A “phantom stockout” is a retail stockout phenomenon caused either by inventory shrinkage or by shelf execution failure. Unlike the conventional stockout which can be corrected by inventory replenishment, a phantom stockout persists and requires human interventions. In this paper, we propose two partially-observable Markov decision models: one for the shrinkage problem and the other for the shelf execution failure problem. In the shrinkage model, the actual inventory level is not known unless an inspection is performed. We derive a probabilistic belief about the actual inventory level based on the system inventory records and historical sales data. We then formulate a joint inspection and replenishment problem and partially characterize the optimal policy. To simplify computation, we further consider a decoupled problem in which the inspection and replenishment decisions can be determined separately by the state-dependent thresholds based on the number of consecutive zero-sales periods. Our simulation study reveals that the joint and decoupled policies outperform the existing policies in the literature in most cases. For the shelf execution failure problem, we model the shelf execution process as a deteriorating process that requires occasional inspection. We show that there exists a closed-form condition to determine the optimal inspection threshold based on the number of consecutive zero-sales periods. Interestingly, we find that the inspection frequency in this problem is not always increasing in the deterioration probability.

Key words: Phantom Stockout, Phantom Inventory, Shelf Inspection, Joint Inspection and Inventory Replenishment, Deterioration Process, Partially-Observable Markov Decision Process.

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1. Introduction

Shelf stockouts are a major source of lost revenue and poor customer satisfaction for retailers. It is estimated that, on average, shelf stockouts cost retailers 4% of their annual sales (Gruen et al. 2002). A recent industry study identifies inventory record inaccuracies and shelf execution failures as the two main causes of shelf stockouts (Gruen and Corsten 2007). Inventory record inaccuracies occur when the store encounters inventory shrinkage and transaction errors (Raman et al. 2001ab). Specifically, inventory shrinkage, such as theft and spoilage, can cause a retailer’s inventory records to indicate greater inventory than is physically available in the store, which, in turn, can freeze the store replenishment system and result in prolonged stockouts (Kang and Gershwin 2005). This system-freezing effect is termed “phantom inventory” by Gruen and Corsten (2007). Shelf execution failures occur when there is inventory in the backroom, but store associates fail to move it to the shelf. As a result, the shelf remains empty unless and until someone intervenes. Fisher and Raman (2010, p. 175) term this phenomenon a “phantom stockout.” In this paper, we use phantom stockout to broadly refer to a shelf stockout situation caused either by inventory shrinkage or by shelf execution failure.

Phantom stockouts are different from conventional “planned” stockouts (due to service level settings) in that human interventions are required to correct the problem. Intuitively, one could track the number of consecutive zero-sales periods for a product and use this signal to trigger an inspection. For example, some retailers used an ad hoc seven-day (zero-sales) rule to trigger inspection for fast-moving products. Fisher and Raman (2010, pp. 177-178) proposed a more sophisticated statistical process control (SPC) method that triggers inspection based on the probability of consecutive zero-sales periods. These policies are intuitive and easy to implement. However, they do not factor in cost implications and therefore may not be cost-effective. In a different problem context, Kok and Shang (2007) suggested a cycle count (CC) inspection policy and DeHoratius et al. (2008) proposed an expected value of perfect information (EVPI) inspection policy to address the inventory record inaccuracy problem. It is not clear how these policies would perform in the phantom stockout problem. In this paper, we seek to design new solutions for the phantom stockout problem and to compare our solutions with the existing solutions (e.g., the aforementioned SPC, CC, and EVPI policies) in a simulated system that is subject to phantom stockouts.

Specifically, we develop two partially-observable Markov decision models for the phantom stockout problem: one for the inventory shrinkage problem and the other for the shelf execution failure problem. In the shrinkage model, the actual inventory level is not known unless an inspection is
performed; and in the execution failure model, the status of shelf execution is not observed before an inspection. As will be evident below, the shrinkage problem is much more challenging than the execution failure problem. Thus, we focus primarily on making algorithmic and computational contributions for the shrinkage problem in this paper; the analysis of the execution failure problem is included at the end of the paper.

To tackle the shrinkage problem, we first derive a Bayesian belief of the actual inventory level based on the system inventory records and historical sales data. We then simplify the Bayesian belief by approximating it with a combination of three intuitive sufficient statistics: the system inventory record, the number of periods since the last inspection, and the number of consecutive zero-sales periods (leading up to the current period). Thus, our Bayesian belief is more sophisticated than that used in the traditional cycle-count policy (which relies on a single sufficient statistic, i.e., the number of periods since the last inspection). Based on these sufficient statistics, we formulate a joint inspection and replenishment problem. We show that, if the system starts right after an inspection, it is optimal to either inspect or replenish inventory whenever the number of consecutive zero-sales periods reaches a state-dependent threshold. To determine which action to take, we can apply the standard value iteration method to solve the problem.

Shelf inspection and store inventory replenishment are often managed by different business functions in a retail store. This separation of job responsibilities motivates us to seek for solutions that decouple the inspection and replenishment decisions and in the meantime retain the simple \((s, S)\) replenishment policy structure commonly used in practice (the base-stock policy is a special case of the \((s, S)\) policy). To this end, we divide the original problem into three subproblems: 1) compute the optimal \((s, S)\) policy by ignoring shrinkage and use it for regular replenishment based on the system inventory record; 2) when the system inventory record falls between \(s + 1\) and \(S - 1\), determine additional replenishment based on the Bayesian inventory belief; and 3) find the optimal inspection policy under a simplified model. We show that the inspection and replenishment decisions can be determined (separately) by the state-dependent thresholds based on the number of consecutive zero-sales periods.

We create a simulated inventory system in which the actual inventory is depleted by both demand and unobserved shrinkage. With this simulated system, we can have a controlled comparison of our solutions and the existing inspection polices (e.g., the SPC, CC, and EVPI policies). To our knowledge, this is the first simulation study to compare all the existing inspection policies in the literature.

We obtain the following insights from the simulation study. First, we find that our joint inspec-
tion and replenishment policy consistently outperforms all other policies. The decoupled inspection and replenishment policy has a fairly stable performance across all scenarios. Although not as good as the joint policy, the decoupled policy performs better than the SPC, CC, and EVPI policies in most cases. The joint and decoupled policies differ from the existing inspection policies in that they allow for additional replenishment based on the Bayesian inventory belief. Intuitively, replenishment brings extra inventory into the system, and can proactively prevent phantom stockouts from occurring (even though the inventory record may remain inaccurate). Our simulation results suggest that replenishment can indeed be a cost-effective way to substitute (costly) inspection. Second, when the shrinkage rate is low, the potential gain from an optimal policy is small. In this case, the simple SPC policy may just be sufficient. On the other hand, when the shrinkage rate is high, the potential value from an optimal policy increases significantly. In this case, more advanced inspection and replenishment policies (such as the joint and decoupled policies) should be used. Third, we conduct additional robustness tests for each policy under misspecified inspection cost parameters. We find that our joint and decoupled policies are robust to the specification error. In contrast, the EVPI and CC policies are more sensitive to the specification errors (the SPC policy is independent of the inspection cost). These results suggest that allowing for additional replenishment also makes the system performance less sensitive to misspecified inspection cost parameters.

Finally, for the shelf execution failure problem, we assume that inventory shrinkage is negligible and the store backroom has ample inventory. We model the shelf execution process as a deteriorating process. At the end of each period, the execution process can deteriorate into a failure state with a certain probability. When the execution process enters the failure state, the sales become zero and human inspection is required to fix the problem (which is analogous to a broken vending machine). Similar system deterioration models have been used in the machine maintenance and quality control literature (see a discussion in §2). We show that there exists a closed-form condition to determine the optimal inspection threshold based on the number of consecutive zero-sales periods. The calculation can be easily implemented in practice. Interestingly, we find that the inspection frequency is not always increasing in the deterioration probability. Specifically, when inspection cost is high, if the execution process deteriorates to the failure state very soon, it may be cost-effective to reduce the inspection frequency, so as to save on the inspection cost.

The rest of this paper is organized as follows. We provide a literature review in §2, and then study the shrinkage problem in §3. In §4, we report results from an extensive simulation study. We further consider the shelf execution failure problem in §5. §6 contains our concluding remarks. All proofs are presented in the Appendix.
2. Literature Review

Our paper is closely related to three streams of research in the literature. The first stream concerns inventory management under inventory record inaccuracy (see Chen and Mersereau 2014 for a comprehensive review). This stream of research can be divided into two categories. In the first category, inventory replenishment problems without inspection are considered, such as Fleisch and Tellkamp (2004), Kang and Gershwin (2005), Atali et al. (2006), Heese (2007), Lee and Ozer (2007), Bensoussan et al. (2007, 2008, 2011), and Mersereau (2011). In the second category, inspection decision is included in the problem formulation. Our paper belongs to this second category. Two types of inspection policy have been studied in the literature: an infrequent store-level inventory cycle-count policy (e.g., Iglehart and Morey 1972, Lee and Rosenblatt 1985, Kok and Shang 2007, 2010, DeHoratius et al. 2008, and Huh et al. 2009) and a daily walk-through inspection called a “zero balance walk” (e.g., Fisher et al. 2000, Bensoussan et al. 2007). Unlike the inspection problem considered in this paper, the daily zero balance walk inspection practice does not target any specific products \textit{a priori} (Fisher et al. 2000). As a result, it may not catch every phantom-stockout situation and it may require too much manpower.

Kok and Shang (2007) and DeHoratius et al. (2008) are the two most related papers to our work. Our phantom stockout model differs from theirs in the following ways. First, Kok and Shang (2007) assumed that unmet demand is fully backlogged and the inventory errors do not depend on the actual inventory level. As a result, their inventory belief only depends on the total number of periods since the last inspection. In our model, unmet demand is lost and inventory shrinkage depends on the available inventory level. Therefore, our inventory belief also depends on the system inventory record and the number of consecutive zero sales periods. Second, while DeHoratius et al. (2008) accounted for lost sales in their inventory belief, they did not formulate the inspection problem as a dynamic program. Instead, they proposed a heuristic rule based on the expected value of perfect information (see a discussion of their decision rule in §4). In our model, we formulate the joint inspection and replenishment problem as a dynamic program and further propose a decoupled policy to simplify the computation. Third, both Kok and Shang (2007) and DeHoratius et al. (2008) assumed that inventory errors can be both positive and negative and that there is no fixed ordering cost. In our model, we consider only the shrinkage process. Therefore, inventory errors in our model are nonnegative. Furthermore, our model considers the more general setting with a fixed ordering cost.

The second stream of research related to our paper is the recent research work on store ex-
ution problems. In an empirical study with Borders Group, Ton and Raman (2010) found that higher store inventory tends to create more chaos in execution and thus leads to more phantom stockouts. To tackle this problem, Fisher and Raman (2010, p. 175) suggested a just-in-time (JIT) store execution approach, i.e., ship to the store only as much inventory as fits the shelf capacity and eventually eliminate inventory storage in the backroom. Our solution for the shelf execution problem complements their JIT approach in the way that we propose a data-driven approach to help fix the shelf execution problem.

The third stream of research related to our paper is the machine maintenance and quality control literature. For comprehensive surveys of the machine maintenance literature, we refer readers to Pierskalla and Voelker (1976), Monahan (1982), and Valdez-Flores and Feldman (1989). Eckles (1968) first studied the machine replacement problem with incomplete information and proposed an age-dependent machine replacement policy that has essentially the same threshold policy structure as in our problem. Kaplan (1969) studied investigation strategies for financial report irregularity. Specifically, he considered a two-action, two-state deterioration model similar to ours. However, his problem structure does not yield a closed-form solution as does our model. Ross (1971) studied a three-action, two-state deterioration model and characterized the optimal policy structure. Again, closed-form solutions are not available in his model. More recently, Grosfeld-Nir (1996) studied a two-action, two-state deterioration model and was able to obtain a closed-form solution under uniformly distributed observations. However, his result does not apply in our problem because the observation process in our model is not uniformly distributed. Our paper contributes to this literature in two ways. First, we introduce the seemingly unrelated shelf execution problem as a form of inspection problem with partially observable information. Second, we derive an easy-to-implement closed-form formula for determining the inspection threshold in our model.

3. The Shrinkage Problem

Consider selling a single product over an infinite horizon. The demand occurs in discrete time periods (e.g., daily). Let us assume that the demand in each period $t$, denoted by $D_t$, is an independent and identically distributed (i.i.d.) nonnegative random variable that takes value in \{0, 1, 2, ...\}. Let $\varphi$ denote the zero demand probability in a period, i.e., $\varphi = P(D_t = 0)$. Let $R_t \geq 0$ denote the inventory replenishment quantity in period $t$. The inventory replenishment leadtime is assumed to be negligible. Thus, $R_t$ is immediately available for meeting demand in period $t$.

In addition to the demand process, there is an unobserved inventory shrinkage process, denoted by $U_t$. The shrinkage $U_t$ in each period is also an i.i.d. nonnegative random variable that takes
value in \( \{0, 1, 2, \ldots\} \). We assume that \( U_t \) is independent of \( D_t \) and occurs after the demand process \( D_t \) in each period. Let \( I_t \) be the actual physical inventory level at the beginning of period \( t \) before receiving the replenishment quantity. Thus, the sales in period \( t \) is given by \( Z_t = \min\{I_t + R_t, D_t\} \), where we assume the unmet demand is lost, as is the case in many retail environments. Under these assumptions, it is straightforward to verify that the actual inventory level is evolving according to the following equation:
\[
I_{t+1} = (I_t + R_t - Z_t - U_t)^+,
\]
where \((\cdot)^+ = \max\{\cdot, 0\}\).

Because the shrinkage process is unobservable, the computerized inventory system only tracks the logical inventory based on the replenishment quantities and sales transactions. Let \( \tilde{I}_t \) denote the logical inventory level at the beginning of period \( t \) before receiving the replenishment quantity. The evolution of the logical inventory level \( \tilde{I}_t \) is given by
\[
\tilde{I}_{t+1} = \tilde{I}_t + R_t - Z_t.
\]
Comparing equations (1) and (2), we observe that, if the system starts with \( I_1 = \tilde{I}_1 \) with probability one and let \( t_0 = 1 \). Without loss of generality, suppose that replenishment activities occur in periods \( t_1, \ldots, t_r \), with \( 1 = t_0 \leq t_1 < \ldots < t_r \leq t \). If \( Z_t > 0 \), then the following holds:
\[
I_{t+1} = (\tilde{I}_{t+1} - \tilde{I}_{t_r} + I_{t_r} - U_{t_r}^t)^+,
\]
where \( I_{t_j} = (\tilde{I}_{t_j} - \tilde{I}_{t_{j-1}} + I_{t_{j-1}} - U_{t_{j-1}}^{t_{j-1}})^+ \) for \( 1 \leq j \leq r \).

Clearly, from (3), the actual inventory level \( I_{t+1} \) can be inferred based on the logical inventory levels \( \tilde{I}_1, \ldots, \tilde{I}_{t_r}, \tilde{I}_{t+1} \), and the cumulative shrinkage process. In the special case when there is no replenishment between period 1 and period \( t \), we have \( I_{t+1} = (\tilde{I}_{t+1} - U_1^t)^+ \), where \( I_{t+1} \) can be
simply inferred from $\tilde{I}_{t+1}$ and the cumulative shrinkage $U^t_1$. The recursive relationship specified in Proposition 1 can be further simplified by leveraging the following observation:

Lemma 1 Suppose that replenishment activities occur in periods $1 \leq t_1 < ... < t_r \leq t$ and $Z_t > 0$. If $\tilde{I}_{t+1} < \tilde{I}_{t_r}$, then $I_{t_r} > 0$. Similarly, for any $1 \leq j < j' < r$, if $\tilde{I}_{t_{j'}} < \tilde{I}_{t_j}$, then $I_{t_j} > 0$.

The intuition behind the above result is that, if there is a net decrease in logical inventory level between two periods, then the actual inventory in the earlier period cannot be zero. With this observation, let $\mathcal{T} = \{t_1, ..., t_r\}$. In the set $\mathcal{T}$, we can search for the period with the lowest logical inventory level, i.e., $\tau_1 = \arg\min_{r \in \mathcal{T}} \{\tilde{I}_r\}$. When there are multiple minimal points, we take the one with the smallest time index. If $\tau_1 < t_r$, we can search for the next lowest logical inventory level in $\mathcal{T}$ from $\tau_1$ onward, i.e., $\tau_2 = \arg\min_{r \in \mathcal{T}, r > \tau_1} \{\tilde{I}_r\}$. Repeat the above procedure until we reach the point $\tau_l = t_r$. This search procedure is illustrated in Figure 1.

![Figure 1: Illustration of the search procedure for $\tau_1, ..., \tau_l$.](image)

Given the above procedure, it is clear that $1 \leq \tau_1 < ... < \tau_l = t_r$. Moreover, we have $\tilde{I}_{\tau_1} < ... < \tilde{I}_{\tau_l} = \tilde{I}_{t_r}$. By Lemma 1, we can establish the following result:

Corollary 1 Suppose that $I_1 = \tilde{I}_1$ with probability one and let $\tau_0 = 1$. If $Z_t > 0$, then

$$I_{t+1} = (\tilde{I}_{t+1} - \tilde{I}_{t_r} + I_{t_r} - U^t_1)^+,$$

where $I_{t_r} = I_{\tau_l}$ can be expressed by the following recursive relationship: for $1 \leq j \leq l$,

$$I_{\tau_j} = (\tilde{I}_{\tau_j} - \tilde{I}_{\tau_{j-1}} + I_{\tau_{j-1}} - U^t_{\tau_{j-1}})^+.$$  (4)
From the above proposition, we observe that the actual physical inventory level $I_{t+1}$ depends only on the logical inventory levels $\tilde{I}_{t_1}, ..., \tilde{I}_{t_{\tau}}$ and the cumulative shrinkage process. We can leverage this result to derive the probabilistic belief of $I_{t+1}$.

### 3.1 Bayesian Inventory Belief

Define $Y = \tilde{I}_{t+1} - \tilde{I}_{t_{r}} + I_{t_{r}}$, where $I_{t_{r}}$ is given by (4) from Corollary 1. Thus, we have $I_{t+1} = (Y - U_{t_{r}}')^+$. Suppose that $Z_t = z > 0$. If the actual inventory level $I_{t+1} > 0$, then the event \{ $D_t = z, I_{t+1} = Y - U_{t_{r}}' > 0$\} must have occurred. On the other hand, if the actual inventory level $I_{t+1} = 0$, there are two possible scenarios. Scenario 1): The event \{ $I_t + R_t - z = 0, D_t \geq z$\} occurred. Scenario 2): The event \{ $I_t + R_t - z > 0, D_t = z, U_{t_{r}}' \geq Y$\} occurred. Because $Z_t = z > 0$, it must be true that $I_t + R_t > 0$. Consider two cases. If $R_t = 0$, from Corollary 1 we have $I_t = \tilde{I}_{t} - \tilde{I}_{t_{r}} + I_{t_{r}} - U_{t_{r}}' = \tilde{I}_{t} - \tilde{I}_{t_{r}+1} + Y - U_{t_{r}}'^{-1}$. Thus, $I_t + R_t - z = \tilde{I}_{t+1} - \tilde{I}_{t_{r}+1} + Y - U_{t_{r}}'^{-1} = Y - U_{t_{r}}'^{-1}$. On the other hand, if $R_t > 0$, we have $t = t_{r}$. For definition of $Y$, we have $I_t = Y - \tilde{I}_{t_{r}+1} + \tilde{I}_{t}$.

Thus, $I_t + R_t - z = Y - \tilde{I}_{t_{r}+1} + \tilde{I}_{t_{r}+1} = Y$. Recall that $U_{t_{r}}'^{-1} = 0$. We can thus unify these two cases by writing $I_t + R_t - z = Y - U_{t_{r}}'^{-1}$. Based on the above analysis, the likelihood function of having an actual inventory level $I_{t+1} = i$ in this case is given by

$$L(i|z > 0, \tilde{I}_{t+1}) = \begin{cases} P(D_t = z) \cdot P(U_{t_{r}}' = Y - i) & \text{if } 0 < i \leq \tilde{I}_{t+1}, \\ P(D_t \geq z) \cdot P(U_{t_{r}}'^{-1} = Y) + P(D_t = z) \cdot P(U_{t_{r}}'^{-1} < Y, U_{t_{r}}' \geq Y) & \text{if } i = 0. \end{cases}$$

Thus, according to Bayes’ rule, the belief probability of $I_{t+1} = i$ for $0 \leq i \leq \tilde{I}_{t+1}$, denoted by $\pi_{t+1}(i)$, is given by,

$$\pi_{t+1}(i|z > 0) = \frac{L(i|z > 0, \tilde{I}_{t+1})}{\sum_{j=0}^{\tilde{I}_{t+1}} L(j|z > 0, \tilde{I}_{t+1})}. \quad (6)$$

Next suppose that $Z_t = z = 0$ and $R_t > 0$. In this case, if the actual inventory level $I_{t+1} > 0$, then the event \{ $D_t = 0, I_{t+1} = Y - U_{t_{r}}' > 0$\} must have occurred. On the other hand, if the actual inventory level $I_{t+1} = 0$, the event \{ $D_t = 0, U_{t_{r}}' \geq Y$\} must have occurred. Therefore, we have

$$L(i|z = 0, R_t > 0, \tilde{I}_{t+1}) = \begin{cases} P(D_t = 0) \cdot P(U_{t_{r}}' = Y - i) & \text{if } 0 < i \leq \tilde{I}_{t+1}, \\ P(D_t = 0) \cdot P(U_{t_{r}}' \geq Y) & \text{if } i = 0. \end{cases}$$

Thus, according to Bayes’ rule, the belief probability of $\tilde{I}_{t+1} = i$ is given by

$$\pi_{t+1}(i|z = 0, R_t > 0) = \begin{cases} P(U_{t_{r}}' = Y - i) & \text{if } 0 < i \leq \tilde{I}_{t+1}, \\ P(U_{t_{r}}' \geq Y) & \text{if } i = 0. \end{cases} \quad (7)$$
Finally, suppose that $Z_t = z = 0$ and $R_t = 0$. In this case, we have $\tilde{I}_{t+1} = \tilde{I}_t$. If the actual inventory level $I_{t+1} > 0$, then the event $\{D_t = 0, I_{t+1} = I_t - U_t\}$ must have occurred. On the other hand, if the actual inventory level $I_{t+1} = 0$, either the event $\{I_t = 0\}$ or the event $\{D_t = 0, U_t \geq I_t\}$ must have occurred. Thus, the likelihood function of having an actual inventory level $I_{t+1} = i$ in this case is given by

$$L(i | z = 0, R_t = 0, \tilde{I}_{t+1}) = \begin{cases} \varphi \cdot \sum_{j=0}^{\tilde{I}_{t+1}} \pi_t(j) P(U_t = j - i) & \text{if } 0 < i \leq \tilde{I}_{t+1}, \\ \pi_t(0) + \varphi \cdot \sum_{j=1}^{\tilde{I}_{t+1}} \pi_t(j) P(U_t \geq j) & \text{if } i = 0, \end{cases}$$

where $\pi_t(i)$ is the belief probability of $I_t = i$ in period $t$. Thus, according to Bayes’ rule, the belief probability of $\tilde{I}_{t+1} = i$ is given by

$$\pi_{t+1}(i | z = 0, R_t = 0) = \frac{L(i | z = 0, R_t = 0, \tilde{I}_{t+1})}{\sum_{j=0}^{\tilde{I}_{t+1}} L(j | z = 0, R_t = 0, \tilde{I}_{t+1})} = \begin{cases} \frac{\varphi \cdot \sum_{j=0}^{\tilde{I}_{t+1}} \pi_t(j) P(U_t = j - i)}{\varphi + (1 - \varphi) \pi_t(0)} & \text{if } 0 < i \leq \tilde{I}_{t+1}, \\ \frac{\pi_t(0) + \varphi \cdot \sum_{j=1}^{\tilde{I}_{t+1}} \pi_t(j) P(U_t \geq j)}{\varphi + (1 - \varphi) \pi_t(0)} & \text{if } i = 0, \end{cases}$$

where we use the fact that $\sum_{j=0}^{\tilde{I}_{t+1}} L(j | z = 0, R_t = 0, \tilde{I}_{t+1}) = \varphi + (1 - \varphi) \pi_t(0)$.

We note that the above Bayesian inventory belief differs from that of DeHoratius et al. (2008) in two ways: 1) our belief is derived based on an inventory shrinkage process, whereas their Bayesian inventory belief allows for both positive and negative inventory errors; and 2) leveraging on the structure of the shrinkage process, we are able to simplify the updating process to a few key logical inventory points, whereas their updating process requires the entire history of sales, replenishment quantities and logical inventory levels.

From the above analysis, it is clear that, when sales $Z_t = 0$ and replenishment $R_t = 0$, the inventory belief depends on the belief of the previous period, and in this case the number of consecutive zero-sales periods can be used as a sufficient statistic. However, when sales $Z_t > 0$ or replenishment $R_t > 0$, the inventory belief is reset. Although it is quite straightforward to compute the Bayesian inventory belief from the above formula, there are two main difficulties in formulating and solving a Bayesian dynamic program based on the above belief system. First, the Bayesian inventory belief depends on the sales process $Z_t$, which is dependent on the (unobserved) actual inventory level in the previous period. Second, the Bayesian inventory belief depends on the logical inventory points $\tilde{I}_{\tau_1}, \ldots, \tilde{I}_{\tau_l}$ (through the definition of $Y$), which significantly increases the problem dimensionality and makes the resulting dynamic program intractable.
3.1.1 Simplification

In order to make our problem tractable, we resort to some approximation techniques to simplify the Bayesian inventory belief. In what follows, we propose a two-step approximation procedure to simplify the Bayesian inventory belief. In the first step of approximation, from expression (5), we observe that

\[
L(0 \mid z > 0, \tilde{I}_{t+1}) = P(D_t \geq z) \cdot P(U_{t}^{t-1} = Y) + P(D_t = z) \cdot P(U_{t}^{t-1} < Y, U_{t}^{t} \geq Y)
\]

\[
\geq P(D_t = z) \cdot P(U_{t}^{t-1} = Y) + P(D_t = z) \cdot P(U_{t}^{t-1} < Y, U_{t}^{t} \geq Y)
\]

\[
= P(D_t = z) \cdot P(U_{t}^{t} \geq Y).
\]

Suppose that we use the lower bound to approximate the original likelihood function, i.e.,

\[
L(0 \mid z > 0, \tilde{I}_{t+1}) \approx P(D_t = z) \cdot P(U_{t}^{t} \geq Y).
\]

The Bayesian inventory belief \( \pi_{t+1}(i \mid z > 0) \) can be simplified as:

\[
\pi_{t+1}(i \mid z > 0) \approx \begin{cases} 
P(U_{t}^{t} = Y - i) & \text{if } 0 < i \leq \tilde{I}_{t+1}, \\
P(U_{t}^{t} \geq Y) & \text{if } i = 0,
\end{cases}
\]

where the inventory belief is disentangled from the sales process. We note that this approximation step underestimates the phantom stockout (zero-inventory) probability. Moreover, the above expression is the same as (7) in the case of \( Z_t = 0 \) and \( R_t > 0 \). Thus, under this approximation, we can use the above expression to unify the inventory beliefs of cases: \( \{Z_t = z > 0\} \) and \( \{Z_t = 0, R_t > 0\} \). The union of these two cases can be written as \( \{Z_t = z > 0 \text{ or } R_t > 0\} \).

In the second step, we further make the following approximation for equation (4) in Corollary 1: for \( 1 \leq j \leq l \),

\[
I_{\tau_j} = (\bar{I}_{\tau_j} - \bar{I}_{\tau_{j-1}} + I_{\tau_{j-1}} - U_{\tau_{j-1}}^{\tau_{j-1}}) + \bar{I}_{\tau_j} - \bar{I}_{\tau_{j-1}} + I_{\tau_{j-1}} - U_{\tau_{j-1}}^{\tau_{j-1}}.
\]

Substituting these approximated expressions into \( Y \), we obtain \( Y \approx \bar{I}_{t+1} - U_{t}^{t-1} \). Therefore, \( Y \) is no longer dependent on the historical logical inventory levels \( \bar{I}_{\tau_1}, \ldots, \bar{I}_{\tau_l} \). The Bayesian inventory belief \( \pi_{t+1}(i \mid z > 0 \text{ or } R_t > 0) \) is thus simplified as

\[
\pi_{t+1}(i \mid z > 0 \text{ or } R_t > 0) \approx \begin{cases} 
P(U_{t}^{t} = \bar{I}_{t+1} - i) & \text{if } 0 < i \leq \bar{I}_{t+1}, \\
P(U_{t}^{t} \geq \bar{I}_{t+1}) & \text{if } i = 0,
\end{cases}
\]

where the inventory belief only depends on the current logical inventory level \( \bar{I}_{t+1} \) and the number of periods since the last inspection. We note that the above approximation is exact when no
replenishment occurs since the last inspection. Furthermore, it is easy to verify that \( Y \geq \tilde{I}_{t+1} - U_1^{t-1} \). Thus, this second approximation step compensates the first step in that it overestimates the phantom stockout (zero-inventory) probability.

Let \( \tilde{I}_{t+1} \) denote the logical inventory level after receiving the replenishment quantity, i.e., \( \tilde{I}_{t+1} = \tilde{I}_{t+1} + R_{t+1} \). We can further translate the inventory beliefs before and after replenishment as follows. When \( R_{t+1} = 0 \), we have \( \tilde{I}_{t+1} = \tilde{I}_{t+1} \) and the inventory belief remains the same. When \( R_{t+1} > 0 \) (implying \( Z_t = z > 0 \) based on our assumption), from (9), the inventory belief after replenishment, denoted by \( \pi_i' \), can be written as

\[
\pi_i'(i|z > 0) \approx \begin{cases} 
P(U_1^t = \tilde{I}_{t+1} + R_{t+1} - i) & \text{if } R_{t+1} < i \leq \tilde{I}_{t+1} + R_{t+1}, \\
P(U_1^t \geq \tilde{I}_{t+1}) & \text{if } i = R_{t+1}, \\
0 & \text{if } 0 \leq i < R_{t+1}, \\
P(U_1^t = \tilde{I}_{t+1} - i) & \text{if } 0 < i \leq \tilde{I}_{t+1} + R_{t+1}, \\
P(U_1^t \geq \tilde{I}_{t+1}) & \text{if } i = 0,
\end{cases}
\]

(10)

where we make a further approximation to make \( \pi_i'(i|z > 0) \) dependent only on \( \tilde{I}_{t+1} \). Without this additional approximation, the Bayesian inventory belief would depend on the replenishment quantity \( R_{t+1} \), which introduces an additional dimensionality to the problem and complicates the analysis. We note that this approximation takes place only when a replenishment occurs and it also compensates the first-step approximation by overestimating the phantom stockout (zero-inventory) probability.

3.1.2 Sufficient Statistics

Based on the simplified Bayesian inventory belief, we can define the following two sufficient statistics: Let \( m + n \) be the total number of periods since the last inspection, with \( n \) being the number of consecutive zero-sales periods (leading up to the current period) during which neither replenishment nor inspection takes place.

With some abuse of notation, we can express the belief of the actual inventory before receiving replenishment as \( \pi(i|\tilde{I}, m, n) \), where \( \tilde{I} \) is the logical inventory level before receiving the replenishment quantity in a period (we suppress the time index for ease of notation). Specifically, \( \pi(i|\tilde{I}, m, n) \)
can be expressed recursively as follows: for \( n > 0 \),

\[
\pi(i|\tilde{I}, m, n) = \begin{cases}
\varphi \cdot \sum_{j=i}^{\tilde{I}} \pi(j|\tilde{I}, m, n-1) P(U_t = j - i) & \text{if } 0 < i \leq \tilde{I}, \\
\varphi + (1 - \varphi) \pi(0|\tilde{I}, m, n-1) & \text{if } i = 0,
\end{cases}
\]

with

\[
\pi(i|\tilde{I}, m, 0) = \begin{cases}
P(U_1^m = \tilde{I} - i) & \text{if } 0 < i \leq \tilde{I}, \\
P(U_1^m \geq \tilde{I}) & \text{if } i = 0.
\end{cases}
\]

Let “\( \geq_{st} \)” denote first order stochastic dominance (see Muller and Stoyan 2002). We can establish the following result:

**Proposition 2** Given a logical inventory level \( \tilde{I} \), for any \( m, n \geq 0 \), \( \pi(\cdot|\tilde{I}, m, n) \geq_{st} \pi(\cdot|\tilde{I}, m+1, n) \), \( \pi(\cdot|\tilde{I}, m, n) \geq_{st} \pi(\cdot|\tilde{I}, m+1, n) \), and \( \pi(\cdot|\tilde{I}, m+1, n) \geq_{st} \pi(\cdot|\tilde{I}, m, n+1) \). Specifically, \( \pi(0|\tilde{I}, m, n) \) is increasing in \( n \), with \( \lim_{n \to \infty} \pi(0|\tilde{I}, m, n) = 1 \).

The above result shows that the inventory belief is skewed more towards zero as the number of periods since the last inspection increases (due to the shrinkage process). Moreover, as the number of consecutive zero sales periods increases, the inventory belief is skewed further towards zero. We note that, in stating the above result, we use “increasing” in the weak sense such that it has the same meaning as “non-decreasing” (the same applies for “decreasing”). We adopt this convention throughout the paper.

Finally, suppose that the logical inventory level after replenishment is \( \tilde{I}' \). When there is no replenishment, we have \( \tilde{I}' = \tilde{I} \) and thus the inventory belief remains the same as \( \pi(\cdot|\tilde{I}, m, n) \). On the other hand, when there is inventory replenishment, we have \( \tilde{I}' > \tilde{I} \). In this case, based on (10), the Bayesian inventory belief after replenishment becomes \( \pi(\cdot|\tilde{I}', m + n, 0) \).

### 3.2 Joint Inspection and Replenishment

To fix the phantom stockout problem caused by inventory shrinkage, in this section we formulate a joint inspection and inventory replenishment problem based on the simplified Bayesian inventory belief derived above. For ease of notation, we shall suppress the period index whenever there is no confusion within the context.

Specifically, we assume that there is a fixed cost \( k \) for inventory inspection and that the system inventory is reset to the actual inventory level after inspection. For the inventory replenishment problem, we assume that there is fixed cost \( k_r \) for replenishment; when there is no shrinkage, an
\((s, S)\) policy is known to be optimal.\(^1\) Let \(h\) and \(b\) denote the unit inventory holding and stockout penalty costs, respectively. Given the inventory belief \(\pi(i|I, m, n)\), we can define the expected inventory costs in a period as

\[
G(I, m, n) = \sum_{i=0}^{\tilde{i}} [hE(i - D)^+ + bE(D - i)^+] \cdot \pi(i|I, m, n).
\]

We further assume that there is a finite inventory storage capacity \(C\) at the store, so the logical inventory level cannot exceed the capacity \(C\). Let \(V(I, m, n)\) denote the cost-to-go function for the joint inspection and inventory replenishment problem. Because the inventory decision space is finite, the expected cost in each period is also finite. It follows that the optimal policy for this problem is stationary (see Bertsekas 2001). Moreover, the optimal policy can be obtained by solving the following dynamic program optimality equations: for \(0 \leq I \leq C\), \(m, n \geq 0\) and \(m + n \geq 1\),

\[
V(I, m, n) = \min \left\{ H^0(I, m, n), \min_{I < I' \leq C} \left\{ k_r + H(I', m + n, 0) \right\}, k + \sum_{i=0}^{I} V(i, 0, 0)\pi(i|I, m, n) \right\}, \tag{12}
\]

with \(V(i, 0, 0) = \min \left\{ H(i, 0, 0), \min_{i < i' \leq C} \left\{ k_r + H(i', 0, 0) \right\} \right\}\), where

\[
H^0(I, m, n) = G(I, m, n) + \delta P(Z(I, m, n) = 0)V(I, m, n + 1)
\]

\[
+ \delta \sum_{i=1}^{I} P(Z(I, m, n) = i)V(I - i, m + n + 1, 0),
\]

\[
H(I, m, 0) = G(I, m, 0) + \delta \sum_{i=0}^{I} P(Z(I, m, 0) = i)V(I - i, m + 1, 0),
\]

and \(Z(I, m, n) = \min\{I, D\}\) is the sales in a period, with \(I\) following the inventory belief \(\pi(\cdot|I, m, n)\).

In the above problem formulation, there are three decision options at the beginning of each period: 1) do nothing; 2) replenish inventory; and 3) inspect inventory. If one chooses to do nothing, the expected cost is given by \(H^0(I, m, n)\). If one chooses to replenish inventory, the logical inventory level is brought up to a new level \(I'\) and the inventory belief is reset to \(\pi(\cdot|I', m + n, 0)\). The resulting cost is given by \(k_r + H(I', m + n, 0)\). If one chooses to inspect inventory, the inventory record is reset to the actual inventory level, based on which one immediately determines whether to replenish inventory or not. As a result, the expected cost is given by \(k + \sum_{i=0}^{I} V(i, 0, 0)\pi(i|I, m, n)\).

We note that in the formulation above, when an action (either replenishment or inspection) is taken, the state variable \(n\) in the next period remains zero regardless of the sales observation; this

\(^1\)Another common replenishment policy used in practice is the \((r, nQ)\) policy (see Zipkin 2000). We note that our problem formulation under the fixed cost assumption can be easily extended to the \((r, nQ)\) policy case with some simple modifications. We omit the discussion of this alternative formulation for brevity.
is due to the fact that the consecutive zero-sales period is defined over periods when no action is taken (see the previous section).

The problem formulation (12) generalizes the classic inventory problem with fixed setup cost in two ways: 1) it incorporates the Bayesian inventory belief into the problem, and 2) it allows for an additional inspection decision. It is generally difficult to characterize the optimal policy structure for this problem. To see this, from the expression of $H(\tilde{I}, m, 0)$, the inventory decision can affect the underlying Bayesian inventory belief, which will, in turn, affect the probability of sales in a period. As a result, the well-known $k$-convex property no longer holds under the Bayesian inventory belief. The fixed inspection cost also introduces additional complexity to the problem. Nevertheless, we can still partially characterize the policy as follows:

**Proposition 3** Let $n^*(\tilde{I}, m)$ be the smallest $n$ such that it is optimal to either inspect or replenish at state $(\tilde{I}, m, n)$. Further suppose that the system starts right after an inspection. Then, the optimal policy is to do nothing if the number of consecutive zero-sales periods $n$ is less than $n^*(\tilde{I}, m)$, and to either inspect or replenish if $n$ reaches $n^*(\tilde{I}, m)$.

The above result guarantees that the optimal policy has a threshold-type structure as long as the system starts right after an inspection. Thus, this result can be viewed as a weak version of the (state-dependent) threshold policy for whether to take an action or not. To determine which action to take, we can apply the standard value iteration method to solve the problem (see Bertsekas 2001). In particular, we can leverage the above result to trim the state space required for policy evaluation.

From our numerical experiences, we observe that, once the threshold for taking an action is reached, the action choice is dependent on the logical inventory level—if the logical inventory level is low, the optimal action tends to be replenishment; on the other hand, if the logical inventory level is high, the optimal action tends to be inspection. We offer the following intuition. When the logical inventory level is low, the actual inventory level is also low and the magnitude of inventory error is limited. As a result, it is better off to ignore the potential inventory error and simply replenish inventory to avoid phantom stockouts. When the logical inventory level is high, the magnitude of inventory error could also be high. In this case, it is better off to inspect inventory to correct the potential system inventory error.

### 3.3 Decoupled Inspection and Replenishment

Implementing the above joint inspection and replenishment policy may require some substantial changes of the existing inventory system. The system needs to track the relevant sufficient statistics...
and update the Bayesian inventory belief in every period. More importantly, it destroys the original simple \((s, S)\) replenishment policy structure. Below we propose another solution approach that decouples the inspection and replenishment decisions and in the meantime retains the simple \((s, S)\) replenishment policy structure.

Specifically, we first ignore inventory shrinkage and solve the classic inventory replenishment problem with fixed cost to obtain an optimal \((s, S)\) policy. This policy can be easily computed; we omit the problem formulation here. We then assume the regular system replenishment is based on this \((s, S)\) policy, that is, whenever the logical inventory level \(\tilde{I}\) drops to or below \(s\), the system will replenish inventory and bring the logical inventory level to \(S\). Besides the regular replenishment, we allow for additional replenishment based on the Bayesian inventory belief when the logical inventory level \(\tilde{I}\) falls in the range of \(s + 1 \leq \tilde{I} \leq S - 1\). The additional replenishment is determined as follows: The logical inventory level will be brought up to \(S\) if \(k_r + G(S, m + n, 0) \leq G(\tilde{I}, m, n)\). Based on this condition, we can determine the threshold for replenishment \(n^*_r(\tilde{I}, m)\) as follows: For \(s + 1 \leq \tilde{I} \leq S - 1\),

\[
\min \left\{ n \geq 0 : G(\tilde{I}, m, n) - G(S, m + n, 0) \geq k_r \right\}.
\]

Because replenishment is not needed when \(\tilde{I} = S\), we set \(n^*_r(S, m) = \infty\) for notational convenience.

Next, we determine the inventory inspection decision by assuming a fixed \((S - 1, S)\) replenishment policy with \(k_r = 0\) in problem (12). In this case, the starting logical inventory level is \(S\) in all periods. Thus, we can suppress the logical inventory state in the dynamic program formulation (12). Let \(\alpha(m, n)\) denote the probability of zero sales in a period, which can be written as

\[
\alpha(m, n) = P(Z(S, m, n) = 0) = \varphi + (1 - \varphi) \cdot \pi(0|S, m, n). \tag{13}
\]

The dynamic program formulation (12) can be rewritten as follows: for \(m, n \geq 0\) and \(m + n \geq 1\),

\[
V(m, n) = \min \left\{ G(S, m, n) + \delta V(m, n+1) \cdot \alpha(m, n) + \delta V'(m+n+1, 0) \cdot (1 - \alpha(m, n)) , \quad k + V(0, 0) \right\},
\]

with \(V(0, 0) = G(S, 0, 0) + \delta V(1, 0)\), where

\[
V'(m, 0) = \min \left\{ G(S, m, 0) + \delta V(m + 1, 0), \quad k + V(0, 0) \right\}.
\]

In the above formulation, \(V'(m, 0)\) is the value function after a replenishment event under the \((S - 1, S)\) policy. In this case, the state variable \(n\) in the next period remains zero regardless of the sales observation because the consecutive zero-sales period is defined over periods when no action is taken (see the previous section).
We further simplify the above formulation by approximating \( V^r(m, 0) \) by \( V(m, 0) \). The resulting dynamic program formulation can be written as follows: for \( m, n \geq 0 \) and \( m + n \geq 1 \),

\[
\tilde{V}(m, n) = \min \left\{ G(S, m, n) + \delta \tilde{V}(m, n+1) \cdot \alpha(m, n) + \delta \tilde{V}(m+n+1, 0) \cdot (1 - \alpha(m, n)), \quad k + \tilde{V}(0, 0) \right\},
\]

with \( \tilde{V}(0, 0) = G(S, 0, 0) + \delta \tilde{V}(1, 0) \).

We establish the following result for the above dynamic program problem:

**Proposition 4** Suppose that \( G(S, m, n) \) is increasing in \( m \) and \( n \), and \( G(S, m, n+1) \geq G(S, m+1, n) \) for any \( m \geq 0 \) and \( n \geq 0 \). Then the cost-to-go function \( \tilde{V}(m, n) \) is increasing in \( m \) and \( n \), and \( \tilde{V}(m, n+1) \geq \tilde{V}(m+1, n) \). There exists a set of state-dependent thresholds \( n^*_i(m) \) for \( m \geq 0 \), such that the optimal policy is to perform inspection whenever the number of consecutive zero-sales periods reaches the threshold \( n^*_i(m) \). Moreover, the optimal threshold \( n^*_i(m) \) is decreasing in \( m \).

The above threshold policy is intuitively appealing. If \( m \) is large enough, the optimal threshold \( n^*_i(m) \) could decrease to zero. In this case, a shelf inspection is triggered regardless whether the sales are zero or not, which can be viewed as a form of cycle count policy (see Kok and Shang 2007). Thus, our model framework unifies the shelf inspection policy for the phantom stockout problem with the traditional inventory cycle count policy. By applying the stochastic dominance results obtained in Proposition 2 and the property of first order stochastic dominance (Muller and Stoyan 2002), we can show that the condition for the above proposition is met if \( S \leq \arg \min_{\tilde{I} \geq 0} G(\tilde{I}, 0, 0) \).

Based on our numerical experiences, the condition is usually met for larger \( S \) values in our problem.

Finally, combining the above inspection and replenishment thresholds together, we have the following inspection and replenishment policy:

\[
\begin{cases}
\text{Replenish inventory to } S & \text{if } 0 \leq \tilde{I} \leq s, \\
\text{Do nothing} & \text{if } s < \tilde{I} \leq S \text{ and } n < \min\{n^*_r(\tilde{I}, m), n^*_i(m)\}, \\
\text{Replenish inventory to } S & \text{if } s < \tilde{I} \leq S \text{ and } n = n^*_r(\tilde{I}, m) < n^*_i(m), \\
\text{Inspect inventory} & \text{if } s < \tilde{I} \leq S \text{ and } n = n^*_i(m) \leq n^*_r(\tilde{I}, m).
\end{cases}
\]

(15)

The above policy is easy to implement as it builds on the commonly used \((s, S)\) policy structure. All the policy parameters can be predetermined easily. However, we should note that this simplicity comes with some performance losses. First, the decoupled policy only replenish inventory up to a fixed level \( S \), while the policy determined by (12) allows for inventory optimization up to capacity \( C \). Second, the decoupled inspection policy is independent of the logical inventory level, while the inspection policy determined by (12) is dependent on the logical inventory level. Therefore, we are essentially trading off some cost performance for simplicity in implementation here.
4. Numerical Results

In this section, we present numerical results to compare the two policies proposed in the previous section and three other inspection policies known in the literature. The first policy, termed “JIR,” is the joint inspection and inventory replenishment policy based on the dynamic program (12). The second policy, termed “DIR,” is the decoupled inspection and replenishment policy given in (15). The third policy is the EVPI policy proposed by DeHoratius et al. (2008). We adapt their policy to our problem setting as follows. The EVPI under an \((s,S)\) replenishment policy is determined by

\[
\Delta(\tilde{I}', m, n) = G(\tilde{I}', m, n) - G(S, 0, 0) \cdot \sum_{i=0}^{s} \pi(i|\tilde{I}', m, n) - \sum_{i=s+1}^{\tilde{I}'} G(i, 0, 0) \cdot \pi(i|\tilde{I}', m, n).
\]

Thus, the EVPI inspection threshold for our problem can be determined by

\[
\min \left\{ n \geq 0 : \gamma + 1 \sum_{j=0}^{m + n - 1} \delta^j [G(S, m + n, 0) - G(S, j, 0)] > k \right\},
\]

where \(0 < \gamma \leq 1\) is a control parameter for the decision rule. We set \(\gamma = 0.5\) based on Figure 1 in DeHoratius et al. (2008, p. 265). We have tried other values for \(\gamma\), the results are all similar.

The fourth policy, termed “CC,” is based on a further simplified inventory belief that treats the case of zero sales the same as the case of positive sales. In other words, the Bayesian inventory belief \(\pi(\cdot|\tilde{I}', m, n)\) derived in (11) is further approximated by \(\pi(\cdot|\tilde{I}', m, n) \approx \pi(\cdot|\tilde{I}', m + n, 0)\). Substituting this simplified inventory belief into problem (14), we obtain the following one-dimensional dynamic program:

\[
\hat{V}(m + n) = \min \left\{ G(S, m + n, 0) + \delta \hat{V}(m + n + 1), \quad k + G(S, 0, 0) + \delta \hat{V}(1) \right\}.
\]

It is easy to verify that the optimal inspection threshold for the above problem is given by

\[
\min \left\{ n \geq 0 : \sum_{j=0}^{m+n-1} \delta^j [G(S, m + n, 0) - G(S, j, 0)] > k \right\}.
\]

We note that this policy can be viewed as an adaptation of the cycle count policy proposed by Kok and Shang (2007, p. 197). In their model, unmet demand is fully backlogged, so their inventory belief only depends on the total number of periods since the last inspection (the same as the approximated inventory belief above).

The final policy is based on an adaptation of the SPC policy of Fisher and Raman (2010). Under the SPC policy, an inspection is triggered when the probability of consecutive zero-demand
periods $\varphi^n$ drops to a certain threshold. Specifically, we set the threshold to be 0.1% as suggested in Fisher and Raman (2010, p. 178).

In the numerical study, we assume the daily demand follows a Poisson distribution with mean $E[D]$. We have run multiple experimental tests with different demand mean values. The numerical insights are all similar. For brevity, below we report the case of $E[D] = 2$ (the zero-demand probability is $\varphi = 0.1353$ in this case). We fix the discount factor at $\delta = 0.96$ and the maximum store capacity at $C = 10$. We set the inventory holding cost $h = 1$ and stockout penalty cost $b = 20$, representing an implied service level of 95%. We vary the fixed cost for replenishment cost $k_r$ among values of 2, 4, and 8. Based on these parameter, the optimal replenishment policy without shrinkage is given by $(s, S) = (3, 5), (2, 6), \text{ and } (2, 8)$, respectively.\(^2\) We further assume the daily shrinkage follows a Poisson distribution with mean $\rho E[D]$. We vary $\rho$ among values of 1%, 3%, and 5%, representing low, medium, and high shrinkage rate, respectively.\(^3\) We also vary the inspection cost $k$ among values of 10, 50, and 100.

We create a computer program that simulates the above inventory system with the specified demand and shrinkage processes. For each policy scenarios, we simulate the system for 1,000 periods in each repetition, with 3,000 repetitions (under the 0.96 discount factor, the discounted cost after 1,000 periods becomes negligible). To reduce simulation variation, we start the system with the logical inventory level equal to the actual inventory level at $S$ across all policy scenarios and all repetitions. Moreover, we use the same 3,000 demand and shrinkage random sample paths across all policy scenarios. After simulation, we compute the discounted cost for each repetition and then average the discounted cost across the 3,000 repetitions for each policy scenarios.

We first simulate the system under the $(s, S)$ policy with full information, i.e., the actual inventory level (which is subject to the shrinkage process) is fully observable. This case is equivalent to setting the inspection cost $k = 0$ in our problem. We compute the system cost under this full information case and denote it as $V^{FI}$. We also compute the system cost when no inspection is performed. This case serves as the worst case scenario in which the system may freeze and stop replenishing after a certain number of periods. We denote the system cost in this worst case as $V^{NI}$. The difference $V^{NI} - V^{FI}$ measures the value of full information and we can use it as the

\(^2\)We note that these $(s, S)$ policies are also optimal if we inflate the demand mean $E[D] = 2$ by 1% or 3% to account for the potential shrinkage. When we inflate the demand mean by 5%, the optimal policy for the case $k_r = 4$ becomes $(3, 6)$, but the optimal policies for the other two cases remain the same.

\(^3\)According to the 2012 National Retail Security Survey (www.lpportal.com/academic-viewpoint), the average shrinkage across the retail industry is about 1.47% of the annual sales in 2012. Because we focus on the individual items that are prone to shrinkage, the conditional shrinkage rate could be higher than the unconditional aggregate shrinkage rate. Therefore, we believe it is safe to assume that 1%, 3%, and 5% represent low, medium, and high shrinkage rate, respectively.
performance benchmark (this metric is included in the column of “Value of FI” in Table 1). We then apply each of the five policies and compute the resulting system cost \( V^P \). We measure the performance of each policy by comparing \( V^{NI} - V^P \) against the value of full information benchmark \( V^{NI} - V^{FI} \) based on the following formula:

\[
\text{Relative % of value captured} = \frac{V^{NI} - V^P}{V^{NI} - V^{FI}} \times 100%.
\]

We put this relative percentage metric under each policy in Table 1 below.\(^4\)

Table 1: Comparison of cost performance under various policies.

<table>
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<th>(s, S)</th>
<th>( \rho )</th>
<th>( k )</th>
<th>Value of FI</th>
<th>JIR</th>
<th>DIR</th>
<th>EVPI</th>
<th>CC</th>
<th>SPC</th>
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From Table 1, we observe that the JIR policy consistently outperforms all other policies. The DIR policy has a fairly stable performance across all scenarios. Although not as good as the JIR

\(^4\)We have run multiple experimental tests with different shrinkage rates, such as \( \rho = 10\% \) and 20\%. The numerical insights are all similar to what we observe in Table 1; we omit them for brevity.
policy, the DIR policy performs better than the EVPI, CC, and SPC policies in most cases. When inspection cost is relatively high and the gap of $S - s$ is relatively large, the JIR and DIR policies perform significantly better than the EVPI, CC, and SPC policies. This is because our policies allow for additional replenishment based on the Bayesian inventory belief, which can help substitute costly inspection to some extent. When the gap of $S - s$ becomes smaller, the performance of the DIR policy becomes close to that of the EVPI and CC policies. In this case, the magnitude of the additional replenishment under the DIR becomes smaller (because it only replenishes up to $S$ when the logical inventory level falls between $s + 1$ and $S - 1$). Thus, the advantage of the DIR policy over the EVPI and CC policies diminishes. Finally, when the shrinkage rate is low, the value of full information is small. In this case, the simple SPC policy may just be sufficient. On the other hand, when the shrinkage rate is high, the value of full information increases significantly. In this case, more advanced inspection and replenishment policies (such as JIR and DIR) should be used to achieve better performance.

To gain additional insight into the behavior of each policy, we compute the inspection hit rates of each policies and present the results in Table 2. We define the inspection hit rate as follows. Under each policy, we count the number of instances in which an inspection is triggered and the logical inventory record is found to be inaccurate ("a hit"), and then divide this number by the total number of inspections triggered to obtain the inspection hit rate. For brevity, in Table 2, we only present the inspection hit rate results for the case of $(s, S) = (2, 6)$ (the hit rate results for other cases are all similar to this case). We also put the average total number of inspection triggered for each policy during 1,000 periods in the parenthesis next to the hit rate.

Table 2: Comparison of inspection hit rate under various policies.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$k$</th>
<th>JIR</th>
<th>DIR</th>
<th>EVPI</th>
<th>CC</th>
<th>SPC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>10</td>
<td>65.6% (17)</td>
<td>70.0% (15)</td>
<td>26.5% (57)</td>
<td>69.8% (14)</td>
<td>98.4% (5)</td>
</tr>
<tr>
<td>50</td>
<td></td>
<td>97.8% (5)</td>
<td>84.4% (10)</td>
<td>43.8% (31)</td>
<td>83.7% (9)</td>
<td>98.4% (5)</td>
</tr>
<tr>
<td>100</td>
<td></td>
<td>99.4% (5)</td>
<td>90.6% (8)</td>
<td>53.0% (23)</td>
<td>90.0% (7)</td>
<td>98.4% (5)</td>
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<tr>
<td>3%</td>
<td>10</td>
<td>79.6% (36)</td>
<td>71.1% (45)</td>
<td>43.7% (92)</td>
<td>73.7% (39)</td>
<td>99.5% (13)</td>
</tr>
<tr>
<td>50</td>
<td></td>
<td>98.5% (14)</td>
<td>85.4% (29)</td>
<td>64.0% (52)</td>
<td>86.9% (26)</td>
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<tr>
<td>100</td>
<td></td>
<td>99.5% (11)</td>
<td>91.1% (23)</td>
<td>73.9% (40)</td>
<td>91.9% (21)</td>
<td>99.5% (13)</td>
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<tr>
<td>5%</td>
<td>10</td>
<td>87.8% (45)</td>
<td>70.6% (76)</td>
<td>53.0% (116)</td>
<td>78.0% (58)</td>
<td>99.8% (21)</td>
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<tr>
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<td></td>
<td>98.9% (21)</td>
<td>86.2% (47)</td>
<td>73.7% (67)</td>
<td>89.1% (39)</td>
<td>99.8% (21)</td>
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<tr>
<td>100</td>
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<td>99.6% (17)</td>
<td>91.3% (38)</td>
<td>82.5% (52)</td>
<td>93.7% (32)</td>
<td>99.8% (21)</td>
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</table>

From Table 2, we observe that the SPC policy achieves the highest hit rate (about the same as
the 99.9% target). However, the SPC policy also triggers considerably less number of inspections than other policies, especially when the inspection cost is low. This is due to the fact that the SPC policy by design is agnostic to the underlying inspection cost. In general, as inspection cost increases, hit rate increases and less number of inspections is triggered across all other policies. The JIR policy has better hit rates than the EVPI and CC policies when the inspection cost is high. The DIR policy has comparable hit rate with the CC policy, but is much better than the EVPI policy. These hit rate results are consistent with the cost performance results shown in Table 1.

Figure 2: Illustration of inspection and replenishment thresholds for various policies.

To further illustrate and compare the structure of different policies, we plot the decision threshold for each policies for the case of \((s, S) = (2, 6), \rho = 3\% \text{ and } k = 10\) in Figure 2. Specifically, in Figure 2(a), we compare the JIR policy with the EVPI, CC, and SPC policies when the logical inventory level \(\tilde{I} = 3\); in Figure 2(b), we compare the DIR policy with the EVPI, CC, and SPC policies when the logical inventory level \(\tilde{I} = 3\). In Figures 2 (c) and (d), we change the logical inventory level to \(\tilde{I} = 6\). From the figure, we observe that both the CC and SPC policies are
independent of the logical inventory level and differ significantly from the JIR and DIR policies. The EVPI policy has a similar structure to the JIR and DIR policies, but it only triggers inspection. When the logical inventory level is low (such as $\bar{I} = 3$ in this case), both the JIR and DIR policies trigger replenishment instead of inspection in most cases, differentiating them from the EVPI policy. When the logical inventory level is high (such as $\bar{I} = 6$ in this case), the JIR policy triggers replenishment if $m$ is small, while the DIR policy triggers only inspection. This explains why the JIR policy performs better than the DIR policy—the former relies more on replenishment to substitute costly inspection.

Table 3: Comparison of cost performance under misspecified inspection cost.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Misspec. %</th>
<th>Value of FI</th>
<th>JIR</th>
<th>DIR</th>
<th>EVPI</th>
<th>CC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>-40%</td>
<td>30.2</td>
<td>79.0%</td>
<td>64.4%</td>
<td>-7.1%</td>
<td>15.6%</td>
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<td>-20%</td>
<td>30.2</td>
<td>79.2%</td>
<td>64.8%</td>
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<td>12.9%</td>
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<td>20%</td>
<td>30.2</td>
<td>79.3%</td>
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In practice, it may be hard to estimate the inspection cost $k$ precisely. To address this issue, we conduct additional robustness tests for each policy except for the SPC policy by assuming that the inspection cost $k$ may be misspecified (this is a nonissue for the SPC policy because it does not depend on the cost $k$). Specifically, we consider the case of $(s, S) = (2, 6)$ with the true inspection cost $k = 50$. We compute the policies using misspecified inspection cost $k' = 30, 40, 60, 70$, representing $\pm 20\%$ and $\pm 40\%$ specification errors, respectively. We then apply the resulting policies to the simulated system and evaluate the cost performance based on the true cost $k$. The results are presented in Table 3. From the table, we observe that the JIR and DIR policies are robust to the specification errors of the inspection cost $k$. On the other hand, the EVPI and CC polices are more sensitive to the specification errors. Interestingly, the performance of the EVPI policy improves under an overestimated inspection cost, while the performance of the CC policy improves under an underestimated inspection cost. This observation suggests that the EVPI policy
might be too aggressive in triggering inspections and that the CC policy might be too conservative, which is consistent with what we observe in Table 2 and in Figure 2.

5. The Shelf Execution Failure Problem

In this section we propose an alternative model to address the shelf execution failure problem. Shelf execution failures occur when there is inventory in the backroom, but store associates fail to move it to the shelf. As a result, the product is unavailable on the shelf and no sales are recorded; the shelf remains empty unless and until someone intervenes. To address this problem, we assume that inventory shrinkage is negligible and that the store backroom has ample inventory. The shelf execution process can be in one of two states: the normal state or the failure state. In reality, execution failure may be self-recovered. For example, store associates may forget to bring inventory from the backroom to the shelf for a few periods, but may later recover from the failure to restock the shelf. In what follows, we shall, however, consider the extreme case in which once execution enters the failure state, it stays in that state unless an inspection is performed. In a sense, we model the shelf execution process as if it were a vending machine: when the machine is broken, the sales become zero and human intervention is required to fix the problem.

In practice, retail store inventory records are only kept at the store level, not at the shelf level. As a result, the system inventory records do not reveal how much inventory is available on the shelf. Therefore, the system inventory records do not provide useful information about the shelf execution state. One can only rely on the sales data to infer the shelf execution state. Specifically, if zero sales are observed in a period, it can be due to one of the following two reasons: 1) there is an execution failure, or 2) the execution process is in the normal state but there is zero demand in the period. On the other hand, if positive sales are observed in a period, we assume that the execution process was in the normal state during that period.

Let $\pi(n)$ denote the probability of execution failure after observing $n$ consecutive zero-sales periods. When $n = 0$, we assume that the store manager forms an initial prior about the probability of execution failure as $\pi(0) = q$, with $0 \leq q < 1$. Suppose that the store manager observes one period of zero sales. Then the posterior probability of execution failure, according to Bayes’ rule, is given by

$$
\pi'(1) = \frac{q}{q + (1 - q)\varphi}.
$$

We further assume that within a zero-sales cycle, the execution process deteriorates from the normal
state to the failure state at the end of a period according to the following transition matrix:

\[
\begin{pmatrix}
1 - p & p \\
0 & 1
\end{pmatrix}
\]

Here we assume the deterioration probability \( p \) is known; we will discuss the estimation method of this parameter in §5.1. We note that similar system deterioration models have also been used in machine maintenance and quality control problems (see Eckles 1968 and Ross 1971).

According to the deterioration transition matrix, the posterior probability of \( \pi'(1) \) can be updated to

\[
\pi(1) = \pi'(1) + (1 - \pi'(1))p = \frac{q + (1 - q)p}{q + (1 - q)\varphi}.
\]

In general, we have the following result:

**Lemma 2** Under the Bayesian deterioration model, given \( n \) consecutive zero-sales periods since the execution process was last observed in the normal state, the posterior probability of the execution process being in the failure state is given by

\[
\pi(n) = \frac{q + (p - q)\varphi - (1 - q)p\varphi[(1 - p)\varphi]^n}{q + (p - q)\varphi + (1 - q)(1 - \varphi)[(1 - p)\varphi]^n}.
\]

Furthermore, \( \pi(n) \) is increasing in \( n \) and \( \lim_{n \to \infty} \pi(n) = 1 \).

Now let \( \alpha(n) \) denote the probability of zero sales in a period following \( n \) consecutive zero-sales periods, which can be written as \( \alpha(n) = \pi(n) + (1 - \pi(n))\varphi \). Let \( b' \) denote the penalty cost when the execution process is in the failure state. Let \( V(n) \) denote the cost-to-go function, we can formulate the shelf inspection problem as the following dynamic program: for \( n \geq 0 \),

\[
V(n) = \min \left\{ \pi(n)b' + \delta \alpha(n)V(n + 1) + \delta(1 - \alpha(n))V(0), \ k + \delta V(0) \right\}.
\]

By solving the above dynamic program, we can obtain the following result:

**Proposition 5** Under the Bayesian deterioration model, the optimal inspection threshold \( n^*(\varphi, p, q) \) is determined by

\[
\min \left\{ n \geq 0 : \frac{q + (p - q)\varphi - (1 - q)p\varphi[(1 - p)\varphi]^n}{q + (p - q)\varphi + (1 - q)(1 - \varphi)[(1 - p)\varphi]^n} \delta \Delta(n) > \frac{k}{b'} \right\},
\]

where \( \Delta(n) = \sum_{j=0}^{n-1}[(1 - p)\varphi]^j - [(1 - p)\varphi]^n \sum_{j=0}^{n-1} \delta^j \). Moreover, the following results hold:

(a) \( n^*(\varphi, p, q) \) is monotonically increasing in \( \varphi \) if and only if \( q = p \). If \( q < p \), there exists a finite \( n_1 > 0 \), such that \( n^*(\varphi, p, q) \) is increasing in \( \varphi \) for \( n^*(\varphi, p, q) \geq n_1 \). However, if \( q > p \), there exists a finite \( n_2 > 0 \), such that \( n^*(\varphi, p, q) \) is decreasing in \( \varphi \) for \( n^*(\varphi, p, q) \geq n_2 \).
(b) $n^*(\varphi, p, q)$ is not monotone in $q$ and $p$. However, there exists a finite $n_3 > 0$, such that $n^*(\varphi, p, q)$ is increasing in $q$ for $n^*(\varphi, p, q) \geq n_3$. Similarly, there exists a finite $n_4 > 0$, such that $n^*(\varphi, p, q)$ is increasing in $p$ for $n^*(\varphi, p, q) \geq n_4$.

When $q = p$, the above result confirms the intuition that as the zero-demand probability increases, the optimal inspection threshold increases and one would inspect less frequently. When $q < p$, this intuition continues to hold if $n^*(\varphi, p, q)$ becomes sufficiently large. However, when $q > p$, this monotonicity result reverses—one would inspect more frequently as the zero-demand probability increases. In other words, if the store manager strongly believes the system will deteriorate into the failure state right after inspection (i.e., $q > p$), then she would increase the inspection frequency as it becomes more difficult for her to infer the execution failure probability from the sales data (due to the increase of $\varphi$). Moreover, when the optimal inspection threshold is high, a further increase in $p$ or $q$ will lead to a higher inspection threshold and one would inspect less frequently. A high inspection threshold usually implies a high inspection-to-penalty cost ratio. In such cases, if the system deteriorates to the failure state very soon, it may be economic to reduce the inspection effort.

We have assumed that the zero-demand probability $\varphi$ and the deterioration probability $p$ are stationary. In practice, these parameters may be time-varying. For example, there may be seasonality or day-of-week effects in the demand pattern. As a result, the zero-demand probability could be lower during the peak demand period than during the regular demand period. Similarly, the deterioration probability could also be different at different times in a selling season. For example, at the beginning of a selling season, store inventory level may be high and the store may be more prone to execution errors (see Ton and Raman 2010). We can generalize the above shelf execution failure model to account for this kind of nonstationarity. Specifically, under a finite-horizon setting, we can show that the optimal inspection policy is to perform inspection when the number of consecutive zero-sales periods reaches a state-dependent threshold (a formal proof of this result is available from the author).

5.1 Estimation of Deterioration Probability

To apply the formula given in Proposition 5, we still need to estimate the parameters $p$ and $q$. The initial prior probability $q$ can either be elicited from the store manager as a subjective belief or jointly estimated with $p$ from the historical data.

Suppose that we have the following historical data: $(n_1, X_1), ..., (n_J, X_J)$, where $n_j$ is the number of consecutive zero-sales periods, and $X_j \in [0, 1]$ is the observed system state ("0" denotes
the normal state and “1” denotes the failure state). From Lemma 2, we have

\[
\Pr(X_j = 0) = \frac{1-q}{q + (p-q)\varphi + (1-q)(1-\varphi)(1-p)\varphi^{n_j}},
\]

\[
\Pr(X_j = 1) = \frac{q + (p-q)\varphi - (1-q)p\varphi((1-p)\varphi^{n_j})}{q + (p-q)\varphi + (1-q)(1-\varphi)(1-p)\varphi^{n_j}}.
\]

Rearrange the data such that \(X_j = 0\) for \(1 \leq j \leq J'\) and \(X_j = 1\) for \(J' + 1 \leq j \leq J\). We can write the joint likelihood function \(L(X_1, ..., X_J)\) as

\[
\prod_{j=1}^{J'} \frac{(1-q)[(1-(1-p)\varphi)(1-p)\varphi]^{n_j}}{q + (p-q)\varphi + (1-q)(1-\varphi)(1-p)\varphi^{n_j}} \cdot \prod_{j=J'+1}^{J} \frac{q + (p-q)\varphi - (1-q)p\varphi((1-p)\varphi^{n_j})}{q + (p-q)\varphi + (1-q)(1-\varphi)(1-p)\varphi^{n_j}}.
\]

Thus, if \(q\) is pre-specified, then \(p\) can be estimated as \(\hat{p} = \arg \max_{0 \leq p < 1} \log L(X_1, ..., X_J|q)\). Alternatively, \(p\) and \(q\) can be jointly estimated as \((\hat{p}, \hat{q}) = \arg \max_{0 \leq p, q < 1} \log L(X_1, ..., X_J)\).

6. Concluding Remarks

In this paper, we have developed two models for the phantom stockout problem: one for the shrinkage problem and the other for the shelf execution failure problem. We formulate the shrinkage problem as a joint inspection and inventory replenishment problem, and partially characterize the optimal policy. We further simplify the problem by decoupling the inspection and replenishment decisions. Our simulation study reveals that the joint and decoupled policies outperform existing policies in the literature in most cases. Particularly, when the shrinkage rate is high, more advanced inspection and replenishment policies (such as the joint and decoupled policies) should be used to avoid phantom stockouts. Our simulation results also suggest that allowing for additional replenishment not only saves on inspection cost, but also makes the system performance less sensitive to misspecified inspection cost parameters. For the shelf execution failure problem, we model the shelf execution process as a deteriorating process. We show that there exists a closed-form condition to determine the optimal inspection threshold based on the number of consecutive zero-sales periods.

A couple of extensions of the current study merit further investigation. We have assumed that the inventory replenishment leadtime is zero in this paper. One could further extend the model by considering a positive leadtime. With a positive leadtime, the replenishment problem becomes more complicated. Even without the shrinkage process, the lost sales problem with a positive leadtime is a known hard problem in the literature. Therefore, we would need to develop new approximation techniques to solve the problem. Another extension is to formulate a multiple-product problem. The labor cost of inspecting additional products on a specific inspection route is likely to be marginal. In this case, one can schedule the store associate to inspect multiple
products on a route. To further improve efficiency, one can also incorporate the store plan-o-gram
information to optimally sequence the products to be inspected along an inspection route. This can
be used together with the zero balance walk practice to maximize efficiency (Fisher et al. 2000).

Retail execution, described as the “missing link” in retail operations (Raman et al. 2001b), has
been relatively overlooked in the operations management literature. A failure in retail execution
is like a failure at the finish line—all the hard work invested by everyone along the supply chain,
from manufacturers to retailers to operations analysts, becomes wasted. Thus, we hope our work
can stimulate further research aimed at tackling this important problem.

Appendix

Proof (Proposition 1) Let us first consider a simple case in which there is no replenishment
from period one to \( t \). Because \( Z_t > 0 \), it implies \( I_t \geq Z_t > 0 \). Because there is no replenishment
since period one, the actual inventory must decrease from period one to period \( t \). Thus, we have
\( I_t \geq I_t > 0 \) for all \( 1 \leq \tau \leq t \). From equation (1), it follows that \( I_{\tau+1} = I_{\tau} - Z_{\tau} - U_{\tau} > 0 \) for all \( 1 \leq \tau \leq t - 1 \). Substituting \( I_{\tau} \) repeatedly based on the relationship, we have
\( I_t = I_1 - \sum_{\tau=1}^{t-1} Z_{\tau} - U_{1}^{t-1} \).

Recall that the logical inventory level is given by \( \tilde{I}_t = \tilde{I}_1 - \sum_{\tau=1}^{t-1} Z_{\tau} = I_1 - \sum_{\tau=1}^{t-1} Z_{\tau} \). Thus,
we obtain \( I_t = \tilde{I}_t - U_1^{t-1} \). Substituting this equation into (1), we arrive at the following simple
relationship: \( I_{t+1} = (\tilde{I}_{t+1} - U_1^{t})^+ \).

Now let us consider the general case in which there are replenishment activities taking place
from period one to period \( t \). If \( t_r = t \), the equation (3) holds from definitions (1) and (2). Let
us consider the case of \( t_r < t \). In this case, there is no replenishment occurring from period
\( t_r + 1 \) to period \( t \). Because \( Z_t > 0 \), with the same argument used above, it must be true that
\( I_t = I_r - \sum_{\tau=t_r}^{t-1} Z_{\tau} - U_{t_r}^{t-1} \). Also note that the logical inventory level is given by \( \tilde{I}_t = \tilde{I}_{t_r} - \sum_{\tau=t_r}^{t-1} Z_{\tau} \).
Thus, we obtain \( I_t = \tilde{I}_t - \tilde{I}_{t_r} + I_{t_r} - U_{t_r}^{t-1} \). From equation (1), we have

\[
I_{t+1} = (I_t - Z_t - U_t)^+ = (\tilde{I}_t - \tilde{I}_{t_r} + I_{t_r} - U_{t_r}^{t-1} - Z_t - U_t)^+ = (\tilde{I}_{t+1} - \tilde{I}_{t_r} + I_{t_r} - U_{t_r}^{t})^+,
\]

where we use the fact that \( \tilde{I}_{t+1} = \tilde{I}_t - Z_t \). Now from period \( t_{j-1} \) to \( t_j \) for \( 1 < j \leq r \), by our replenishment assumption that \( Z_{t_{j-1}} > 0 \), and with the same logic,
\( I_{t_j} = (\tilde{I}_{t_j} - \tilde{I}_{t_{j-1}} + I_{t_{j-1}} - U_{t_{j-1}}^{t_{j-1}})^+ \).

Finally, from period one to \( t_1 \), as shown above, we have \( I_{t_1} = (\tilde{I}_{t_1} - U_1^{t_1})^+ \) (because \( I_1 = \tilde{I}_1 \)).

Proof (Lemma 1) Suppose that \( t_r = t \). Because \( \tilde{I}_{t+1} < \tilde{I}_{t_r} = \tilde{I}_t \), from (2), we have
\( \tilde{I}_t - \tilde{I}_{t+1} = Z_t - R_t > 0 \). But recall that \( Z_t = \min\{I_t + R_t, D_t\} \leq I_t + R_t \). It follows that \( I_t = I_{t_r} > 0 \).
Now suppose that \( t_r < t \). Because \( Z_t > 0 \), it must be true that \( I_t > 0 \). From Proposition 1, we have \( I_t = \tilde{I}_t - \tilde{I}_{t_r} + I_{t_r} - U_{t_r}^{-1} \). Suppose that \( I_{t_r} = 0 \). Then we have

\[
I_t - Z_t = \tilde{I}_t - \tilde{I}_{t_r} - U_{t_r}^{-1} - Z_t = \tilde{I}_{t+1} - \tilde{I}_{t_r} - U_{t_r}^{-1} < 0,
\]

where the inequality follows from the assumption that \( \tilde{I}_{t+1} < \tilde{I}_{t_r} \). But based on the definition, we have \( Z_t = \min\{I_t, D_t\} \leq I_t \). We arrive at a contradiction. Therefore, it must be true that \( I_{t_r} > 0 \).

By applying a similar argument to a simple induction, we can show that, for any \( 1 \leq j < j' < r \), if \( \tilde{I}_{t,j'} < \tilde{I}_{t,j} \), then \( I_{t,j} > 0 \). \( \square \)

**Proof (Corollary 1)** If \( \tau_1 = 1 \), we have \( I_{\tau_1} = \tilde{I}_{\tau_1} = (\tilde{I}_{\tau_1})^+ \). The result holds trivially. Now let us assume \( \tau_1 > 1 \). Let \( t_j \) be a replenishment period between period one and \( \tau_1 \). Based on the definition of \( \tau_1 \), we have \( \tilde{I}_{\tau_1} < \tilde{I}_{t_j} \). Thus, by Lemma 1, we have \( I_{t_j} > 0 \). From Proposition 1, it follows that \( I_{t_j} = \tilde{I}_{t_j} - \tilde{I}_{t_j-1} + I_{t_j-1} - U_{t_j}^{-1} \). Since this applies to all \( t_j \) between period one and \( \tau_1 \), substituting this result into the expression of \( I_{\tau_1} \) given in Proposition 1, we obtain \( I_{\tau_1} = (\tilde{I}_{\tau_1} - U_{t_1}^{-1})^+ \). Repeating the same logic for the replenishment periods between period \( \tau_j \) and \( \tau_j' \) (note that \( \tau_j = t_r \) by definition), we obtain \( I_{\tau_j} = (\tilde{I}_{\tau_j} - \tilde{I}_{\tau_j-1} + I_{\tau_j-1} - U_{\tau_j}^{-1})^+ \). Thus, the result follows. \( \square \)

**Proof (Proposition 2)** The proof below is based on the definition of \( \pi(i|\tilde{I}, m, n) \) given in (11).

It is easy to verify that, for \( 0 \leq j \leq \tilde{I} \),

\[
\sum_{i=0}^{j} \pi(i|\tilde{I}, m, n + 1) = \frac{\pi(0|\tilde{I}, m, n) + \varphi \sum_{i=1}^{j} \pi(i|\tilde{I}, m, n) + \varphi \sum_{i=j+1}^{\tilde{I}} \pi(i|\tilde{I}, m, n) P(U_t \geq i-j)}{\varphi + (1-\varphi)\pi(0|\tilde{I}, m, n)}
\]

\[
\geq \frac{\pi(0|\tilde{I}, m, n) + \varphi \sum_{i=1}^{j} \pi(i|\tilde{I}, m, n)}{\varphi + (1-\varphi)\pi(0|\tilde{I}, m, n)} \geq \sum_{i=0}^{j} \pi(i|\tilde{I}, m, n),
\]

where the last inequality follows from some algebra and the fact that \( \sum_{i=0}^{j} \pi(i|\tilde{I}, m, n) \leq 1 \). Thus, by the definition of first order stochastic dominance, we have \( \pi(\cdot|\tilde{I}, m, n) \geq_{st} \pi(\cdot|\tilde{I}, m, n+1) \). Next we prove \( \pi(\cdot|\tilde{I}, m, n) \geq_{st} \pi(\cdot|\tilde{I}, m+1, n) \) by induction. By the definition of \( \pi(\cdot|\tilde{I}, m, 0) \), it is also straightforward to verify that \( \pi(\cdot|\tilde{I}, m, 0) \geq_{st} \pi(\cdot|\tilde{I}, m+1, 0) \) because \( U_t \) is an i.i.d. random variable.

Now assume that \( \pi(\cdot|\tilde{I}, m, n) \geq_{st} \pi(\cdot|\tilde{I}, m+1, n) \). By definition we have, for \( 0 \leq j \leq \tilde{I} \),

\[
\sum_{i=0}^{j} \pi(i|\tilde{I}, m+1, n+1) = \frac{\pi(0|\tilde{I}, m+1, n) + \varphi \sum_{i=1}^{j} \pi(i|\tilde{I}, m+1, n) + \varphi \sum_{i=j+1}^{\tilde{I}} \pi(i|\tilde{I}, m+1, n) P(U_t \geq i-j)}{\varphi + (1-\varphi)\pi(0|\tilde{I}, m+1, n)}
\]

\[
= \frac{(1-\varphi)\pi(0|\tilde{I}, m+1, n) + \varphi X}{\varphi + (1-\varphi)\pi(0|\tilde{I}, m+1, n)} \geq \frac{(1-\varphi)\pi(0|\tilde{I}, m, n) + \varphi X}{\varphi + (1-\varphi)\pi(0|\tilde{I}, m, n)},
\]

29
where \( X = \sum_{l=j}^{l-1} \sum_{i=0}^{l} \pi(i|\bar{I}, m + 1, n) P(U_l = l - j) + \sum_{i=0}^{l} \pi(i|\bar{I}, m + 1, n) P(U_l \geq \bar{I} - j) \) and the inequality follows from the induction assumption \( \pi(0|\bar{I}, m, n) < \pi(0|\bar{I}, m + 1, n) \). Now by applying the induction assumption again on \( X \), we obtain

\[
X \geq \sum_{l=j}^{l-1} \sum_{i=0}^{l} \pi(i|\bar{I}, m, n) P(U_l = l - j) + \sum_{i=0}^{l} \pi(i|\bar{I}, m, n) P(U_l \geq \bar{I} - j).
\]

Substituting this relationship of \( X \) into the above inequality, we obtain

\[
\sum_{i=0}^{j} \pi(i|\bar{I}, m + 1, n + 1) \geq \sum_{i=0}^{j} \pi(i|\bar{I}, m, n + 1).
\]

Thus, we conclude \( \pi(·|\bar{I}, m, n + 1) \geq_{st} \pi(·|\bar{I}, m + 1, n + 1) \). This completes the induction proof.

Finally, we prove \( \pi(·|\bar{I}, m + 1, n) \geq_{st} \pi(·|\bar{I}, m, n + 1) \) by induction. When \( n = 0 \), by definition we have, for \( 0 \leq j \leq \bar{I} \),

\[
\begin{align*}
\sum_{i=0}^{j} \pi(i|\bar{I}, m, 1) &= \pi(0|\bar{I}, m, 0) + \varphi \sum_{i=1}^{j} \pi(i|\bar{I}, m, 0) + \varphi \sum_{i=j+1}^{\bar{I}} \pi(i|\bar{I}, m, 0) P(U_l \geq i - j) \\
&= (1 - \varphi)\pi(0|\bar{I}, m, 0) + \varphi \left[ \sum_{i=0}^{j} \pi(i|\bar{I}, m, 0) + \sum_{i=j+1}^{\bar{I}} \pi(i|\bar{I}, m, 0) P(U_l \geq i - j) \right] \\
&\geq \sum_{i=0}^{j} \pi(i|\bar{I}, m, 0) + \sum_{i=j+1}^{\bar{I}} \pi(i|\bar{I}, m, 0) P(U_l \geq i - j) \\
&= \pi(0|\bar{I}, m, 0) + \sum_{i=1}^{\bar{I}} \pi(i|\bar{I}, m, 0) P(U_l \geq i) + \sum_{i=1}^{j} \sum_{j=i}^{\bar{I}} \pi(j|\bar{I}, m, 0) P(U_l = j - i) \\
&= \sum_{i=0}^{j} \pi(i|\bar{I}, m + 1, 0),
\end{align*}
\]

where the inequality follows from the fact that \( \pi(0|\bar{I}, m, 0) \geq 0 \) and the expression in the bracket is less than one, and the last equality follows from the definition of \( \pi(i|\bar{I}, m + 1, 0) \). Thus, we have \( \pi(·|\bar{I}, m + 1, 0) \geq_{st} \pi(·|\bar{I}, m, 1) \). Now assume that \( \pi(·|\bar{I}, m + 1, n - 1) \geq_{st} \pi(·|\bar{I}, m, n) \). By definition
we have, for $0 \leq j \leq \bar{l}$,

\[
\sum_{i=0}^{j} \pi(i|\bar{l}, m, n + 1) = \frac{\pi(0|\bar{l}, m, n) + \varphi \sum_{i=0}^{j} \pi(i|\bar{l}, m, n) + \varphi \sum_{i=j+1}^{\bar{l}} \pi(i|\bar{l}, m, n) \Pr(U_t \geq i - j)}{\varphi + (1 - \varphi) \pi(0|\bar{l}, m, n)} = \frac{(1 - \varphi) \pi(0|\bar{l}, m, n) + \varphi \left[ \sum_{i=0}^{j} \pi(i|\bar{l}, m, n) + \sum_{i=j+1}^{\bar{l}} \pi(i|\bar{l}, m, n) \Pr(U_t \geq i - j) \right]}{\varphi + (1 - \varphi) \pi(0|\bar{l}, m, n)} = \frac{(1 - \varphi) \pi(0|\bar{l}, m, n) + \varphi \left[ \sum_{l=j}^{\bar{l}-1} \sum_{i=0}^{l} \pi(i|\bar{l}, m, n) \Pr(U_l = l - j) + \Pr(U_t \geq \bar{l} - j) + \Pr(U_t \geq i - j) \right]}{\varphi + (1 - \varphi) \pi(0|\bar{l}, m, n)}.
\]

Let $X = \sum_{l=j}^{\bar{l}-1} \sum_{i=0}^{l} \pi(i|\bar{l}, m, n) \Pr(U_l = l - j) + \Pr(U_t \geq \bar{l} - j)$. By the induction assumption that $\pi(\cdot|\bar{l}, m + 1, n - 1) \leq_{st} \pi(\cdot|\bar{l}, m, n)$, we have $\pi(0|\bar{l}, m, n) \geq \pi(0|\bar{l}, m + 1, n - 1)$. Also note that $X \leq 1$. It follows that

\[
\sum_{i=0}^{j} \pi(i|\bar{l}, m, n + 1) \geq \frac{(1 - \varphi) \pi(0|\bar{l}, m + 1, n - 1) + \varphi X}{\varphi + (1 - \varphi) \pi(0|\bar{l}, m + 1, n - 1)} .
\] (A17)

Furthermore, by applying the induction assumption again to $X$, we have

\[
X \geq \sum_{l=j}^{\bar{l}-1} \sum_{i=0}^{l} \pi(i|\bar{l}, m + 1, n - 1) \Pr(U_l = l - j) + \Pr(U_t \geq \bar{l} - j) = \sum_{i=0}^{j} \pi(i|\bar{l}, m + 1, n - 1) + \sum_{i=j+1}^{\bar{l}} \pi(i|\bar{l}, m + 1, n - 1) \Pr(U_t \geq i - j).
\]

Substituting the above inequality into (A17), with some term arrangement, we obtain

\[
\sum_{i=0}^{j} \pi(i|\bar{l}, m, n + 1) \geq \frac{\pi(0|\bar{l}, m + 1, n - 1) + \varphi \sum_{i=1}^{j} \pi(i|\bar{l}, m + 1, n - 1) + \varphi \sum_{i=j+1}^{\bar{l}} \pi(i|\bar{l}, m + 1, n - 1) \Pr(U_t \geq i - j)}{\varphi + (1 - \varphi) \pi(0|\bar{l}, m + 1, n - 1)} = \sum_{i=0}^{j} \pi(i|\bar{l}, m + 1, n),
\]

where the last equality follows from the definition of $\pi(i|\bar{l}, m + 1, n)$. By the definition of first order stochastic dominance, we conclude $\pi(\cdot|\bar{l}, m + 1, n) \leq_{st} \pi(\cdot|\bar{l}, m, n + 1)$. This completes the induction proof. By the property of first order stochastic dominance, it is straightforward to verify that $\pi(0|\bar{l}, m, n)$ is increasing in $n$, with $\lim_{n \to \infty} \pi(0|\bar{l}, m, n) = 1$. □
Proof (Proposition 3) It suffices to show that, if the system starts from the state \((\bar{I}, m, n^*(\bar{I}, m) + 1)\), the system will never reach the state of \((\bar{I}, m, n^*(\bar{I}, m) + 1)\): zero sales following a do-nothing decision. However, before observing the zero sales, \(n = n^*(\bar{I}, m)\), which implies do-nothing is not an optimal decision. Hence, it is impossible to reach \((\bar{I}, m, n^*(\bar{I}, m) + 1)\) through this avenue. Therefore, we conclude that, when the system starts right after an inspection, the optimal policy is to do nothing if \(n < n^*(\bar{I}, m)\), and to either inspect or replenish if \(n\) reaches \(n^*(\bar{I}, m)\)

\[\square\]

Proof (Proposition 4) From the stochastic dominance result in Proposition 2 and the definition (13), it is easy to verify that \(\alpha(m, n)\) is increasing in \(m\) and \(n\), and \(\alpha(m, n + 1) \geq \alpha(m + 1, n)\). Let \(\tilde{V}_0(m, n) = 0\), and define the mapping \(T\): for \(m, n \geq 0\) and \(m + n \geq 1\),

\[
(T\tilde{V}_0)(m, n) = \min \left\{ G(S, m, n), \quad k + G(S, 0, 0) \right\},
\]

\[
(T^i\tilde{V}_0)(m, n) = \min \left\{ G(S, m, n) + \delta \alpha(m, n)(T^{i-1}\tilde{V}_0)(m, n + 1)
\right.
\]

\[
\left. + \delta(1 - \alpha(m, n))(T^{i-1}\tilde{V}_0)(m + n + 1, 0), \quad k + G(S, 0, 0) + \delta(T^{i-1}\tilde{V}_0)(1, 0) \right\}.
\]

Because the cost function in each period is bounded, from Proposition 1.2.1 of Bertsekas (2001, p. 10), we have \(\tilde{V}(m, n) = \lim_{i \to \infty}(T^i\tilde{V}_0)(m, n)\). By the property of \(G(S, m, n)\), we know that \((T\tilde{V}_0)(m, n)\) is increasing in \(m\) and \(n\), and \((T\tilde{V}_0)(m, n + 1) \geq (T\tilde{V}_0)(m + 1, n)\). By the induction argument and using the properties of \(\alpha(m, n)\) shown above, it is easy to verify that \((T^i\tilde{V}_0)(m, n)\) is increasing in \(m\) and \(n\), and \((T^i\tilde{V}_0)(m, n + 1) \geq (T^i\tilde{V}_0)(m + 1, n)\) for any \(i \geq 1\). Thus, these results also hold at its limit \(\tilde{V}(m, n)\). Therefore, we conclude that \(\tilde{V}(m, n)\) is increasing in \(m\) and \(n\) and \(\tilde{V}(m, n + 1) \geq \tilde{V}(m + 1, n)\). Given a fixed \(m\), there must exist an optimal threshold \(n^*_i(m)\), such that the optimal policy is to perform inspection whenever \(n\) reaches the threshold \(n^*_i(m)\). Moreover, because \(\tilde{V}(m, n)\) is increasing in \(m\), the optimal threshold \(n^*_i(m)\) must be decreasing in \(m\)

\[\square\]

Proof (Lemma 2) Prove by induction. Because \(\pi(0) = q\), it is easy to verify that the result is true for \(n = 0\). Now assume that it is true for the case of \(n - 1\). For the case of \(n\), we have

\[
\pi'(n) = \frac{\pi(n - 1)}{\pi(n - 1) + (1 - \pi(n - 1))\varphi}.
\]
Following the deterioration transition matrix, the updated posterior probability is given by

\[
\pi(n) = \frac{\pi(n-1) + (1 - \pi(n-1))\varphi p}{\pi(n-1) + (1 - \pi(n-1))\varphi} = \frac{q + (p-q)\varphi - (1-q)p\varphi[(1-p)\varphi]^{n-1} + (1-q)p\varphi[(1-p)\varphi]^{n-1}(1 - \varphi + p\varphi)}{q + (p-q)\varphi - (1-q)p\varphi[(1-p)\varphi]^{n-1} + (1-q)p\varphi[(1-p)\varphi]^{n-1}(1 - \varphi + p\varphi)}
\]

where the second equality follows from the induction assumption. This completes the induction proof. Now we note that

\[
\pi(n) = \pi(n-1) + (1 - \pi(n-1))\varphi p - \pi(n-1) = \frac{(1 - \pi(n-1))\varphi p}{\pi(n-1) + (1 - \pi(n-1))\varphi} = \frac{(1 - \pi(n-1))(1 - \varphi + p\varphi)}{\pi(n-1) + (1 - \pi(n-1))\varphi} \geq 0.
\]

Hence, \(\pi(n)\) is increasing in \(n\). Finally, it is straightforward to verify that \(\lim_{n \to \infty} \pi(n) = 1\).

**Proof (Proposition 5)** By the similar argument used in Proposition 4, we can show that \(V(n)\) increases in \(n\) and that an threshold inspection policy based on on \(n\) is optimal. Let us assume the optimal inspection threshold is \(n^*\). From (16), we have

\[
\alpha(n^*) \left( \delta V(n^* + 1) - \delta V(0) + \frac{b'}{1 - \varphi} \right) - \frac{\varphi b'}{1 - \varphi} - k > 0, \quad (A18)
\]

with \(V(n^* + 1) = k + \delta V(0)\). It thus remains to determine \(V(0)\) under the optimal threshold \(n^*\). For \(n = 0, 1, ..., n^* - 1\), we have the following recursive difference equations:

\[
V(n) = \alpha(n)\delta V(n + 1) + (1 - \alpha(n)) \left( \delta V(0) - \frac{b'}{1 - \varphi} \right) + b'.
\]

Let \(\beta(j) = \delta^{j} \prod_{i=0}^{j-1} \alpha(i)\) and \(\beta(0) = 1\). From the above difference equations, we obtain

\[
V(0) = \frac{\beta(n^*) k - \sum_{j=0}^{n^*-1} \beta(j)(1 - \alpha(j)) \cdot \frac{b'}{1 - \varphi} + \sum_{j=0}^{n^*-1} \beta(j)b'}{1 - \beta(n^*)\delta - \sum_{j=0}^{n^*-1} \beta(j)(1 - \alpha(j))\delta}
\]

\[
= \frac{\beta(n^*) k - \sum_{j=0}^{n^*-1} \beta(j) \cdot \frac{\varphi b'}{1 - \varphi} + \sum_{j=0}^{n^*-1} \beta(j) + 1 \cdot \frac{b'}{1 - \varphi}}{(1 - \delta) \sum_{j=0}^{n^*} \beta(j)}
\]

Now substitute the \(V(0)\) expression back into (A18). With some algebra, we arrive at the following optimality condition for \(n^*\):

\[
n^* = \min \left\{ n \geq 0 : \frac{\alpha(n) - \varphi \sum_{j=0}^{n} \beta(j) + \varphi \alpha(n) \sum_{j=0}^{n-1} \beta(j)}{1 - \varphi \left( \sum_{j=0}^{n} \beta(j) - \delta \alpha(n) \sum_{j=0}^{n-1} \beta(j) \right)} > \frac{k}{b'} \right\}.
\]
Now from Lemma 2, we have
\[
\alpha(n) = \pi(n) + (1 - \pi(n))\varphi = \frac{q + (p - q)\varphi + (1 - q)(1 - \varphi)}{q + (p - q)\varphi + (1 - q)(1 - \varphi)} \varphi^{n+1},
\]
\[
\beta(n) = \delta^n \prod_{i=0}^{n-1} \alpha(i) = \frac{q\delta^n + (p - q)\varphi\delta^n + (1 - q)(1 - \varphi)\varphi\delta^n}{1 - \varphi + p\varphi}.
\]
Substitute the the expressions of \( \alpha(n) \) and \( \beta(n) \) into the optimality condition. With some algebra, the left-hand side of the optimality condition becomes
\[
\frac{\alpha(n) - \varphi \sum_{j=0}^{n} \beta(j) + \varphi \delta \alpha(n) \sum_{j=0}^{n-1} \beta(j)}{(1 - \varphi) \left( \sum_{j=0}^{n} \beta(j) - \delta \alpha(n) \sum_{j=0}^{n-1} \beta(j) \right)} = \frac{q + (p - q)\varphi - (1 - q)p\varphi[(1 - p)\varphi]^n + (1 - q)[q + (p - q)\varphi]\varphi \Delta(n)}{q + (p - q)\varphi + (1 - q)(1 - \varphi)[(1 - p)\varphi]^n - (1 - q)[q + (p - q)\varphi](1 - \varphi)\Delta(n)},
\]
where \( \Delta(n) = \sum_{j=0}^{n-1} [(1 - p)\varphi\delta]^j - [(1 - p)\varphi]^n \sum_{j=0}^{n-1} \delta^j \). Thus, we arrive at the desired threshold condition.

To show property (a), we let \( g(n) \) denote the left-hand side of the optimality condition. Let \( \gamma = q + (p - q)\varphi \) and \( x = (1 - p)\varphi \). With some rearrangement, we have
\[
g(n) = \frac{(1 - q)[\varphi \Delta(n) - x^n(1 - x)]}{\gamma + (1 - q)(1 - \varphi)x^n - (1 - q)(1 - \varphi)\gamma \Delta(n)} + 1.
\]
Let \( \Delta_\varphi(n) = \partial \Delta(n)/\partial \varphi \). We can show that
\[
\frac{\partial g(n)}{\partial \varphi} = \frac{(1 - q)\varphi \Delta_\varphi(n) - (1 - q)\Delta_\varphi(n)^2 + x^{n-1}z(n)}{\varphi + (1 - q)(1 - \varphi)x^n - (1 - q)(1 - \varphi)\varphi \Delta_\varphi(n)^2}.
\]
where \( z(n) = -(1 - q)(1 - \varphi)(1 - p)x\delta(\delta - 1)\gamma \Delta(n) - \varphi \gamma \Delta_\varphi(n) + q \Delta(n)] - n(1 - p)(1 - x)\gamma + p(1 - q)x(1 - x^n) + (1 + p)(1 - q)x\gamma \varphi \Delta(n) \). When \( q = p, \) we have
\[
\frac{\partial g(n)}{\partial \varphi} = \frac{-p(1 - p)^2 \left( 1 + p \sum_{i=0}^{n+1} \delta^i \right) x^{n-1}(1 - x)^2 \left[ \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} (x \delta^j x^i) \right]}{[p + (1 - p)(1 - \varphi)x^n - p(1 - p)(1 - \varphi)\varphi \Delta(n)]^2} \leq 0.
\]
Therefore, \( n^*(\varphi, p, q) \) is monotonically increasing in \( \varphi \) if \( q = p \).

When \( p \neq q, \) it is easy to verify that \( \lim_{n \to \infty} x^{n-1}z(n) = 0 \) and
\[
\lim_{n \to \infty} [\Delta_\varphi(n) - (1 - q)\Delta_\varphi(n)^2] = \frac{(q - p)\delta}{(1 - x\delta)^2}.
\]
Hence, if \( q < p, \) \( \lim_{n \to \infty} \partial g(n)/\partial \varphi < 0. \) Now consider a special case with \( n = 1 \) and \( q = 0. \) In this case, we have
\[
\lim_{\varphi \to 1} \frac{\partial g(1)}{\partial \varphi} \bigg|_{q=0} = p\delta[1 - (1 + \delta)p].
\]
Thus, for $0 = q < p$, the above limit can be either positive or negative, depending on the values of $p$ and $\delta$. Therefore, $n^*(\varphi, p, q)$ is not monotone in $\varphi$ when $q < p$. However, there exists a finite $n_1 > 0$, such that $n^*(\varphi, p, q)$ is increasing in $\varphi$ for $n^*(\varphi, p, q) \geq n_1$.

On the other hand, if $q > p$, $\lim_{n \to \infty} \partial g(n)/\partial \varphi > 0$. Now consider another special case with $n = 1$ and $p = 0$. In this case, we have

$$\frac{\partial g(1)}{\partial \varphi} \bigg|_{p=0} = \frac{-(1-q)q\{1+q\delta+(1-q)\varphi^2\delta+q(1-q)(1-\varphi)^2\delta^2\}}{[q+(1-q)\varphi-q(1-q)(1-\varphi)^2\delta]^2} \leq 0.$$ 

Thus, for any $q > p = 0$, it follows that $n^*(\varphi, p, q)$ is increasing in $\varphi$ in the neighborhood of $n^* = 1$. Therefore, $n^*(\varphi, p, q)$ is not monotone in $\varphi$ when $q > p$. However, there exists a finite $n_2 > 0$, such that $n^*(\varphi, p, q)$ is decreasing in $\varphi$ for $n^*(\varphi, p, q) \geq n_2$.

For property (b), recall that $g(0) = q$. Thus, $\partial g(0)/\partial q > 0$, and it follows that $n^*(\varphi, p, q)$ is decreasing in $q$ in the neighborhood of $n^* = 0$. In general, we can derive that

$$\frac{\partial g(n)}{\partial q} = \frac{-\gamma^2\Delta(n) + x^{n}[(1-x)^2 + (1-q)^2(1-\varphi)^2x\delta\Delta(n)]}{[\gamma+(1-q)(1-\varphi)x-\gamma(1-q)(1-\varphi)^2\delta\Delta(n))^2}.$$ 

Note that $\lim_{n \to \infty} \Delta(n) = 1/(1-x\delta) > 0$, and $\lim_{n \to \infty} x^n[(1-x)^2 + (1-q)^2(1-\varphi)^2x\delta\Delta(n)] = 0$. Therefore, $\lim_{n \to \infty} \partial g(n)/\partial q < 0$. It follows that $n^*(\varphi, p, q)$ is not monotone in $q$. However, there exists a finite $n_3 > 0$, such that $n^*(\varphi, p, q)$ is increasing in $q$ for $n^*(\varphi, p, q) \geq n_3$.

It is easy to verify that

$$\frac{\partial g(1)}{\partial p} = \frac{(1-q)(1-x)\varphi\gamma + [1-x-n(1-q)(1-\varphi)]\gamma\delta + (1-q)(\varphi-\varphi^2)(1-p)(1+x\delta)}{[\gamma+(1-q)(1-\varphi)x-\gamma(1-q)(1-\varphi)^2\delta(1-x)]^2} \geq 0.$$ 

Thus, $n^*(\varphi, p, q)$ is decreasing in $p$ in the neighborhood of $n^* = 1$. In general, let $\Delta_p(n) = \partial \Delta(n)/\partial p$. We have

$$\frac{\partial g(n)}{\partial p} = \frac{(1-q)[\gamma^2\Delta_p(n) + x^{n-1}Z'(n)]}{[\gamma+(1-q)(1-\varphi)x-\gamma(1-q)(1-\varphi)^2\delta\Delta(n))^2},$$ 

where $Z'(n) = n(1-x)\varphi\gamma + (1-q)(1-\varphi)x\{\varphi(1-x^n) + [(n+1)\gamma + x]\varphi\delta\Delta + x\varphi\delta\Delta (n)\}$. Note that

$$\Delta_p(n) = -\varphi \left( \frac{\delta - nx^{n-1}\delta^n + (n-1)x^n\delta_{n+1}}{(1-x\delta)^2} - \frac{nx^{n-1}(1-\delta^n)}{1-\delta} \right).$$ 

Therefore, $\lim_{n \to \infty} \Delta_p(n) = -\varphi\delta/(1-(1-p)\varphi\delta) < 0$. Also note that $\lim_{n \to \infty} x^{n-1}Z'(n) = 0$. Hence, $\lim_{n \to \infty} \partial g(n)/\partial p < 0$. It follows that $n^*(\varphi, p, q)$ is not monotone in $p$. However, there exists a finite $n_4 > 0$, such that $n^*(\varphi, p, q)$ is increasing in $p$ for $n^*(\varphi, p, q) \geq n_4$. 

\[ \square \]
References


