Single-stage Approximations for Optimal Policies in Serial Inventory Systems with Non-stationary Demand

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Companies often face non-stationary demand due to product life cycles and seasonality, and this non-stationary demand complicates supply-chain managers’ inventory decisions. The present paper suggests a simple heuristic for determining stocking levels in a serial inventory system. Unlike the exact optimization algorithm, the heuristic can generate a near-optimal solution by solving independent, single-stage systems with the original problem data. This result enables us to reduce the computation time and complexity by allowing each location to simultaneously solve its own inventory problem. We further examine myopic solutions for these single-stage systems. Specifically, in a numerical study, we find that the change of the myopic solution is consistent with that of the optimal solution. We then derive a closed-form expression for the myopic solution and use it to approximate the optimal local base-stock level and to gain insights into how to manage safety stocks. The closed-form approximation shows how the local base-stock level is affected by future demand; it also explains the observation that the safety stock at an upstream stage is often stable and may not increase when demand variability increases. Finally, we discuss how the heuristic leads to a coordination scheme that enables a decentralized supply chain to achieve the near-optimal solution.

1. Introduction

Customer demand is often non-stationary in practice. Causes of non-stationary demand include product life cycles, seasonality, trends, and economic conditions. Non-stationary demand makes it difficult for managers to determine optimal stocking levels in a supply chain. One difficulty is computational – finding the optimal system-wide solution often requires solving several interrelated, recursive cost functions between stages across time. The other difficulty concerns implementation: if a supply chain is composed of independent firms, these firms may not be willing to implement the optimal solution without appropriate incentives. In a non-stationary demand environment, the optimal solution is often time-varying. Thus, designing an incentive scheme that can induce each stage to choose the optimal stocking level in each time period poses a difficult challenge.

This paper provides a heuristic that can simplify the computation and help facilitate the implementation of the optimal solution. To illustrate the approach, we consider an $N$-stage, serial
inventory model with a finite horizon. Materials flow from stage $N$ to stage $N-1$, $N-1$ to $N-2$, etc. until stage 1, where in each period a random, non-stationary demand occurs. Clark and Scarf (1960) first study this model with $N = 2$. They show that (time-varying) echelon base-stock policies are optimal. Although the structure of the policy is simple, obtaining the optimal solution is quite complex, especially for the upstream stage. This is because the optimal value function for the upstream stage depends on the downstream stage’s optimal base-stock level. Thus, one has to solve two sets of optimal value functions simultaneously. Clearly, the complexity will grow quickly when the chain becomes longer because the optimal value function of an upstream stage depends on all of its downstream solutions.

The heuristic we propose can generate an approximation for stage $j$ ($1 \leq j \leq N$) without knowing stage $i$’s base-stock level, $i < j$. More specifically, we show that the optimal base-stock level for stage $j$ is bounded by the optimal solutions of two single-stage systems with the original problem data. This result is established in three steps. First, we show that the optimal value function for stage $j$ is bounded above and below by that of a revised $j$-stage system. We refer to these revised systems as the upper-bound system and the lower-bound system, respectively. The upper-bound system is constructed by requiring stage $i(< j)$ to always order up to stage $i+1$’s echelon inventory level in each period. On the other hand, the lower-bound system is constructed by regulating stage $i$’s holding and order cost parameters. Second, we show that the optimal base-stock level for stage $j$ in the upper-bound (lower-bound) system is a lower (upper) bound to that in the original system. Finally, we show that solving these revised $j$-stage systems is equivalent to solving a single-stage system with parameters obtained from the original problem data. A numerical study suggests that the gap between these two solution bounds is generally quite small. This result motivates us to propose a heuristic solution for each stage by solving a single-stage system with a weighted average of the cost parameters obtained from the upper- and the lower-bound systems. In a numerical study, we find that the heuristic generates 58% of the optimal solutions, and that 98% of the heuristic solutions are within the ±1 unit range of the optimal solutions.

The results described in this paper make three contributions. First, the suggested heuristic can generate a near-optimal solution for a stage without knowing the base-stock levels of its downstream stages. This separation feature not only simplifies the computation, but also shortens the computation time by allowing each stage to solve its own problem in parallel. In other words, the heuristic can generate a solution at least $N$ times faster than the exact algorithm, provided that parallel processing is possible. Second, the effectiveness of the heuristic solution motivates us to
investigate the corresponding myopic solution (an upper bound to the heuristic solution; see, for example, Zipkin 2000). We find that the change of the myopic solution obtained from the heuristic single-stage system is consistent with that of the optimal solution when the system parameters vary. Thus, we can provide a closed-from expression for the myopic solution to approximate the optimal local base-stock level. The closed-form expression clearly shows how the system parameters affect the local base-stock level and the safety stock. It also explains why the safety stock of an upstream stage is often stable and may not increase when the variance of the demand increases over time. Finally, for a supply chain consisting of independent, self-interested firms (i.e., decentralized control system), we show that our heuristic can lead to a remarkably simple, time-consistent contract that induces each stage to choose the near-optimal solution.

Several researchers have provided methods to simplify the computation for the Clark and Scarf model. Federgruen and Zipkin (1984) consider an infinite-horizon version of the model with i.i.d. demand and show that the optimal policy can be obtained by recursively solving two cost functions that have the form of a single-period problem. Chen and Zheng (1994) reinterpret Federgruen’s and Zipkin’s results, simplify the optimality proof, and present an optimization algorithm to facilitate the computation. Although the computation effort is much reduced with these results, finding an upstream solution still is not easy because it depends on the downstream solutions. Thus, there is a stream of research that aims to further simply the computation and to reveal insights by solving single-stage problems. Noteworthy examples include Dong and Lee (2003), Shang and Song (2003), Gallego and Özer (2005), and Chao and Zhou (2007). To our knowledge, the existing heuristic solutions for the Clark-Scarf model are all for the infinite-horizon problem. Given that the Clark-Scarf model is a building block for many supply chain models and that non-stationary demand is prevalent in practice, the solution approach presented here has wide applicability.

Several papers have derived solutions for practical issues in supply chains under non-stationary demand. Erkip et al. (1990), for example, consider a one-depot-multi-warehouse system in which the warehouses’ demands are correlated. They derive an expression for the optimal safety stock as a function of the level of correlation through time. Ettl et al. (2000) consider a supply chain network that implements base-stock policies subject to service level requirements. They approximate the lead time demand for each location and suggest a rolling-horizon approach to find the base-stock levels for the non-stationary demand case. Abhyankar and Graves (2001) consider a two-stage serial system with a Markov-modulated Poisson demand process. They implement an inventory hedging policy to protect against cyclic demand variability. Graves and Willems (2008) consider a problem
of allocating safety stocks in a supply chain network where the demand is bounded and there is a guaranteed service time between stages and customers. They propose an algorithm to determine safety stocks under a constant service time policy. Similar to the above papers, the present work aims to provide a simple control policy.

Finally, our paper is also related to the coordination literature. Most coordination papers consider an infinite-horizon model with stationary demand. Because of the regenerative process, these infinite-horizon models are equivalent to single-period problems. These coordination papers often analyze a decentralized Nash equilibrium solution and provide contracts to induce the system to achieve the centralized (first best) solution. Noteworthy examples include Lee and Whang (1999), Chen (1999), Cachon and Zipkin (1999), and Shang et al. (2009). However, when the system fails to form a regenerative process, studying the decentralized behaviors becomes more difficult. Donohue (2000) studies a two-period model with demand forecasts. She suggests using time-varying contract terms to coordinate the system. Parker and Kapuscinski (2010) consider a two-stage serial inventory system with capacity limits, where each stage aims to minimize its own costs. They show that there exists a Markov equilibrium policy for a dynamic game in the decentralized control system. In general, it is very difficult to derive a coordination contract in a finite-horizon model. Were such a contract to exist, it would be too difficult to implement because the contract terms are often time-varying.

2. Model and Main Results

We consider an $N$-stage serial inventory system, where stage 1 orders from stage 2, stage 2 from stage 3, etc. until stage $N$, which orders from an ample outside supplier. There is a lead time $\tau_j$ between stage $j$ and stage $j+1$, and $\tau_j$ is a positive integer. Denote $\tau[i,j] = \sum_{k=i}^{j} \tau_k$ and $\tau[i,j] = 0$ if $i > j$. Let $h_j$ be the echelon holding cost rate at stage $j$ and let $b$ be the backorder cost rate at stage 1. Let $h[i,j] = \sum_{k=i}^{j} h_k$. Define $p_j$ as the unit order cost for stage $j$. We use $t$ to index the time period and count the time backwards. That is, if $t$ is the current period, $t - 1$ will be the next period, etc. Let $T$ be the planning horizon. Denote $D(t)$ the demand in period $t$. The demands are independent between periods, but the demand distributions may differ from period to period. Let $D[t, s] = \sum_{i=t}^{s} D(i)$, representing the total demand in period $t$, $t - 1$, $t - 2$, ..., $s$, where $t \geq s$. The sequence of events in a period is as follows: (1) each stage receives a shipment sent one period ago from its upstream stage at the beginning of the period; (2) each stage places
an order at the beginning of the period; (3) each stage sends a shipment to its downstream stage; (4) demand occurs at stage 1 during the period; (5) inventory cost is evaluated at the end of the period.

Clark and Scarf (1960) show that time-varying, echelon base-stock policies are optimal. Let $s_j(t)$ be the optimal echelon base-stock level for stage $j$. The echelon base-stock policy is executed as follows: stage $j$ reviews $x_j$ at the beginning of each period, where $x_j$ is the echelon inventory level for stage $j$ (= inventory at stage $j$ + inventory in-transit to and at stage $i (< j)$ - backorders). The stage orders up to $s_j(t)$ if $x_j < s_j(t)$, and does not order otherwise. It is well known that $s_j(t)$ depends on the downstream base-stock levels and can be found by solving $j$ sets of dynamic programs sequentially. More specifically, finding $s_1(t)$ is equivalent to solving a single-stage system. With the known $s_1(t)$, one can form a dynamic program to compute the functional equation for stage 2, assuming that stage 2 has ample supply. The optimal base-stock level $s_2(t)$ is the optimal solution obtained from the functional equation. Continuing this procedure, with the known $s_i(t)$, $i < j$, one can compute the functional equation for stage $j$ and the corresponding optimal solution $s_j(t)$. See Appendix A for the detailed algorithm. In other words, when finding the optimal solution for stage $j$, we can only focus on echelon $j$ that includes stage 1 to stage $j$ by viewing the echelon has an ample supply from its upstream stage.

Below we demonstrate that $s_j(t)$ is bounded by the solution obtained from two single-stage systems with the original problem data. The lower-bound (upper-bound) solution is generated from an upper-bound (lower-bound) system presented in §2.1 (§2.2). We only present the main results and refer the reader to Appendix B for proofs.

2.1 Upper-Bound System

Consider echelon $j$ with a more restrictive policy: stage $i$ always orders up to $x_{i+1}$ in each period except $t \leq \tau[1,i]$ for $i < j$. (When $t$ is in this interval, stage $i$ would not order because of the end of the horizon.) Let $s_j^f(t)$ be the resulting optimal echelon base-stock level for stage $j$. Clearly, such a policy is suboptimal and the resulting total cost is an upper bound to that of the original system. For this reason, we call this restrictive system the upper-bound system. It is not clear, however, whether there exists an order relationship between $s_j^f(t)$ and the optimal solution $s_j(t)$. To see this, consider a two-stage system. Under this restrictive ordering policy, stage 1 will order

\[ s_j^f(t) \]
no less than it does in the original system. To minimize the cost in the upper-bound system, stage 2 should order less. Thus, it is not clear whether the net effect of the restrictive policy will be to make the resulting echelon base-stock level $s_2^k(t)$ smaller than $s_2(t)$. This uncertainty is particularly great when the demand changes drastically between periods. Nevertheless, we show that the order relationship exists.

**Theorem 1** If $x_{j-1} < s_{j-1}(t)$, $s_j(t) \geq s_j^k(t)$ for $t \geq \tau[1,j]$.

Let us take a closer look at the upper-bound system. To construct the upper-bound system for echelon $j$, $j = 2, ..., N$, we regulate stage $i(< j)$ to always order up to $x_{i+1}$ in each period. By doing so, any unit ordered by stage $j$ will eventually arrive at stage 1 in $\tau[1,j]$ periods. Thus, we can see each stage $i, i = 2, ..., j$ is a transit point and that the echelon $j$ is effectively a single-stage system with a lead time of $\tau_j = \tau[1,j]$ periods. Next, let us consider the unit order cost. For each unit ordered by stage $j$ in period $t$, $p_j$ is incurred. This unit will arrive at stage $j$ in period $t - \tau_j$, during which stage $j - 1$ will order it with an order cost of $p_{j-1}$. In addition, stage $j$ will incur a holding cost $h_j$ for this unit in period $t - \tau_j$. Thus, the effective order cost is $(p_{j-1} + h_j)$. Continuing this logic, this unit will arrive at stage 2 in period $t - \tau[2,j]$, during which stage 1 will incur an order cost of $p_1$ and stage 2 will incur a holding cost $h_2$ for this unit. In summary, the unit order cost for the upper-bound system for echelon $j$, $j = 2, ..., N$, is

$$p_j^u = p_j + \sum_{i=2}^{j} \alpha^{i,j} \left(p_{i-1} + h_i \right).$$

Once this unit arrives at stage 1, it will kept as inventory. Thus, it will incur either the local holding cost rate $h_j^u = h[1,j]$ or the local effective backorder cost rate $b_j = b + h[j+1,N]$ (see Shang and Song (2003) for an explanation of the formulation of $b_j$).

To summarize, let $S_j^u[p_j^u, h_j^u, b_j, \tau_j]$ denote the upper-bound system for echelon $j$, $j = 2, ..., N$, where $p_j^u$ is the unit order cost, $h_j^u$ the holding cost rate, $b_j$ the backorder cost rate, and $\tau_j$ the lead time. Set $h_{j+1} = 0$.

**Proposition 2** The lower-bound solution $s_j^l(t)$ can be obtained by solving $S_j^u[p_j^u, h_j^u, b_j, \tau_j]$, for $j = 1, ..., N$.

### 2.2 Lower-Bound System

The approach of constructing the lower-bound system for echelon $j$ is different from that of constructing the upper-bound system. More specifically, we set $h_i = 0$ and $p_i = 0$ for $i < j$. Clearly,
the resulting cost is a lower-bound to echelon $j$’s cost. Under this construction, stage $i$ would choose to order up to $x_{i+1}$ as there is no intention to carry inventory at stage $i+1$. Thus, it turns out that the optimal order policies are the same for the upper- and the lower-bound systems. However, as in the upper-bound system, it is not clear whether an order relationship exists between the resulting solution, $s^u_j(t)$, and the optimal solution $s_j(t)$. Consider a two-stage system. Under such new cost parameters, stage 1 will always order up to $x_2$. If we follow the same logic as before, stage 2 should order less. Thus, there is little intuition that $s^u_2(t)$ is larger than $s_2(t)$. The following theorem shows that, indeed, the order relationship holds.

**Theorem 3** $s^u_j(t) \geq s_j(t)$, for $t > \tau[1,j]$.

Since the downstream stages use the same optimal policy, a similar logic can be applied to show that the system is equivalent to a single-stage system. We can simply set $h_i = 0$ and $p_i = 0$ for $i < j$ to obtain the cost parameters for the lower-bound system. More specifically, let $S^l_j[p^l_j, h^l_j, b_j, T_j]$ denote the lower-bound system for echelon $j$, where the order cost $p^l_j = p_j + \alpha\tau[j,j]h_j$, the holding cost rate $h^l_j = h_j$, and the effective backorder cost rate $b_j$, and the lead time $T_j$ periods.

**Proposition 4** The upper-bound solution $s^u_j(t)$ can be obtained by solving $S^l_j[p^l_j, h^l_j, b_j, T_j]$, for $j = 1, \ldots, N$.

### 3. Myopic Solution

For managers, it is crucial to learn how the system parameters affect the stocking decision in a supply chain. This section provides a closed-form expression for the optimal local base-stock level and the safety stock at each stage.

We propose using a myopic solution. More specifically, in §4, we suggest a heuristic that solves a single-stage system with a weighted average of the cost parameters obtained from the upper- and lower-bound systems. More specifically, let $p^\alpha_j = wp^u_j + (1-w)p^l_j$ and $h^\alpha_j = wh^u_j + (1-w)h^l_j$, where $0 \leq w \leq 1$. We call the resulting single-stage system heuristic system $j$, denoted by $S^\alpha_j(p^\alpha_j, h^\alpha_j, b_j, T_j)$ and we define the resulting optimal solution as $s^\alpha_j(t)$. In §4, we numerically show that $s^\alpha_j(t)$ is an effective approximation to the optimal solution $s_j(t)$. Furthermore, it is well known that the myopic solution is an upper bound to the optimal solution of a single-stage system (Zipkin 2000, p. 378-379). These two results together motivate us to derive a closed-form approximation, referred to as $s^m_j(t)$, for $s^\alpha_j(t)$. In our numerical study, we find that in all cases $s^m_j(t)$ moves in the same direction
as the optimal solution $s_j(t)$. Thus, we can use the myopic solution to derive an approximation for the optimal local base-stock level to examine the system behaviors. Below we lay out the detailed steps.

Let the myopic solution for $S^a_j(p^a_j, h^a_j, b_j, T_j)$ be $s^m_j(t)$.

**Proposition 5** For $t > T_j$,

$$s^m_j(t) = \arg \min_{s_j} \left\{ P\left(D[t, t - T_j] \leq s_j \right) > \beta_j \right\},$$

where

$$\beta_j = \frac{\alpha^{\xi_j}b_j - p^a_j(1 - \alpha)}{\alpha^\xi_j(b_j + h^a_j)}.$$  

Note that when $t < T_j$, stage $j$ will not order. When $t = T_j + 1$, $s^m_j(t)$ is equal to the solution obtained from the above equation except for the removal of the term $(1 - \alpha)$ in the numerator due to the termination value being equal to zero.

To obtain a closed-form expression, we apply normal approximation on $D[t, t - T_j]$. Let the mean of $D[t, t - T_j]$ be $\lambda[t, t - T_j]$ and the standard deviation be $\sigma[t, t - T_j] = \sqrt{\text{Var}[D[t, t - T_j]]}$.

We can form a closed-form expression for $s^m_j(t)$:

$$s^m_j(t) = \lambda[t, t - T_j] + \sigma[t, t - T_j]\Phi^{-1}(\beta_j).$$

Thus, the local base-stock level is $s^m_1(t) = s^m_1(t)$, and for $j = 2, ..., N$,

$$s^m_j(t) = s^m_j(t) - s^m_{j-1}(t) = \lambda[t - \xi_{j-1} - 1, t - \xi_j] + \sigma[t, t - \xi_j]\Phi^{-1}(\beta_j) - \sigma[t, t - \xi_{j-1}]\Phi^{-1}(\beta_{j-1}).$$  \hspace{1cm} (1)

The first term in Equation (1) is the average pipeline inventory in period $t$, which depends on the average $\tau_j$ periods of future demand in period $t - \xi_{j-1} - 1, t - \xi_{j-1} - 2, ..., t - \xi_j$. The second term is the safety stock for stage $j$ in period $t$, denoted as $ss^m_j(t)$, which depends on the cost ratios $\beta_j$ and $\beta_{j-1}$, and the variability of the demand in period $[t, t - \xi_j]$.

Equation (1) allows us to analytically investigate how the system parameters affect the optimal base-stock level and the safety stock at each stage. For example, if we are interested in the change to the amount of safety stock of the upstream stage in a two-stage system, we can define the change to stage 2’s safety stock in period $t$ as

$$\Delta ss^m_2(t) = ss^m_2(t - 1) - ss^m_2(t)$$

8
\[
\begin{align*}
&= \left( \sigma_{t-1, t-1 - \mathcal{I}_2} - \sigma_{t, t - \mathcal{I}_2} \right) \Phi^{-1}(\beta_2) \\
&\quad - \left( \sigma_{t-1, t-1 - \mathcal{I}_1} - \sigma_{t, t - \mathcal{I}_1} \right) \Phi^{-1}(\beta_1).
\end{align*}
\] (2)

From the above equation, we can see that \( \Delta s_{\text{ss}}^2(t) \) will be fairly small unless there is a significant difference between \( \text{Var}\left[D(t-1 - \mathcal{I}_2)\right] \) and \( \text{Var}\left[D(t-1 - \mathcal{I}_1)\right] \). This implies that the safety stock at the upstream stage should be fairly stable. (A similar conclusion is observed in Graves and Willems (2008) in their numerical study.) In addition, \( \Delta s_{\text{ss}}^2(t) \) may not be positive even if \( \text{Var}\left[D(t)\right] < \text{Var}\left[D(t-1)\right] \), \( \forall t \). More specifically, when either \( p_2 \) is large, or \( h_2 \) is large, or \( b \) is small, \( \Phi^{-1}(\beta_2) \) tends to be smaller than \( \Phi^{-1}(\beta_1) \), causing the difference in (2) to become negative even when the demand variance increases over time. This suggests that the safety stock at an upstream stage may not increase with the demand variability.

**Example.** We consider a two-stage system with \( \tau_1 = \tau_2 = 1, p_2 = 6, h_2 = 1, p_1 = 4, h_1 = 1, \) and \( b = 15 \). The demand follows a Poisson distribution with mean rate shown in Table 1. We report the optimal echelon, heuristic, and myopic base-stock levels in each period. We also report the optimal local base-stock level as well as the corresponding safety stock for stage 2. As can be seen, the safety stock may decrease although the demand rate increases (e.g., from \( t = 10 \) to \( t = 9 \)).

<table>
<thead>
<tr>
<th>Period (t)</th>
<th>10</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
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<tr>
<td>Demand rate</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>9</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>( s_1(t) )</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>24</td>
<td>26</td>
<td>22</td>
<td>16</td>
<td>10</td>
<td>5</td>
<td>–</td>
</tr>
<tr>
<td>( s_2(t) )</td>
<td>16</td>
<td>22</td>
<td>29</td>
<td>33</td>
<td>31</td>
<td>24</td>
<td>16</td>
<td>6</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>( s_2^2(t) )</td>
<td>16</td>
<td>23</td>
<td>29</td>
<td>33</td>
<td>31</td>
<td>24</td>
<td>16</td>
<td>7</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>( \hat{s}_2^1(t) )</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>24</td>
<td>26</td>
<td>23</td>
<td>18</td>
<td>12</td>
<td>7</td>
<td>–</td>
</tr>
<tr>
<td>( \hat{s}_2^2(t) )</td>
<td>16</td>
<td>23</td>
<td>29</td>
<td>33</td>
<td>32</td>
<td>26</td>
<td>19</td>
<td>7</td>
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<td>–</td>
</tr>
<tr>
<td>( \Delta s_2(t) = s_2(t) - s_1(t) )</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>5</td>
<td>2</td>
<td>0</td>
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<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>( s_{\text{ss}}^2(t) = s_2(t) - \mathbb{E}[D[t-2, t-2]] )</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>-2</td>
<td>-3</td>
<td>-3</td>
<td>-5</td>
<td>–</td>
<td>–</td>
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</tbody>
</table>

Table 1: A two-stage example with the optimal, heuristic, and myopic solutions, as well as the local base-stock levels and the resulting safety stocks.

### 4. Numerical Study

We first conduct a numerical study to report the gap between \( s^j_2(t) \) and \( s^u_2(t) \) and examine where within this interval the optimal solution \( s_j(t) \) is located. We seek to determine an effective weight \( w \) that will be used for generating the heuristic solution.
We consider two-stage systems with a horizon of \( T = 10 \) periods. Assume that the period demand follows a Poisson distribution with rate \( \lambda(t) \) in period \( t \). We test the following demand patterns: constant (C), linear increasing (I), linear decreasing (D), concave (V), and convex (X) forms. For the constant demand, we set \( \lambda(t) = 5.5 \), for \( 1 \leq t \leq 10 \); for the increasing demand, \( \lambda(t) = 11 - t \), for \( 1 \leq t \leq 5 \) and \( \lambda(t) = 2t - 10 \) for \( 6 \leq t \leq 10 \); finally, for the concave demand, \( \lambda(t) = 11 - 2t \) for \( 1 \leq t \leq 5 \), and \( \lambda(t) = 22 - 2t \) for \( 6 \leq t \leq 10 \).

We first fix the cost parameters at stage 1 and change the cost parameters at stage 2. Specifically, we set \( h_1 = 1 \), \( p_1 = 4 \), and \( h_2 \in \{0.5, 1, 1.5\} \), \( p_2 \in \{2, 6\} \). The other parameters are \( \tau_1 = \tau_2 = 1 \) and \( b \in \{15, 50\} \). The total number of instances is 60. We then swap the stage index in the above order and holding cost parameters to generate another 60 instances. The total number of instances is 120 and the total number of optimal base-stock levels for stage 2 is 960.

To examine the overall effectiveness of the bounds, we define the following two measures. Let

\[
\xi = \frac{s_u^2(t) - s^2(t)}{s_2(t)}, \quad \theta = \frac{s^2(t) - s_l^2(t)}{s^2_l(t) - s^2(t)}.
\]

The first measure \( \xi \) signifies the size of the gap between the solution bounds with respect to the optimal base-stock level; the second measure \( \theta \) signifies the position of the optimal base-stock level between the solution bounds. When \( s_u^2(t) = s^2_l(t) \), we set \( \theta = 0.5 \).

We find that a total of 533 lower bound solutions and a total of 122 upper bound solutions are equal to \( s_2(t) \). Overall, \( s_2(t) \) tends to be closer to \( s_u^2(t) \). But when \( b \) increases, \( s_2(t) \) tends to move toward \( s_u^2(t) \). Also, the average gap is smaller when \( b \) is large. The average \( \xi \) and \( \theta \) for the 960 solutions are 8.50\% and 0.27, respectively. Because \( \theta \) is roughly equal to 0.3, we then set \( w = 0.7 \) to generate the heuristic solution.

To examine the performance of the resulting heuristic solution \( s_u^2(t) \), and the myopic solution \( s_m^2(t) \), we solve the same 120 two-stage instances above. Let

\[
\epsilon_i = \left| \frac{s_i^2(t) - s_2(t)}{s_2(t)} \right| \times 100\%, \quad i \in \{a, m\},
\]

which denotes the performance of the heuristic and myopic solution, respectively. Table 2 summarizes the average \( \epsilon \). The parenthesis shows the number of optimal solutions generated from the heuristic and the myopic solutions. Overall, the heuristic generates 769 optimal solutions (80.1\%). Interestingly, the myopic solution is also quite effective - it generates 649 optimal solutions (67.6\%). The myopic solution is particularly effective when demand rate is increasing or constant. This is
intuitive. When demand rate increases, it is more likely that the inventory will be consumed at the end of a period. Thus, the manager can determine an effective order quantity without considering the long-term effect of the left-over inventory. On the other hand, we observe two situations in which the myopic solution is least effective: (1) when the demand is decreasing over time and (2) a few periods before the end of the horizon.

To test the effectiveness of the heuristic solution and the resulting myopic solution under a more general system, we consider a four-stage system with $T = 20$. We fix $\sum_{j=1}^{4} h_j = 1$ and consider different holding cost forms: $(h_1, h_2, h_3, h_4) \in \{(0.25, 0.25, 0.25, 0.25), (0.2, 0.2, 0.2, 0.2), (0.1, 0.1, 0.4, 0.4), (0.2, 0.2, 0.4, 0.2)\}$, representing linear, affine, kink and jump forms, respectively. Similarly, we fix $\sum_{j=1}^{4} \tau_j = 6$ and let $(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(2, 2, 1, 1), (1, 2, 2, 1), (1, 1, 2, 2)\}$, representing long lead time at the downstream, middle, and upstream stages, respectively. For the order costs, we consider two scenarios: $(p_1, p_2, p_3, p_4) \in \{(1, 1, 2, 2), (2, 2, 1, 1)\}$, representing high order costs at the upstream and downstream stages, respectively. For the backorder cost, let $b \in \{9, 39\}$, representing low and high customer service levels, respectively. Finally, for the demand form, we consider both convex demand $(D(1), D(2), ..., D(20)) = (10, 9, \ldots, 1, 1, 2, \ldots, 10)$ and concave demand $(D(1), D(2), ..., D(20)) = (1, 2, \ldots, 10, 10, 9, \ldots, 1)$. There are a total of 96 instances, with 1632 optimal solutions at stage 2, 1472 at stage 3, and 1344 at stage 4. Again, we set the weight $w = 0.7$. Table 3 below summarizes the percentage performance of both solutions at each stage. At a first glance, it seems that the heuristic is less effective for the upstream stage when the chain becomes longer. A more careful observation reveals that the optimal solution tends to be closer to the lower-bound solution at the upstream stages. In other words, one should use a larger $w$ for the upstream stages. For example, if we are using $w = 0.8$ for stage 4 parameters, we can reduce the percentage error to less than 2%. The myopic solution behaves the same - it is most effective when demand rate is increasing, and is least effective when demand rate is decreasing or near the end of the horizon.

<table>
<thead>
<tr>
<th>b</th>
<th>Demand Form</th>
<th>C</th>
<th>I</th>
<th>D</th>
<th>V</th>
<th>X</th>
<th>Overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>Avg $\epsilon_a$ (#)</td>
<td>1.27% (76)</td>
<td>1.76% (69)</td>
<td>2.56% (80)</td>
<td>1.85% (76)</td>
<td>1.98% (67)</td>
<td>1.88% (368)</td>
</tr>
<tr>
<td></td>
<td>Avg $\epsilon_m$ (#)</td>
<td>1.97% (70)</td>
<td>1.91% (66)</td>
<td>6.75% (52)</td>
<td>5.18% (48)</td>
<td>2.34% (60)</td>
<td>3.63% (296)</td>
</tr>
<tr>
<td>50</td>
<td>Avg $\epsilon_a$ (#)</td>
<td>0.66% (81)</td>
<td>0.52% (86)</td>
<td>0.84% (79)</td>
<td>0.67% (80)</td>
<td>1.26% (75)</td>
<td>0.79% (401)</td>
</tr>
<tr>
<td></td>
<td>Avg $\epsilon_m$ (#)</td>
<td>0.89% (76)</td>
<td>0.52% (86)</td>
<td>3.11% (57)</td>
<td>2.46% (62)</td>
<td>1.34% (72)</td>
<td>1.66% (353)</td>
</tr>
</tbody>
</table>

Table 2: Summary of the effectiveness of the solution bounds


<table>
<thead>
<tr>
<th>Solution</th>
<th>stage 2</th>
<th>stage 3</th>
<th>stage 4</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^h_j(t)$</td>
<td>1.50%</td>
<td>1.57%</td>
<td>2.03%</td>
<td>1.70%</td>
</tr>
<tr>
<td>$s_j(t)$</td>
<td>1.57%</td>
<td>1.68%</td>
<td>2.28%</td>
<td>1.84%</td>
</tr>
</tbody>
</table>

Table 3: Summary of the effectiveness of the heuristic and the myopic solutions for the four-stage system.

5. Implementation Issues

So far, we have focused the discussion on the centralized control system, i.e., the supply chain belongs to a single firm. We have demonstrated how our heuristic can help the owner reduce the computational complexity and duration of computation by allocating the computation to each stage.

Now, let us consider a decentralized control system, i.e., the supply chain is composed of independent firms, each pursuing its own best interest. There are two questions of interest. First, is there an equilibrium solution between these individual firms in the finite horizon? If so, by how much the equilibrium solution does deviate from the centralized (first-best) solution? To answering this question, one has to analyze a dynamic game; in general, it is difficult to show the existence of a Markov equilibrium solution (Parker and Kapuscinski 2010). Second, how does one design a contract (or mechanism) that can coordinate the firms to achieve the centralized solution? Because the centralized solution is often time-varying, a coordination contract, if it existed, would have non-stationary parameters. Thus, it is more difficult to implement the contract.

Here, we focus only on the second question. As we shall demonstrate below, our heuristic leads to a simple, time-consistent contract that can induce the supply chain partners to choose the heuristic solution $s_j^h(t)$ in each period.

To demonstrate the coordination scheme, let us consider a two-stage system, where stage $j$ has full information and is responsible for determining its echelon base-stock level. (This is the echelon information scenario described in, e.g., Cachon and Zipkin (1999), Shang et al. (2009) and Parker and Kapuscinski (2010).) We assume that stage $j$ chooses an arbitrary base-stock level $s_j^0(t)$ in period $t, t = 1, ..., T$. Denote the resulting supply chain cost incurred in the T-period horizon as $f_j^0(x_1, x_2)$, where $x_j$ is the echelon inventory level for stage $j$ at the beginning of period $T$. Clearly, $f_j^0(x_1, x_2)$ is the sum of the cost incurred at stage 1, $c_j^0(x_1)$, and at stage 2, $v_j^0(x_2)$. (Here, stage 1 and stage 2 are cost centers. In practice, there can be different ways to calculate the cost for the cost center. For example, a natural way is to charge $p_j$ per unit of inventory ordered, $h_j$ per unit
of the echelon on-hand inventory for stage \( j, j = 1, 2 \), and \( b \) per unit of the backorders for stage 1 in each period. In any case, the bottom line is \( f_I^0(x_1, x_2) = c_T^0(x_1) + g_T^0(x_2) \).

We assume that there is an integrator who knows the centralized solution and is responsible for payment transfers between the firms. The integrator can be one of these firms, a team of them, or a third-party organization. The integrator designs a contract for each stage \( j \) with three cost parameters \((\theta_j, p_j^0, \theta_j, h_j^0, \theta_j, b_j)\), aiming to induce the stage manager to choose \( s_j^1(t) \) in each period, \( t = 1, 2, ..., T \), where \( \theta_j \) is an adjustment factor for stage \( j \), \( 0 \leq \theta_j \leq 1 \). Let the resulting near-optimal cost be \( f_I^0(x_1, x_2) \) and assume \( f_I^0(x_1, x_2) < f_T^0(x_1, x_2) \).

The contract is implemented as follows: in each period, the integrator first compensates the actual cost incurred for stage \( j \). Then, stage \( j \) pays the integrator based on the accounting echelon inventory level \( \bar{x}_j \) according to the cost terms. (The accounting echelon inventory level is defined as the echelon inventory level by assuming that there is an ample supply from upstream.) More specifically, with the perceived lead time \( T_j \) periods, stage \( j \) pays the integrator \( \theta_j p_j^0 \) for each unit ordered, \( \theta_j h_j^0 \) per unit of \((\bar{x}_j)^+\) and \( \theta_j b_j \) per unit of \((\bar{x}_j)^-\) in each period, where \((x)^+ = \max\{x, 0\}\) and \((x)^- = \max\{0, -x\}\). From stage \( j \)'s perspective, because his actual cost will be covered by the integrator, he only needs to minimize the payment to the integrator in the \( T \) period. Thus, the problem for stage \( j \) is exactly the same as solving the heuristic system \( j \), and stage \( j \) will therefore choose \( s_j^0(t) \) as the optimal solution. \((\theta_j \) is a constant, which will not affect the optimal solution.\)

To see why such a contract is implementable, we have to demonstrate that each player, including the integrator, is better off in this game. Let us denote stage 1’s payment (respectively, stage 2’s payment) to the integrator as \( \theta_1 c_T^0(\bar{x}_1) \) (respectively, \( \theta_2 g_T^0(\bar{x}_2) \)). Clearly, the stages are willing to accept the contract if the outflow payment is smaller than the inflow payment, i.e.,

\[
\theta_1 c_T^0(\bar{x}_1) < c_T^0(x_1), \quad \theta_2 g_T^0(\bar{x}_2) < g_T^0(x_2).
\]  

Similarly, the integrator is willing to perform this function if she can make profits after both stages implement the solution \( s_j^0(t) \), i.e.,

\[
\theta_1 c_T^0(\bar{x}_1) + \theta_2 g_T^0(\bar{x}_2) > f_T^0(x_1, x_2).
\]  

Because \( c_T^0(x_1) + g_T^0(x_2) = f_T^0(x_1, x_2) > f_T^0(x_1, x_2) \), there exist \( \theta_1 \) and \( \theta_2 \) such that Equations (3) and (4) hold. Thus, the contract is implementable.

In fact, the resulting solution \((s_1^1(t), s_2^1(t))\) is a Markov equilibrium in this dynamic game. (This means that each player will choose \( s_j^1(t) \) in each period without deviation.) This can be verified
by the fact that each firm optimizes its inventory decision independently. Thus, the solution 
\((s^1_1(t), s^2_1(t))\) is a Nash equilibrium in each period.

Although the suggested coordination scheme can only achieve a near-optimal solution, it has
the advantage of easy implementation because the contract terms are stationary. A key enabler
of the coordination scheme is the integrator, who decouples the players’ decisions in each period.
In practice, the integrator can be one of the firms, a subset of the firms, or a third-party financial
institute. We refer the reader to Shang et al. (2009) for examples of the integrator.

6. Concluding Remarks

This paper presents an approach to generating bounds for the optimal echelon base-stock levels for
a serial inventory system with a finite horizon. The demands between periods are independent but
are allowed to be nonidentical. The solution bounds are obtained by solving two revised problems.
The revised problem that generates the lower bound solution is formulated by regulating stages’
order decisions; the revised problem that generates the upper bound solution is constructed by
regulating stages’ holding cost and order cost parameters. We prove that solving these revised
problems is equivalent to solving single-stage models with the original problem data. We suggest
a heuristic by solving a series of single-stage systems with a weighted average of cost parameters.
A numerical study suggests that the heuristic is near-optimal. We therefore derive the myopic
solution for the local base-stock level in the heuristic system to reveal insights concerning the local
inventories and safety stocks. We also demonstrate how our heuristic can lead to a coordination
scheme that achieves the near-optimal solution.

The solution bounds and the heuristic can be extended to a one-warehouse-multi-retailer dis-
tribution system in which the retailers are identical. Under the so-called balance assumption (i.e.,
the inventory levels of the retailers can be freely and instantaneously re-distributed as needed),
the distribution system is equivalent to a two-stage serial system, where the downstream stage can
be viewed as a composite stage that includes all retailers’ demands. As shown in Federgruen and
Zipkin (1984), one can apply the Clark and Scarf serial algorithm to obtain the echelon base-stock
level for the warehouse and the composite stage. Then, one could apply the myopic allocation rule
in each period to the retailers to determine the retailers’ base-stock levels. Federgruen and Zipkin
reports that this approach can generate a very effective solution. Clearly, we can apply the same
technique to generate single-stage approximations for the resulting two-stage system, and then ap-
ply the myopic allocation for the system. It will be interesting to examine whether our results can be applied to the non-identical retailer system. We leave this for the future research.

Finally, our solution bounds can be extended to a system with Markov-modulated demand. More specifically, assume that the demand process is driven by a homogeneous, discrete-time Markov chain $W$ with $K$ states. It is known that a state-dependent echelon base-stock policy is optimal (e.g., Chen and Song 2000, Muharremoglu and Tsitsiklis 2008). Let $s_j(k, t)$ be the optimal echelon base-stock level for stage $j$ when the demand state is $k$ in period $t$, $k = 1, ..., K$. Following a similar analysis, we can derive single-stage bounds $s_L^j(k, t)$ and $s_U^j(k, t)$ for each demand state $k$ in each period $t$ such that $s_L^j(k, t) \leq s_j(k, t) \leq s_U^j(k, t)$. The proof is available from the author.

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**References**


Appendix A: Optimal Value Functions

To facilitate the analysis of the proofs in Appendix B, we provide dynamic program formulations for the Clark-Scarf model, the upper-bound system, and the lower-bound system. Note that our formulation for the Clark-Scarf model is different from that in Clark and Scarf (1960), who decouple the total cost by introducing an induced-penalty cost function. With the penalty cost function, Clark and Scarf decouple the total cost into each stage. We, on the other hand, decouple the total cost into each echelon. This alternative decomposition scheme is necessary for our analysis because the functional equation for each stage in the upper- and the lower-bound systems turns out to be the echelon cost. Also, to simplify the notation and the analysis, we only focus on the two-stage system with $\tau_1 = \tau_2 = 1$. A similar but more tedious analysis can be carried out for the general system.

The Clark and Scarf Model

Let $x_j$ be the echelon inventory level at stage $j$ after a shipment is received, and let $y_j$ be the echelon inventory in-transit position at stage $j$ after an order is placed. Define $L_j(x_j, t)$ the inventory cost incurred for stage $j$ in period $t$ when the echelon inventory level is $x_j$, namely,

$$
L_1(x_1, t) = \mathbb{E}[h_1(x_1 - D(t)) + (b + h_1 + h_2)(x_1 - D(t))^-],
$$

$$
L_2(x_2, t) = \mathbb{E}[h_2(x_2 - D(t))],
$$

where $(x)^- = \max\{0, -x\}$. The total inventory holding and backorder cost incurred in period $t$ is then $L_1(x_1, t) + L_2(x_2, t)$. Let $f_t(x_1, x_2)$ be the optimal total discounted cost for the system with initial echelon inventory levels $(x_1, x_2)$ when there are $t$ periods to go. An alternative dynamic program based on the echelon system for the Clark-Scarf model is as follows: Let $f_0(x_1, x_2) = 0$. For $t \geq 1$,

$$
f_t(x_1, x_2) = \min_{x_1 \leq y_1 \leq x_2 \leq y_2} \left\{ p_1(y_1 - x_1) + p_2(y_2 - x_2) + L_1(x_1, t) + L_2(x_2, t) + \alpha\mathbb{E}[f_{t-1}(y_1 - D(t), y_2 - D(t))] \right\}
$$
\[
= C_t(x_1) + G_t(x_2),
\]

where

\[
C_t(x_1) = L_1(x_1, t) - p_1 x_1 + [U_t(\max\{x_1, s_1(t)\}) - U_t(s_1(t))],
\]

(5)

\[
G_t(x_2) = U_t(\min\{x_2, s_1(t)\}) + L_2(x_2, t) - p_2 x_2 + V_t(\max\{x_2, s_2(t)\}),
\]

(6)

\[
U_t(y_1) = p_1 y_1 + \alpha \mathbb{E}[C_{t-1}(y_1 - D(t))],
\]

\[
V_t(y_2) = p_2 y_2 + \alpha \mathbb{E}[G_{t-1}(y_2 - D(t))],
\]

\[
s_1(t) = \arg \min_{y_1} \{U_t(y_1)\},
\]

\[
s_2(t) = \arg \min_{y_2} \{V_t(y_2)\}.
\]

Note that Equations (5) and (6) should be revised for \(t = 1\) and \(t = 2\). When \(t = 1\), both stages will not order, so \(C_1(x_1) = L_1(x_1, 1)\) and \(G_1(x_2) = L_2(x_2, 1)\); when \(t = 2\), stage 2 will not order, so \(G_2(x_2)\) is the same as (6) except that the last term is changed to \(V_2(x_2)\).

Under the above cost decomposition scheme, the total system cost \(f_t(x_1, x_2)\) consists of two echelon cost functions. \(G_t(x_2)\) is the cost for echelon 2, which includes the costs directly and indirectly determined by \(x_2\), assuming that stage 1 will always order up to its optimal base-stock level in each period. \(C_t(x_1)\) includes the remaining costs directly determined by \(x_1\).

**The Upper Bound System**

We first define the following single-period inventory holding and backorder cost function:

\[
L(x_1, t) = \mathbb{E}[(h_1 + h_2)(x_1 - D(t)) + (b + h_1 + h_2)(x_1 - D(t)^-)].
\]

Let \(\overline{f}_t(x_1, x_2)\) denote the optimal total cost when the state is \((x_1, x_2)\) at the beginning of period \(t\), and \(\overline{f}_0(x_1, x_2) = 0\). For any \(t \geq 1\),

\[
\overline{f}_t(x_1, x_2) = \min_{x_2 \leq y_2} \left\{ p_1(x_2 - x_1) + p_2(y_2 - x_2) + L(x_1, t) + L_2(x_2, t) \\
+ \alpha \mathbb{E}[\overline{f}_{t-1}(x_2 - D(t), y_2 - D(t)) \] \\
= \overline{C}_t(x_1) + \overline{G}_t(x_2),
\]

where

\[
\overline{C}_t(x_1) = L_1(x_1, t) - p_1 x_1,
\]
\[
\begin{align*}
\overline{C}_t(x_2) &= \overline{U}_t(x_2) + L_2(x_2, t) - p_2x_2 + \overline{V}_t(\max\{s'_2(t), x_2\}), \\
\overline{U}_t(x_2) &= p_1x_2 + \alpha E[\overline{C}_{t-1}(x_2 - D(t))], \\
\overline{V}_t(y_2) &= p_2y_2 + \alpha E[\overline{C}_{t-1}(y_2 - D(t))], \\
s'_2(t) &= \arg\min_{y_2} \{\overline{V}_t(y_2)\}.
\end{align*}
\]

(We have to revise the above \(\overline{C}_t(x_1)\) and \(\overline{G}_t(x_2)\) when \(t = 1\) and \(t = 2\) in a similar fashion as in the above Clark and Scarf formulation.)

The Lower Bound System

Let \(f_t(x_1, x_2)\) denote the optimal discounted total cost with initial echelon inventory levels \((x_1, x_2)\) when there are \(t\) periods before termination, and \(f_0(x_1, x_2) = 0\). We define the one-period inventory cost as follows:

\[
\begin{align*}
L_1(x_1, t) &= E[(b + h_2)(x_1 - D(t))], \\
L_2(x_2, t) &= E[h_2(x_2 - D(t))].
\end{align*}
\]

For any \(t \geq 1\),

\[
\begin{align*}
\hat{f}_t(x_1, x_2) &= \min_{x_1 \leq y_1 \leq x_2 \leq y_2} \left\{p_2(y_2 - x_2) + L_1(x_1, t) + L_2(x_2, t) + \alpha E[f_{t-1}(y_1 - D(t), y_2 - D(t))]\right\} \\
&= \overline{C}_t(x_1) + \overline{G}_t(x_2),
\end{align*}
\]

where

\[
\begin{align*}
\overline{C}_t(x_1) &= L_1(x_1, t), \\
\overline{G}_t(x_2) &= \overline{U}_t(x_2) + L_2(x_2, t) - p_2x_2 + \overline{V}_t(\max\{x_2, s_2^t(t)\}), \\
\overline{U}_t(x_1) &= \alpha E[\overline{C}_{t-1}(x_2 - D(t))], \\
\overline{V}_t(y_2) &= p_2y_2 + \alpha E[\overline{G}_{t-1}(y_2 - D(t))], \\
s_2^t(t) &= \arg\min_{y_2} \{\overline{V}_t(y_2)\}.
\end{align*}
\]

(We have to revise the above \(\overline{C}_t(x_1)\) and \(\overline{G}_t(x_2)\) when \(t = 1\) and \(t = 2\) in a similar fashion as in the above Clark and Scarf formulation.)
Appendix B: Proofs

Theorem 1

We use the notation “$\Delta$” to represent the difference of a function, i.e.,

$$\Delta f(x) = f(x + 1) - f(x).$$

Part (1) can be shown by a simple induction.

To show part (2), we need to show $\Delta V_t(y_2) \geq \Delta V_t(y_2)$ for all $y_2$ when $x_1 < s_1(t)$. We prove this result by induction. Notice that under the assumption of $x_1 < s_1(t)$, $C_t(x_1) = L(x_1, t) - p_1x_1$, which is equal to $\overline{C}_t(x_1)$. Thus, in each period $t$, $U_t(\cdot) = \overline{U}_t(\cdot)$.

We start the induction from $t = 3$. When $t = 3$,

$$\Delta V_3(y_2) = p_2 + \alpha E\left[G_2(y_2 + 1 - D(3)) - G_2(y_2 - D(3))\right]$$

$$= p_2 + \alpha E\left[U_2(\min\{y_2 + 1 - D(3), s_1(2)\}) - U_2(\min\{y_2 - D(3), s_1(2)\}) + (h_2 - p_2) + \Delta V_2(y_2 - D(3))\right]$$

$$\leq p_2 + \alpha E[U_2(y_2 + 1 - D(3)) - U_2(y_2 - D(3)) + (h_2 - p_2) + \Delta V_2(y_2 - D(3))]$$

$$= \Delta V_3(y_2).$$

Thus, $s_2(3) \geq s_2'(3)$. (In fact, the condition is not required for $t = 3$.)

Suppose $t - 1$ holds true, i.e., $\Delta V_{t-1}(y_2) \leq \Delta V_{t-1}(y_2)$ and $s_2(t-1) \geq s_2'(t-1)$. For period $t$, we have

$$\Delta V_t(y_2) = p_2 + \alpha E[G_{t-1}(y_2 + 1 - D(t)) - G_{t-1}(y_2 - D(t))]$$

$$= p_2 + \alpha E\left[U_{t-1}(\min\{y_2 + 1 - D(t), s_1(t - 1)\}) - U_{t-1}(\max\{y_2 + 1 - D(t), s_2(t - 1)\}) - U_{t-1}(\min\{y_2 - D(t), s_1(t - 1)\}) + U_{t-1}(\max\{y_2 - D(t), s_2(t - 1)\}) + (h_2 - p_2)\right]$$

$$\leq p_2 + \alpha E\left[U_{t-1}(y_2 + 1 - D(t)) - U_{t-1}(y_2 - D(t)) + (h_2 - p_2) + \Delta V_{t-1}(\max\{y_2 - D(t), s_2(t - 1)\})\right]$$

$$= p_2 + \alpha E\left[U_{t-1}(y_2 + 1 - D(t)) - U_{t-1}(y_2 - D(t)) + (h_2 - p_2) + \Delta V_{t-1}(\max\{y_2 - D(t), s_2'(t - 1)\})\right]$$

$$\leq p_2 + \alpha E[\overline{V}_{t-1}(y_2 + 1 - D(t)) - \overline{V}_{t-1}(y_2 - D(t)) + (h_2 - p_2) + \Delta \overline{V}_{t-1}(\max\{y_2 - D(t), s_2'(t - 1)\})]$$

$$= p_2 + \alpha E[\overline{V}_{t-1}(y_2 + 1 - D(t)) - \overline{V}_{t-1}(y_2 - D(t))]$$

$$= \Delta \overline{V}_t(y_2).$$

The last inequality is due to $\Delta V_{t-1}(y_2) \leq \Delta \overline{V}_{t-1}(y_2)$ and the third equality is due to $s_2'(t - 1) \leq s_2(t - 1)$. 

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Proposition 2

The single-period inventory holding and backorder cost for the two-stage system can be expressed as

\[ L_1(x_1, t) + L_2(x_2, t) = L(x_1, t) + h_2(x_2 - x_1). \]

One can view \( L(x_1, t) \) as the local inventory holding and backorder cost for stage 1 and \( h_2(x_2 - x_1) \) as the inventory holding cost incurred by the inventory held at stage 2 and in transit to stage 1.

It is more clear to prove this result by redefining the state variables. Let \( m = x_2 - x_1 \) and \( z = y_2 - x_2 \). With slight abuse of notation, the optimality equation for the two-stage upper-bound system is

\[
\begin{align*}
\bar{f}_t(x_1, m) &= \min_{z \geq 0} \left\{ p_2 z + (p_1 + h_2) m + L(x_1, t) + \alpha \mathbb{E}[\bar{f}_{t-1}(x_1 + m - D(t), z)] \right\}, \\
\end{align*}
\]

where \( \bar{f}_0(x_1, m) = 0, \bar{f}_1(x_1, m) = L(x_1, 1) + h_2 m, \) and

\[
\begin{align*}
\bar{f}_2(x_1, m) &= \min_{z \geq 0} \left\{ p_2 z + (p_1 + h_2) m + L(x_1, 2) + \alpha \mathbb{E}[\bar{f}_{t-1}(x_1 + m - D(2), z)] \right\} \\
&= (p_1 + h_2) m + L(x_1, 2) + \alpha \mathbb{E}[L(x_1 + m - D(2), 1)] + \min_{z \geq 0} \left\{ (p_2 + \alpha h_2) z \right\} \\
&= (p_1 + h_2) m + L(x_1, 2) + \alpha \mathbb{E}[L(x_1 + m - D(2), 1)].
\end{align*}
\]

For \( t \geq 3 \), we have

\[
\bar{f}_t(x_1, m) = (p_1 + h_2) m + L(x_1, t) + \alpha \mathbb{E}[L(x_1 + m - D(t), t - 1)] + \bar{G}_t(x_1 + m),
\]

where

\[
\begin{align*}
\bar{G}_t(x_1 + m) &= \min_{z \geq 0} \left\{ [p_2 + \alpha (p_1 + h_2)] z + \alpha^2 \mathbb{E}[L(x_1 + m + z - D(t) - D(t - 1), t - 2)] \\
&\quad + \alpha \mathbb{E}[\bar{G}_{t-1}(x_1 + m + z - D(t)))] \right\}.
\end{align*}
\]

This dynamic program has the same structure as that of the single-stage problem (Karlin and Scarf 1958) with lead time of two periods and with the specified cost parameters in the proposition.

Theorem 3

Part (1) is straightforward and omitted.

We show Part (2) by induction. To show \( s_2^u(t) \geq s_2(t) \), we need to show \( \Delta V_t(y_2) \geq \Delta V_1(y_2) \) for all \( y_2 \). Denote \( (x)^+ = \max\{x, 0\} \).

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Let us first consider $t = 3$.

\[
\Delta V_3(y_2) = p_2 + \alpha \mathbb{E}[G_2(y_2 + 1 - D(3)) - G_2(y_2 - D(3))]
\]

\[
= p_2 + \alpha \mathbb{E} \left[ U_2(\min\{y_2 + 1 - D(3), s_1(2)\}) - U_2(\min\{y_2 - D(3), s_1(2)\}) + (h_2 - p_2) + \Delta V_2(y_2 - D(3)) \right]
\]

\[
= p_2 + \alpha \mathbb{E} \left[ p_1(\min\{y_2 + 1 - D(3), s_1(2)\}) - p_1(\min\{y_2 - D(3), s_1(2)\}) + \alpha \mathbb{E}[C_1(\min\{y_2 + 1 - D(3), s_1(2)\} - D(2))] - \alpha \mathbb{E}[C_1(\min\{y_2 - D(3), s_1(2)\} - D(2))] + (h_2 - p_2) + \Delta V_2(y_2 - D(3)) \right]
\]

\[
\geq p_2 + \alpha \mathbb{E} \left[ \alpha \mathbb{E}\left[ h_1(\min\{y_2 + 1 - D(3), s_1(2)\} - D(2) - D(1))^+ \right]
\right]
\]

\[
+ (b + h_2)(\min\{y_2 + 1 - D(3), s_1(2)\} - D(2) - D(1))^-
\]

\[
- \alpha \mathbb{E}\left[ h_1(\min\{y_2 - D(3), s_1(2)\} - D(2) - D(1))^+ \right]
\]

\[
+ (b + h_2)(\min\{y_2 - D(3), s_1(2)\} - D(2) - D(1))^-
\]

\[
+ (h_2 - p_2) + \Delta V_2(y_2 - D(3)) \right]
\]

\[
\geq p_2 + \alpha \mathbb{E} \left[ \alpha \mathbb{E}\left[ (b + h_2)(\min\{y_2 + 1 - D(3), s_1(2)\} - D(2) - D(1))^- \right]
\right]
\]

\[
- \alpha \mathbb{E}\left[ (b + h_2)(\min\{y_2 - D(3), s_1(2)\} - D(2) - D(1))^- \right]
\]

\[
+ (h_2 - p_2) + \Delta V_2(y_2 - D(3)) \right]
\]

\[
\geq p_2 + \alpha \mathbb{E} \left[ \alpha \mathbb{E}\left[ L_1(y_2 + 1 - D(3) - D(2), 1) - L_1(y_2 - D(3) - D(2), 1) \right]
\right]
\]

\[
+ (h_2 - p_2) + \Delta V_2(y_2 - D(3)) \right] \quad \text{(because } \mathbb{V}_2(\cdot) = \mathbb{V}_2(\cdot))
\]

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\[
\begin{align*}
\geq & \quad p_2 + \alpha \mathbb{E} \left[ U_3(y_2 + 1 - D(3)) - U_2(y_2 - D(3)) + (h_2 - p_2) + \Delta V_2(y_2 - D(3)) \right] \\
= & \quad \Delta V_1(y_2).
\end{align*}
\]

Thus, we have \( s_2^y(3) \geq s_2(3) \).

Suppose \( t - 1 \) holds true, that is, \( \Delta V_{t-1}(y_2) \geq \Delta V_{t-1}(y_2) \) for all \( y_2 \), and \( s_2^y(t - 1) \geq s_2(t - 1) \). Then, for period \( t \),

\[
\begin{align*}
\Delta V_t(y_2) & = \quad p_2 + \alpha \mathbb{E} [G_{t-1}(y_2 + 1 - D(t)) - G_{t-1}(y_2 - D(t))] \\
& = \quad p_2 + \alpha \mathbb{E} \left[ U_{t-1}(\min\{y_2 + 1 - D(t), s_1(t - 1)\}) - U_{t-1}(\min\{y_2 - D(t), s_1(t - 1)\}) + (h_2 - p_2) + \Delta V_{t-1}(\max\{y_2 - D(t), s_2(t - 1)\}) \right] \\
& = \quad p_2 + \alpha \mathbb{E} \left[ p_1(\min\{y_2 + 1 - D(t), s_1(t - 1)\}) - p_1(\min\{y_2 - D(t), s_1(t - 1)\}) + \alpha \mathbb{E} [C_{t-2}(\min\{y_2 + 1 - D(t), s_1(t - 1)\} - D(t - 1))] \\
& \quad - \alpha \mathbb{E} [C_{t-2}(\min\{y_2 - D(t), s_1(t - 1)\} - D(t - 1))] + (h_2 - p_2) + \Delta V_{t-1}(\max\{y_2 - D(t), s_2(t - 1)\}) \right] \\
& \geq \quad p_2 + \alpha \mathbb{E} \left[ \alpha \mathbb{E} [L_1(\min\{y_2 + 1 - D(t), s_1(t - 1)\} - D(t - 1), t - 2) \\
& \quad - L_1(\min\{y_2 - D(t), s_1(t - 1)\} - D(t - 1), t - 2)] + (h_2 - p_2) + \Delta V_{t-1}(\max\{y_2 - D(t), s_2(t - 1)\}) \right] \\
& \geq \quad p_2 + \alpha \mathbb{E} \left[ \alpha \mathbb{E} [L_1(\min\{y_2 + 1 - D(t), s_1(t - 1)\} - D(t - 1), t - 2) \\
& \quad - L_1(\min\{y_2 - D(t), s_1(t - 1)\} - D(t - 1), t - 2)] + (h_2 - p_2) + \Delta V_{t-1}(\max\{y_2 - D(t), s_2(t - 1)\}) \right] \\
& = \quad p_2 + \alpha \mathbb{E} \left[ \alpha \mathbb{E} [L_1(\min\{y_2 + 1 - D(t), s_1(t - 1)\} - D(t - 1), t - 2) \\
& \quad - L_1(\min\{y_2 - D(t), s_1(t - 1)\} - D(t - 1), t - 2)] + (h_2 - p_2) + \Delta V_{t-1}(\max\{y_2 - D(t), s_2^y(t - 1)\}) \right] \\
& = \quad \Delta V_t(y_2).
\end{align*}
\]

Thus, we have \( s_2^y(t) \geq s_2(t) \).
Propositions 4

The proof is similar to that of proposition 2, and thus omitted.

Proposition 5

The dynamic program for a single-stage system with lead time $L$ periods, order cost $p$, holding cost rate $h$, and backorder cost rate $b$ is as follows:

$$f_t(x) = \min_{y \geq x} \left\{ p(y - x) + \alpha^L \mathbb{E}[L(y - D[t, t - L + 1], t - L)] + \alpha \mathbb{E}[f_{t-1}(y - D(t))] \right\}$$

$$= \min_{y \geq x} \{ u_t(y) \} - px.$$

where $f_0(x) = 0$. Here, $x$ and $y$ are the inventory order position before and after ordering at the beginning of period $t$, respectively. $L(x, t)$ is the one-period inventory holding and backorder cost in period $t$ when the initial inventory order position is $x$. The optimal base-stock level $s(t)$ can be found by minimizing $u_t(y)$ in each period: $s(t) = \arg\min_y \{ u_t(y) \}$, and the resulting optimal value function is $f_t(x) = u_t(\max\{x, s(t)\}) - px$.

Let $s^m(t)$ denote the myopic solution in period $t$ for the above inventory problem. For $t = L + 1$,

$$s^m(L + 1) = \arg\min_y \{ py + \alpha^L \mathbb{E}[L(y - D[L + 1, 1])] \}.$$

Thus, $s^m(L + 1)$ is the smallest $y$ such that $\mathbb{P}(D[L + 1, 1]) \leq y \geq (\alpha^L b - p)/\alpha^L (b + h)$.

For $t > L + 1$,

$$f_t(x) = \min_{y \geq x} \left\{ p(y - x) + \alpha^L \mathbb{E}[L(y - D[t, t - L + 1], t - L)] + \alpha \mathbb{E}[u_{t-1}(\max\{y - D(t), s(t - 1)\}) - p(y - D(t))] \right\}.$$

Thus, $s^m(t) = \arg\min_y \{ p(1 - \alpha)y + \alpha^L \mathbb{E}[L(y - D[t, t - L + 1], t - L)] \}$, or equivalently, $s^m(t)$ is the smallest $y$ such that $\mathbb{P}(D[L + 1, 1]) \leq y \geq (\alpha^L b - p(1 - \alpha))/\alpha^L (b + h)$.