). The function \( g_j(T_j) \) is the lower bound to the allocated retailer cost.

Define \( \zeta_j(T_j) = (K_j/T_j) + g_j(T_j) \) for \( j = 0, 1, ..., N \). From above derivation, we summarize the lower bound result.

**Theorem 4** \( C(S, T) \geq \sum_{j=0}^N \zeta_j(T_j) \).

Note that the lower bound to the system cost is a sum of \( N + 1 \) separable functions. This result establishes the optimization procedure discussed in the next section.

### 4.3 Bounds on the Optimal Reorder Intervals

We can use Theorem 4 to derive the solution bounds for \( T_j^* \), \( j = 0, 1, ..., N \). Suppose that we can find the minimizer of \( \zeta_j(T_j) \) and the corresponding minimum cost, \( \xi_j \), for \( j = 0, 1, ..., N \). Let \( C^h \) be the cost obtained from any heuristic policy. (In §5.2, we will suggest a simple and effective heuristic.) Then,

\[
C^h \geq C(S^*, T^*) \geq \sum_{j=0}^N \zeta_j(T_j^*) \geq \zeta_j(T_j^*) + \sum_{i \neq j} \zeta_i.
\]

An upper bound \( \bar{T}_j \) and a lower bound \( \tilde{T}_j \) to \( T_j^* \) can be obtained by solving the following inequalities:

\[
\bar{T}_j = \max \left\{ T_j \big| \ z_j(T_j) \leq C^h - \sum_{i \neq j} \zeta_i \right\}, \quad \tilde{T}_j = \min \left\{ T_j \big| \ z_j(T_j) \leq C^h - \sum_{i \neq j} \zeta_i \right\}.
\] (22)

The challenge of using this approach is that, unfortunately, \( \zeta_j(T_j) \) is not quasi-convex in \( T_j \) for all \( j \) (Liu and Song 2010). Thus, it is very difficult to obtain the minimizer and the minimum value \( \zeta_j \). To overcome this difficulty, we construct a quasi-convex function that bounds the \( \zeta_j(T_j) \) function.

Define

\[
\tilde{c}_0(T_0) = \frac{K_0}{T_0} + \min_y \left\{ \frac{1}{T_0} \sum_{r=0}^{T_0-1} \left[ h_0(y - (L_0 + r + 1)) + \min_{\sum_j y_j(r) \leq y - (L_0 + r)} \sum_{j=1}^N g_{0j}(y_j(r)) \right] \right\}.
\]

For \( j = 1, ..., N \),

\[
\tilde{c}_j(T_j) = \frac{K_j}{T_j} + \min_y \left\{ \frac{1}{T_j} \sum_{r=0}^{T_j-1} h_{jj}(y - \lambda_j(L_j + r + 1)) + (h_{jj} + b_{jj})(y - \lambda_j(L_j + r + 1)) \right\} + g_{0j}(S_j).
\]

**Proposition 5** For \( j = 0, 1, ..., N \), (1) \( \zeta_j(T_j) \geq \tilde{c}_j(T_j) \) for all \( T_j \); (2) \( \tilde{c}_j(T_j) \) is quasi-convex in \( T_j \).
Proposition 5(1) states that the lower bound function $\tilde{c}_j(T_j)$ is bounded below by the function $\check{c}_j(T_j)$, which is obtained from Jensen’s inequality. Proposition 5(2) shows that this lower bound function $\check{c}_j(T_j)$ is quasi-convex in $T_j$. Figure 2 illustrates these functions.

We can use $\check{c}_j(T_j)$ to search for the minimum value of $\xi_j(T_j)$, or $\xi_j$. Specifically, we evaluate both $\xi_j(T_j)$ and $\tilde{c}_j(T_j)$ sequentially from $T_j = 1, 2, \ldots$ until $T_j$ reaches a value, say $s$, such that $\check{c}_j(s) \geq \min_{t \in \{1, 2, \ldots, s\}} \xi_j(t)$. From Proposition 5(1), we have $\xi_j(s) \geq \check{c}_j(s) \geq \min_{t \in \{1, 2, \ldots, s\}} \xi_j(t)$, which implies that we cannot find a reorder interval greater than $s$ such that the resulting cost is smaller than $\min_{t \in \{1, 2, \ldots, s\}} \xi_j(t)$. Thus, the minimum value will be $\xi_j = \min_{t \in \{1, 2, \ldots, s\}} \xi_j(t)$.

After we obtain all $\xi_j$, we can use $\check{c}_j(T_j)$ to find the solution bound $T_j$ and $\bar{T}_j$ through (22). We start with an empty set $\mathcal{S}$. We evaluate both $\xi_j(T_j)$ and $\check{c}_j(T_j)$ sequentially from $T_j = 1, 2, \ldots$, etc. If $\xi_j(T_j) \leq C^h - \sum_{i \neq j} \xi_i$, we add this $T_j$ to the set $\mathcal{S}$. Continue this procedure until for some $T_j = s$ such that both conditions $\check{c}_j(s) \geq \check{c}_j(s - 1)$ and $\check{c}_j(s) > C^h - \sum_{i \neq j} \xi_i$ are satisfied. In this case, we do not need to consider any $T_j > s$ because $\xi_j(T_j) \geq \check{c}_j(T_j) \geq C^h - \sum_{i \neq j} \xi_i$. Thus, $T_j$ ($\bar{T}_j$) is the minimum (maximum) value in $\mathcal{S}$.

Notice that $\xi_j(T_j)$ is constructed by a combination of $(h_{j0}, h_{jj})$ and $(b_{j0}, b_{jj})$ with $h_{j0} + h_{jj} = h_j$ and $b_{j0} + b_{jj} = b_j$. In other words, by splitting $h_j$ and $b_j$ into different combinations, we can generate different solution bounds from the above procedure. Our final upper bound $\bar{T}_j$ (lower bound $T_j$) will be the minimum (maximum) value of all upper (lower) bounds generated from these different combinations. In §5, we shall report our choices of splitting $h_j$ and $b_j$.

Finally, to find the optimal reorder intervals, we need to enumerate the policies with $T_j \in [T_j, \bar{T}_j]$ and their corresponding optimal base-stock levels. Below we prove a property that can reduce the
number of feasible solutions.

**Proposition 6** If $T_j^* / T_i^*$ or $T_i^* / T_j^*$ $\in \mathbb{N}$ for any $i, j \in \{0, 1, ..., N\}$, then $T_0^* \geq \min\{T_1^*, ..., T_N^*\}$.

Proposition 6 states that if the optimal reorder intervals form integer-ratio relationships, the reorder interval of the warehouse must be no less than the minimum of those for the retailers. This is intuitive because if the above condition does not hold, the warehouse will carry inventory that cannot immediately be shipped to the retailer, causing unnecessary inventory holding costs. Thus, the system cost (both holding and fixed order costs) can always be improved by increasing $T_0$. This condition is more restrictive than that of the integer-ratio policy. Nonetheless, this property can help us avoid evaluating non-optimal policies. For example, we do not need to evaluate the policy with $(T_0, T_1, T_2) = (2, 4, 8)$. We notice that this property corresponds to the “last-minute” property, i.e., the warehouse orders only when at least one retailer orders, established by Schwarz (1973) for the deterministic model.

## 5 Numerical Study and Heuristic

In §5.1, we examine the optimal reorder intervals and the POT solution. We aim to observe properties of the optimal solution and identify conditions under which the deterministic POT solution performs less effectively. These observations are useful to suggest an effective heuristic for the stochastic OWMR model, which is presented in in §5.2. In §5.3, we report the performance of the suggested heuristic. In §5.4, we report the quality of the solution bounds under the proposed heuristic. A complete numerical result of optimal solutions as well as the solution bounds is available in Appendix C in the online companion.

### 5.1 The Optimal and POT Policies

Since there are many parameters in the system, we first conduct a pre-test to exclude the parameters that do not have a direct impact on the reorder intervals. This pre-test includes 128 two-retailer instances. Our finding indicates that the optimal reorder intervals are insensitive to the change of lead times. This is intuitive as the lead time should have a direct impact on the optimal base-stock level, which is observed in this pre-test. Thus, we fix lead times equal to one for the subsequent studies.

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3These 128 instances are generated from the following parameters: We fix the retailer 1’s parameters and vary the parameters for the warehouse and retailer 2. More specifically, the parameters for retailer 1 are $K_1 = h_1 = L_1 = \lambda_1 = 1$ and $b_1 = 3$. For retailer 2, we set $h_2 = 1$. The rest of the parameters for the warehouse and the retailer 2 are chosen from the following sets: $h_0 \in \{1, 2\}$, $K_0, K_2 \in \{0.25, 16\}$, $L_0, L_2 \in \{1, 3\}$, $b_2 \in \{3, 18\}$, $\lambda_2 \in \{0.5, 1\}$. 

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The test bed includes 432 one-warehouse-two-retailer systems, which are generated from the following parameter sets.

\[ L_0 = L_1 = L_2 = 1, \quad K_0, K_1, K_2 \in \{2, 8, 32\}, \quad h_1 = 1, h_0, h_2 \in \{0.5, 2\}, \]
\[ b_1 = 25, b_2 \in \{25, 50\}, \quad \lambda_1 = 3, \lambda_2 \in \{3, 6\}. \]

In these examples, we search for the optimal reorder intervals as well as the POT solution.

Roundy (1985, pp. 1419-1420) provides an algorithm to obtain a power-of-two (POT) solution that guarantees 94\% effectiveness when the base period is an integer. The algorithm includes two steps: clustering and minimization. In the clustering step, the algorithm will cluster the warehouse and retailers into three groups: G, E, and L. The G (E, L, respectively) set contains the retailers whose reorder intervals are larger than (equal to, less than, respectively ) the warehouse’s. In the minimization step, a POT solution will be generated for each facility in the cluster.

We use the Poisson demand rate as the demand rate for the deterministic model, and apply Roundy’s (1985) algorithm to generate power-of-two reorder intervals, denoted as \( T_{d} = (T_{d0}, T_{d1}, ..., T_{dN}) \), and find their corresponding optimal base-stock levels by using the algorithm shown in Appendix A. Let the resulting cost be \( C(T_d) \). To find the optimal \((S, T)\) policy, we first optimize the base-stock levels \( S \) for fixed reorder intervals \( T \) and then search for the optimal \( T \). Denote \((S^*, T^*)\) and \( C(S^*, T^*) \) the optimal solution and cost, respectively. Define the percentage error as

\[ \frac{C(T_d) - C(S^*, T^*)}{C(S^*, T^*)} \times 100\%. \]

The average percentage error is 5.35\% with maximum error of 36.64\%. Although the overall performance is reasonably effective, to our surprise, the POT solution could be very inefficient. More specifically, there are 80 instances (18.5\% of the 432 instances) whose percentage errors are more than 10\%. This shows that using Roundy’s POT solution directly in the stochastic model may result in an unsatisfactory performance. Below we provide several observations on the optimal reorder intervals and the POT solution.

(1) Although we do not restrict the inventory policy to satisfy the “last minute property”, we find that the optimal reorder intervals always satisfy \( T_0^* \geq \min_{j \in \{1,...,N\}} T_j^* \). We also find that in most scenarios the optimal policy satisfies the integer-ratio property, but there are exceptions. For example, when \((K_0, K_1, K_2) = (8, 32, 8), (h_0, h_1, h_2) = (0.5, 1, 2), (b_1, b_2) = (25, 50), (\lambda_1, \lambda_2) = (3, 6)\), the optimal reorder intervals are \((T^*_0, T^*_1, T^*_2) = (2, 3, 1)\). Nevertheless, an integer-ratio policy is still a good candidate when designing a heuristic for the reorder
intervals. Among the 432 instances we tested, there are only 7 instances whose optimal solutions are not an integer-ratio policy.

(2) We find that the cost ratio $K_j/(h_j \lambda_j)$ is a key driver that affects the effectiveness of the POT solution. More specifically, when $K_0/(h_0 \lambda_0)$ is significantly smaller than one of $K_j/(h_j \lambda_j)$, $j = 1, 2$, the POT solution tends to perform worse. For example, for the 98 instances with

$$\frac{\max\{K_1/(h_1 \lambda_1), K_2/(h_2 \lambda_2)\}}{K_0/(h_0 \lambda_0)} > 20,$$

the average percentage error is 10.85% with a maximum error of 36.64%.

Under the condition of $K_0/(h_0 \lambda_0) < K_j/(h_j \lambda_j)$ for any $j \in \{1, 2\}$, it is likely that the POT algorithm will cluster the warehouse and the retailer with the smaller $K_j/(h_j \lambda_j)$, say, retailer 1, into the $E$ cluster, leaving retailer 2 in the $G$ cluster. Consequently, $T^d_0 = T^d_1 < T^d_2$.

However, we observe that the optimal solution $T^*_0$, $T^*_1$, and $T^*_2$ tend to be the same in this case. One reason that leads to this difference is the order (demand) pooling effect. In the deterministic model, since there is no demand variability, the retailers are clustered only based on the cost ratios. However, with stochastic demand, the warehouse can choose a larger reorder interval to consolidate the retailers’ orders to take advantage of order/demand pooling. Thus, the optimal reorder intervals tend to be the same in the stochastic demand model even when $K_j/(h_j \lambda_j)$ are quite different among the retailers.

(3) It is conceivable that when the backorder cost $b_j$ increases, retailer $j$’s optimal reorder interval $T^*_j$ will decrease. This is because a shorter reorder interval makes the retailer more responsive to the demand, which will reduce the backorders. Interestingly, we find that a larger backorder cost at one retailer may also shorten the optimal reorder intervals of the other retailers. For example, in the case with $(K_0, K_1, K_2) = (8, 8, 32), (h_0, h_1, h_2) = (0.5, 1, 0.5), b_1 = 25$, when $b_2$ increases from 25 to 50, the optimal reorder intervals are changed from $(T^*_0, T^*_1, T^*_2) = (4, 4, 4)$ to $(3, 3, 3)$. This phenomenon is due to the benefit of order coordination. From the system’s perspective, although reducing $T^*_1$ from 4 to 3 may increase the fixed ordering cost, this additional cost is offset by the benefit received from order coordination between these two retailers (due to demand consolidation). We find that the demand rate has a similar effect: When the demand rate at a retailer increases, the optimal reorder intervals of all retailers become smaller.
(4) Overall, compared with the optimal base-stock levels, the optimal reorder intervals are relatively insensitive to the change of the system parameters. In other words, if a supply chain manager chooses an effective set of reorder intervals, the system can achieve a high efficiency by adjusting the base-stock levels. From the management perspective, the supply chain manager can view the reorder interval decision as a medium-term tactical planning whereas the inventory decision as a short-term operational planning that can be changed more frequently according to the system states.

5.2 The Stochastic OWMR Heuristic

Based on the above observations, we conclude that an integer-ratio policy is a good candidate for a heuristic policy and that the POT clustering step based on the ratio $K_j/(h_j \lambda_j)$ is an effective method for clustering facilities, except when $K_0/(h_0 \lambda_0) < K_j/(h_j \lambda_j)$. We should also utilize the lower bound cost function to incorporate the stochastic demand effect.

We suggest the following two-step algorithm to generate a heuristic solution. The first step is to cluster facilities according to two clustering schemes. The first scheme is to follow the POT algorithm where we cluster the retailers into one of the three sets, $G, E,$ and $L$. The set $G$ ($E, L,$ respectively) contains the facilities whose reorder intervals are larger than (equal to, less than, respectively) the warehouse’s. The warehouse by default is assigned to the set $E$. The second scheme is to simply cluster all facilities into the set $E$ (that is, all facilities use the same reorder interval). This takes into account the benefit of pooling when all facilities use the same reorder interval. Thus, after the first step, we have generated two clustering outputs.

The second step is to generate an integer-ratio policy for each clustering output by solving a series of single-stage problems. More specifically, assume that there are $|E|$ facilities in the set $E$. We solve the problems $\arg \min_T \{ \sum_{j \in E} c_j(T) \}$ and $\arg \min_{T, j} T_j c_j(T_j)$ for facility $j \notin E$. In other words, let $i$ represent the number of problems we need to solve in the second step. Then, $i = (N + 1) - (|E| + 1) = N - |E| + 2$. After we solve these $i$ problems, we sort the resulting solutions from the smallest to the largest: $T_{(1)} \leq T_{(2)} \leq \ldots \leq T_{(i)}$. Here, $(m)$ represents the set that includes facility $j$ with the solution $T_{(m)}$, $m = 1, 2, \ldots, i$. Now, we generate the integer-ratio

\[\text{Notice that } \sum_{j \in E} c_j(T) \text{ and } c_j(T) \text{ for } j \notin E \text{ may not be convex in } T. \text{ In our numerical study, these cost functions tend to have a quasi-convex shape and we therefore set the solution to be the first minimizer of these cost functions.}\]
solution as follows: Let \( \tilde{T}_{(1)} = T_{(1)} \). For \( m = 2, \ldots, i \), we set

\[
\tilde{T}_{(m)} = \begin{cases}
\arg\min_{T = \tilde{T}_{(m-1)}} \left\{ \sum_{j \in (m)} \xi_j(T) \right\}, & \text{for } (m) = E, \\
\arg\min_{T = \tilde{T}_{(m-1)}} \left\{ \xi_j(T) \right\}, & \text{for } j \in (m) \neq E,
\end{cases}
\]

where \( q \) is a positive integer such that \( q \tilde{T}_{(m-1)} \) is the first minimizer for the considered cost function. Then, we can get an integer-ratio solution \( \tilde{T}_j = \tilde{T}_{(m)} \) for \( j \in (m), m = 1, \ldots, i \). Since we have two clustering results, our final heuristic solution will be the one with a smaller cost.

### 5.3 Effectiveness of the Heuristic

We test the effectiveness of the heuristic solution \( \tilde{T} = (\tilde{T}_0, \ldots, \tilde{T}_N) \) and its corresponding best base-stock levels \( \tilde{S} = (\tilde{S}_0, \ldots, \tilde{S}_N) \) for the above 432 two-retailer systems. We split \( h_j \) and \( b_j \) with \( h_j^0 = 0, h_j^1 = h_j, b_j^0 = 0.5b_j, b_j^1 = 0.5b_j \) to generate the lower bound functions \( \xi_j(T_j), j = 0, \ldots, N \).

We define the percentage error of the heuristic as

\[
\frac{C(\tilde{S}, \tilde{T}) - C(S^*, T^*)}{C(S^*, T^*)} \times 100\%.
\]

The average percentage error of our heuristic is 0.34\% with a maximum error of 5.95\%. The heuristic solution significantly outperforms the POT solution. Remarkably, 88\% of the instances have a percentage error less than 1\%. Figure 3 shows the distribution of the percentage error. We observe that the heuristic performs less effectively when \( K_0/(\lambda_0h_0) \) is close to one of \( K_j/(\lambda_jh_j), j = 1, 2 \) and is significantly smaller than the other one. For example, the instance with the largest percentage error has the following parameters: \( K_0 = 2, K_1 = 32, K_2 = 2, h_0 = 0.5, h_1 = 1, h_2 = 0.5, b_1 = 25, b_2 = 25, \lambda_1 = 3, \lambda_2 = 6 \). The optimal solution is \( (T_0^*, T_1^*, T_2^*) = (1, 3, 1) \) and the heuristic solution is \( (\tilde{T}_0, \tilde{T}_1, \tilde{T}_2) = (1, 5, 1) \). In such case, our heuristic tends to cluster the warehouse and retailer 2 into the set \( E \), leaving retailer 1 to the set \( G \). This behavior is the same as the optimal solution. However, the clustering result of our heuristic does not enjoy the benefit of pooling, making the resulting heuristic reorder interval \( \tilde{T}_1 \) larger than \( T_1^* \).

To further investigate the robustness of the heuristic, we test larger systems with \( N = 4 \) and \( N = 8 \). We choose the parameters as follows:

\[
L_0 = L_1 = L_2 = 1, \quad K_0 = 32, K_1, K_2 \in \{2, 8\}, \quad h_1 = 1, h_0, h_2 \in \{0.5, 2\},
\]

\[
b_1 = 25, b_2 \in \{25, 50\}, \quad \lambda_1 = 3, \lambda_2 \in \{3, 6\},
\]

and we keep retailer 2 to retailer \( N \) identical. The total number of tested instances for both the four- and eight-retailer system is 64. The average (maximum) heuristic errors for \( N = 4 \) and \( N = 8 \)
are 0.52% (4.25%) and 0.41% (4.92%). The corresponding error distribution is also shown in Figure 3. It is clear that the performance of our heuristic does not deteriorate when \( N \) increases.

![Percentage error distributions for \( N = 2, 4, 8 \).](image)

Our heuristic can generate a solution very efficiently: The average running time for one instance of two-, four- and eight-retailer system is 99.7 seconds, 176.4 seconds and 588.8 seconds respectively, with a 3.0GHz CPU. The running time includes finding the heuristic reorder intervals and the corresponding optimal base stock levels as well as evaluating the heuristic cost. This is significantly shorter than the average running time for solving the exact optimal solution\(^5\).

### 5.4 Effectiveness of the Optimal Solution Bounds

This section examines the quality of the solution bounds \( T_j^* \) and \( T_j \) obtained in \( \S 4 \) based on the 432 two-retailer instances. We split \( h_j \) and \( b_j \) with \( h_j^0 = 0, h_j^1 = h_j \) and \( b_j^0 = \alpha b_j, b_j^1 = (1 - \alpha) b_j \), where \( \alpha \in \{0.2, 0.4, 0.6, 0.8\} \). For each \( \alpha \), we can get an upper bound and a lower bound for optimal \( T_j^* \). Then, we choose the maximum of the lower bounds and the minimum of the upper bounds as the final solution bounds.

We denote \( \Delta_j = T_j^* - T_j \) as the gap for the solution bounds. The average gap for each facility among the 432 instances is shown in the following table. We find the bounds are quite tight on average; in some cases we have \( T_j = T_j^* \), which directly gives the optimal \( T_j^* \).

To further analyze the impact of system parameters on the effectiveness of bounds, we conduct a parametric analysis on the system parameters. Table 2 summaries the result.

We observe that \( \Delta_j \) tends to be increasing in \( K_j \) and decreasing in \( h_j \) and \( \lambda_j \). For example, when \( K_2 \) increases from 8 to 32, \( \Delta_2 \) increases from 4.9 to 8.8, and when \( \lambda_2 \) increases from 3 to 6, \( \Delta_2 \)

---

\(^5\)The average running time for finding optimal solution for \( N = 2, 4 \) and 8 is over 0.5 hours, 1.5 hours and 5 hours, respectively.
Table 1: Summary for the gap of solution bounds for $N = 2$

<table>
<thead>
<tr>
<th></th>
<th>$\Delta_0$</th>
<th>$\Delta_1$</th>
<th>$\Delta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>max</td>
<td>11</td>
<td>21</td>
<td>28</td>
</tr>
<tr>
<td>min</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>average</td>
<td>3.2</td>
<td>6.8</td>
<td>5.7</td>
</tr>
</tbody>
</table>

Table 2: Impact of parameters on the gap of the solution bounds

<table>
<thead>
<tr>
<th>$K_0$</th>
<th>$(\Delta_0, \Delta_1, \Delta_2)$</th>
<th>$K_1$</th>
<th>$(\Delta_0, \Delta_1, \Delta_2)$</th>
<th>$K_2$</th>
<th>$(\Delta_0, \Delta_1, \Delta_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(2.4, 7.3, 6.4)</td>
<td>2</td>
<td>(2.8, 3.9, 5.2)</td>
<td>2</td>
<td>(3.1, 6.6, 3.4)</td>
</tr>
<tr>
<td>8</td>
<td>(3.0, 6.4, 5.5)</td>
<td>8</td>
<td>(2.7, 5.7, 5.1)</td>
<td>8</td>
<td>(2.9, 6.2, 4.9)</td>
</tr>
<tr>
<td>32</td>
<td>(4.1, 6.5, 5.3)</td>
<td>32</td>
<td>(4.0, 10.7, 6.8)</td>
<td>32</td>
<td>(3.5, 7.5, 8.8)</td>
</tr>
<tr>
<td>$h_0$</td>
<td>$(\Delta_0, \Delta_1, \Delta_2)$</td>
<td>$h_2$</td>
<td>$(\Delta_0, \Delta_1, \Delta_2)$</td>
<td>$\lambda_2$</td>
<td>$(\Delta_0, \Delta_1, \Delta_2)$</td>
</tr>
<tr>
<td>0.5</td>
<td>(4.2, 4.6, 3.9)</td>
<td>0.5</td>
<td>(3.0, 6.8, 9.0)</td>
<td>3</td>
<td>(3.7, 6.3, 7.3)</td>
</tr>
<tr>
<td>2</td>
<td>(2.1, 8.9, 7.6)</td>
<td>2</td>
<td>(3.3, 6.7, 2.5)</td>
<td>6</td>
<td>(2.6, 7.2, 4.1)</td>
</tr>
</tbody>
</table>

We provide an explanation of why the gap of the solution bounds of retailer $j$ increases in $K_j$ and decreases in $h_j$ and $\lambda_j$, or simply speaking, increases in the ratio of $K_j/(h_j\lambda_j)$. Recall that the bounds are obtained by using $\tilde{c}_j(T_j)$ to search over $T_j$ within the region that the resulting cost is lower than the cost difference, $(C^h - \sum_{i \neq j} \xi_i)$. Thus, if the cost difference is larger, the gap of the solution bounds will tend to be bigger. Intuitively, when the cost ratio $K_j/(h_j\lambda_j)$ is larger, the corresponding reorder interval tends to be longer, making the resulting cost $\xi_j$ bigger. This implies that the cost difference $(C^h - \sum_{i \neq j} \xi_i)$ will be bigger as well.

We finally test the solution bound for the cases with $N = 4$ and $N = 8$. The parameters are the same as those in the previous subsection and Table 3 shows the result. Since retailer $j \geq 2$ are identical, the bounds for $T_j^*, j = 2, \ldots, N$ are the same. Thus, we only report the gaps for retailer 1 and retailer 2.
\begin{table}
\centering
\begin{tabular}{lcccc}
\hline
 & \multicolumn{3}{c}{$N = 4$} & \multicolumn{3}{c}{$N = 8$} \\
\hline
\multicolumn{1}{c}{} & $\Delta_0$ & $\Delta_1$ & $\Delta_2$ & $\Delta_0$ & $\Delta_1$ & $\Delta_2$ \\
\hline
max & 6 & 14 & 20 & 4 & 29 & 29 \\
min & 1 & 3 & 1 & 0 & 5 & 1 \\
average & 3.1 & 7.7 & 6.5 & 2.02 & 11.9 & 10.3 \\
\hline
\end{tabular}
\caption{Summary for the gap of the solution bounds for $N = 4$ and $N = 8$}
\end{table}

6 Concluding Remarks

This paper studies a one-warehouse, multi-retailer system in which the echelon $(S,T)$ policy is implemented. When the demand is deterministic, it is known that there exists a simple POT solution that yields a near-optimal cost. However, fewer results are known for the corresponding stochastic model. In this paper, we first derive a simple bottom-up recursion to evaluate the system cost for any given $(S,T)$ policy. We then construct a series of lower bound cost functions that lead to a method of obtaining the optimal reorder intervals and an efficient heuristic. In the numerical study, among others, we find that the optimal reorder intervals tend to be an integer-ratio policy and the deterministic POT solution can be very ineffective under some conditions. This conclusion is in contrast to a common belief that a deterministic POT solution in general is an effective approximation for the stochastic model.

Our model assumes that the warehouse implements the virtual allocation rule. This allocation rule not only permits significant tractability and leads to near-optimal inventory policies, but also is commonly implemented in practice. It will be interesting to extend the current analysis to the other allocation policies. (In general, it would be very difficult to characterize the exact optimal policy for the distribution system, which depends on the inventory states of facilities.) We also assume deterministic lead times in our model. This assumption facilitates the synchronized scheduling rule. It remains an open question of what would be a plausible and effective scheduling rule for the system with stochastic lead times. We leave these studies for future research.

References


Appendix A: Optimization of Base-Stock Levels with Fixed T

This appendix shows how to optimize the optimal echelon base-stock levels when the reorder intervals $T$ are fixed. It is worth mentioning that Axsäter (1993) provides an approach (which is a generalization of Axsäter (1990) on base-stock systems) for finding the optimal base-stock levels with fixed reorder intervals. Axsäter’s model is a special case of ours because he assumes that the retailers have the same reorder interval and that the reorder interval of the warehouse is an integer multiple of those of the retailers (i.e., the so-called nested integer-ratio policies). Below we scratch the idea of how his approach can be generalized to our model. The proofs for Propositions 7-9 are shown in Appendix B.

We present the analysis from the local policy perspective, because $B_0$ is a function of $s_0$, and $S_j$ is the same as the local base-stock levels $s_j$.

Define the local inventory holding and backorder cost for retailer $j$ as follows:

$$f_j(y, r) = \mathbb{E}[H_j(y - D_j[L_j + M_{T_j}(r)])] + (b_j + H_j)(y - D_j[L_j + M_{T_j}(r)])^-,$$

and

$$f_0(y, r) = h_0 \mathbb{E} \left[ y - D_0[L_0 + M_{T_0}(r)] + \sum_{j=1}^{N} D_j[L_j + M_{T_j}(r)] + \sum_{j=1}^{N} \frac{\lambda_j}{\lambda_0} (y - D_0[L_0 + M_{T_0}(r)])^- \right].$$

**Proposition 7**

$$C(S, T) = \sum_{j=0}^{N} \frac{K_j}{T_j} + \frac{1}{T} \sum_{r=0}^{T-1} \left( f_0(s_0, r) + \sum_{j=1}^{N} \mathbb{E}[f_j(s_j - B_{0j}(L_0 + r_j(r)), r)] \right) + h_0 \sum_{j=1}^{N} (\lambda_j L_j).$$

Note that $f_0(s_0, r)$ is the warehouse inventory holding cost; $f_j(s_j - B_{0j}(L_0 + r_j(r)), r)$ is retailer $j$’s inventory holding and backorder cost; the last term is the average holding cost of pipeline inventory per period, which is constant.

For convenience, let us define

$$\hat{f}_0(s_0) = \frac{1}{T} \sum_{r=0}^{T-1} f_0(s_0, r), \quad \hat{f}_j(s_0, s_j) = \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E}[f_j(s_j - B_{0j}(L_0 + r_j(r)), r)].$$

Here, $\hat{f}_j(\cdot, \cdot)$ is a function of $s_0$ because $B_{0j}$ is a function of $s_0$.

**Proposition 8** For fixed $T$ and $s_0$, $\hat{f}_j(s_0, s_j)$ is convex in $s_j$. 

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With Proposition 8, we can find the best local base-stock level \( s_j(s_0) \) for each retailer \( j \). That is,
\[
s_j(s_0) = \arg \min_{s_j} \hat{f}_j(s_0, s_j).
\]
Substituting \( s_j(s_0) \) for \( s_j \) in \( C(S, T) \), the objective function becomes a function of \( s_0 \), i.e., \( C(s_0) = \hat{f}_0(s_0) + \sum_{j=1}^{N} \hat{f}_j(s_0, s_j(s_0)) \). Unfortunately, \( C(s_0) \) is not convex in \( s_0 \), so we have to construct bounds for the optimal \( s_0 \), denoted as \( s_0^* \), and conduct a search over the feasible interval.

Following Axsäter’s (1990) approach, we next provide bounds for \( s_0^* \). Let \( s_j^\ell = s_j(\infty) \) and \( s_j^u = s_j(0) \). Define
\[
s_0^u = \arg \min_{s_0} \left\{ \hat{f}_0(s_0) + \sum_{j=1}^{N} \hat{f}_j(s_0, s_j^\ell) \right\}
\]
and
\[
s_0^\ell = \arg \min_{s_0} \left\{ \hat{f}_0(s_0) + \sum_{j=1}^{N} \hat{f}_j(s_0, s_j^u) \right\}.
\]
Then,

**Proposition 9**

1. \( s_j^\ell \leq s_j^* \leq s_j^u \);
2. \( s_0^\ell \leq s_0^* \leq s_0^u \).

With Proposition 9, we can search over all possible \( s_0 \) between \( s_0^\ell \) and \( s_0^u \). After \( s_0^* \) is found, the optimal base-stock level for retailer \( j \) is \( s_j^* = s_j(s_0^*) \).

**Appendix B: Proofs**

**Proposition 2**

Note that
\[
g_0(S_0, T_0) = \frac{1}{T_0} \sum_{t=0}^{T_0-1} \mathbb{E} \left[ h_0 I_L(t + L + r) + \sum_{y_j(r) \leq I_L(t + L + r)} \min_{j=1}^{N} g_j^0(y_j(r)) \right]
\]
\[
= \frac{1}{T_0} \sum_{t=0}^{T_0-1} \mathbb{E} \left[ h_0 (S_0 - D[L + r]) + \sum_{y_j(r) \leq S_0 - D[L + r - 1]} \min_{j=1}^{N} g_j^0(y_j(r)) \right],
\]
where the first term in the brackets is clearly convex in \( S_0 \); the second term is also convex in \( S_0 \) because, with any realized \( D[L + r - 1] \), it is the minimum of a summation of convex functions subject to a linear constraint (that depends on \( S_0 \)). Thus \( g_0(S_0, T_0) \) is convex in \( S_0 \) for fixed \( T_0 \).
Proposition 5

To show (1), we see that for \( j = 1, \ldots, N \),

\[
\xi_j(T_j) = K_j \frac{T_j}{T_j} + \min_y \frac{1}{T_J} \mathbb{E} \sum_{r=0}^{T_j-1} \left[ h_{jy}(y - D_j[L_j + r]) + (h_{jy} + b_{jy})(y - D_j[L_j + r]) \right] + g^0_j(S_j).
\]

For fixed \( T_j \) and \( y \), \( \frac{K_j}{T_j} + \frac{1}{T_j} \mathbb{E} \sum_{r=0}^{T_j-1} \left[ h_{jy}(y - D_j[L_j + r]) + (h_{jy} + b_{jy})(y - D_j[L_j + r]) \right] + g^0_j(S_j) \)
is convex in \( D_j[L_j + r] \), therefore by Jensen’s inequality,

\[
\frac{K_j}{T_j} \frac{T_j}{T} \mathbb{E} \sum_{r=0}^{T_j-1} \left[ h_{jy}(y - D_j[L_j + r]) + (h_{jy} + b_{jy})(y - D_j[L_j + r]) \right] + g^0_j(S_j) \geq \frac{K_j}{T_j} \mathbb{E} \sum_{r=0}^{T_j-1} \left[ h_{jy}(y - \lambda_j(L_j + r)) + (h_{jy} + b_{jy})(y - \lambda_j(L_j + r)) \right] + g^0_j(S_j).
\]

Optimize over \( y \) on both side of the above inequality will lead to \( \xi_j(T_j) \geq \hat{c}_j(T_j) \). Similar arguments apply to \( \xi_0(T_0) \).

To prove (2), we will show that for any continuous convex function \( f(x) : \mathbb{R} \to \mathbb{R}, h(T) = \frac{K}{T} + \min_x \frac{1}{T} \sum_{r=0}^{T-1} f(x - ry) : \mathbb{Z}^+ \to \mathbb{R} \) is quasi-convex in \( T \) for any \( y > 0 \) and \( K > 0 \). We see that \( \hat{c}_j(T_j) \) can be represented in such a form. Let

\[
x_1 = \arg \min_x \sum_{r=0}^{T-2} f(x - ry), \\
x_2 = \arg \min_x \sum_{r=0}^{T-1} f(x - ry), \\
x_3 = \arg \min_x \sum_{r=0}^{T} f(x - ry),
\]

By the optimality of \( x_3 \), we must have \( f'(x_3 - 3y) \leq 0 \), otherwise \( f'(x_3 - 3y) \geq f'(x_3 - Ty) > 0 \) for all \( r = 0, \ldots, T \) and therefore \( \sum_{r=0}^{T} f'(x_3 - ry) > 0 \), which contradicts the optimality of \( x_3 \). Similarly, we must have \( f'(x_3) \geq 0, f'(x_2) \geq 0, f'(x_1) \geq 0 \) and \( f'(x_2 - (T - 1)y) \leq 0, f'(x_1 - (T - 2)y) \leq 0 \).

It then follows that

\[
h(T + 1) - h(T) = \frac{K}{T + 1} + \frac{1}{T + 1} \sum_{r=0}^{T} f(x_3 - ry) - \frac{K}{T} \frac{1}{T} \sum_{r=0}^{T-1} f(x_2 - ry)
\]

\[
= \frac{1}{T(T + 1)} \left[ T \sum_{r=0}^{T} f(x_3 - ry) - (T + 1) \sum_{r=0}^{T-1} f(x_2 - ry) - K \right],
\]

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and
\[ h(T) - h(T - 1) = \frac{1}{T(T - 1)} \left[ (T - 1) \sum_{r=0}^{T-1} f(x_2 - ry) - T \sum_{r=0}^{T-2} f(x_1 - ry) - K \right]. \]

We only need to show that
\[ T \sum_{r=0}^{T} f(x_3 - ry) - (T + 1) \sum_{r=0}^{T-1} f(x_2 - ry) \geq (T - 1) \sum_{r=0}^{T-1} f(x_2 - ry) - T \sum_{r=0}^{T-2} f(x_1 - ry), \]
or equivalently
\[ \sum_{r=0}^{T} f(x_3 - ry) + \sum_{r=0}^{T-2} f(x_1 - ry) \geq 2 \sum_{r=0}^{T-1} f(x_2 - ry). \]

If (23) is true, then it implies that if \( h(T) \geq h(T-1) \) then \( h(T+1) \geq h(T) \), i.e., \( h(T) \) is quasi-convex.

If
\[ f(x_3 - Ty) \geq f(x_1 - (T - 1)y), \]
then
\[ \sum_{r=0}^{T} f(x_3 - ry) + \sum_{r=0}^{T-2} f(x_1 - ry) \geq \sum_{r=0}^{T-1} f(x_3 - ry) + \sum_{r=0}^{T-1} f(x_1 - ry) \]
\[ \geq 2 \sum_{r=0}^{T-1} f(x_2 - ry), \]
where the second inequality is due to the optimality of \( x_2 \). On the other hand, if
\[ f(x_3 - Ty) < f(x_1 - (T - 1)y), \]
then \( x_3 - Ty \geq x_1 - (T - 1)y \) because \( f'(x_1 - (T - 1)y) \leq f'(x_1 - (T - 2)y) \leq 0 \) (see the discussion after the definition of \( x_1, x_2, x_3 \)). Then we have
\[ x_3 \geq x_1 + y. \]
Furthermore, because \( f'(x_1 + y) \geq f'(x_1) \geq 0, f(x_3) \geq f(x_1 + y) \). Henceforth
\[ \sum_{r=0}^{T} f(x_3 - ry) + \sum_{r=0}^{T-2} f(x_1 - ry) \geq \sum_{r=1}^{T} f(x_3 - ry) + \sum_{r=1}^{T-2} f(x_1 - ry) \]
\[ = \sum_{r=0}^{T-1} f(x_3 - y - ry) + \sum_{r=0}^{T-1} f(x_1 + y - ry) \]
\[ \geq 2 \sum_{r=0}^{T-1} f(x_2 - ry), \]
where the second inequality follows again from the optimality of \( x_2 \). Therefore, we have proved that \( h(T) \) is quasi-convex in \( T \in \mathbb{Z}^+ \).
Proposition 6

Let the optimal reorder intervals be \((T_0, T_1, ..., T_N)\). Given \(T_i\) satisfies integer ratio relations, we want to show \(T_0 \geq \min_{j \geq 1} \{T_j\}\). Consider two retailers with \(T_1 \leq T_2\). Due to integer ratio constraint, \(M_{T_1}(T_2) = 0\). Suppose \(T_0 < T_1\) and so \(M_{T_0}(T_1) = 0\) and denote \(n_1 = T_1/T_0\). And it can be seen that \(M = T_2\) by its definition. We compare the costs of two systems: one with \((T_0, T_1, T_2)\) and the other with \((T_1, T_1, T_2)\). \((S_0, S_1, S_2)\) are given echelon base-stock levels.

For \(r = 0, 1, \ldots, T_2 - 1\), we first study how \(B_0(t + L_0 + r_j(r))\) differs in these two systems for all \(j \geq 1\). Recall that

\[
B_0(t + L_0 + r_j(r)) = \left[ s_0 - D_0[t + r_0(r_j(r)), t + L_0 + r_j(r)] \right].
\]

For \(j = 2\), it is clear that with given \(s_0\), \(B_0(t + L_0 + r_j(r))\) is the same in both systems as \(r_j(r) = 0\). So we just consider \(j = 1\). When the warehouse reorder interval is \(T_1\), \(r_0(r_1(r)) = \left\lfloor \frac{r}{T_1} \right\rfloor T_1\); when its reorder interval is \(T_0\),

\[
r_0(r_1(r)) = \left\lfloor \frac{r}{T_0} \right\rfloor T_0 = n_1 \left\lfloor \frac{r}{T_1} \right\rfloor T_0 = \left\lfloor \frac{r}{T_1} \right\rfloor T_1
\]

since \(M_{T_0}(T_1) = 0\) and \(n_1 = T_1/T_0\). So with given \(s_0\), these two systems share the same \(B_0(t + L_0 + r_j(r))\) for each \(r\) and so the same \(B_{0j}(t + L_0 + r_j(r))\).

The system inventory related cost is evaluated as

\[
\frac{1}{M} \sum_{r=0}^{M-1} G_0(S, T, r) = \frac{1}{M} \sum_{r=0}^{M-1} E \left[ h_0(S_0 - D_0[L_0 + M_{T_0}(r)]) + \sum_{j=1}^{N} G_j \left( S_j - B_0j \left( L_0 + r_j(r) \right), T_j, r \right) \right],
\]

where

\[
G_j(y, T_j, r) = E[h_j(y - D_j[L_j + M_{T_j}(r)]) + (b_j + H_j)(y - D_j[L_j + M_{T_j}(r)])].
\]

It can be seen that if one increases \(T_0\) to \(T_1\), both \(M\) and \(E[\sum_{j=1}^{N} G_j \left( S_j - B_0j \left( L_0 + r_j(r) \right), T_j, r \right)]\) remain unchanged. However, the average fixed cost at the warehouse decreases and the holding cost term of the warehouse also decreases as \(D_0[L_0 + M_{T_0}(r)]\) increases. Hence, the system with \((T_0, T_1, T_2)\) incurs a higher cost than the system with \((T_1, T_1, T_2)\) that uses the same base-stock levels. Therefore, the result holds for a two-retailer system. The proof can be readily extended to a system with general multiple retailers and so we omit the details.
Proposition 7

Recall that

\[ G_j(y, T_j, r) = \mathbb{E}[h_j(y - D_j[L_j + \mathcal{M}_T(r)]) + (b_j + H_j)(y - D_j[L_j + \mathcal{M}_T(r)])]. \]

and

\[ G_0(S, T, r) = \mathbb{E} \left[ h_0(S_0 - D_0[L_0 + \mathcal{M}_T(r)]) + \sum_{j=1}^{N} G_j(S_j - B_{0j}(L_0 + r_j(r)), r) \right] \]

Because \( s_j = S_j \) and \( s_0 = S_0 - \sum_{j=1}^{N} S_j, \)

\[ G_0(S, T, r) \]

\[ = \mathbb{E}[h_0(S_0 - D_0[L_0 + \mathcal{M}_T(r)])] + \sum_{j=1}^{N} \mathbb{E}[h_j(s_j - B_{0j}(L_0 + r_j(r)) - D_j[L_j + \mathcal{M}_T(r)])] \]

\[ + (b_j + H_j)(s_j - B_{0j}(L_0 + r_j(r)) - D_j[L_j + \mathcal{M}_T(r)])] \]

\[ = \mathbb{E}[h_0(S_0 - D_0[L_0 + \mathcal{M}_T(r)])] - \sum_{j=1}^{N} \mathbb{E}[h_0(s_j - B_{0j}(L_0 + r_j(r)) - D_j[L_j + \mathcal{M}_T(r)])] \]

\[ + \sum_{j=1}^{N} \mathbb{E}[H_j(s_j - B_{0j}(L_0 + r_j(r)) - D_j[L_j + \mathcal{M}_T(r)])] \]

\[ + (b_j + H_j)(s_j - B_{0j}(L_0 + r_j(r)) - D_j[L_j + \mathcal{M}_T(r)])] \]

\[ = \mathbb{E}[h_0(s_0 - D_0[L_0 + \mathcal{M}_T(r)])] + \sum_{j=1}^{N} \mathbb{E}[h_0(B_{0j}(L_0 + r_j(r)))] - \sum_{j=1}^{N} \mathbb{E}[h_0(-D_j[L_j + \mathcal{M}_T(r)])] \]

\[ + \sum_{j=1}^{N} \mathbb{E}[f_j(s_j - B_{0j}(L_0 + r_j(r)), r)] \]

\[ = h_0 \mathbb{E} \left[ (s_0 - D_0[L_0 + \mathcal{M}_T(r)]) + \sum_{j=1}^{N} (D_j[\mathcal{M}_T(r)])] + \sum_{j=1}^{N} \frac{\lambda_j}{\lambda_0} (s_0 - D_0[L_0 + r_j(r) - r_0(r_j(r))) \right] \]

\[ + \sum_{j=1}^{N} h_0(\lambda_j(L_j)) + \sum_{j=1}^{N} \mathbb{E}[f_j(s_j - B_{0j}(L_0 + r_j(r)), r)]. \]

As \( h_0 = H_0 \), so from Proposition 1, the result follows.

Proposition 8

The convexity on \( s_j \) follows directly from the definition of \( \hat{f}_j(s_0, s_j) \). We omit the detailed proof for brevity.
Proposition 9

We need to first show that \( \tilde{f}_j(s_0, s_j) \) is supermodular in \( s_0 \) and \( s_j \), or

\[
\tilde{f}_j(s_0, s_j + 1) - \tilde{f}_j(s_0, s_j) \leq \tilde{f}_j(s_0 + 1, s_j) - \tilde{f}_j(s_0 + 1, s_j).
\] (24)

For notational simplicity, we denote \( D_j[L_j + M_T(r)] \) by \( D_j \). In addition, to emphasize the dependency of \( B_{0j} \) on \( s_0 \) and for brevity, let \( B_{0j}(s_0) \) denote \( B_{0j}(L_0 + r_j(r)) \). Note that

\[
\tilde{f}_j(s_0, s_j + 1) - \tilde{f}_j(s_0, s_j) = \frac{1}{T} \sum_{r=0}^{T-1} (b_j + H_j) E[(s_j + 1 - B_{0j}(s_0) - D_j)^- - (s_j - B_{0j}(s_0) - D_j)^-]
\]

\[
\leq \frac{1}{T} \sum_{r=0}^{T-1} (b_j + H_j) E[(s_j + 1 - B_{0j}(s_0 + 1) - D_j)^- - (s_j - B_{0j}(s_0 + 1) - D_j)^-]
\]

\[
= \tilde{f}_j(s_0 + 1, s_j + 1) - \tilde{f}_j(s_0 + 1, s_j)
\]

because \((-)^-\) is convex and \( B_{0j}(s_0 + 1) \) is smaller than \( B_{0j}(s_0) \). So the inequality (24) is valid. With this result and the convexity of \( \tilde{f}_j(s_0, s_j) \) is \( s_j \), part (1) follows.

To show part (2), from (24), we have, for any \( s_j \geq s_j^\ell \),

\[
0 \geq \tilde{f}_0(s_\ell^u) + \sum_{j=1}^{N} \tilde{f}_j(s_j^\ell, s_j^u) - [\tilde{f}_0(s_\ell^u + 1) + \sum_{j=1}^{N} \tilde{f}_j(s_j^\ell, s_j^u + 1)]
\]

\[
\geq \tilde{f}_0(s_\ell^u) + \sum_{j=1}^{N} \tilde{f}_j(s_j, s_j^u) - [\tilde{f}_0(s_\ell^u + 1) + \sum_{j=1}^{N} \tilde{f}_j(s_j, s_j^u + 1)]
\]

where the first inequality follows from the definition of \( s_\ell^u \). As the optimal \( s_j^* \) will be greater than \( s_j^\ell \), \( s_\ell^u \) is an upper bound for \( s_j^* \). Similarly, we can verify the lower bound \( s_0^u \).