Joint Inventory and Cash Management for Multi-Divisional Supply Chains

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May 10, 2013

Cash pooling is a powerful tool to consolidate intragroup liquidity for multi-divisional corporations. This paper develops a centralized supply chain model that aims to assess the value of cash pooling. The supply chain is owned by a single corporation with two divisions, where the downstream division (headquarter), facing random customer demand, replenishes materials from the upstream one. The downstream division receives cash payments from customers and determines a system-wide inventory replenishment and cash retention policy. We consider two cash management systems that represent different levels of cash concentration. For cash pooling, the supply chain adopts a financial services platform which allows the headquarter to create a corporate master account that aggregates the divisions’ cash. For transfer pricing, on the other hand, each division owns its cash and pays for the ordered material according to a fixed price. Comparing both systems yields the value of adopting such financial services. We prove that the optimal policy for the cash pooling model has a surprisingly simple structure – both divisions implement a base-stock policy for material control; the headquarter monitors the corporate working capital and implements a two-threshold policy for cash retention. Solving the transfer pricing model is more involved. We derive a lower bound on the optimal cost by connecting the model to an assembly system. Our results show that the value of cash pooling can be very significant when demand is increasing (stationary) and the markup for the upstream division is small (high). Nevertheless, a big portion of the pooling benefit may be recovered if the headquarter can decide the optimal transfer price and the lead time is short.

Key words: multi-echelon, cash pooling, supply chain integration
1 Introduction

The fundamental objective of supply chain management is to efficiently coordinate material, information, and financial flows to reduce mismatches between demand and supply. When financial markets are efficient, i.e., external funding is plentiful and relatively inexpensive, the financial decisions can be decoupled from the logistics decisions (Modigliani and Miller 1958). In this case, a downstream entity pays material it orders from an upstream entity so financial flow becomes the output of logistics decisions. This perspective may explain why the supply chain literature has largely focused on the integration of material and information flows. Nonetheless, with the recent global financial crisis limiting the availability of external funding, many multinational, multi-divisional corporations in their “hunt for cash” have witnessed a significant increase on their intragroup financial transactions (Rogers et al. 2009). The reason is simple: these multinationals realized that they can concentrate the intragroup liquidity for centralized planning to receive more benefit. For example, Hewlett-Packard and General Electric transferred funds from their overseas divisions to domestic ones for the sake of the entire company group (Linebaugh 2013). One common practice of cash concentration is cash pooling (Polak and Klusacek 2010). Under cash pooling, the headquarter creates a corporate master account that aggregates divisions’ cash on a daily basis (Jansen 2011). While the value of cash pooling has been discussed in the finance literature, there is little study that assesses the value from a supply chain perspective. Indeed, the discussion of integrating financial flows into supply chain models is relatively sparse in the supply chain literature. The objective of this paper is to fill this gap.

We consider a corporation that owns a supply chain consisting of two divisions (can be generalized to N divisions). The downstream division, division 1, replenishes inventory from an upstream division, division 2, which further replenishes from an outside ample vendor. There is a positive delivery lead time for both divisions. Division 1 is the headquarter that faces stochastic customer demand. The demands are independent between periods but not necessarily identical. Division 1 receives cash payment from customers who order the material. Similarly, division 2 pays to the outside vendor for the material shipped. To make the consideration of financial flow relevant, we assume that there is no external borrowing. The decision is centralized and the headquarter has to decide a system-wide inventory replenishment and cash retention policy. The cash retention policy refers to how much cash the supply chain should hold for operations, i.e., inventory payments in our context. Typically, firms do not wish to hold excess cash as it loses the potential benefit from external investments; on the other hand, liquidating the invested assets to assist operations incurs transaction costs, or may not be feasible in some cases (Baumol 1952, Miller and Orr 1966). Thus, the headquarter has to find a balance between the cash retained for internal operations and that invested for external assets.
We consider two cash management systems that represent different levels of cash concentration. For the cash pooling system, a financial services platform is adopted, and the entire supply chain is operated under a single account for conducting financial transactions with customers and the outside vendor. For the transfer pricing system, no such platform is installed and the headquarter makes the joint decision under a two-account regime: each division maintains its own cash and division 1 pays exactly what it orders to division 2 according to a fixed internal transfer price, i.e., the price that a selling division charges for a product or service supplied to a buying division of the same corporation (Abdallah 1989). We assume that the transfer price is pre-determined according to a market price (Martini 2011). There are linear holding and backorder costs related to the inventory. In addition, there is opportunity cost of holding cash for internal operations. The objective is to find a joint inventory replenishment and cash retention policy such that the total supply chain cost is minimized within a finite horizon under each of the cash management systems.

The logistics system of the considered supply chain is a seminal model proposed by Clark and Scarf (1960). We incorporate cash flows into this classical model. The transfer pricing model represents a traditional supply chain in the sense that cash flow is driven by the inventory decision (constrained by the available cash). Thus, cash may not be efficiently distributed, leading to a less effective inventory and cash retention policy. For example, cash shortage of the upstream division will affect its normal operations, which, in turn, affects the material supply to the downstream one. This inefficiency can be mitigated under cash pooling because the headquarter can consolidate the cash within the supply chain for better usage. In practice, there are physical pooling and notional (virtual) pooling (i.e., funds are not physically transferred but managed as if they were in a single account). In any case, cash pooling usually requires financial and legal services provided by a third party and installing a costly system-wide technology platform, such as treasury management system, for transferring funds from divisions to the headquarter (Camerinelli 2010). Thus, our study of comparing these two systems can be used to justify the value of adopting such financial services and technology platforms.

We first formulate a dynamic program for the cash pooling model which includes two inventory states and one cash state (the corporate master account). To be consistent with the inventory literature, we name division and stage interchangeably. The problem is difficult to solve as one cannot directly prove a structured joint optimal policy. Nevertheless, by redefining the state variables into echelon terms, we can transform the original two-stage system into a three-echelon system, under which the optimal joint policy can be characterized. The optimal policy is surprisingly simple. The inventory policy has the same structure as that of the traditional multi-echelon system (cf. Clark and Scarf 1960): each stage reviews the echelon inventory position at the beginning of a period and orders up to a target
echelon base-stock level. For the cash retention policy, stage 1 (or the headquarter) reviews the entire system working capital (= inventory on hand at both stages + inventory in transit - backorders at stage 1 + inventory-equivalent total system cash) at the beginning of each period and retains the cash holding within an interval determined by two threshold values. A key technical contribution is that we simplify the computation by decoupling the original dynamic program with three states into three separate dynamic programs, each with one state variable. Thus, the optimal policy parameters can be easily calculated. The decoupling result is based on a set of penalty cost functions, some appearing to be new in the literature.

Solving the transfer pricing model is more involved. Simply speaking, the problem is similar to a serial capacitated system (cf. Parker and Kapuscinski 2004) in the sense that the on-hand cash level at each stage can be viewed as a budgetary constraint that restricts the amount of inventory ordering. However, a major difference between the traditional capacitated system and ours is that the cash constraint is endogenously determined by the inventory and cash retention decisions. Although we are not able to characterize the optimal policy, we provide a lower bound to the optimal cost by connecting the transfer pricing model to an assembly system (c.f. Rosling 1989) with two component flows – stage 1’s cash flow and the system’s material flow.

We obtain several insights. First, the optimal policy of the cash pooling model suggests that the inventory decision can be made separately from the cash retention decision; however, making the cash retention decision has to take into account the entire supply chain inventory. That is, monitoring system working capital level is key to ensuring the system efficiency. In most firms, the cash retention decision is made by a treasurer in the finance department, and the replenishment decision is made by an inventory manager in the operations department. An implication of our finding is that close inter-departmental collaboration is crucial. Second, one of the most important financial decisions for a firm is to decide how much cash to hold in order to cope against the volatility of the external environment (Opler, et al. 1999, Ramirez et al. 2007, Baum et al 2008). We consider demand volatility and find that when the demand becomes more variable, the supply chain will simultaneously increase the amount of cash and inventory holdings. Nonetheless, the change of cash holding is relatively smaller than that of inventory holding. Thus, our analysis is useful for firms to decide an optimal liquid-asset mix (i.e., cash and inventory). Third, comparing the optimal cost of the cash pooling model and the lower bound cost of the transfer pricing model renders the (conservative) value of cash pooling, or equivalently, installing the financial services platform. Our numerical result suggests that the value of cash pooling can be very significant when the markup of the upstream stage is low and the demand is increasing or when the markup is high and the demand tends to be stationary. For the former case, the upstream
division tends to have a cash shortage in the transfer pricing model. Lacking cash at the upstream stage restricts the order quantity, which affects the material supply for the downstream stage. On the other hand, for the latter case, there is excess cash accumulated in the upstream division. Pooling cash together will facilitate the headquarter to invest the excess cash to external assets.

A natural extension of the transfer pricing model is how much benefit can be recovered if the headquarter can determine an optimal transfer price. Determining transfer prices is one of the most important topics for multi-divisional firms in the finance literature (see §2). When an inventory manager attempts to determine the optimal flows of products among divisions, the price of a product is almost always considered as a given parameter. However, this is not the case in real multi-divisional firms since the transfer price is inherently subjective and the headquarter can determine it with some degree of flexibility through *advance pricing agreements* (Martini 2011)\(^1\). Thus, one can treat transfer pricing as a tool of re-distributing cash between divisions (Stewart 1977). We refer to the system with optimized internal transfer price as the *optimal pricing system*. The optimal transfer price can be determined from the optimal order quantity and the *optimal cash payment* between these two divisions in each period (i.e., transfer price equals to cash payment divided by order quantity)\(^2\). In other words, we need to obtain an optimal joint inventory replenishment, cash payment, and cash retention policy for the supply chain. Interestingly, this joint optimal policy can be obtained by extending the solution approach for the cash pooling model. More specifically, the inventory replenishment and cash retention policy structure remains the same as those in the cash pooling system; for the payment between divisions, division 2 monitors its echelon working capital and receives payment up to a target level.

In a numerical study, we find that optimizing the transfer price can recover a big portion of the cash pooling benefit. However, the benefit decreases when the lead time is long or the cash holding cost rates are significantly different between the divisions. The benefit of re-distributing liquidity through the optimal transfer price can be clearly demonstrated in the product life cycle example in §5: during the introduction and growth stages, the upstream division is normally short of cash so a cash subsidy through increasing the transfer price is valuable. On the other hand, during the mature and decline stages, the upstream division has accumulated sufficient funds for the decreased demand, so a reduction of cash payment through decreasing the transfer price is beneficial.

\(^1\)Our focus is to determine the system-wide optimal transfer price. Certainly, an optimal transfer price may not be aligned with each division’s best interest, so there is a separate issue regarding how to implement the optimal transfer prices for the divisions. However, if implementing the optimal transfer price decreases the supply chain’s cost, the headquarter can capture this benefit and design an incentive compatible compensation, such as side payment, that induces the division to accept the transfer price.

\(^2\)In our context, the optimal cash payment between two divisions can broadly include intragroup loans or financial subsidies.
2 Literature Review

Our work is related to four streams of research in the literature: cash management, multi-echelon inventory models, capacitated inventory models, and inventory models with financial issues.

For cash management in single firms, most papers treat cash as inventory and use inventory control tools to find the optimal cash balance for firms. Baumol (1952) studied the optimal cash level for a firm that uses cash either for paying transactions or for investment. We have a similar setup for the headquarter in our model. This line of research was further extended by Tobin (1956) and Miller and Orr (1966). For dynamic, periodic-review cash balance problems, Girgis (1968) modeled the selection of a cash level in anticipation of future net expenses as a single-product, multi-period inventory system. Heyman (1973) presented a model to minimize the average cash balance subject to a constraint on the probability of stock-out. The difference between these studies and ours is that we specifically model the cash and inventory dynamics as two inter-related flows.

For cash management in multi-divisional corporations, our model is related to resource allocation from a centralized planning perspective. This literature can be categorized into two groups. The first group is related to cash pooling. Most literature focuses on how cash pooling is implemented for multinational firms and discusses its potential benefits, e.g., Wündisch (1973), Cooper (2004), Polak and Klusacek (2010), Jansen (2011). Eijje and Westerman (2002) suggested that the reduction of financial imperfections in transferring cash in the euro zone diminishes the need for separate local cash holdings and facilitates the cash concentration and headquarter’s financial control. Building an analytical model to access the value of cash pooling is new and our paper contributes to this topic. The second group concerns obtaining transfer prices to maximize the profit for a multi-divisional corporation, e.g., Merville and Petty (1978), Vidal and Goetschalckx (2001), Gjerdrum et al. (2002), Lakhal (2006), Curtis (2008), Villegas and Oueniche (2008), and Perron et al. (2010). Our model is different from these papers in that we consider a supply chain setting in a finite horizon and characterize the optimal control policy. There is another stream of research regarding how to design transfer prices to coordinate decentralized divisions (i.e., each division has high autonomy and is treated as a profit center). Since the focus is different from our paper, we refer the interested reader to Ronen and McKinney (1970), Yoon et al. (2000), and references therein.

Our research is also related to the multi-echelon literature. In particular, our model incorporates cash flows into the seminal supply chain model developed by Clark and Scarf (1960), who proved that an echelon base-stock policy is optimal. Furthermore, they showed that the problem can be decoupled into a series of one-dimensional dynamic programs by introducing the notion of echelon inventories. Federgruen and Zipkin (1984) and Chen and Zheng (1994) streamlined the analysis by considering an
infinite horizon model. Recently, Angelus (2011) considered a multi-echelon model which allows each stage to dispose excess inventory to a secondary market. He introduced a class of heuristic policies, called disposal saturation policies, which can be obtained by using the Clark-Scarf decomposition.

The capacitated inventory problem is related to our model since the cash constraint on inventory replenishment can be viewed as the supply capacity. For single-stage systems, Federgruen and Zipkin (1986) showed that the modified base-stock policy is optimal. Angelus and Porteus (2002) derived the optimal joint capacity adjustment and production plan with and without carryover of unsold inventory units. Their capacity adjustment decision is similar to our cash retention decision, but our cash holding amount is also affected by payment decisions and random sales. For serial systems, Parker and Kapuscinski (2004) demonstrated that a modified echelon base-stock policy is optimal in a two-stage system where there is a smaller capacity at the downstream facility. Glasserman and Tayur (1995) and Huh et al. (2010) studied the stability issue of the system. The main difference between the serial capacitated models and ours is that the cash constraint is endogenously determined by the inventory and cash decisions.

Finally, there have been several recent studies to incorporate financial decisions or budget constraints into inventory models. Most of these papers are based on single-stage systems. Buzacott and Zhang (2004) incorporated asset-based financing into production decisions. They demonstrated the importance of joint consideration of production and financing decisions to capital constrained firms. Chao et al. (2008) considered a self-financing retailer who replenishes inventory under a cash budget constraint. They characterized the optimal inventory control policy. Gupta and Wang (2009) presented a discrete-time inventory model with trade credit and showed that the problem can be converted into a single-stage system model with refined holding cost rates. Babich (2010) studied a manufacturer's joint inventory and financial subsidy decisions when facing a supplier whose financial state is governed by a firm-value model. He showed that an order-up-to policy and subsidize-up-to policy are optimal for the manufacturer. Yang and Birge (2011) modeled a Stackelberg game between a retailer and a supplier with the use of a trade credit contract. They demonstrated that an effective trade credit contract can enhance supply chain efficiency. Bendavid et al. (2012) analyzed the material management practices of a self-financing firm under working capital requirement. Song and Tong (2012) provided a new accounting framework to study inventory systems with different payment times. Tanrisever et al. (2012) built a two-period model to study a start-up firm's trade-off between process investment and survival. Li et al. (2013) studied a dynamic model in which inventory and financial decisions are made simultaneously in the presence of uncertain demand. They characterized the policy that maximizes the expected present value of dividends. Ding et al. (2007) studied an integrated operational
and financial hedging decision faced by a global firm which sells to both home and foreign markets. For multi-echelon models, Hu and Sobel (2007) studied a serial inventory model with the objective of optimizing the expected present value of dividends. They showed that there is no optimal echelon base-stock policy if there are financial constraints. Protopappa-Sieke and Seifert (2010) conducted a simulation study on a two-stage supply chain to reveal qualitative insights on the allocation of working capital between the supply chain partners. Chou et al. (2013) studied a one-warehouse-multi-retailer system with trade credits. They showed that a longer trade credit term received from the external supplier may not lead to a longer trade credit term provided to the retailers.

The rest of this paper is organized as follows. §3 studies the cash pooling model and formulates the corresponding dynamic program. §4 focuses on the transfer pricing model. We provide lower bounds to the optimal cost. §5 discusses the qualitative insights through a numerical study. §6 concludes. Appendix provides proofs. Throughout this paper, we define $x^+ = \max(x, 0)$, $x^- = -\min(x, 0)$, $a \lor b = \max(a, b)$, and $a \land b = \min(a, b)$.

3 Cash Pooling System

We consider a periodic-review, two-stage serial supply chain where stage 1 orders from stage 2, which orders from an outside ample vendor. The supply chain is owned by a single corporation, with stage 1 being the headquarter and stage 2 the subsidiary. Stage 1 faces a stochastic customer demand $D_t$ in period $t$. The demands are independent between periods, but the demand distributions may differ from period to period. We assume that unsatisfied demand is fully backlogged, and the material lead time is one period for both stages (without loss of generality).

This section focuses on the cash pooling (CP) system, in which the headquarter (stage 1) creates a corporate master account that aggregates the divisions’ cash. Here and in the sequel, we use prime to indicate local (stage specific) variables and parameters. After receiving the customer’s payment, the headquarter decides the amount of cash that will be used for external investments, such as money and bond markets, facility expansion, or R&D etc. The remaining cash will be used for operations, that is, paying inventory ordered to the outside vendor. Figure 1 shows the material and cash flows in solid and dashed arrows, respectively. The circle in Figure 1 represents the external investment portfolio on assets; the top white rectangle represents the master account. We assume that the external investment portfolio has a return rate of $\eta'$. Since holding cash for operations has zero return rate, $\eta'$ can be viewed as cash holding cost rate, which represents the opportunity cost of holding cash (Allen and Hafer 1984)\textsuperscript{3}. Moreover, we assume that the headquarter can liquidate its portfolio assets to assist inventory

\textsuperscript{3}Allen and Hafer (1984) conducted an empirical study that shows the cash holding cost rate is positively correlated to a company’s interest return on short-term money markets and long-term on bond markets.
payment, if necessary. Nevertheless, how much cash can flow into the pooled account depends on an
exogenous market condition described by a limit $K'(\geq 0)$ in each period\(^4\). Let $\beta'_i$ and $\beta'_o$ denote the
unit transaction cost charged on the cash transferred to and from the pooled cash account, respectively.
In practice, these transaction costs can be regarded as brokerage fees. Here, $\beta'_i$, $\beta'_o$ and $K'$ represent
the level of easiness of liquidating the portfolio assets into cash.

We now introduce the other cost parameters. Following the inventory literature, we charge a linear
local holding cost $h'_i$ for each unit of inventory held at stage $i$ in each period, and a backorder cost $b$
for each unit of backorder incurred at stage 1 in each period. Here, we assume that $h'_1 > h'_2 > \eta'c$, i.e.,
holding a unit of inventory at downstream is more costly than that at upstream, and holding a unit of
inventory is more costly than holding the same value amount of cash. The later is generally true since
inventory holding cost consists of both the financial opportunity cost and the physical shelf cost.

The inventory replenishment and cash retention decision is made centrally by the headquarter. The
sequence of events in a period is as follows: At the beginning of the period, (1) shipments are received
at both stages; (2) payment is made to the outside vendor; (3) cash retention decision is made; (4)
orders are placed at both stages. During this period, demand is realized and sales revenue is collected.
At the end of the period, all inventory and cash related costs are calculated. The planning horizon is
$T$ periods, and the objective is to minimize the supply chain’s total expected discounted cost within
the entire horizon.

We now define state and decision variables. For stage $i = 1, 2$ and period $t$, let

$$ x'_{1,t} = \text{net inventory level at stage 1 after Event (1)}; $$

$$ x'_{2,t} = \text{on hand inventory level at stage 2 after Event (1)}; $$

$$ w'_t = \text{cash balance in the pooled account after Event (2)}; $$

$$ v_t = \text{amount of cash transferred into the pooled account in Event (3)}; $$

$$ z_{i,t} = \text{order quantity for stage $i$ made in Event (4)}; $$

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\(^4\)The sequence of $K'_i$ can be generalized to a Markov chain that captures the stochastic liquidity level according to
the market condition.
Note that $v_t^+$ is the cash amount that flows into the pooled account and $v_t^-$ is the cash amount that flows out for investment. Clearly, $v_t$ cannot exceed $K'$. Let $p_1$ be the unit selling price to the end customer and $c$ be the unit procurement cost from the outside vendor. We assume $c < p_1$ to ensure profitability. The system dynamics are shown below:

$$x'_{1,t+1} = x'_{1,t} + z_{1,t} - D_t, \quad (1)$$
$$x'_{2,t+1} = x'_{2,t} + z_{2,t} - z_{1,t}, \quad (2)$$
$$w'_{t+1} = w'_t + v_t - cz_{2,t} + p_1 D_t. \quad (3)$$

We assume that the actual payment transaction occurs upon the receipt of shipments. That is, the outside vendor will not receive the payment determined in period $t$ until period $t + 1$, when stage 2 receives the shipment (placed in period $t$). This payment practice is similar to a Letter of Credit (LC). In other words, we can view that there is a one-period lead time for the cash payment.

For the cash dynamic in (3), we assume that the customer will pay at the order epoch. This assumption is reasonable as all demand will be filled under the backorder model. It is also commonly seen in practice (such as iPhone) and in the dynamic pricing literature, e.g., Federgruen and Heching (1999). We do not include inventory holding and backorder costs in (3) because inventory holding cost is usually not incurred in the periodic cash transactions, and backorder cost usually represents loss of goodwill, which is a non-monetary cost.

Define $\mathbf{x}' = (x'_1, x'_2)$, and $\mathbf{z} = (z_1, z_2)$. The constraint set in each period is

$$\tilde{S}(x'_2, w') = \{ \mathbf{z}, v \mid 0 \leq z_1 \leq x'_2, 0 \leq z_2 \leq (w' + v)/c, v \leq K' \}.$$  

The first constraint states that stage 1’s order quantity cannot exceed stage 2’s on-hand inventory; the second constraint states that stage 2’s order quantity is constrained by the cash balance in the pooled account, which also implies that the investment amount in each period cannot exceed its on-hand cash level, i.e., $v \geq -w'$. Finally, the last constraint imposes a limit $K'$ on the amount of cash that can be injected into the pooled cash account.

The single-period expected cost function is

$$\tilde{G}_t(\mathbf{x}', w', z_2, v) = E_{D_t} \left[ h'_1(x'_1 - D_t)^+ + b(x'_1 - D_t)^- \right] + h'_2 x'_2 + c z_2 + \eta E_{D_t} (w' + v + p_1 D_t) + \beta'_i v^+ + \beta'_o v^- . \quad (4)$$

The first line in the cost function is the inventory-related cost, which includes inventory holding, backlogging and procurement costs. By convention, we charge $h'_2$ to the pipeline inventory so $h'_2 x'_2$ is the cost for the inventories held at stage 2 plus those in the pipeline. The second line is the cash-related
cost, which includes cash holding and transaction costs. As shown, we charge \( \eta' \) for \( w' + v + p_1D_t \) because the inventory payment to the outside vendor is held until the receipt of goods.

Let \( \alpha \) be the single-period discount rate. Denote \( \hat{J}_t(x', w', z, v) \) as the expected cost over period \( t \) to \( T + 1 \), given states and decisions \((x', w', z, v)\). Denote \( \hat{V}_t(x', w') \) as the minimum expected cost over period \( t \) to \( T + 1 \) over all feasible decisions. The dynamic program is

\[
\hat{J}_t(x', w', z, v) = \hat{G}_t(x', w', z_2, v) + \alpha E_{D_t}[\hat{V}_{t+1}(x'_1 + z_1 - D_t, x'_2 + z_2 - z_1, w' + v - cz_2 + p_1D_t)],
\]

(5)

\[
\hat{V}_t(x', w') = \min_{z, v \in S(x'_2, w')} \hat{J}_t(x', w', z, v),
\]

(6)

with \( \hat{V}_{T+1}(x', w') = 0 \). Here we assume a zero terminating cost for simplicity. In the sequel, we omit the terminating cost from the dynamics program if it equals to zero.

The local formulation in (5) and (6) is difficult to solve. Specifically, one can show the joint convexity of \( \hat{J}_t(\cdot) \) and derive a state-dependent global minimum solution. However, computing the solution is quite hard due to the curse of dimensionality. In the next section, we transform the original problem into a new system, from which the exact optimal joint policy can be shown to have a surprisingly simple structure.

### 3.1 Echelon Formulation

We transform the original two-stage system into a three-stage serial model by introducing new system variables. First, define the following echelon variables:

\[
x_1 = x'_1, \quad x_2 = x'_1 + x'_2, \quad w = x'_1 + x'_2 + w'/c.
\]

Let \( x = (x_1, x_2) \). We refer to \( x \) as the echelon net inventory level, and \( w \) as the net working capital level measured in inventory unit, which is obtained by converting cash to inventory at the value of \( c \).

This state transformation explicitly treats cash as inventory. More specifically, the financial flow in the system can be seen as an extension of the material flow after “flipping” the corporate master account to upstream. We define the corresponding echelon decision variables:

\[
y_1 = x'_1 + z_1, \quad y_2 = x'_1 + x'_2 + z_2, \quad r = x'_1 + x'_2 + (w' + v)/c.
\]

Let \( y = (y_1, y_2) \). Figure 2 shows the transformed CP system. With this transformation, the cash account becomes stage 3 in the new system, directly supplying stage 2. We hereby call echelon 3 (with state variable \( w \)) as the system working capital.

Similar to the multi-echelon inventory model, we derive the echelon holding cost rate as follows:
Figure 2: The three-stage transformed cash pooling system.

\( \eta = \eta c, \ h_2 = h_2' - \eta c, \) and \( h_1 = h_1' - h_2'. \) Since \( h_1' > h_2' > \eta c \) by assumption, we have \( h_1 > 0 \) and \( h_2 > 0. \) Furthermore, let \( \beta_i = \beta_i' c, \ \beta_o = \beta_o' c, \ \theta = p_1/c - 1 > 0, \) and \( K = K'/c. \) With these echelon terms, the state dynamics in (1)-(3) become

\[
x_{1,t+1} = y_{1,t} - D_t, \quad x_{2,t+1} = y_{2,t} - D_t, \quad w_{t+1} = r_t + \theta D_t,
\]

and the constraint set becomes

\[
S(x, w) = \{ y, r \mid x_1 \leq y_1 \leq x_2 \leq y_2 \leq r \leq w + K \}.
\]

We further specify the holding and backorder cost associated with each echelon:

\[
H_{1,t}(x_1) = E_{D_t}
\left[
(\eta c + h_2 + \eta + b)(D_t - x_1)^+ + h_1(x_1 - D_t)
\right],
\]

\[
H_{2,t}(x_2) = E_{D_t} h_2(x_2 - D_t),
\]

\[
H_{3,t}(r) = E_{D_t} \eta(r + \theta D_t).
\]

Then, we can rewrite the dynamic program in (5) and (6) as follows:

\[
J_t(x, w, y, r) = G_t(x, w, y_2, r) + \alpha E_{D_t} V_{t+1}(y_1 - D_t, y_2 - D_t, r + \theta D_t), \tag{7}
\]

\[
V_t(x, w) = \min_{y, r \in S(x, w)} J_t(x, w, y, r), \tag{8}
\]

where the single-period cost function can be shown as

\[
G_t(x, w, y_2, r) = H_{1,t}(x_1) + H_{2,t}(x_2) + H_{3,t}(r)
\]

\[+ c(y_2 - x_2) + \beta_i(r - w)^+ + \beta_o(r - w)^-.
\]

We refer to (7) and (8) as the echelon formulation of the CP model.

### 3.2 The Optimal Policy

We first state the optimal joint policy for the CP model, which includes two types of decisions made through four control parameters \( (y_1^*, y_2^*, l^*, u^*) \) in each period. For the inventory ordering decisions,
each stage implements an echelon base-stock policy. That is, stage $i$ reviews its $x_i$ at the beginning of each period. If $x_i < y_i^*$, it orders up to $y_i^*$ or as close as possible if its upstream does not have sufficient stock; otherwise, it does not order. For the cash retention decision, stage 1 reviews $w$: if $w > u^*$, it disposes cash down to the maximum of $u^*$ and $x_2$; if $w < l^*$, it retrieves cash up to $l^*$ or as close as possible (due to the upper bound $K$); otherwise, it does not transfer cash.

For the traditional multi-echelon inventory model, there exists an equivalence result between echelon and local base-stock policies. Namely, each stage will generate exactly the same inventory orders based on the local and echelon policies\(^5\); see, e.g., Chapter 8 of Zipkin 2000. This result can dramatically simplify the implementation of the optimal policy as each stage can monitor its local information to execute the optimal policy. We have a similar result here: the optimal echelon policy $(y_1^*, y_2^*, l^*, u^*)$ can be converted back to the local term $(y_1'^*, y_2'^*, l'^*, u'^*)$, where $y_1'^* = y_1^*$, $y_2'^* = y_2^* - y_1^*$, $l'^* = l^* - y_2^*$, and $u'^* = u^* - y_2^*$. In this way, the procurement department of stage $i$ can implement a local base-stock policy based on its local inventory level $x_i$; the accounting department of the headquarter can implement a local two-threshold policy based on the master account cash position $w'$.

We next explain how the optimal policy is derived and how to calculate these policy parameters. This is done by transforming a three-state dynamic program into three, single-dimensional dynamic programs. We summarize the main result in the following proposition. \(^5\)

**Proposition 1.** For all $t$ and $(x, w)$, $V_t(x, w) = f_{1,t}(x_1) + f_{2,t}(x_2) + f_{3,t}(w)$, where $f_{i,t}()$ is convex.

We define $f_{i,t}()$ as the expected optimal cost for echelon $i$ in period $t$. Starting from echelon 1, we have

$$f_{1,t}(x_1) = H_{1,t}(x_1) + \min_{x_1 \leq y_1} \left\{ \alpha E_{D_t} f_{1,t+1}(y_1 - D_t) \right\}. \quad (9)$$

Let $g_{1,t}(y_1) = \alpha E_{D_t} f_{1,t+1}(y_1 - D_t)$. Then, the optimal control parameter $y_{1,t}^*$ can be obtained by solving the minimization problem:

$$y_{1,t}^* = \arg \min_{y_1} \left\{ g_{1,t}(y_1) \right\}.$$ 

Now, we express the expected optimal cost functions of echelon 2 as follows:

$$f_{2,t}(x_2) = H_{2,t}(x_2) + \Gamma_{2,t}(x_2) + \Lambda_{2,t}(x_2) + \min_{x_2 \leq y_2} \left\{ c(y_2 - x_2) + \alpha E_{D_t} f_{2,t+1}(y_2 - D_t) \right\}. \quad (10)$$

Similar to echelon 1, let $g_{2,t}(y_2) = cy_2 + \alpha E_{D_t} f_{2,t+1}(y_2 - D_t)$ and $y_{2,t}^* = \arg \min_{y_2} \left\{ g_{2,t}(y_2) \right\}$. For

\(^5\)The equivalence holds when each stage places an order in each period. It is possible, although very rare, that no order is placed under the echelon policy while the corresponding local policy suggests ordering. In such a case, the echelon and local policy are not equivalent. This can be easily fixed by modifying the rule of placing an order. That is, under the local policy, the upstream stage places an order only when it receives an order from its downstream stage and when its inventory state is lower than the target level.
echelon 3,
\[
f_{3,t}(w) = \Lambda_{3,t}(w) + \begin{cases} L_t(w), & \text{if } w \leq l_t^* \\
H_{3,t}(w) + \Gamma_{3,t}(w) + \alpha E_{D_t} f_{3,t+1}(w + \theta D_t), & \text{if } l_t^* < w \leq u_t^* \\
U_t(w), & \text{if } u_t^* < w \end{cases}
\]
(11)

Note that $\Gamma_{2,t}(\cdot)$ and $\Gamma_{3,t}(\cdot)$ are the so-called induced penalty cost functions defined in Clark and Scarf (1960), i.e.,
\[
\Gamma_{2,t}(x_2) = \begin{cases} \alpha E_{D_t} \left[ f_{1,t+1}(x_2 - D_t) - f_{1,t+1}(y_{1,t}^* - D_t) \right], & x_2 \leq y_{1,t}^*, \\
0, & \text{otherwise.} \end{cases}
\]
(12)
\[
\Gamma_{3,t}(r) = \begin{cases} c(r - y_{2,t}^*) + \alpha E_{D_t} \left[ f_{2,t+1}(r - D_t) - f_{2,t+1}(y_{2,t}^* - D_t) \right], & r \leq y_{2,t}^*, \\
0, & \text{otherwise.} \end{cases}
\]
(13)

Here, $\Gamma_{2,t}(\cdot)$ represents the penalty cost charged to echelon 2 if stage 2 cannot ship up to stage 1’s target base-stock level $y_{1,t}^*$. Although bearing the same structure, $\Gamma_{3,t}(\cdot)$ has a different economic meaning: it represents the penalty cost charged to the headquarter’s accounting department (which manages the master account), if it fails to hold sufficient cash to pay for the inventory procurement up to the target echelon base-stock level $y_{2,t}^*$.

There are new penalty cost functions $\Lambda_{2,t}(\cdot)$ and $\Lambda_{3,t}(\cdot)$ in (10) and (11). To illustrate their meanings, we define
\[
g_{3,t}(w) = H_{3,t}(w) + \Gamma_{3,t}(w) + \alpha E_{D_t} f_{3,t+1}(w + \theta D_t),
\]
(14)
\[
L_t(w) = -\beta_t (w - l_t^*) + g_{3,t}(l_t^*),
\]
(15)
\[
U_t(w) = \beta_t (w - u_t^*) + g_{3,t}(u_t^*).
\]
(16)

One can view $g_{3,t}(w)$ as the optimal cost for echelon 3 when the system working capital $w$ is in $[l_t^*, u_t^*]$. Under the optimal policy, when $w < l_t^*$, stage 1 should retrieve cash until $w$ reaches $l_t^*$. Thus, $L_t(w)$ can be viewed as the optimal cost when $w < l_t^*$. Similarly, $U_t(w)$ can be viewed as the optimal cost when $w > u_t^*$ because in this case stage 1 should dispose cash down to $u_t^*$. With these explanations, the two new penalty cost functions can be defined as follows:
\[
\Lambda_{2,t}(x_2) = \begin{cases} 0, & \text{if } x_2 \leq u_t^*, \\
g_{3,t}(x_2) - U_t(x_2), & \text{otherwise,} \end{cases}
\]
(17)
\[
\Lambda_{3,t}(w) = \begin{cases} g_{3,t}(w + K) + \beta_t K - L_t(w), & w \leq l_t^* - K, \\
0, & \text{otherwise.} \end{cases}
\]
(18)

Let us first consider $\Lambda_{2,t}(x_2)$ in (17). This is a penalty cost charged to echelon 2 if the system carries too much inventory. Intuitively, if echelon inventory $x_2$ is less than or equal to $u_t^*$, echelon 3 (or the headquarter accounting department) can always maintain a system working capital between $l_t^*$ and $u_t^*$.

However, if $x_2 > u_t^*$, the best that stage 1 can do is to dispose all cash on hand, making $w = x_2$. The remaining cash can be used.
In such a case, the extra cost $g_{3,t}(x_2) - U_t(x_2)$ incurred at echelon 3 should be charged to echelon 2 due to its excess inventory. For this reason, we call $\Lambda_{2,t}(x_2)$ the *excess inventory penalty*. (Recall that $\Gamma_{2,t}(x_2)$ is the penalty cost charged to echelon 2 due to insufficient inventory holding.) The cash retention control thresholds can be obtained from the following equations:

$$l^*_t = \sup \left\{ w : \frac{\partial}{\partial w} g_{3,t}(w) \leq -\beta_i \right\}, \quad u^*_t = \sup \left\{ w : \frac{\partial}{\partial w} g_{3,t}(w) \leq \beta_0 \right\}.$$  

With a similar logic, $\Lambda_{3,t}(w)$ in (18) can be explained: this is a self-induced penalty cost charged to echelon 3 if the system working capital $w$ is less than $l^*_t - K$ due to too much cash disposal in the previous period. In this case, stage 1 is penalized with the extra cost $g_{3,t}(w + K) + \beta_i K - L_t(w)$ for over-disposing cash.

![Diagram](image)

Figure 3: Induced penalty functions of the cash pooling model.

Figure 3(a) depicts functions $L(\cdot), U(\cdot), g_3(\cdot), f_3(\cdot)$, as well as induced penalty functions $\Lambda_2(\cdot)$ and $\Lambda_3(\cdot)$ created while decoupling echelon 2 and 3 (with time subscripts suppressed). The optimal control threshold $l^* (u^*)$ derived as the tangent point of curve $g_{3,t}(\cdot)$ and a line with slope $-\beta_i (\beta_0)$. Function $f_3(\cdot)$ is shown as the bold convex curve connected by four different functions, which are, from the right to the left, the linear function $U(\cdot)$, the convex function $g_3(\cdot)$, the linear function $L(\cdot)$, and the convex function $g_3$ shifted from point $(l^*, g_3(l^*))$ to point $(l^* - K, L(l^* - K))$; the induced penalty function $\Lambda_3(w)$ is the difference between $f_3(w)$ and $L(w)$ to the left of $l^* - K$; the induced penalty function $\Lambda_2(x_2)$ is the difference between $g_3(x_2)$ and $U(x_2)$ to the right of $u^*$. Figure 3(b) illustrates the relationship between four echelons and five penalty cost functions in our problem. The direction of the arrow indicates to which echelon that the penalty cost is charged.
4 Transfer Pricing System

Let us now consider the transfer pricing (TP) system. In this setting, stage $i$ holds its own, separate cash account $w_i'$, and stage 1 pays stage 2 for the ordered material according to a fixed transfer price $p_2$. The investment function is held by the headquarter (stage 1). Thus, the cash retention decision directly affects the dynamics of stage 1’s cash balance $w_1'$. Similar to the cash pooling system, we denote $\eta_i'$ the cash holding cost rate for stage $i$’s, which represent the opportunity cost of holding cash at the divisions. We make no ex ante assumption on the order of $\eta_1'$ and $\eta_2'$. The rest of the notation remains the same as that in §3. Figure 4(a) shows the material and financial flows of the TP model.

The inventory dynamics of the TP model are identical to the CP model, as in equation (1) and (2). Due to separate accounts, the cash dynamics of the TP model become

\[
\begin{align*}
  w_{2,t+1}' &= w_{2,t}' + p_2z_{1,t} - cz_{2,t}, \\
  w_{1,t+1}' &= w_{1,t}' + v_t - p_2z_{1,t} + p_1D_t. 
\end{align*}
\]

(19) (20)

We again assume the payment to stage 2 occurs upon the receipt of shipment. Define $w' = (w_2', w_1')$. The constraint set for the TP model is

\[
\hat{S}(x_2', w') = \left\{ z, v \mid 0 \leq z_1 \leq \min \left( \frac{w_1' + v}{p_2}, x_2' \right), 0 \leq z_2 \leq w_2'/c, v \leq K' \right\}.
\]

(21)

As shown in the first inequality, $p_2 z_1$ cannot exceed the available cash $(w_1' + v)$.

The single-period expected cost function is

\[
\hat{C}_t(x', w', z_2, v) = E_{D_t} \left[ h_1'(x_1' - D_t)^+ + b(x_1' - D_t)^- + h_2'x_2' + cz_2 
\right. \\
\left. + \eta_2'w_2' + \eta_1'\bar{E}_{D_t} \left( w_1' + v + p_1D_t \right) + \beta_1'v^+ + \beta_2'v^- \right].
\]

(22)

The dynamic program of the TP model can be expressed as follows:

\[
\begin{align*}
  \hat{J}_t(x', w', z, v) &= \hat{G}_t(x', w', z_2, v) + \alpha \bar{E}_{D_t} \left[ \hat{V}_{t+1}(x_1' + z_1 - D_t, x_2' + z_2 - z_1, \\
  w_2' + p_2z_1 - cz_2, w_1' + v - p_2z_1 + p_1D_t) \right], \\
  \hat{V}_t(x', w') &= \min_{z, v \in \hat{S}(x_2', w')} \hat{J}_t(x', w', z, v).
\end{align*}
\]

(23) (24)

The TP model is essentially a serial inventory problem with capacities (in the form of cash constraints) at both stages. However, these constraints are random and endogenous, which are different from those assumed in the traditional capacitated inventory model (e.g., Parker and Kapuscinski 2004).

We are not able to obtain the exact optimal joint policy for the TP model. Nonetheless, we can
obtain a lower bound to the optimal cost of the TP model. In the subsequent sections, we shall introduce a different echelon notion from that of the CP model. From this new echelon formulation, we can connect the TP problem to an assembly system from which the lower bound cost is derived.

### 4.1 Echelon Formulation

We shall create a different echelon transformation scheme for the TP model. Define

\[
x_1 = x'_1, \quad y_1 = x'_1 + z_1, \quad x_2 = x'_1 + x'_2, \quad y_2 = x'_1 + x'_2 + z_2,
\]
\[
w_1 = x'_1 + w'_1/p_2, \quad r_1 = x'_1 + (w'_1 + v)/p_2, \quad w_2 = x'_1 + x'_2 + w'_2/c.
\]

Here, \(x\) and \(y\) are the same as in the CP model; \(w_1\) is defined to be stage 1's working capital (in inventory units); \(w_2\) is defined as stage 2's echelon working capital, which includes inventory at both stages and stage 2's cash balance (in inventory units). With these state transformations, we redefine the echelon holding cost parameters for the TP model: \(\eta_2 = \eta'_2 c\), \(h_2 = h'_2 - \eta'_2 c\), \(\eta_1 = \eta'_1 p_2\), and \(h_1 = h'_1 - h'_2 - \eta'_1 p_2\). Also redefine \(\beta_i = p_2 \beta'_i\), \(\beta_0 = p_2 \beta'_0\), \(\theta = p_1/p_2 - 1 > 0\), \(K = K'/p_2\), and finally \(\rho = p_2/c\). With the new echelon terms, the feasible set becomes

\[
S(x, w) = \{y, r_1 \mid x_1 \leq y_1 \leq r_1 \leq w_1 + K, \ x_1 \leq y_1 \leq x_2 \leq y_2 \leq w_2\}.
\]

As shown in Figure 4(b), the transformed TP system is similar to an assembly system. We further redefine the holding and backorder cost associated with each echelon as

\[
H_{1,t}(x_1) = E_{D_1}[(h_1 + h_2 + \eta_2 + \eta_1 + b)(D_t - x_1)^+ + h_1(x_1 - D_t)],
\]
\[
H_{2,t}(x_2) = E_{D_1}h_2(x_2 - D_t), \quad H_{3,t}(w_2) = E_{D_1}h_2(w_2 - D_t), \quad H_{4,t}(r_1) = E_{D_1}\eta_1(r_1 + \theta D_t).
\]
The echelon formulation of the TP model becomes

\[ J_t(x, w, y, r_1) = G_t(x, w, y_2, r_1) + E_{D_t} V_{t+1}(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t, r_1 + \theta D_t), \quad (25) \]

\[ V_t(x, w) = \min_{y, r_1 \in S(x, w)} J_t(x, w, y, r_1), \quad (26) \]

where the single-period cost function can be shown as

\[ G_t(x, w, y_2, r_1) = H_{1,t}(x_1) + H_{2,t}(x_2) + H_{3,t}(w_2) + H_{4,t}(r_1) + c(y_2 - x_2) + \beta_t(r_1 - w_1)^+ + \beta_0(r_1 - w_1)^-. \quad (27) \]

After the new transformation, some of the complexities caused by the endogenous constraints disappear. More specifically, the dynamics of the new echelon variable \( w_1 \) no longer depend on \( z_1 \). However, the dynamics of echelon \( w_2 \) still depend on the decision \( y_1 - x_1 \) associated with echelon 1, as shown in (25). This unique property undermines the decomposition structure in the CP model and differentiates the TP model from the traditional assembly system (Rosling 1989). Below, we derive lower bounds to the optimal cost of the TP model.

### 4.2 Lower Bounds

This subsection establishes two lower bounds to the optimal cost for the TP model. Recall that the TP model is similar to an assembly system. The main idea of constructing these lower bounds is to decompose this assembly system. Specifically, the expression of \( S(x, w) \) indicates that stage 1’s decision \( y_1 \) is subject to two constraints: one is \( y_1 \leq r_1 \leq w_1 + K \), which represents the cash constraint on the order quantity; the other is \( y_1 \leq x_2 \leq y_2 \leq w_2 \), which can be viewed as a material order constraint in a two-stage system with an endogenous, random capacity \( w_2 \) at the upstream stage 2. Figure 5(a) shows these two sets of constraints.

Now, imagine that the final product sold at stage 1 consists of two components: a physical component (depicted by triangles) supplied from stage 2’s stock, and a “cash” component (depicted by circles) supplied from stage 1’s operating account. The constraint \( 0 \leq z_1 \leq \min\{(w_1 + v)/p_2, x_2\} \) in (21) (or, equivalently, \( x_1 \leq y_1 \leq \min\{r_1, x_2\} \)) implies a similar structure to an assembly system: the same amount of inventory and cash equivalent are matched through replenishment at stage 1.

To derive a lower bound to the optimal cost, we relax the above matching constraint by assuming that the components can be ordered and sold separately. As a result, the original system is decoupled into two independent subsystems as shown in Figure 5(b) – Subsystem 1 represents the cash flows; Subsystem 2 represents the material flow. The sum of the minimum costs of subsystems is a lower bound on the minimum cost of the original system.
Figure 5: Decomposition of the transfer pricing system.

We specify the total cost function for each of the subsystems. Let \( h_1 \) and \( h_2 \) be the inventory holding cost for Subsystem 1 and 2, respectively, where \( h_1 + h_2 = h_1 \). Let \( b_1 \) and \( b_2 \) be the backorder cost for Subsystem 1 and 2, respectively, where \( b_1 + b_2 = b \).

\[
H_{1,t}(x_1) = E_{D_t} \left[ (h_1 + \eta_1 + b_1)(D_t - x_1)^+ + h_1(x_1 - D_t) \right], \tag{28}
\]

\[
H_{2,t}(x_1) = E_{D_t} \left[ (h_1^2 + h_2 + \eta_2 + b_2)(D_t - x_1)^+ + h_1^2(x_1 - D_t) \right]. \tag{29}
\]

Now, let us define

\[
G_1^1(x_1, w_1, r_1) = H_{1,t}(x_1) + H_{4,t}(r_1) + \beta_i(r_1 - w_1)^+ + \beta_o(r_1 - w_1)^-, \tag{30}
\]

\[
G_2^2(x_1, x_2, w_2, y_2) = H_{2,t}(x_1) + H_{2,t}(x_2) + H_{3,t}(w_2) + c(y_2 - x_2). \tag{31}
\]

Note that \( H_{1,t}(x_1) = H_{1,t}^1(x_1) + H_{2,t}^2(x_1) \), hence \( G_1^1(x_1, w_1, r_1) + G_2^2(x_1, x_2, w_2, y_2) = G_t(x, w, y_2, r_1) \).

With this cost allocation, the dynamic program for Subsystem 1 can be expressed as

\[
V_{t}^1(x_1, w_1) = \min_{x_1 \leq y_1 \leq r_1 \leq w_1 + K} \left\{ G_1^1(x_1, w_1, r_1) + \alpha E_{D_t} V_{t+1}^1(y_1 - D_t, r_1 + \theta D_t) \right\}. \tag{32}
\]

And the dynamic program for Subsystem 2 is

\[
V_{t}^2(x_1, x_2, w_2) = \min_{x_1 \leq y_1 \leq x_2 \leq y_2 \leq w_2} \left\{ G_2^2(x_1, x_2, w_2, y_2) \right. \right.

\left. + \alpha E_{D_t} V_{t+1}^2(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t) \right\}. \tag{33}
\]

**Proposition 2.** \( V_t(x, w) \geq V_t^1(x_1, w_1) + V_t^2(x_1, x_2, w_2) \) for all \( (x, w) \) and \( t \).

Proposition 2 shows that for any combination of \( (h_1^1, h_1^2) \) and \( (b_1, b_2) \), the sum of the two subsystems forms a cost lower bound to the original system. Maximizing expected cost over all combinations of \( h_1 \) and \( b \) yields the best lower bound.

The remaining question is how to find the optimal cost of these subsystems. A careful examination of Subsystem 1 described in (32) reveals that it is the echelon transformation for a single-stage system with a joint inventory and cash retention decision. Thus, we can characterize the optimal joint policy,
i.e., using the base-stock policy to control the inventory replenishment and the two-threshold policy to manage the working capital level.

Solving Subsystem 2 is more difficult. The dynamic problem described in (33) is the echelon expression of a two-stage inventory model with random, endogenous capacity at the upstream stage. There exists no known optimal policy for this model. Thus, we provide two approaches to further develop a lower bound to the optimal cost for Subsystem 2.

Constraint Relaxation (CR) Bound

We form the lower bound by relaxing the constraint \( y_2 \leq w_2 \) at stage 2. Once \( w_2 \) is removed from the constraint set, it only appears in the single-period cost function in the dynamic program. The following lemma characterizes the expected value of \( w_2 \) through the flow conservation.

**Lemma 1.** Given the initial states \( w_{2,1} \) and \( x_{1,1} \), for any policy we have

\[
E_{D_{1},\ldots,D_{t-1}} w_{2,t} = \rho \cdot E_{D_{1},\ldots,D_{t-1}} x_{1,t} + B_t,
\]

where \( B_t = (\rho - 1) \sum_{s=1}^{t-1} \mu_s + w_{2,1} - \rho x_{1,1} \).

Recall that in the single-period cost function (31), the function \( H_{3,t}(w_2) \) is a linear function of \( w_2 \). Therefore, by using Lemma 1, we can replace \( H_{3,t}(w_2) \) with \( H_{3,t}(\rho x_1 + B_t) \) without affecting the optimal decision in each period. With this construction, \( w_2 \) can be replaced by \( x_1 \) and Subsystem 2 becomes a classical two-stage serial system in which Clark and Scarf’s algorithm can be applied to find the optimal echelon base-stock levels for both stages. The CR bound generally works well when the constraint \( y_2 \leq w_2 \) is not binding, i.e., when stage 2 holds sufficient cash. This occurs if the stage 2’s markup \( (p_2/c_2 - 1) \) is high and demand tends to be stationary. However, under increasing demand, it is optimal for stage 2 to order more in anticipation of future demand uprise. In this case, stage 2’s cash constraint could become binding, especially if its markup is low. Thus, we need another lower bound to complement the performance of the CR bound.

Sample Path (SP) Bound

The difficulty of solving Subsystem 2 comes from keeping track of the state \( w_{2,t} \) as the current period’s \( w_{2,t} \) depends on the previous period’s demand and order quantity. However, if we consider a specific demand sample path, \( w_{2,t} \) can be fully characterized by flow conservation.

**Lemma 2.** Let \( d_t(\omega) \) represent the demand realization in period \( t \) given a demand sample path \( \omega \). With initial states \( w_{2,1} \) and \( x_{1,1} \), we have \( w_{2,t} = \rho x_{1,t} + B_t(\omega) \), where

\[
B_t(\omega) = (\rho - 1) \sum_{s=1}^{t-1} d_s(\omega) + w_{2,1} - \rho x_{1,1}.
\]
The proof of Lemma 2 is similar to that of Lemma 1, and thus omitted. Given the initial states and a demand sample path, \( B_t(\omega) \) is a constant. If we replace \( w_{2,t} \) (according to Lemma 2) in both the constraint set and the periodic cost function, Subsystem 2 can be reduced to a two-stage serial system with deterministic demand subject to the following constraint (at time \( t \)):

\[
S^d_t(x_1, x_2 | \omega) = \{ y_1, y_2 | x_1 \leq y_1 \leq x_2 \leq y_2 \leq \rho x_1 + B_t(\omega) \}.
\]

The constraints state that stage 1’s order decision \( y_1 \) is affected by stage 2’s echelon inventory level \( x_2 \); stage 2’s order decision \( y_2 \) is affected by a linear function of stage 1’s inventory level \( x_1 \). The optimal \( y_1^* \) and \( y_2^* \) can be obtained by solving a two-dimensional convex program in each period. To facilitate the computation, we prove that this problem can be decoupled into two one-dimensional convex programs. Let \( V^d_t(x_1, x_2 | \omega) \) represent the optimal cost for Subsystem 2 for any demand sample path \( \omega \) after \( w_{2,t} \) is substituted with \( \rho x_{1,t} + B_t(\omega) \). The following proposition shows the decoupling result.

**Proposition 3.** \( V_t^d(x_1, x_2 | \omega) = v^1_t(x_1 | \omega) + v^2_t(x_2 | \omega) \), where \( v^i_t(x_i | \omega) \) is a convex function.

We refer the reader to the proof for the detailed formulation of \( v^1_t \) and \( v^2_t \) functions. A lower bound to the optimal cost of the Subsystem 2 under the SP approach can be found by averaging total costs over all demand sample paths. In summary, we are able to generate two lower bounds – the sum of the optimal cost obtained from Subsystem 1 and the optimal cost obtained from either the CR approach or the SP approach.

### 4.3 Optimal Transfer Pricing Model

For some multi-divisional corporations with a powerful headquarter, it is possible that the headquarter can determine transfer price to efficiently distribute liquidity. This section extends the TP model to optimize the transfer price between the divisions. Notice that the optimal transfer price can be obtained by the optimal order quantity and the optimal cash payment in each period. Thus, we modify the transfer pricing model to incorporate the inter-division cash payment decision. For period \( t \), define

\[ m_t = \text{amount of cash payment paid from stage 1 to stage 2 before the demand occurs.} \]

Then, the optimal transfer pricing (OP) model can be obtained by replacing \( p_{2,z_{1,t}} \) with \( m_t \) in the TP model, as shown in Figure 6(a).

Interestingly, solving the OP model is no harder than solving the CP model. More specifically, we can follow the same logic of solving the CP model by defining a set of new echelon variables and cost parameters, making the original two-stage model transformed into a four-stage serial system. Figure 6(b) shows the transformed system with division 2’s and division 1’s cash account being stage 3 and
stage 4, respectively. Similarly, we can decompose the resulting four-state dynamic program into four separable, single-state dynamic programs. We refer the reader to Luo and Shang (2012) for the detailed analysis. In summary, we can obtain the exact optimal joint inventory, cash payment and retention policy for the OP model. The optimal transfer price is equal to the optimal cash payment divided by the optimal order quantity.

**Proposition 4.** The optimal policy for the OP model can be described as follows. For inventory replenishment, both stages implement an echelon base-stock policy; for cash payment, stage 2 monitors its echelon working capital \((x'_1 + x'_2 + w'_2/c)\) and receives payment up to a target level; for cash retention, stage 1 monitors the system working capital and maintains it within an interval.

![Diagrams](image)

**Figure 6:** The transformed optimal transfer pricing model.

5 Numerical Study

5.1 Value of Cash Pooling

We assess the value of cash pooling by comparing the optimal cost of the CP model, \(C^*\), with the lower bound cost of the TP model, \(C_L = \max\{C_R, C_S\}\), where \(C_R\) and \(C_S\) represent the cost of the constraint relaxation bound and the sample path bound, respectively. Note that the value we obtain is a lower bound to the actual value. We define the value of cash pooling as

\[
\% \text{ value} = \frac{C_L - C^*}{C_L} \times 100\%.
\]

This represents the percentage of cost reduction of the TP model if cash pooling is implemented.

We conduct a numerical study by starting with a test bed which has the time horizon of 10 periods. We fix parameters \(\alpha = 0.95, c = 1, \eta'_1 = 0.05, h'_1 = 1\), and vary the other parameters with each taking two values: \(p_2 = (1.2, 2), p_1 = (2.5, 4), b = (5, 10), \eta'_2 = (0.05, 0.2), h'_2 = (0.25, 0.75), \beta'_o = (0.05, 0.15),\) and \(\beta'_i = (0.05, 0.2)\). In addition, two demand forms are considered. For the i.i.d. demand case, \(D_t\) is Poisson distributed with mean \(\mu_t = 10\) for all \(t\); for the increasing demand case, \(D_t\) is Poisson
distributed with the first period mean $\mu_1 = 10$ and $\mu_t$ increasing at a rate of 1.2 per period. In both demand cases, we fix the liquidity level $K_t = \mu_t$. For each demand form, we generate 128 instances. The total number of instances in the test bed is 256. The combination of these parameters covers a wide range of different system characteristics. For example, when $(p_1, p_2) = (2.5, 2) ((4, 1.2)$, respectively) the transfer price is high (low, respectively) compared to the retail price. For all cases we assume the initial on-hand inventory and cash level $(x_{1,1}, x_{2,1}, w'_{2,1}, w'_{1,1}) = (16, 10, 10, 10)$, roughly equal to the steady-state inventory/cash level under the i.i.d. demand. When computing $C_s$, we run a simulation of 1000 iterations for each instance. When computing $C^*$, we assume $\eta' = \min\{\eta'_1, \eta'_2\}$, and set the initial balance of the cash pool as $w'_1 = w'_{1,1} + w'_{2,1}$.

In this test bed of 256 cases, the average cost reduction of adopting cash pooling is 29.29%. More specifically, the cost reduction is 13.62% when demand is i.i.d and 44.96% when demand is increasing. Table 1 (left) summarizes the value of cash pooling under the i.i.d. demand (128 cases). The results are further aggregated into 4 quadrants, each displaying the average value of 32 cases with the same $p_2$ and $\eta'_2$. As shown in Table 1 (left), cash pooling does not add much value if the transfer price is low (e.g., $p_2 = 1.2$). This is because under the i.i.d. demand, division 2 has to purchase inventory in each period to cover the (stationary) order received from division 1. With a low transfer price, division 2’s average inventory procurement cost per period will be close to the average payment received per period. Thus, division 1 will not accumulate too much cash that leads to system inefficiency (hence the value of cash pooling is small). On the other hand, with a high transfer price (e.g., $p_2 = 2$), cash pooling will then play a significant role – it will be better off to allocate more cash to division 1 so less cash will be accumulated at division 2. The value of cash pooling is more significant when division 2 cash holding cost $\eta'_2$ increases as the excess cash will be charged with a higher rate.

<table>
<thead>
<tr>
<th>Cash holding cost $\eta'_2$</th>
<th>Transfer price $p_2$</th>
<th>Transfer price $p_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1.2$</td>
<td>$2$</td>
</tr>
<tr>
<td>0.05</td>
<td>4.68%</td>
<td>9.56%</td>
</tr>
<tr>
<td>0.2</td>
<td>12.87%</td>
<td>27.38%</td>
</tr>
</tbody>
</table>

Table 1: Value of cash pooling - i.i.d. demand (left) and increasing demand (right)

Under the increasing demand, the TP system will perform poorly when the transfer price is low. More specifically, as division 1 order size increases with the demand, ideally division 2 should in turn increase its inventory stocking to prepare for the future increasing order sizes. However, under fixed transfer pricing, division 2 will not have sufficient cash to do so due to its low markup ($p_2 = 1.2$). Table 1 (right) demonstrates this inefficiency. As shown, when the transfer price is low and demand is increasing, the value of cash pooling can be very significant. Interestingly, this finding is consistent with
Caterpillar’s strategy: After the financial crisis, many suppliers of Caterpillar have difficulty getting funds from external banks to stock up the raw material for the expected soaring demand. Thus, Caterpillar took a more proactive role to subsidize their suppliers (Aeppel 2010). The value of cash pooling is clearly higher when backorder cost is larger. Figure 7(a) summarizes the conditions under which cash pooling has a significant value.

![Diagram of demand, transfer price, and CP value](image)

(a) Demand form, transfer price, and CP value.

![Graph of downstream liquidity and CP value](image)

(b) Downstream liquidity and CP value.

Figure 7: Value of cash pooling.

Figure 7(b) illustrates the impact of downstream liquidity level $K$ on the value of cash pooling when demand is increasing under different selling prices $p_1$. Here, we set the transfer price $p_2 = 1.05$, $b = 5$, $\eta' = 0.2$, $h'_2 = 0.25$, $\beta'_o = \beta'_t = 0.05$ and the other parameters are the same as in the test bed. For a fixed selling price, the value of cash pooling is increasing in $K$, but the marginal benefit of cash pooling decreases in $K$. Thus, cash pooling makes a more significant cost improvement when $K$ is small. In addition, we find that $p_1$ and $K$ complement each other’s role as a liquidity source. For example, $(K, p_1) = (8, 1.2)$ and $(4, 1.5)$ yield a similar CP value.

We next examine the impact of demand volatility on the optimal mix between cash and inventory in a simulation study for the cash pooling model. The instances have the following parameters: $p_1 = 2.5$, $b = 10$, $h'_2 = 0.25$, $\eta' = 0.05$, $\beta'_o = \beta'_t = 0.05$, and $K'_t = 0$. All the other parameters are same as those in the test bed. We assume that the demand is negative binomial with mean equal to 10 and variance increasing from 15 to 55 with an increment of 10. We calculate the ratio of cash holding amount to the total system inventory level for each simulated instance and then take an average over these ratios. While both cash and inventory holding amount increase in the demand variability, the ratio of cash to inventory decreases. More specifically, when the variances are 15, 25, 35, 45, and 55, the corresponding ratios are 0.32, 0.30, 0.28, 0.26, 0.25, respectively. This result suggests that cash

---

6Figure 7(a) also shows that CR (SP) bound performs better in the upper left (lower right) quadrant.
holding is less sensitive to the change of demand variability than the inventory holding.

Finally, notice that the optimal base-stock levels obtained from the Clark-Scarf model is suboptimal to the TP model. Thus, our study suggests that ignoring the impact of financial flow on the inventory decision will lead to a significant cost increase when the financial markets are not perfect.

5.2 Optimal Transfer Pricing System

While we have demonstrated significant value of cash pooling, one interesting question to investigate is how much benefit can be recovered by optimizing the transfer price. To answer this question, we compare the optimal cost from the CP model, $C^*$, with the optimal cost from the OP model, $C_O$. We define the percentage cost reduction as $(C_O - C^*)/C_O \times 100\%$. We test the same 256 instances. The average (maximum, minimum) percentage cost reduction of adopting cash pooling is 6.38\% (9.98\%, 3.34\%) for the i.i.d. demand and 9.33\% (14.75\%, 4.65\%) for the increasing demand case. The cost reduction is more significant when the cash holding cost $\eta_2'$ or the backorder cost $b$ is high. Two reasons lead to this cost difference. First, cash pooling can consolidate the entire system cash to a single account that has a smaller cash holding cost rate. (This explains why the cost saving is more significant when $\eta_2'$ is larger.) Second, cash pooling eliminates the lead time for the payment and allows cash to move bi-directionally to upstream or downstream, making the supply chain more responsive and leading to a smaller number of backorders. (Thus, the benefit of cash pooling is more significant when the backorder cost is high.)

Compared with the cost reduction with the fixed transfer price tested in the previous section, the percentage cost reduction is relatively small when the transfer price is optimized. This suggests that optimizing the transfer price can retain a big portion of the benefit achieved by adopting cash pooling. This indeed is useful for firms if adopting cash pooling is not possible due to legal issues or cash shortage for investing in the costly financial services platform.

It is interesting to see how the optimized transfer price helps to re-distribute the supply chain cash between these two divisions for firms facing a product life cycle demand. More specifically, we consider a time horizon of 22 periods with Poisson demand in each period. The demand mean starts from 6, ramps up at a peak of 36, declines to 14 and remains there for the last 5 periods. These demand rates represent introduction, growth, maturity, and decline stages in a product life cycle (see Figure 8). To illustrate the transfer pricing dynamics, we consider a instance with $p_2 = 1.25$, $p_1 = 1.5$, $b = 55$, $\eta_2 = 0.15$, $h_2' = 0.2$, $\beta_o' = \beta_1' = 0.05$, $K' = 0$, and the other fixed parameters in the test bed. We obtain the optimal transfer price as the optimal cash payment, $m$, divided by the optimal order quantity, $z_1$ in a simulation study and plot the average optimal transfer price in each period. Figure 9 shows the dynamics of the corresponding optimal transfer price from period to period. Notice that if the optimal
transfer price $p_2^s$ is larger than $p_1 = 1.5$, the price difference can be viewed as financial subsidy offered by stage 1; on the other hand, if $p_2^s$ is smaller than the purchase cost $c = 1$, the price difference can be viewed as delayed payment made by stage 1.

![Product life cycle demand](image)

**Figure 8:** Product life cycle demand.

![Optimal transfer price](image)

**Figure 9:** Optimal transfer price under product life cycle demand.

Figure 9 provides an interesting insight on how to set up the optimal transfer price. Recall that this is a case with $K' = 0$, i.e., the source of stage 1’s liquidity is completely from the sales revenue. From the figure, we find that during the introduction stage, the transfer price should be set to a value close to the selling price $p_1$. This implies that division 1’s cash should be moved to division 2, as the latter needs to spare more cash for material ordering. The transfer price declines slightly but ramps up quickly during the growth stage, reflecting the fact that division 1 should even subsidize division 2 for material ordering. Finally, during the maturity and decline stages, division 2 does not need to reserve excess cash for material ordering, so the transfer price declines gradually. During the periods 13 to 17, the transfer price can be lower than the purchase cost $c$. We can view this as a payment reduction received by division 1 to compensate the cash subsidy offered to division 2 during the growth stage.

6 Concluding Remarks

This paper studies the benefit of cash pooling for a corporation that owns a supply chain with two divisions. We quantify the value of cash pooling by comparing two cash management systems, repre-
senting different levels of cash concentration due to the existence of the financial services platform. We prove the exact optimal inventory and cash retention policy for the cash pooling model and construct a lower bound to the optimal cost for the transfer pricing model. We quantify the conditions under which investing in a financial services platform that achieves cash pooling provides the most benefit. Our results can be extended to the system with general lead times, general number of stages, and Markov modulated system parameters.

Our results contribute to different business disciplines as follows. For general management, we show that an inter-departmental collaboration between accounting, finance and operations is crucial to ensure supply chain efficiency. In addition, we characterize the conditions under which cash pooling will provide the most benefits. For operations, we show that the inventory decision can be determined in the same fashion as that of the traditional inventory system if the cash can be pooled and managed centrally by the headquarter. Finally, for accounting and finance, we show that it is necessary to take inventory into accounting when making the cash retention decision.

The focus of this paper is to derive a centralized solution for supply chains with different cash management systems. This perspective is appropriate for a single-owner supply chain or a virtually integration supply chain (i.e., the entities in a supply chain share a common goal of optimizing the supply chain performance; see Porter (1985)). Nonetheless, there are decentralized supply chains in which the entities are individual firms, each with its own interests. An important question for the decentralized supply chain is to design an incentive scheme in which each individual firm would choose the first best solution. The centralized solution we obtain can be viewed as the first best solution for this purpose. We leave this decentralized control issue for future research.

References


**Appendix: Proofs**

Lemma 3 (Karush 1959) shows the additive separation of a function value.

**Lemma 3.** If a function $f(y)$ is convex on $(-\infty, \infty)$ and attains its minimum at $y^*$, then

$$
\min_{a \leq y \leq b} f(y) = f_L(a) + f_U(b),
$$

where $f_L(a) = \min_{a \leq y} f(y)$ is convex non-decreasing in $a$, and $f_U(b) = f(b) - f(b \vee y^*)$ is convex non-increasing in $b$. 

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Proposition 1.

Proof. We prove by induction. The claim trivially holds for $t = T + 1$. Assume $V_{t+1}(x, w) = f_{1,t+1}(x_1) + f_{2,t+1}(x_2) + f_{3,t+1}(w)$, then

$$ V_t(x, w) = \min_{y,r \in S(x,w)} \left\{ \begin{array}{l} H_{1,t}(x_1) + \alpha E_{D_t} f_{1,t+1}(y_1 - D_t) \\
+ H_{2,t}(x_2) + c(y_2 - x_2) + \alpha E_{D_t} f_{2,t+1}(y_2 - D_t) \\
+ H_{3,t}(r) + \beta_i (r - w)^+ + \beta_o (r - w)^- + \alpha E_{D_t} f_{3,t+1}(r + \theta D_t) \end{array} \right\}. \quad (34) $$

Let $g_{1,t}(y_1) = \alpha E_{D_t} f_{1,t+1}(y_1 - D_t)$, and $g_{2,t}(y_2) = cy_2 + \alpha E_{D_t} f_{2,t+1}(y_2 - D_t)$. Since $f_{i,t+1}(\cdot)$ is convex (from induction), by Lemma 3 we can decompose the cost function of echelon 1 and 2:

$$ \min_{y_1 \leq x_1 \leq x_2} g_{1,t}(y_1) = \min_{y_1 \leq x_1} \{ \alpha E_{D_t} f_{1,t+1}(y_1 - D_t) \} + \Gamma_{2,t}(x_2), $$

$$ \min_{y_2 \leq x_2 \leq r} g_{2,t}(y_2) = \min_{x_2 \leq y_2} \{ cy_2 + \alpha E_{D_t} f_{2,t+1}(y_2 - D_t) \} + \Gamma_{3,t}(r), $$

where the induced penalty functions $\Gamma_{2,t}(x_2)$ and $\Gamma_{3,t}(r)$ are expressed in (12) and (13), respectively.

Now, let us define $f_{1,t}(x_1)$ as in (9), and

$$ g_{3,t}(r) = H_{3,t}(r) + \Gamma_{3,t}(r) + \alpha E_{D_t} f_{3,t+1}(r + \theta D_t). $$

Plugging the expressions of $f_{1,t}(x_1)$ and $g_{3,t}(r)$ in (34), we have

$$ V_t(x, w) = f_{1,t}(x_1) + H_{2,t}(x_2) + \Gamma_{2,t}(x_2) + \min_{x_2 \leq y_2} \{ \alpha E_{D_t} f_{2,t+1}(y_2 - D_t) \} $$

$$ + \min_{x_2 \leq r \leq w + \kappa} \{ g_{3,t}(r) + \beta_i (r - w)^+ + \beta_o (r - w)^- \}. \quad (35) $$

Let $\tilde{r}_t = \arg\min_r \{ g_{3,t}(r) \}$, and $r_t^* = \arg\min_r \{ g_{3,t}(r) + \beta_i (r - w)^+ + \beta_o (r - w)^- \}$. The convexity of $g_{3,t}(r)$ implies the existence of the one-sided derivative $\partial g_{3,t}(r)/\partial r$. Define

$$ l_t^* = \sup \{ r : \frac{\partial g_{3,t}(r)}{\partial r} \leq -\beta_i \}, \quad u_t^* = \sup \{ r : \frac{\partial g_{3,t}(r)}{\partial r} \leq \beta_o \}. $$

The monotonicity of $\partial g_{3,t}(r)/\partial r$ implies $l_t^* \leq \tilde{r}_t \leq u_t^*$. Using Proposition B-7 in Heyman and Sobel (1984), we have

$$ r_t^* = \begin{cases} l_t^*, & \text{if } w \leq l_t^*, \\ w, & \text{if } l_t^* < w \leq u_t^*, \\ u_t^*, & \text{if } u_t^* < w. \end{cases} \quad (36) $$

Define $L_t(w) = -\beta_i (w - l_t^*) + g_{3,t}(l_t^*)$, $U_t(w) = \beta_o (w - u_t^*) + g_{3,t}(u_t^*)$, and let

$$ W_t(w) = \begin{cases} L_t(w), & \text{if } w \leq l_t^*, \\ g_{3,t}(w), & \text{if } l_t^* < w \leq u_t^*, \\ U_t(w), & \text{if } u_t^* < w. \end{cases} \quad (37) $$

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From (36) it can be easily shown that
\[ W_t(w) = \min_r \{ g_{3,t}(r) + \beta_1(r-w)^+ + \beta_o(r-w)^- \}. \]

Now, we impose the constraint \( x_2 \leq r \leq w + K \). First, let
\[ r_t^{**} = \arg \min_{x_2 \leq r \leq w + K} \{ g_{3,t}(r) + \beta_1(r-w)^+ + \beta_o(r-w)^- \}, \tag{38} \]
and define the induced penalty functions \( \Lambda_{2,t}(x_2) \) and \( \Lambda_{3,t}(w) \) according to (17) and (18), respectively. Then, we define \( f_{3,t}(w) \) as in (11). The convexity of \( f_{3,t}(w_1) \) can be easily proved by showing that \( \partial f_{3,t}(w)/\partial w \) is non-decreasing in \( w \). Next, we prove the decomposition of echelon 3:
\[ \min_{x_2 \leq r \leq w + K} \{ g_{3,t}(r) + \beta_1(r-w)^+ + \beta_o(r-w)^- \} = f_{3,t}(w) + \Lambda_{2,t}(x_2). \tag{39} \]

The echelon system dynamics and constraint guarantee that \( x_2 \leq w \) holds for all periods. To prove (39), we consider all possible relationships between \( x_2, w, l_t^* \) and \( u_t^* \), as extensively described in the four cases below.

**Case 1.** When \( w \leq l_t^* - K \), we have \( r_t^{**} = w + K \leq l_t^* \), \( f_{3,t}(w) = g_{3,t}(w + K) + \beta_1 K \), and \( \Lambda_{2,t}(x_2) = 0 \). Thus, \( f_{3,t}(w) + \Lambda_{2,t}(x_2) = g_{3,t}(w + K) + \beta_1 K = g_{3,t}(r_t^{**}) + \beta_1 (r_t^{**} - w)^+ + \beta_o (r_t^{**} - w)^- \), i.e., (39) holds.

**Case 2.** When \( l_t^* - K < w \leq l_t^* \), we have \( r_t^{**} = l_t^* \), \( f_{3,t}(w) = L_t(w) \) and \( \Lambda_{2,t}(x_2) = 0 \). Clearly, (39) holds.

**Case 3.** When \( l_t^* < w \), and \( x_2 \leq u_t^* \), we have \( r_t^{**} = r_t^* = \tilde{r}_t \), \( f_{3,t}(w) = g_{3,t}(w) \), and \( \Lambda_{2,t}(x_2) = 0 \). Clearly, (39) holds.

**Case 4.** When \( u_t^* < x_2 \leq w \), we have \( r_t^{**} = x_2 \), \( f_{3,t}(w) = U_t(w) \) and \( \Lambda_{2,t}(x_2) = g_{3,t}(x_2) - U_t(x_2) \). Thus, \( f_{3,t}(w) + \Lambda_{2,t}(x_2) = g_{3,t}(x_2) + U_t(w) - U_t(x_2) = g_{3,t}(x_2) + \beta_1 (w-x_2) = g_{3,t}(r_t^{**}) + \beta_1 (r_t^{**} - w)^+ + \beta_o (r_t^{**} - w)^- \), i.e., (39) holds.

Therefore, we verified that (39) holds in all cases. Substituting (39) into (35), and defining \( f_{2,t}(x_2) \) as in (10), we complete the induction \( V_t(x, w) = f_{1,t}(x_1) + f_{2,t}(x_2) + f_{3,t}(w) \). Using Lemma 3, all induced penalty functions are convex, thus, \( f_{i,t}(\cdot) \) is convex \((i = 1, 2, 3)\). \( \square \)

**Proposition 2.**

*Proof.* We prove by induction. \( V_{T+1}(x, w) = 0 = V^1_{T+1}(x_1, w_1) + V^2_{T+1}(x_1, x_2, w_2) \). Suppose \( V_{t+1}(x, w) \geq
\[ V^{1}_{t+1}(x_1, w_1) + V^{2}_{t+1}(x_1, x_2, w_2) \] for all \((x, w)\), then

\[
V_t(x, w) = \min_{y, r_1 \in S(x, w)} \{G_t(x, w, y_2, r_1) + \alpha \mathbb{E}_{D_t} V_{t+1}^1(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t, \rho \theta D_t) \}
\]

\[
\geq \min_{y, r_1 \in S(x, w)} \{G_t(x, w, y_2, r_1) + \alpha \mathbb{E}_{D_t} V_{t+1}^1(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t) \}
\]

\[
\geq \min_{y, r_1 \in S(x, w)} \{G_t^1(x_1, w_1, r_1) + \alpha \mathbb{E}_{D_t} V_{t+1}^1(y_1 - D_t, r_1 + \theta D_t) \}
\]

\[
+ \min_{y, r_1 \in S(x, w)} \{G_t^2(x_2, w_2, y_2) + \alpha \mathbb{E}_{D_t} V_{t+1}^2(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t) \}
\]

\[
= V_t^1(x_1, w_1) + V_t^2(x_1, x_2, w_2).
\]

The inequality in (41) and (43) are due to induction and constraint relaxation, respectively. The above relationship holds for all \((x, w)\) in period \(t\), completing the induction. \(\square\)

**Lemma 1.**

**Proof.** We write out the flow conservation of \(w_2\) and \(x_1\) from \(s = 1\) to \(s = t - 1\).

\[
\mathbb{E}_{D_1, \ldots, D_{t-1}} w_{2,t} - w_{2,1} = \sum_{s=1}^{t-1} \rho z_{1,s} - \sum_{s=1}^{t-1} \mu_s,
\]

\[
\mathbb{E}_{D_1, \ldots, D_{t-1}} x_{1,t} - x_{1,1} = \sum_{s=1}^{t-1} z_{1,s} - \sum_{s=1}^{t-1} \mu_s.
\]

The result is shown by subtracting \(\rho \times (46)\) from (45). \(\square\)

**Proposition 3.**

**Proof.** We first specify the cost functions when demand is deterministic. Define

\[
H_{1,t}^2(x_1) = (h_1^2 + h_2 + \eta_2 + b_2)(d_t(\omega) - x_1) + h_1^2(x_1 - d_t(\omega)),
\]

\[
H_{2,t}^2(x_2) = h_2(x_2 - d_t(\omega)), \quad H_{3,t}^d(a) = \eta_2(a - d_t(\omega)).
\]

We then prove by induction. The claim trivially holds for \(t = T + 1\). Now, we assume \(V_{t+1}^d(x_1, x_2 | \omega) = v_{t+1}^1(x_1 | \omega) + v_{t+1}^2(x_2 | \omega)\). Let \(g_{1,t}^d = \alpha v_{t+1}^1(y_1 - d(\omega) | \omega)\) and \(g_{2,t}^d = c_2 y_2 + \alpha v_{t+1}^1(y_2 - d(\omega) | \omega)\). From the convexity of \(v_{t+1}^d(\cdot)\) and Lemma 3, we can decompose the cost functions of echelon 1 and 2.
as follows:

\[
\begin{align*}
\min_{x_1 \leq y_1 \leq x_2} g_{1,t}^d(y_1) &= \min_{x_1 \leq y_1} \{ \alpha v_{t+1}^1(y_1 - d(\omega) \mid \omega) \} + \Gamma_{2,t}^d(x_2), \\
\min_{x_2 \leq y_2 \leq a} g_{2,t}^d(y_2) &= \min_{x_2 \leq y_2} \{ c y_2 + \alpha v_{t+1}^2(y_2 - d(\omega) \mid \omega) \} + \Gamma_{1,t}^d(a),
\end{align*}
\]

where \( a = \rho x_1 + B_t(\omega) \). Let \( y_{1,t}^* \) minimize \( g_{i,t}^d(y_i) \), the induced penalty functions are

\[
\Gamma_{2,t}^d(x_2) = \begin{cases} 
\alpha [v_{t+1}^1(x_2 - d_t(\omega) \mid \omega) - v_{t+1}^1(y_{1,t}^* - d_t(\omega) \mid \omega)], & x_2 \leq y_{1,t}^*, \\
0, & \text{otherwise}, 
\end{cases}
\]

\[
\Gamma_{1,t}^d(x_1) = \begin{cases} 
c(a - y_{2,t}^*) + \alpha [v_{t+1}^2(a - d_t(\omega) \mid \omega) - v_{t+1}^2(y_{2,t}^* - d_t(\omega) \mid \omega)], & a \leq y_{2,t}^*, \\
0, & \text{otherwise}. 
\end{cases}
\]

From Lemma 3, the functions above are convex. Therefore, the following functions are convex:

\[
\begin{align*}
v_t^1(x_1 \mid \omega) &= H_{1,t}^d(x_1) + H_{3,t}^d(a) + \Gamma_{1,t}^d(a) + \min_{x_1 \leq y_1} \{ \alpha v_{t+1}^1(y_1 - d_t(\omega) \mid \omega) \}, \\
v_t^2(x_2 \mid \omega) &= H_{2,t}^d(x_2) + \Gamma_{2,t}^d(x_2) + \min_{x_2 \leq y_2} \{ c(y_2 - x_2) + \alpha v_{t+1}^2(y_2 - d_t(\omega) \mid \omega) \}.
\end{align*}
\]

Furthermore, \( V_t^d(x_1, x_2 \mid \omega) = v_t^1(x_1 \mid \omega) + v_t^2(x_2 \mid \omega) \), completing the proof.