In this paper we consider the inventory control problem for serial supply chains with Markov-modulated demand (MMD). Our goal is to simplify the computational complexity by resorting to certain approximation techniques, and, in doing so, to gain a deeper understanding of the problem. To this end, we analyze the problem in several new ways. We first perform a derivative analysis of the problem’s optimality equations, and develop general bounds for the optimal policy. Based on the bound results, we derive a simple procedure for computing near-optimal heuristics for the problem. We further perform asymptotic analysis with long replenishment leadtime. We show that the relative errors between our heuristics and the optimal solutions converge to zero as the leadtime becomes sufficiently long, with the rate of convergence being the square root of the leadtime. Our numerical results reveal that our heuristics can achieve near-optimal performance even under relatively short leadtimes. In addition, we show that, by leveraging the Laplace transformation, the optimal policy becomes computationally tractable under the gamma distribution family. Under the optimal policy, we observe that the internal fill rate and demand variability propagation in an optimally controlled supply chain under MMD exhibit behaviors different from those under stationary demand.

Key words: Serial Multi-Echelon Systems, Markov-Modulated Demand, Bounds and Approximations, Derivative Analysis, Asymptotic Analysis.

History: Revised on October 18, 2014.
1. Introduction

The increasingly open global economy has made it possible for companies to seek the best available resources and technologies worldwide and to serve new markets. As a result, many supply chains have been stretched long and thin, often consisting of multiple production and distribution stages across different countries and continents. This new supply chain configuration, however, brings new challenges to managers. For example, demand shocks in one region, be they political, economic, technological, or climatic, can quickly propagate to other regions, causing shortages in some stages but oversupplies in others along the chain. Thus, companies that traditionally operate their supply chains in a relatively stable environment, such as within one country, must now learn how to effectively rationalize inventory along the global supply chain to hedge against greater environmental uncertainties.

To help managers to meet this new challenge, in this paper, we examine a serial supply chain model that explicitly takes into account demand uncertainties driven by such dynamic environmental factors. Our goal is two-fold: First, we aim to develop easy-to-compute solution techniques to enable swift decision making in choosing the appropriate inventory levels along the supply chain. Second, we aim to link the simple solutions to system parameters to shed light on both the qualitative and quantitative effects of the environmental uncertainties. These insights can help guide managers to invest wisely to obtain the right information (or problem parameters) and to react quickly in the right direction.

Specifically, we assume that customer demand in each period is influenced by a “world” state that evolves according to a discrete-time Markov chain. This demand model is known in the literature as the Markov modulated demand (MMD) process (see Iglehart and Karlin 1962; Song and Zipkin 1993). One advantage of MMD is the structured data requirement and the modeling flexibility of Markov chains. To construct the demand state process, one can first identify discrete scenarios, such as different phases in a product development process and different stages in a product life cycle. One can then assess the likelihood of each scenario as well as the transition probabilities between the scenarios, much as in the construction of a decision tree (see Song and Zipkin 1996b).

Chen and Song (2001) showed that, for the series system considered here, a world-state-dependent echelon base-stock policy is optimal. Computing the optimal policy, however, is a nontrivial task. Chen and Song (2001) showed that the optimal policy can be obtained by a nested optimization algorithm with multiple state and stage iterations. While this exact algorithm provides a significant improvement over the standard dynamic programming algorithms, there remains
a computational challenge, especially in dealing with continuous demand distributions. In particular, in each iteration one needs to solve a set of integral renewal equations. The iterative nature of the algorithm also renders the solution approach less intuitive for practitioners, who may perceive it as a numbers-in-and-numbers-out black box.

In this paper, we seek to simplify the computational complexity of the problem by resorting to certain approximation techniques and at the same time to make the “black box” more transparent. To this end, we analyze the problem in the following new ways.

**Derivative Analysis and Solution Bounds**

The first approach is to perform a derivative analysis of the optimality equations of the problem and develop general solution bounds. These bounds can be computed directly without solving the integral renewal equations required in the exact algorithm. They also reveal a closer relationship with the model parameters, which was not apparent in the exact algorithm.

It has been recognized that deriving solution bounds for serial inventory systems facing non-stationary demand processes is a challenging problem. A common approach is to assume that the (echelon) base-stock level is achievable in each period; see Dong and Lee (2003) for a serial system subject to time-correlated demand, and Shang (2012) for independent but nonidentical demand. In contrast, our derivative analysis approach does not require such a simplifying assumption.

In the special case of the independent and identically-distributed (i.i.d.) demand process (where the Markov chain reduces to a single state), our bounds coincide with those obtained by Shang and Song (2003). Our analysis thus provides an alternative proof of their results. More importantly, our method provides new insights into why their bounds work. In addition, their bounding approach does not extend to the general MMD process to obtain the lower bound, so our approach is more general and flexible.

Based on our solution bound results, we develop an even simpler procedure to compute near-optimal heuristics for the problem that requires no iterations. Specifically, we show that the effect of the underlying Markov chain transition probabilities can be mapped to a linear weight associated with each demand state, and that obtaining the heuristic solutions only requires knowledge of the leadtime demand distribution for each demand state. Such a computation method is easy to implement and also easy for practitioners to understand.

To our knowledge, our heuristics represent the first set of approximations of the optimal policies for both the single- and multi-stage inventory systems with MMD. For a two-state demand case, we further show that, when it is highly likely to transit from a high demand state to a low demand
state, one should reduce the optimal inventory level for the high demand state, and our heuristic solution moves in the same direction as the optimal inventory level. Moreover, we prove that our heuristic solution is close to the optimal solution when there is a high chance of transiting from a low demand state to a high demand state.

Asymptotic Analysis

Our second approach is to perform an asymptotic analysis for the problem with long replenishment leadtime. We begin by establishing a central limit theorem for the leadtime demands under MMD. We show that, when the leadtime is sufficiently long, the leadtime demand follows a normal distribution, whose variance has a closed-form expression. It is the sum of the single-period demand variance and the covariance across different periods, where the latter depends on the demand mean of each state in a linear product form. The intuition behind this result is that there is an inherent averaging effect of the leadtime demand under MMD—the leadtime demand distributions with different initial states become more congruent to each other as leadtime increases. Therefore, when leadtime is sufficiently long, we can obtain a closed-from expression for inventory planning along the supply chain.

We show that the relative errors between our heuristics and the optimal solutions converge to zero as the leadtime becomes sufficiently long, with the rate of convergence being the square root of the leadtime. Thus, our simple heuristics are guaranteed to yield near-optimal performance under long leadtimes. Our numerical results further reveal that the near-optimal performance can be achieved even under relatively short leadtimes. Therefore, in a serial inventory system with many stages, the most challenging problem of determining the optimal policy for the upstream stages (which requires multiple state and stage iterations) has a surprisingly simple near-optimal solution, due to the fact that the total leadtime of an upstream stage (i.e., the sum of the lead-times of all downstream stages) is longer than that of a downstream stage.

Exact Evaluation and Observations

Moreover, by leveraging the Laplace transformation and its inverse, we show that when demand belongs to the gamma distribution family, the optimal policy for a serial inventory system with MMD becomes computationally tractable. This enables us to numerically compare the performance of our bounds and heuristics against the optimal policy under the gamma demand distribution family. To the best of our knowledge, our paper presents the first numerical study of the optimal policy for serial inventory systems with MMD under continuous demand. From the numerical
study, we observe that our heuristics yield near-optimal performance in almost all cases.

With the optimal policy under gamma demand distribution, we can obtain numerical observations of system behaviors under an optimally controlled serial inventory system with MMD. Specifically, we investigate the system behavior of the internal fill rate, i.e., the service level of one location to its immediate downstream location. The general understanding in the literature is that internal fill rates are much lower than the fill rate at the customer-facing stage when demand is i.i.d. However, we observe that this is not always true under the general MMD process. In some numerical cases, the internal fill rate can be very high, suggesting that the serial system can be effectively decoupled into separate single-stage systems (e.g., Abhyankar and Graves 2001).

Another system behavior we investigate is the demand variability propagation effect, or the bullwhip effect (see Chen and Lee 2009, 2012 and the references therein). Interestingly, we observe that the bullwhip effect can be significantly dampened under the optimal policy, suggesting that the state-dependent inventory policy under MMD may inherently smooth demand variability propagation in the system.

The rest of this paper is organized as follows. We provide a brief literature review in §2. We then analyze single-stage inventory systems in §3. This lays the foundation for the analysis of serial inventory systems in §4. §5 presents the numerical results, and §6 concludes the paper. All proofs are presented in the Appendix.

2. Literature Review

The serial system structure we consider is the classic multiechelon inventory model, first developed and analyzed by Clark and Scarf (1960), and further studied by many researchers in various dimensions (e.g., see Zipkin 2000 for a review). It serves as an important baseline model and a key building block for more complex supply chain structures. Much of the literature on this classic model assumes an i.i.d. demand process. Under this assumption, it has been shown that a stationary echelon base-stock policy is optimal. Earlier work focused on the computational efficiency of the optimal policy; e.g., see Federgruen and Zipkin (1984) and Chen and Zheng (1994). More recently, researchers have developed simple bounds and heuristics that aim to increase the transparency of the factors that determine the optimal policy. See, for example, Gallego and Zipkin (1999), Dong and Lee (2003), Shang and Song (2003), Gallego and Ozer (2005), and Chao and Zhou (2007). Our work extends their efforts to systems with MMD.

The MMD process, which extends the i.i.d. demand process to incorporate dynamic state evolution, was first introduced by Iglehart and Karlin (1962) to study a single-stage inventory model, but
only gained popularity in the inventory literature in the last two decades; see, for example, Song and Zipkin (1993), Ozeciki and Parlar (1993), Beyer and Sethi (1997), and Sethi and Cheng (1997). An important special case of MMD is the cyclic demand model, which various authors explored; see, for example, Karlin (1960), Zipkin (1989), Aviv and Federgruen (1997), and Kapuscinsky and Tayur (1998). Several authors also adopted MMD in serial inventory systems, but focused on different aspects of the problem than we do. For example, Song and Zipkin (1992, 1996a, 2009) and Abhyankar and Graves (2001) analyzed specific types of policies, and Angelus (2011) considered the complexity of incorporating secondary market sales. The most closely related works to ours are Chen and Song (2001) and Muharremoglu and Tsitsiklis (2008). These authors showed that, for a serial inventory system, a state-dependent echelon base-stock policy is optimal. They also presented exact algorithms to compute the optimal policy.

There has also been research in the literature on deriving bounds and heuristics for serial inventory systems with nonstationary demand processes. For example, Dong and Lee (2003) developed lower bounds for optimal policies for serial systems with time-correlated demand. The time-series demand model requires past demand information, whereas the MMD model we employ here focuses on external factors. Shang (2012) extended the single-stage approximation approach of Shang and Song (2003) to derive solution bounds and heuristics for serial inventory systems with independent but nonidentical demands over a finite horizon. The demand process considered in Shang (2012) is a special case of MMD. Our paper complements this literature by showing general solution bounds for serial inventory systems with MMD.

3. Single-Stage Inventory Systems

Consider a single-stage inventory control problem for a product. Demand in each period is met with on-hand inventory; when there is stockout, we assume the unmet demand is fully backlogged. Inventory is replenished from an external source with ample supply. The replenishment leadtime is a constant of $L$ periods. At the end of each period, unit inventory holding cost $h$ and unit backlog penalty cost $b$ are charged on the on-hand inventory and backorders, respectively. The planning horizon is infinite, and the objective is to minimize the long-run average cost of the inventory system. Because the linear ordering cost under the long-run average cost is a constant, we can assume the ordering cost is zero without loss of generality (see Federgruen and Zipkin 1984).

The demand process is nonstationary and has an embedded Markov chain structure (see Iglehart and Karlin 1962, Chen and Song 2001). Specifically, demand in each period is a nonnegative random variable denoted by $D_k$, where $k$ is the demand state that determines the continuous demand
distribution with density function \( f_k(\cdot) \) and the cumulative density function \( F_k(\cdot) \). The demand state in period \( t \) follows a Markov chain \( W = \{ W(t), t = 1, 2, \ldots \} \), with a total of \( K \) states, denoted by \( \{1, 2, \ldots, K\} \). We assume that the Markov chain is time-homogeneous, and we let \( p_{ij} \) denote the one-step transition probability from state \( i \) to \( j \). Let \( P = (p_{ij}) \) denote the transition probability matrix. Without loss of generality, we assume the Markov chain is irreducible, which implies that all states communicate with each other. (When the Markov chain is reducible, under the long run average cost criterion, the problem is equivalent to the one involving only the irreducible class of the chain.) Also let \( D_k[t, t'] \) denote the total demand in periods \( t, \ldots, t' \) with the demand state being \( k \) in period \( t \), and let \( D_k[t, t'] \) denote the total demand in periods \( t, \ldots, t' - 1 \).

In each period, the sequence of events is as follows: 1) at the beginning of a period, the demand state \( k \) is observed; 2) an inventory replenishment order is placed with the supplier; 3) a shipment ordered \( L \) periods ago is received from the supplier; 4) demand arrives during the period; and 5) at the end of the period, holding and backorder costs are assessed.

### 3.1 Preliminaries

It has been shown (see Iglehart and Karlin 1962, Beth and Sethi 1997, Chen and Song 2001) that the optimal policy for this problem is a state-dependent base-stock policy \( s^*(k) \) for \( k = 1, \ldots, K \). It works as follows: When the demand state is \( k \), if the inventory position is below the base-stock level \( s^*(k) \), order up to this level; otherwise, do not order.

Given the demand state \( k \) and inventory position \( y \), the single-period expected cost is

\[
G(k, y) = h \cdot E\{(y - D_k[t, t + L])^+\} + b \cdot E\{(D_k[t, t + L] - y)^+\},
\]

where \((x)^+ = \max\{x, 0\}\). The above function is the well-known newsvendor cost function, which is convex in \( y \). Its minimizer, denoted by \( \bar{s}(k) = \arg \min_y G(k, y) \), is termed the myopic base-stock level, which we also refer to in this paper as the newsvendor solution. Specifically, \( \bar{s}(k) \) solves the following first-order condition:

\[
\partial_y G(k, y) = -b + (b + h)F_{k, L}(y) = 0, \quad \text{or} \quad F_{k, L}(y) = \frac{b}{b + h},
\]

where we use “\( \partial_y \)” to denote the first-order derivative with respect to \( y \) (we shall adopt this notation throughout the paper) and \( F_{k, L}(\cdot) \) is the cumulative distribution of the leadtime demand \( D_k[t, t+L] \). We shall also refer to the function \( \partial_y G(k, y) \) as the newsvendor derivative function throughout the paper. For later usage, we define the smallest myopic newsvendor solution among all states as

\[
s^{\min} = \min_{1 \leq k \leq K} \{ \bar{s}(k) \}.
\]
The above newsvendor solution \( \bar{s}(k) \) is easy to compute. However, it is not the optimal base-stock level for state \( k \): The demand state in the next period may result in a (stochastically) much smaller demand distribution than the one in the current period; therefore, stocking up to the myopic level may cause an overstock in the next period. To compute the optimal policy, we need more sophisticated methods. Chen and Song (2001) showed that the optimal policy can be obtained by a \( K \)-iteration algorithm. Their algorithm is simpler and more efficient than the standard dynamic programming algorithms. However, as will be evident in the next section, because each iteration builds on the results obtained in the previous iterations, the \( K \)-iteration algorithm remains quite complicated.

### 3.1.1 Exact Algorithm

To facilitate our analysis, we first provide a sketch of the exact algorithm by Chen and Song (2001). The algorithm starts with a partition of the state space of \( W \). In each iteration \( i \), let \( V^i \) denote the set that contains the states for which we have not found the optimal solution yet, and \( U^i \) the complementary set of \( V^i \). Specifically, in the first iteration \( i = 1 \), let \( V^1 = \{1, \ldots, K\} \) and \( U^1 = \phi \). Define

\[
G^1(k, y) = G(k, y)
\]

as given in (1), solve the single-period problem and let \( s^1(k) = \bar{s}(k) \) denote the solution for state \( k \). Find the demand state that gives the smallest solution among \( V^1 \). Denote such a state as \( \hat{s}^1(k) = \arg \min_{1 \leq k \leq K} \{\bar{s}(k)\} \). Then the optimal solution for state \( \hat{s}^1(k) \) is just \( \hat{s}(\hat{s}^1(k)) = \bar{s}(\hat{s}^1(k)) \). Next, update the state space partition by \( V^2 = V^1 \setminus \{\hat{s}^1(k)\} \) and \( U^2 = U^1 \cup \{\hat{s}^1(k)\} \), and proceed to iteration \( i = 2 \). In iteration \( i + 1 \), \( i = 1, \ldots, K - 1 \), for each \( k \in V^{i+1} \), solve the cost minimization problem

\[
G^{i+1}(k, y) = G^i(k, y) + \sum_{u \in U^{i+1}} p_{ku} E\{R^i(u, y - D_{k})\},
\]

and for any \( u \in U^{i+1} \),

\[
R^i(u, y) = \left\{ \begin{array}{ll}
\sum_{u' \in U^{i+1}} p_{uu'} E\{R^i(u', y - D_u)\} & \text{if } u \in U^i, \\
G^i(i^*, y) + \sum_{u' \in U^{i+1}} p_{i^*u'} E\{R^i(u', y - D_{i^*})\} & \text{if } u = i^*,
\end{array} \right.
\]

with \( G^i(i^*, y) = G^i(i^*, \max\{y, s^i(i^*)\}) \). It can be shown that \( G^{i+1}(k, y) \) is convex in \( y \). Let \( s^{i+1}(k) = \arg \min_y G^{i+1}(k, y) \). Also let \( (i + 1)^* = \arg \min_{k \in V^{i+1}} s^{i+1}(k) \), i.e., the demand state that gives the
smallest $s^{i+1}(k)$ among $V^{i+1}$. Then the optimal solution for state $(i + 1)^*$ is given by $s^*((i + 1)^*) = s^{i+1}((i + 1)^*)$. Update the state sets by $V^{i+2} = V^{i+1} \setminus \{(i + 1)^*\}$ and $U^{i+2} = U^{i+1} \cup \{(i + 1)^*\}$. Repeat the above procedure until reaching the final iteration $K$.

In summary, the above algorithm finds the optimal base-stock level for each demand state by sorting the demand states based on the solutions to the convex cost minimization problems $\min_y \{G^{i+1}(k, y)\}$ in each iteration. The main difficulty of the algorithm is in determining the function $R^i$ at each iteration, as given in (4). When the demand distribution is discrete (such as a Poisson demand), computing $R^i$ is relatively easy, as it involves solving a set of linear equations (see Chen and Song 2001). However, when the demand distribution is continuous, determining $R^i$ becomes much more convoluted, as it requires solving a set of integral renewal equations.

### 3.1.2 Our Approach

Based on the exact algorithm, it is useful to let $[k]$ denote the iteration in which the optimal solution $s^*(k)$ for state $k$ is obtained. In other words, the optimal solution $s^*(k)$ is determined by solving the following cost minimization problem:

$$\min_{y \geq 0} \left\{ G^{[k]}(k, y) \right\}.$$ 

Because $G^{[k]}(k, y)$ is convex in $y$, if we can find bounding functions for its derivative, then the roots of the bounding functions become bounds for the optimal solution. Thus, our approach is to develop bounding functions that involve only the primitive model parameters, instead of depending on the solutions from the previous iterations as in $G^{[k]}(k, y)$. The resulting solution bounds then will also depend only on the primitive problem data and hence will yield simple (near-optimal) solutions for the original problem.

To this end, by repeatedly applying (3) and (4) and then taking derivatives, we first observe that

$$\partial_y G^{[k]}(k, y) = \partial_y G(k, y) + \sum_{i=1}^{[k]-1} \sum_{u \in U^{i+1}} p_{ku} \partial_y E\{R^i(u, y - D_k)\}.$$  \hspace{1cm} (5)

From the above expression, the first term $\partial_y G(k, y)$ is a simple newsvendor derivative function. Thus, it remains to find a simple function to bound the second term. Specifically, we want to bound $\partial_y E\{R^i(u, y - D_k)\}$ by a simple function. The next section accomplishes this by analyzing the derivative of the $R^i$ functions.
3.2 Derivative and Solution Bounds

We note that all functions involved in the exact algorithms are continuously differentiable and all the expectations are finite. Therefore, we can safely interchange the derivative and expectation (or integral) operators in our subsequent derivations. Also, recall from the definition that \( G^i(i^*, y) = G^i(i^*, \max\{y, s^i(i^*)\}) \). It follows that \( \partial_y G^i(i^*, y) = (\partial_y G^i(i^*, y))^+ \).

To derive bounds for \( \partial_y G^{[k]}(k, y) \), we need to establish some auxiliary results and introduce some new notation. First, from equation (4), it is straightforward to show that \( \partial_y R^i(u, y) = 0 \) for \( y \leq s^i(i^*) \). By using a standard result in renewal equations (see Sgibnev 2001), we can establish the following convex properties for functions \( R^i \), \( G^i \) and \( G^i \):

**Lemma 1** For any \( u \in U^i+1 \), \( R^i(u, y) \) is increasing convex in \( y \). Moreover, for any \( k = 1, ..., K \), \( G^i(k, y) \) is convex in \( y \) and \( G^i(i^*, y) \) is increasing convex in \( y \).

From the increasing convex property of \( R^i \), the second term of (5) is nonnegative, so we immediately have

\[
\partial_y G^{[k]}(k, y) \geq \partial_y G(k, y) = -b + (b + h) F_{k,L}(y).
\]

Thus, \( \partial_y G(k, y) \) serves as a simple lower bound function for \( \partial_y G^{[k]}(k, y) \).

We now proceed to derive the upper bound for \( \partial_y G^{[k]}(k, y) \). Let \( m = (m_1, ..., m_K) \) denote a permutation of the sequence of demand states \( \{1, ..., K\} \). Define the following sub-matrices of transition probabilities under \( m \): for \( i = 1, ..., K - 1 \),

\[
P^{(i)} = \begin{pmatrix}
p_{m_1m_1} & \cdots & p_{m_1m_i} \\
\vdots & \ddots & \vdots \\
p_{m_im_1} & \cdots & p_{m_im_i}
\end{pmatrix}.
\]

Let \( I^{(i)} \) be the \( i \times i \) identity matrix, \( e^i \) the unit vector with the \( i \)-th element being one, and \( e \) the \( i \)-dimensional vector with all elements being one. Because the Markov chain \( W \) is irreducible, it follows that \( I^{(i)} - P^{(i)} \) is invertible for any \( i = 1, ..., K - 1 \). Thus, we can define the following parameters: for \( i = 1, ..., K - 1 \),

\[
\beta_i(m) = \frac{|I^{(i-1)} - P^{(i-1)}|}{|I^{(i)} - P^{(i)}|},
\]

where \(| \cdot |\) is the matrix determinant and we adopt the convention that \(|I^{(0)} - P^{(0)}| = 1\). We note that \( \beta_i(m) \) depends only on the transition matrix \( P \) and the sequence \( m \).
Let \( m^* = (1^*, ..., K^*) \) denote the optimal demand state sequence determined by the exact algorithm. Let \( D^{(i)}_{m^*}(x) \) be the following diagonal matrix involving demand densities for states \( 1^*, ..., i^* \) obtained in the first \( i \) iterations of the exact algorithm:

\[
D^{(i)}_{m^*}(x) = \begin{pmatrix}
  f_{1^*}(x) & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & f_{i^*}(x)
\end{pmatrix}.
\] (6)

Denote \( R^i(y) = [R^i(1^*, y), R^i(2^*, y), ..., R^i(i^*, y)]^T \). Then, we can rewrite (4) in matrix form as follows:

\[
R^i(y) = \int_0^\infty D^{(i)}_{m^*}(x) \cdot P^{(i)}_{m^*} \cdot R^i(y - x) dx + e^i \cdot G^i(i^*, y).
\] (7)

Taking the derivative with respect to \( y \) on both sides of the equation (7), we have

\[
\partial_y R^i(y) = \int_0^y f^{(i)}_{m^*}(x) \cdot P^{(i)}_{m^*} \cdot \partial_y R^i(y - x) dx + e^i \cdot \partial_y G^i(i^*, y)
\]

\[
\leq P^{(i)}_{m^*} \cdot \partial_y R^i(y) + e^i \cdot \partial_y G^i(i^*, y),
\]

where the inequality follows from the fact that \( \partial_y R^i(\cdot, y) \) is increasing in \( y \) and the probability distribution is less than one. Therefore, we have

\[
\left( I^{(i)} - P^{(i)}_{m^*} \right) \partial_y R^i(y) \leq e^i \cdot \partial_y G^i(i^*, y).
\] (8)

With some additional arguments, we obtain the upper bound for \( \partial_y R^i(y) \) as follows:

**Lemma 2** For \( i = 1, ..., K - 1, \)

\[
\partial_y R^i(y) \leq \left( I^{(i)} - P^{(i)}_{m^*} \right)^{-1} \cdot e^i \cdot \partial_y G^i(i^*, y) \leq e \cdot \beta_i(m^*) \cdot \partial_y G^i(i^*, y).
\]

Now apply both Lemmas 1 and 2 to (5). We obtain

\[
\partial_y G^{[k]}(k, y) = \partial_y G(k, y) + \sum_{i=1}^{[k]-1} \sum_{u \in U^{i+1}} p_{ku} \partial_y E\{R^i(u, y - D_k)\}
\]

\[
\leq \partial_y G(k, y) + \sum_{i=1}^{[k]-1} \sum_{u \in U^{i+1}} p_{ku} \beta_i(m^*) \partial_y G^i(i^*, y)
\]

\[
\leq \partial_y G(k, y) + (1 - p_{kk}) \sum_{i=1}^{K-1} \beta_i(m^*) \partial_y G^i(i^*, y),
\]

where the last step relaxes the summation range of the second term to make it independent of the set \( U^i \) determined by the exact algorithm. We can further show that (see the detailed derivation in the Appendix)

\[
\sum_{i=1}^{K-1} \beta_i(m^*) \partial_y G^i(i^*, y) \leq \sum_{i=1}^{K-1} \alpha_i(m^*) (\partial_y G(i^*, y))^+,
\]

11
where for $i = 1, ..., K - 1$,

$$\alpha_i(m) = \sum_{j=i+1}^{K-1} \alpha_j(m) \cdot \phi_{j,i}(m) + \beta_i(m), \quad \text{(9)}$$

with $\phi_{j,i}(m) = [p_{m_1}^{m_1}, p_{m_2}^{m_2}, ..., p_{m_i}^{m_i}]^T (I^{(i)} - P^{(i)}_m)^{-1} e^i$. Thus, we have obtained an upper bound function for $\partial_y G^{[k]}(k, y)$ based on a linear combination of simple newsvendor derivative functions. The following proposition summarizes the above bound results:

**Proposition 1** For any state $k = 1, ..., K$, the following holds:

$$\partial_y G(k, y) \leq \partial_y G^{[k]}(k, y) \leq \partial_y G(k, y) + (1 - p_{kk}) \Delta(y),$$

where

$$\Delta(y) = \max_{m \in S} \frac{K-1}{\sum_{i=1}^{K-1} \alpha_i(m) \cdot (\partial_y G(m_i, y))^+}, \quad \text{(10)}$$

with $S$ being the set of all permutations of $\{1, ..., K\}$.

When $K = 1$, $\Delta(y) \equiv 0$, so the inequalities in the above proposition becomes binding and the result reduces to the derivative of the single-period cost function. This is intuitive because when $K = 1$ the demand process reduces to an i.i.d. demand process. Moreover, for $K > 1$, when $p_{kk}$ is close to one—i.e., the state $k$ becomes more like an absorbing state—the bounds become tighter. This is also intuitive because, in this case, state $k$ closely resembles an i.i.d. demand process.

With the derivative lower bound shown above, it follows that the myopic solution $\bar{s}(k)$ is an upper bound for the optimal solution $s^*(k)$. This result is implied by the exact algorithm of Chen and Song (2001). Symmetrically, from the derivative upper bound shown above, we can obtain a lower bound for the optimal solution $s^*(k)$. Specifically, let $\underline{s}(k)$ be the solution to the following equation:

$$\partial_y G(k, y) + (1 - p_{kk}) \Delta(y) = -b + (b + h) F_{k,L}(y) + (1 - p_{kk}) \Delta(y) = 0. \quad \text{(11)}$$

Therefore, we can obtain the following result:

**Corollary 1** For any demand state $k = 1, ..., K$, the following holds:

$$s^{\text{min}} \leq \underline{s}(k) \leq s^*(k) \leq \bar{s}(k),$$

where $\bar{s}(k)$ and $\underline{s}(k)$ are determined by (2) and (11), respectively.
While the bounds $s^{\text{min}}$ and $\bar{s}(k)$ have appeared in the literature (see, e.g., Song and Zipkin 1993 for systems with the discounted cost criterion), to our knowledge, the lower bound $\underline{s}(k)$ obtained above is new to the literature. This new bound provides a tighter lower bound than $s^{\text{min}}$ for the minimum base-stock level required for each state. Moreover, both the upper and lower bounds can be computed directly without solving the integral renewal equations required in the exact algorithm.

3.3 Heuristics

The insights gained through the derivation of the derivative bounds lead us next to construct heuristics that fall between the solution lower and upper bounds. To do so, we augment the derivative lower bound by a positive component that is less than $(1 - p_{kk})\Delta(y)$. Specifically, we first sort the myopic solution $\bar{s}(k)$ in an increasing order and let $\tilde{m} = (\tilde{m}_1, ..., \tilde{m}_K)$ denote the resulting order sequence of the states. Suppose that $k = \tilde{m}_j$. Based on the expression given in (5) and Lemma 2, we can make the following approximation for $\partial_y G^k[k](k, y)$:

$$
\partial_y G^k[k](k, y) \approx \partial_y G(k, y) + \sum_{i=1}^{j-1} p_{k\tilde{m}_i} \cdot \partial_y E\{R^i(\tilde{m}_i, y - D_k)\}
$$

$$
\approx \partial_y G(k, y) + \sum_{i=1}^{j-1} p_{k\tilde{m}_i} \cdot \beta_i(\tilde{m}) \cdot E\{(\partial_y G(\tilde{m}_i, y - D_k))^+\}
$$

$$
\approx \partial_y G(k, y) + \frac{1}{2} \sum_{i=1}^{j-1} p_{k\tilde{m}_i} \cdot \beta_i(\tilde{m}) \cdot F_k(y - \bar{s}(\tilde{m}_i)) (\partial_y G(\tilde{m}_i, y))^+,
$$

where the last step is based on the following integral approximation:

$$
E\{(\partial_y G(\tilde{m}_i, y - D_k))^+\} = \int_0^{y - \bar{s}(\tilde{m}_i)} f_k(x)dx \approx \frac{1}{2} F_k(y - \bar{s}(\tilde{m}_i)) (\partial_y G(\tilde{m}_i, y))^+.
$$

Now let $s^a(k)$ denote a heuristic solution determined by the following equation:

$$
\partial_y G(k, y) + \frac{1}{2} \sum_{i=1}^{j-1} p_{k\tilde{m}_i} \cdot \beta_i(\tilde{m}) \cdot F_k(y - \bar{s}(\tilde{m}_i)) (\partial_y G(\tilde{m}_i, y))^+ = 0. \quad (12)
$$

We note that the above equation involves only a linear combination of the simple newsvendor derivative functions of different demand states. Thus, solving the heuristic solution $s^a(k)$ only requires a direct evaluation of the leadtime demand distribution of different demand states. Moreover, we have the following result:

**Corollary 2** For any demand state $k = 1, ..., K$, the following holds:

$$
s^{\text{min}} \leq \underline{s}(k) \leq s^a(k) \leq \bar{s}(k).
$$

where $\bar{s}(k)$ and $\underline{s}(k)$ are determined by (2) and (11), respectively.
Combining Corollaries 1 and 2, we observe that \( s^*(k) \) and \( s^a(k) \) fall in the same interval \([\underline{s}(k), \overline{s}(k)]\). Therefore, if the interval is tight, then the heuristic solution \( s^a(k) \) becomes a close approximation to the optimal solution \( s^*(k) \). In the next section, we identify sufficient conditions to ensure that such an interval is tight.

### 3.4 Asymptotic Analysis: The Effect of (Long) leadtime

Intuitively, when the replenishment leadtime \( L \) increases, the leadtime demands for different demand states should converge to the same distribution. As a result, the gap between \( s_{\text{min}} \) and \( \overline{s}(k) \) will narrow. Below we formally prove this result by first establishing a central limit theorem for the MMD process.

Consider the case in which the Markov chain \( W \) has a stationary distribution. Let \( \pi = [\pi(1), ..., \pi(K)] \) denote such a stationary distribution. Let \( D_\pi(t) \) be the demand in period \( t \) under the stationary distribution, that is, \( D_\pi(t) \) follows the distribution of \( D_k \) with probability \( \pi(k) \). Let \( \mu_k = E[D_k] \) and \( \sigma_k^2 = \text{var}[D_k] \). The mean of \( D_\pi(t) \) is given by

\[
\mu = E\{D_\pi(t)\} = \sum_{k=1}^{K} \mu_k \pi(k).
\]  

Further define \( \tilde{\mu}_k = \mu_k - \mu \) as the relative difference between \( \mu_k \) and \( \mu \).

Let \( D_\pi[t, t+L] \) denote the leadtime demand under the stationary distribution \( \pi \). It follows that

\[
E\{D_\pi[t, t+L]\} = (L+1)\mu.
\]

We can further show (see the Appendix for the detailed derivation)

\[
\sigma^2 = \lim_{L \to \infty} \frac{\text{var}\{D_\pi[t, t+L]\}}{L+1} = \text{var}\{D_\pi(t)\} + 2 \sum_{l=1}^{\infty} \text{cov}\{D_\pi(t), D_\pi(t+l)\}
\]

\[
= \sum_{k=1}^{K} (\sigma_k^2 + \tilde{\mu}_k^2) \pi(k) + 2 \sum_{s=1}^{\infty} \sum_{k=1}^{K} \sum_{l=1}^{K} \tilde{\mu}_k \tilde{\mu}_l p_{kl}^{(s)} \pi(k),
\]  

where \( p_{kl}^{(s)} \) is the \( s \)-step transition probability from state \( k \) to state \( l \). Thus, the variance of the leadtime demand distribution contains two parts: the single-period demand variance and the covariance across different periods. It is interesting to note that the covariance structure under MMD depends only on the demand mean of each state. Thus, besides the demand variance at each state, another major source of variability under MMD comes from the variation of the demand mean of each state.

Let \( N(0, 1) \) denote the standard normal random variable with distribution function \( \Phi(\cdot) \). We can establish the following result for the leadtime demand distribution under MMD:
Lemma 3 (MMD Central Limit Theorem) Suppose that \( D_k \) has finite moments for any demand state \( k \). If \( \sigma \neq 0 \), then as \( L \to \infty \),
\[
\frac{D_k[t, t + L] - (L + 1)\mu}{\sigma \sqrt{L + 1}} \xrightarrow{\text{dist.}} N(0, 1),
\]
where \( \mu \) and \( \sigma \) are defined in (13) and (14), respectively.

Thus, when the leadtime \( L \) is sufficiently large, the leadtime demand \( D_k[t, t + L] \) can be approximated by a normal distribution with mean \((L + 1)\mu\) and variance \((L + 1)\sigma^2\), which does not depend on the initial demand state \( k \). Consequently, the myopic base-stock level can be approximated by the following formula:
\[
\bar{s}(k) \approx (L + 1)\mu + z^* \sigma \sqrt{L + 1}, \quad k = 1, \ldots, K,
\]
where \( z^* = \Phi^{-1}(b/(b + h)) \). In other words, all myopic base-stock levels \( \bar{s}(k) \) converge to the same value, as does \( s^{\min} \).

Next, consider a cyclic Markov chain \( W \). In this case, \( W \) does not have a stationary distribution. However, due to its cyclic nature, as the leadtime increases, the leadtime demands starting from different states share a growing common component. As a result, we can show that, for any two states \( k \) and \( k' \), the leadtime demand distributions \( F_{k,L}(y) \) and \( F_{k',L}(y) \) converge to the same limit for every \( y \) as \( L \) goes to infinity (see the proof of Proposition 2 in the Appendix). Thus, it remains true that all myopic base-stock levels \( \bar{s}(k) \) converge to the same value in this case. The following proposition summarizes our discussion above and further determines the convergence rate:

Proposition 2 Suppose that the Markov chain \( W \) either has a stationary distribution or is cyclic, and that \( D_k \) has finite moments for any demand state \( k \). For any demand state \( k = 1, \ldots, K \),
\[
\lim_{L \to \infty} \sqrt{L + 1} \frac{\bar{s}(k) - s^{\min}}{s^{\min}} = 0.
\]

The above proposition shows that the relative percentage gap between \( \bar{s}(k) \) and \( s^{\min} \) converges to zero as the leadtime \( L \) goes to infinity. The speed of convergence is at a rate of \( o\left(1/\sqrt{L + 1}\right) \). Thus, when replenishment leadtime is sufficiently large, we can use the close-form expression (15) to approximate \( s^*(k) \). Similarly, according to Corollary 2, the heuristic solution \( s^a(k) \) is guaranteed to be a close approximation to the optimal solution \( s^*(k) \) when the leadtime \( L \) is sufficiently long. Based on our numerical experiences, \( s^a(k) \) outperforms \( \bar{s}(k) \) in all cases. In addition, \( s^a(k) \) becomes a good approximation to \( s^*(k) \) in most cases, even under relatively short leadtime (see §5 for details).
3.5 A Two-State Example

To illustrate our bounds and heuristic solutions, let us consider a two-state case. For ease of exposition, we write the transition probabilities between the two state as \( p_{12} = p \) and \( p_{21} = q \), so the transition matrix of \( W \) is

\[
P = \begin{pmatrix} 1 - p & p \\ q & 1 - q \end{pmatrix}.
\]

Its stationary distribution is \( \pi = (q/(p+q), p/(p+q)) \). According to Corollary 1, the solution upper bounds are just the myopic base-stock levels \( \bar{s}(k) \), which solves (2). Without loss of generality, we assume \( \bar{s}(1) < \bar{s}(2) \). Thus, the optimal solution for state 1 is given by \( s^*(1) = \bar{s}(1) = s^{\min} = \bar{\pi}(1) = s^a(1) \). It remains to determine the solution lower bound for state 2, \( \underline{s}(2) \), and the heuristic solution \( s^a(k) \).

Given the two demand states, there are possible state sequences: \( m_1 = (1, 2) \) and \( m_2 = (2, 1) \). It is easy to verify that \( P_{m_1}^{(1)} = 1 - p \) and \( P_{m_2}^{(1)} = 1 - q \), so that \( \beta_1(m_1) = \alpha_1(m_1) = 1/p \) and \( \beta_1(m_2) = \alpha_1(m_2) = 1/q \). From Corollary 1, \( \underline{s}(2) \) is the solution of (11), which in this case reduces to

\[
q \cdot \partial_y G(1, y) + p \cdot \partial_y G(2, y) = 0.
\]

This is equivalent to

\[
\frac{q}{p+q} \cdot F_{1, L}(y) + \frac{p}{p+q} \cdot F_{2, L}(y) = \frac{b}{b+h} \quad \text{or} \quad F_{\pi, L}(y) = \frac{b}{b+h}.
\]

Comparing the above equation with (2), we can see that \( \underline{s}(2) \) is a newsvendor solution with an “effective” leadtime demand distribution being a weighted average of the leadtime demand distributions of states 1 and 2 (or the “stationary” leadtime demand distribution).

Next, we illustrate how our heuristic solution works. With the assumption \( \bar{s}(1) < \bar{s}(2) \), the heuristic sequence is \( \tilde{m} = (1, 2) \). Therefore, by equation (12), the heuristic solution for state 1, \( s^a(1) \), is given by \( \partial_y G(1, y) = 0 \), confirming the above statement. The heuristic solution for state 2, \( s^a(2) \), is given by

\[
\partial_y G(2, y) + \frac{q}{2p} \cdot F_2(y - \bar{s}(1)) (\partial_y G(1, y))^+ = 0,
\]

which is equivalent to

\[
F_{2, L}(y) + \frac{q}{2p} \cdot F_2(y - \bar{s}(1)) \left( F_{1, L}(y) - \frac{b}{b+h} \right)^+ = \frac{b}{b+h}. \tag{16}
\]

To gain further insight, we next assume the demand distributions for states 1 and 2 are exponential with mean \( 1/\theta_1 \) and \( 1/\theta_2 \), respectively, with \( \theta_1 > \theta_2 \). To obtain explicit expressions, we further assume the replenishment leadtime \( L = 0 \).
Under these assumptions, we have $F_k(y) = F_{k,0}(y) = 1 - e^{-\theta_k y}$ for $k = 1, 2$. It is easy to verify that $\partial_y G(k, y) = h - (h + b)e^{-\theta_k y}$. Hence $\bar{s}(k) = \ln(1 + b/h) / \theta_k$ for $k = 1, 2$, and $\bar{s}(1) < \bar{s}(2)$. Thus, the optimal solution to state 1 coincides with all the solution bounds and the heuristic solution, that is, $s^*(1) = \bar{s}(1) = s(1) = s^a(1) = \ln(1 + b/h) / \theta_1$.

It remains to determine the optimal solution for state 2. We know that $s^*(2) > s^*(1)$. From (3), using the Laplace transform technique described in Appendix B, we can show that $s^*(2)$ is determined by the following first-order condition:

$$\partial_y G^{[2]}(2, y) = h - (h + b)e^{-\theta_2 y} + \frac{qh}{p} \left(1 - \frac{p\theta_1 e^{-\theta_2 (y - s^*(1))} - \theta_2 e^{-p\theta_1 (y - s^*(1))}}{p\theta_1 - \theta_2}\right) = 0.$$ (17)

From (16), we can show that the heuristic $s^a(2)$ solves the following equation: for $y \geq s^*(1)$,

$$h - (h + b)e^{-\theta_2 y} + \frac{qh}{p} \left(1 - e^{-\theta_1 (y - s^*(1))}\right) \left(1 - e^{-\theta_2 (y - s^*(1))}\right) = 0.$$ (18)

The difference between the above two first-order conditions lies in the last terms. With Taylor expansion to the second order, we can show that the last terms in the above first-order conditions are given by

$$1 - \frac{(p\theta_1 e^{-\theta_2 (y - s^*(1))} - \theta_2 e^{-p\theta_1 (y - s^*(1))})}{p\theta_1 - \theta_2} \approx \frac{1}{2} p\theta_1 \theta_2 (y - s^*(1))^2 + o((y - s^*(1))^2),$$

and

$$\left(1 - e^{-\theta_1 (y - s^*(1))}\right) \left(1 - e^{-\theta_2 (y - s^*(1))}\right) \approx \frac{1}{2} \theta_1 \theta_2 (y - s^*(1))^2 + o((y - s^*(1))^2),$$

respectively. Therefore, we observe that $s^a(2)$ is a close approximation to $s^*(2)$ to the second order if $p$ is close to one (implying that if the demand is in state 1, it will jump from state 1 to state 2 with a high probability in the next period). The following proposition further characterizes how $s^a(2)$ and $s^*(2)$ are affected by the Markov chain transition probability $q$:

**Proposition 3** Under a two-state MMD with exponential density and zero leadtime, both the optimal solution $s^*(2)$ and the heuristic $s^a(2)$ are decreasing in transition probability $q$.

The above result shows that, when it is highly likely to transit from a high demand state to a low demand state, one should reduce the optimal inventory level for the high demand state, and our heuristic solution moves in the same direction as the optimal inventory level.

4. Serial Inventory Systems

In this section we extend the single-stage analysis to a serial inventory system with $N > 1$ stages. Specifically, random customer demand arises in every period at Stage 1, Stage 1 orders from Stage
2, ..., and Stage \(N\) orders from an external supplier with ample supply. We also call the external supplier Stage \(N+1\). If there is stockout at Stage 1, we assume the unmet demand is fully backlogged. The production-transportation leadtime from Stage \(n+1\) to Stage \(n\) is a constant of \(L_n\) periods. Let \(L'_n = \sum_{j=1}^{n} L_j\) denote the total leadtime from Stage 1 to Stage \(n\). At the end of each period, a unit (installation) inventory holding cost \(H_n\) is charged for on-hand inventories at Stage \(n\) (1 \(\leq n \leq N\)) and a unit backlog penalty cost \(b\) is charged for backlogs at Stage 1. As is standard in the literature, we define the echelon inventory holding cost at Stage \(n\) as \(h_n = H_n - H_{n+1} > 0\), with \(H_{N+1} = 0\). The demand process at Stage 1 follows the MMD process as specified in §3.

We assume that all the replenishment activities in a period happen at the beginning of the period after observing the demand state. At Stage \(n\) \((n > 1)\), the sequence of events is the following: an order from Stage \(n-1\) is received, an order is placed with Stage \(n+1\), a shipment is received from Stage \(n+1\), and then a shipment is sent to downstream Stage \(n-1\). For Stage 1, an order is placed at the beginning of the period, a shipment is received from Stage 2, and then the demand arrives during the period. The planning horizon is infinite and the objective is to minimize the long-run average cost of the inventory system. As in the single-stage problem, we assume the ordering cost is zero without loss of generality. In what follows, we shall assign a subscript \("n\) to the corresponding functions and variables for Stage \(n\).

### 4.1 Preliminaries

Let \(IL_n(t)\) denote the echelon inventory level at Stage \(n\) by the end of period \(t\), which includes the total on-hand inventory at Stages 1, ..., \(n\) plus the total inventory in transit to Stages 1, ..., \(n-1\) minus the backorders at Stage 1. Also let \(B(t)\) denote the backorder level at Stage 1 by the end of period \(t\). Then, the system inventory holding and penalty cost at the end of period \(t\) can be written as

\[
\sum_{n=2}^{N} H_n[IL_n(t) - IL_{n-1}(t)] + H_1[IL_1(t) + B(t)] + bB(t) = \sum_{n=1}^{N} h_n IL_n(t) + (b + H_1)B(t),
\]

where the righthand side is a sum of a sequence of echelon-related costs from Echelon 1 to \(N\). Consider first the Echelon 1 cost. Under the long-run average cost criterion, we can charge the cost of period \(t + L_1\) to period \(t\) without affecting the cost assessment. Specifically, given the demand state \(k\) and inventory position \(y\) at the beginning of period \(t\), the expected cost at the end of period
$t + L_1$ is given by

$$E\{h_1 LL_1(t + L_1) + (b + H_1)B(t + L_1)\}$$

$$= E\{h_1(y - D_k[t, t + L_1]) + (b + H_1)(D_k[t, t + L_1] - y)^+\}$$

$$= h_1 \cdot E\{(y - D_k[t, t + L_1])^+\} + (b + H_2) \cdot E\{(D_k[t, t + L_1] - y)^+\}.$$

Thus, we can define the single-period expected cost function at Echelon 1 as

$$G^1(k, y) = h_1 \cdot E\{(y - D_k[t, t + L_1])^+\} + (b + H_2) \cdot E\{(D_k[t, t + L_1] - y)^+\}. \quad (19)$$

Recall that when $N = 1$, $h_1 = H_1$ and $H_2 = 0$. Therefore, in the single-stage system, $G^1(k, y)$ reduces to $G(k, y)$ as defined in (1). It is known that the echelon decomposition technique of Clark and Scarf (1960) for the i.i.d. demand case can be extended to the MMD process. That is, the optimal echelon base-stock policy can be determined sequentially from Echelons 1 to $N$, with the aid of an induced penalty function between the successive echelons. Below we provide a sketch of the exact algorithm developed by Chen and Song (2001) for determining the optimal echelon base-stock policy.

### 4.1.1 Exact Algorithm

Start with Stage 1. Apply the $K$-iteration algorithm described in §3.1.1 to $G^1(k, y)$ defined in (19), and obtain the optimal echelon base-stock levels $s^*_n(k)$ for Stage 1. For Stage $n \geq 2$, first define the following induced penalty functions: for $k = 1, ..., K$,

$$G_{n-1,n}(k, y) = G^{[k]}_{n-1}(k, \min\{y, s^*_n(k)\}) - G^{[k]}_{n-1}(k, s^*_n(k)), \quad (20)$$

where $G^{[k]}_{n-1}(k, \cdot)$ are the functions obtained from the $K$-iteration algorithm for Stage $n - 1$ (see §3). With the induced penalty functions, further define the following cost functions for Stage $n$: for $i = 1, ..., K$,

$$G^i_n(k, y) = E\{h_n(y - D_k[t, t + L_n]) + G_{n-1,n}(W_k(t + L_n), y - D_k[t, t + L_n])\}, \quad (21)$$

where $W_k(t + L_n)$ is the demand state at the beginning of period $t + L_n$ given that the state is $k$ in period $t$. Now apply the $K$-iteration algorithm again to $G^1_n(k, y)$, and obtain the optimal echelon base-stock levels $s^*_n(k)$ for Stage $n$. Repeat the above procedure until reaching the last Stage $N$.

Similar to the expression (5) in the single-stage system, we can show that the derivative of the $G^{[k]}_n(k, y)$ function at Stage $n$ is given by

$$\partial_y G^{[k]}_n(k, y) = \partial_y G^1_n(k, y) + \sum_{i=1}^{[k]-1} \sum_{u \in U^*_n+i} p_{ku} \partial_y E\{R^i_n(u, y - D_k)\}. \quad (22)$$
The computation process for the optimal echelon inventory policy for the \( N \)-stage system requires \( N \times K \) iterations. Besides the difficulty in determining the function \( R_i^n \) at each iteration, computing the induced penalty function \( G_{n-1,n} \) for upstream stages as given in (20) presents an additional challenge. Following the same approach described in §3.2, we seek to derive bounds for the optimal policy by conducting a derivative analysis of the two key equations (20) and (21) in the above exact algorithm.

### 4.2 Derivative and Solution Bounds

For illustrative purposes, consider the cost function for Stage 2. From (20) and Proposition 1, we have

\[
\partial_y G_{1,2}(k, y) = \partial_y G_1^k(k, \min\{y, s_1^*(k)\}) \geq \partial_y G_1^1(k, \min\{y, s_1^*(k)\}) = \min\{0, \partial_y G_1^1(k, y)\}.
\]

Because \( H_1 \geq H_2 \),

\[
\partial_y G_1^1(k, y) = -(b + H_2) + (b + H_1) F_{k,L_1}(y) \geq -(b + H_2) + (b + H_2) F_{k,L_1}(y).
\]

Notice that the righthand side of the above inequality is always negative. It follows that

\[
\partial_y G_{1,2}(k, y) \geq -(b + H_2) + (b + H_2) F_{k,L_1}(y).
\]

Applying the above inequality of \( \partial_y G_{1,2}(k, y) \) to \( \partial_y G_2^1(k, y) \), we obtain

\[
\partial_y G_2^1(k, y) = h_2 + E\{\partial_y G_{1,2}(W_k(t + L_2), y - D_k[t, t + L_2])\} \\
\geq h_2 + E\{-(b + H_2) + (b + H_2) F_{W_k(t+L_2),L_1}(y - D_k[t, t + L_2])\} \\
= -(b + H_3) + (b + H_2) F_{k,L_2}(y).
\]

Here, the derivative passes through the expectation because \( \partial_y G_{1,2}(\cdot, y) \) is continuous in \( y \) and the last equality follows from the fact that \( E\{F_{W_k(t+L_2),L_1}(y - D_k[t, t + L_2])\} = F_{k,L_2}(y) \). By arguments analogous to those used in Proposition 1, we have

\[
\partial_y G_2^k(k, y) \geq -(b + H_3) + (b + H_2) F_{k,L_2}(y).
\]  \hspace{1cm} (23)

Repeating the above argument from Stage 2 to Stage \( n \), we can obtain a lower bound for \( \partial_y G_n^k(k, y) \). As in the single-stage system, deriving the derivative upper bound is more complex. We leave the details to the Appendix. The following result generalizes Proposition 1 to the serial system:

**Proposition 4** For any stage \( n = 1, \ldots, N \) and any state \( k = 1, \ldots, K \), the following holds:
(a) \( \partial_y G_n^{[k]}(k,y) \geq -(b + H_{n+1}) + (b + H_n)F_{k,L_n'}(y) \),

(b) \( \partial_y G_n^{[k]}(k,y) \leq -(b + H_{n+1}) + (b + H_1)F_{k,L_n'}(y) + (1 - p_{kk})\Delta_n(y) + \sum_{j=1}^{n-1} \Delta_{j,n}(k,y) \),

where \( \Delta_n(y) \) and \( \Delta_{j,n}(k,y) \) are recursively defined as follows: for \( n = 1, ..., N \),

\[
\Delta_n(y) = \max_{m \in \mathcal{S}} \sum_{i=1}^{K-1} \alpha_i(m) \left[ -(b + H_{n+1}) + (b + H_1)F_{m,\tilde{L}_n'}(y) + \sum_{j=1}^{n-1} \Delta_{j,n}(m_i,y) \right] + ,
\]

with \( \Delta_{j,n}(k,y) = E \left\{ \Delta_j \left( y - D_k[t,t+L_n'-L_j'] \right) \right\} \) for \( 1 \leq j \leq n - 1 \) and \( \alpha_i(m) \) given by (9).

From the above expression, it is straightforward to verify that \( \Delta_1(y) = \Delta(y) \) as defined in (10). Thus, the above result reduces to that of Proposition 1 when \( N = 1 \). When \( N > 1 \), there is an extra term \( \sum_{j=1}^{n-1} \Delta_{j,n}(k,y) \) on the righthand side of the inequality. This term captures the dependence of Echelon \( n \) on all the downstream base-stock levels from Echelons 1 to \( n - 1 \). Thus, computing the derivative upper bound in the serial system becomes more complex than that in the single-stage system.

When the demand process is i.i.d., i.e., \( K = 1 \), the \( \Delta \) terms in Proposition 4 (b) vanish. In this case, \( \partial_y G_n^{[k]}(k,y) \) is bounded below and above by simple newsvendor derivative functions. It is interesting to note that the solution bounds obtained from these simple newsvendor derivative functions coincide with those obtained by Shang and Song (2003). However, their single-stage approximation bounding approach does not extend to the MMD process to obtain the solution lower bound, whereas our derivative analysis approach yields general solution bounds for the MMD process (see Appendix C for a detailed discussion).

Based on Proposition 4, let \( \bar{s}_n(k) \) be the solution to the equation

\[
-(b + H_{n+1}) + (b + H_n)F_{k,L_n'}(y) = 0,
\]

and let \( s_n(k) \) be the solution to the equation:

\[
-(b + H_{n+1}) + (b + H_1)F_{k,L_n'}(y) + (1 - p_{kk})\Delta_n(y) + \sum_{j=1}^{n-1} \Delta_{j,n}(k,y) = 0. \tag{25}
\]

It follows that

**Corollary 3** For any stage \( n = 1, ..., N \) and any state \( k = 1, ..., K \), the following holds:

\[
\bar{s}_n(k) \leq s^*_n(k) \leq \bar{s}_n(k),
\]

where \( \bar{s}_n(k) \) and \( s_n(k) \) are determined by (24) and (25), respectively.
The above result generalizes the single-stage bound result given in Corollary 1 to the serial inventory system. While a similar solution upper bound has been shown in the literature (e.g., Shang 2012 for serial systems with independent but nonidentical demand), to our knowledge, the lower bound $s_n(k)$ is new to the literature. Moreover, our lower bound holds in general for any serial systems facing MMD, whereas the lower bounds proposed by Dong and Lee (2003, Theorem 6) and Shang (2012, Theorem 1) require assumptions to guarantee the echelon base-stock level is achievable in each period (which may not always hold under a state-dependent policy). We will provide numerical examples to illustrate this difference in §5.2. Finally, it is worth commenting that both the upper and lower bounds can be computed directly without solving the integral renewal equations required in the exact algorithm.

4.3 Heuristics

Observe that by ignoring the $\Delta$ terms in the derivative upper bound in Proposition 4, we obtain a newsvendor derivative function
\[-(b + H_{n+1}) + (b + H_1)F_{k,L_n'}(y) + \sum_{i=1}^{j-1} p_{k\tilde{m}_{n,i}} \cdot \beta_i(\tilde{m}_n) \cdot E\{R_n(\tilde{m}_{n,i}, y - D_k)\} \]
Then, it follows that $s_n(k) \leq s_n^u(k) \leq \bar{s}_n(k)$. Thus, $s_n^u(k)$ can serve as a simple heuristic solution for the problem.

We can further improve this heuristic based on the idea used in the single-stage problem in §3.3. Specifically, we can sort $s_n^u(k)$ in an increasing order and let $\tilde{m}_n = (\tilde{m}_{n,1}, ..., \tilde{m}_{n,K})$ denote the resulting order sequence of the states. Suppose that $k = \tilde{m}_{n,j}$. Based on the expression (22) and Lemma 2, we can make the following approximation for $\partial_y G_n^{[k]}(k, y)$:

\[
\partial_y G_n^{[k]}(k, y) \approx -(b + H_{n+1}) + (b + H_1)F_{k,L_n'}(y) + \sum_{i=1}^{j-1} p_{k\tilde{m}_{n,i}} \cdot \beta_i(\tilde{m}_n) \cdot E\{R_n(\tilde{m}_{n,i}, y - D_k)\} \\
\approx -(b + H_{n+1}) + (b + H_1)F_{k,L_n'}(y) + \sum_{i=1}^{j-1} p_{k\tilde{m}_{n,i}} \cdot \beta_i(\tilde{m}_n) \cdot E\{- (b + H_{n+1}) + (b + H_1)F_{\tilde{m}_{n,i},L_n'}(y - D_k)\} + \\
\approx -(b + H_{n+1}) + (b + H_1)F_{k,L_n'}(y) + \frac{1}{2} \sum_{i=1}^{j-1} p_{k\tilde{m}_{n,i}} \cdot \beta_i(\tilde{m}_n) \cdot F_k(y - s_n^u(\tilde{m}_{n,i})) \cdot \left(-(b + H_{n+1}) + (b + H_1)F_{\tilde{m}_{n,i},L_n'}(y)\right)^+,
\]
where the last step follows from the same integral approximation as we used in the single-stage problem in §3.3.
Next, let $s^a_n(k)$ denote a heuristic solution determined by the following equation:

$$
- (b + H_{n+1}) + (b + H_1)F_{k,L_n}(y)
+ \frac{1}{2} \sum_{i=1}^{j-1} p_{k\tilde{m}_{n,i}} \cdot \beta_i(\tilde{m}_n) \cdot F_k(y - s^a_n(\tilde{m}_{n,i})) \cdot (b + H_{n+1}) + (b + H_1)F_{\tilde{m}_{n,i},L_n}(y) = 0, \quad (27)
$$

where the above equation involves only a linear combination of simple newsvendor derivative functions of different demand states. Thus, solving the heuristic solution $s^a_n(k)$ only requires a direct evaluation of the leadtime demand distribution of different demand states. We have the following result:

**Corollary 4** For any stage $n = 1, \ldots, N$ and any state $k = 1, \ldots, K$, the following holds:

$$
\underline{s}_n(k) \leq s^a_n(k) \leq \bar{s}_n(k),
$$

where $\underline{s}_n(k)$ and $\bar{s}_n(k)$ are determined by (24) and (25), respectively.

Combining Corollaries 3 and 4, we observe that $s^a_n(k)$ and the optimal solution $s^*_n(k)$ fall in the same interval $[\underline{s}_n(k), \bar{s}_n(k)]$, suggesting that $s^a_n(k)$ can be a close approximation for the optimal solution $s^*_n(k)$ as long as the interval is tight. In the next section we identify sufficient conditions to ensure that such an interval is tight.

### 4.4 Asymptotic Analysis: The Effect of (Long) leadtime

From the definition of $\underline{s}_n(k)$ given in (25), we further define $s^\text{min}_n = \min_{k \in \{1, \ldots, K\}} s^a_n(k)$. By the MMD central limit theorem shown in Lemma 3, we can extend Proposition 2 to serial systems as follows:

**Proposition 5** Suppose that the Markov chain $W$ either has a stationary distribution or is cyclic, and that $D_k$ has finite moments for any demand state $k$. For any Stage $n = 1, \ldots, N$ and any demand state $k = 1, \ldots, K$,

$$
\limsup_{L'_n \to \infty} \sqrt{L'_n + 1} \cdot \frac{\bar{s}_n(k) - s^\text{min}_n}{s^\text{min}_n} = c
$$

where $c$ is a nonnegative constant.

The above proposition shows that the relative percentage gap between $\bar{s}_n(k)$ and $s^\text{min}_n$ converges to zero as the leadtime $L$ goes to infinity. Convergence occurs at a rate of $O\left(1/\sqrt{L'_n + 1}\right)$, where $L'_n$ is the total leadtime from Stage 1 to Stage $n$. Therefore, according to Corollary 4, the heuristic solution $s^a_n(k)$ is guaranteed to be a close approximation to the optimal solution $s^*_n(k)$ when the total
leadtime $L'_n$ is sufficiently long. In a serial inventory system, the total leadtime at an upstream stage is always longer than the total leadtime at a downstream stage. Thus, the above result suggests that the heuristic solution at the upstream stage tends to be a closer approximation to the optimal solution than does the heuristic solution at the downstream stage. In other words, at the upstream stages, we can safely replace the (complex) $N \times K$-iteration exact algorithm with our simple heuristic solution without much sacrificing system performance.

It is worth commenting that the above convergence result for the serial system is weaker than that for the single-stage system. The intuition is as follows: In the single-stage system, we have $s^{\text{min}} = \min_{k \in \{1, \ldots, K\}} \tilde{s}(k)$, whereas in the serial system we have a less tight lower bound $s^{\text{min}}_n = \min_{k \in \{1, \ldots, K\}} \underline{s}_n(k)$ to work with. This difference results in the weaker convergence rate in the serial system.

5. Numerical Results

In this section we conduct an extensive numerical study that evaluates the performance of our solution bounds and heuristic solution. We assume the demand distributions under different demand states all belong to the gamma distribution family. With gamma demand distribution, the optimal policy becomes computationally tractable by leveraging the Laplace transform technique (see the Appendix B for details). This allows us to evaluate the performance of our heuristic solutions by directly comparing them to the optimal solution.

For brevity, we present numerical results for four demand cases. The first and second cases involve three demand states, with each state following a gamma distribution. The parameters for these gamma distributions are $(n_i, \theta_i) = (1, 1), (2, 2)$, and $(2, 0.2)$, for $i = 1, 2, 3$, representing demand means of 1, 1, and 10, respectively. The transition matrices for these cases are given below:

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0.3 & 0.1 & 0.6 \\
0.5 & 0.2 & 0.3 \\
0.3 & 0.5 & 0.2
\end{pmatrix}.
\]

The third and fourth demand cases involve five demand states, with each state also following a gamma distribution. The parameters for these gamma distributions are $(n_i, \theta_i) = (1, 1), (2, 2), (3, 3), (2, 0.2)$, and $(2, 0.2)$ for $i = 1, 2, 3, 4, 5$, representing demand means of 1, 1, 1, 10, and 10, respectively. We set the demand distributions for states 4 and 5 to be the same in these two cases, to illustrate the fact that the optimal base-stock levels for these two states may be different (because the leadtime demand distributions starting from these two states may be different under MMD). The transition matrices for these two cases are given as follows:
Cyclic demand
\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
Noncyclic demand
\[
\begin{pmatrix}
0.10 & 0.10 & 0.20 & 0.40 & 0.20 \\
0.35 & 0.10 & 0.20 & 0.20 & 0.25 \\
0.10 & 0.20 & 0.20 & 0.10 & 0.40 \\
0.50 & 0.05 & 0.25 & 0.15 & 0.05 \\
0.30 & 0.05 & 0.25 & 0.30 & 0.10
\end{pmatrix}.
\]

5.1 Single-Stage Systems

We first consider single-stage systems. We set the unit holding cost \( h = 1 \) and the unit penalty cost \( b = 50 \), representing an implicit 98% fill rate. We further set the leadtime to be \( L = 0, 2 \) and 4. Thus, we have a total of 12 scenarios.

For each scenario, we compute four solutions: the optimal solution \( s^*(k) \), the lower bound \( s(k) \), the upper bound \( \bar{s}(k) \), and the heuristic \( s^h(k) \) defined by (12). We further compare the latter three solutions to the optimal solution using the following weighted relative solution error percentage metric. Specifically, for a given policy \( s(k) \), the weighted relative solution error percentage is defined as

\[
\text{Weighted relative solution error percentage} = \frac{\sum_{k=1}^{K} |s(k) - s^*(k)|}{\sum_{k=1}^{K} s^*(k)} \times 100%.
\]

We evaluate the long run average system costs under these policies through simulation. For each parameter scenario, we simulate the system for 50,000 periods and compute the average cost over these 50,000 periods. To reduce variance in the simulation results, we use the same random demand sample path for different policies under each scenario. For the myopic policy \( \bar{s}(k) \) and our proposed heuristic solution \( s^h(k) \), we compute the relative cost error percentage with respect to the cost under the optimal policy \( s^*(k) \). The numerical results are presented in Table 1 (for the three-state demand cases) and Table 2 (for the five-state demand cases).

From Tables 1 and 2, we make the following observations. First, our heuristic \( s^h(k) \) provides consistently better approximation to the optimal solution than the myopic solution \( \bar{s}(k) \), with lower percentages in both solution and system cost errors. Second, the solution upper bound \( \bar{s}(k) \) (which is also the myopic solution) is tighter than the solution lower bound \( s(k) \) in most cases, except for the three-state case with zero leadtime, suggesting the upper bound provides a better approximation to the optimal solution than does the lower bound. Third, in both the three-state and five-state demand cases, the base-stock levels for different states are significantly different when the leadtime is zero. However, as the leadtime increases, the base-stock levels for different states become close to each other. Moreover, as the leadtime increases, our heuristic \( s^h(k) \) provides a much closer approximation to the optimal solution, resulting in fewer than 1% solution and system
Cyclic demand | Noncyclic demand
---|---
| | L | k | \(s^*(k)\) | \(s(k)\) | \(\bar{s}(k)\) | \(s^a(k)\) | \(s^*(k)\) | \(s(k)\) | \(\bar{s}(k)\) | \(s^a(k)\)
0 | 1 | 3.69 | 3.28 | 3.93 | 3.73 | 3.89 | 3.33 | 3.93 | 3.90
| 2 | 2.92 | 2.92 | 2.92 | 2.92 | 2.92 | 2.92 | 2.92 | 2.92 | 2.92
| 3 | 23.23 | 20.96 | 29.28 | 27.00 | 23.84 | 20.92 | 29.28 | 26.82 | 26.82
| Soln error | - | 9.0% | 21.1% | 12.8% | - | 11.4% | 17.9% | 9.8%
| Cost error | 18.25 | - | 11.4% | 4.7% | 18.26 | - | 10.9% | 3.8%
| \(L\) | k | \(s^*(k)\) | \(s(k)\) | \(\bar{s}(k)\) | \(s^a(k)\) | \(s^*(k)\) | \(s(k)\) | \(\bar{s}(k)\) | \(s^a(k)\)
2 | 1 | 31.42 | 31.42 | 31.42 | 31.42 | 36.37 | 34.99 | 36.52 | 36.40
| 2 | 31.42 | 31.42 | 31.42 | 31.42 | 33.83 | 33.83 | 33.83 | 33.83
| 3 | 31.42 | 31.42 | 31.42 | 31.42 | 43.39 | 38.99 | 45.10 | 43.65
| Soln error | - | 0.0% | 0.0% | 0.0% | - | 5.1% | 1.6% | 0.3%
| Cost error | 24.86 | - | 0.0% | 0.0% | 31.72 | - | 0.9% | 0.1%
| \(L\) | k | \(s^*(k)\) | \(s(k)\) | \(\bar{s}(k)\) | \(s^a(k)\) | \(s^*(k)\) | \(s(k)\) | \(\bar{s}(k)\) | \(s^a(k)\)
4 | 1 | 33.55 | 33.55 | 33.55 | 33.55 | 52.60 | 51.21 | 52.76 | 52.63
| 2 | 47.70 | 41.94 | 48.70 | 48.70 | 50.09 | 50.09 | 50.09 | 50.09
| 3 | 46.18 | 41.97 | 48.74 | 46.69 | 59.14 | 54.80 | 60.70 | 59.31
| Soln error | - | 7.8% | 2.8% | 1.2% | - | 3.5% | 1.1% | 0.1%
| Cost error | 31.35 | - | 0.8% | 0.3% | 40.06 | - | 0.7% | 0.1%

Table 1: Comparison of policies in a single-stage system under the three-state demand cases.

cost errors. This confirms the insight derived from the convergence result in Proposition 2 that \(s^a(k)\) becomes a good approximation to the optimal solution when leadtime increases. Fourth, our heuristic \(s^a(k)\), as well as the myopic solution \(\bar{s}(k)\), tend to perform better under noncyclic demand than under cyclic demand. Under cyclic demand, our heuristic \(s^a(k)\) performs much better than \(\bar{s}(k)\), suggesting that the insight obtained in the two-state case in Proposition 3 is robust to cases with more than two states. Finally, it is interesting to note that under cyclic demand, when the leadtime satisfies the condition \((L + 1) \mod K = 0\), the base-stock levels are the same across all states. This is because the leadtime demand distributions are exactly the same from different states when the above condition is met.

To further illustrate the performance of our bounds and our heuristic solution, we consider the three-state demand case with leadtime \(L = 0\), which represents the least favorable scenario for our bounds and heuristic solution (see Proposition 2), with the following transition matrix:

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
q & 0 & 1 - q
\end{pmatrix},
\]

(28)

where we vary the transition probability \(q\) from 0.1 to 1. When \(q = 1\), the above transition matrix corresponds to the cyclic demand case shown in Table 1. For demand state 3, we plot the optimal
<table>
<thead>
<tr>
<th>L</th>
<th>k</th>
<th>$s^*(k)$</th>
<th>$g(k)$</th>
<th>$\bar{s}(k)$</th>
<th>$s^a(k)$</th>
<th>$s^*(k)$</th>
<th>$g(k)$</th>
<th>$s(k)$</th>
<th>$s^a(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>3.62</td>
<td>2.81</td>
<td>3.93</td>
<td>3.73</td>
<td>3.78</td>
<td>2.80</td>
<td>3.93</td>
<td>3.82</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2.90</td>
<td>2.60</td>
<td>2.92</td>
<td>2.89</td>
<td>2.92</td>
<td>2.59</td>
<td>2.92</td>
<td>2.91</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.51</td>
<td>2.51</td>
<td>2.51</td>
<td>2.51</td>
<td>2.51</td>
<td>2.51</td>
<td>2.51</td>
<td>2.51</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>27.42</td>
<td>15.10</td>
<td>29.28</td>
<td>29.28</td>
<td>24.17</td>
<td>15.71</td>
<td>29.28</td>
<td>27.01</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>21.71</td>
<td>15.10</td>
<td>29.28</td>
<td>27.00</td>
<td>25.00</td>
<td>15.37</td>
<td>29.28</td>
<td>27.50</td>
</tr>
<tr>
<td>Soln error</td>
<td>-</td>
<td>34.5%</td>
<td>16.8%</td>
<td>12.5%</td>
<td>-</td>
<td>33.2%</td>
<td>16.3%</td>
<td>9.2%</td>
<td></td>
</tr>
<tr>
<td>Cost error</td>
<td>18.34</td>
<td>-</td>
<td>12.1%</td>
<td>5.9%</td>
<td>18.75</td>
<td>-</td>
<td>8.7%</td>
<td>3.1%</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>6.50</td>
<td>6.50</td>
<td>6.50</td>
<td>6.50</td>
<td>36.83</td>
<td>35.86</td>
<td>36.92</td>
<td>36.85</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>31.35</td>
<td>21.63</td>
<td>31.35</td>
<td>31.35</td>
<td>35.25</td>
<td>35.25</td>
<td>35.25</td>
<td>35.25</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>43.85</td>
<td>33.33</td>
<td>46.58</td>
<td>46.58</td>
<td>36.49</td>
<td>35.76</td>
<td>36.57</td>
<td>36.48</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>41.56</td>
<td>33.35</td>
<td>46.63</td>
<td>44.59</td>
<td>45.01</td>
<td>38.48</td>
<td>46.71</td>
<td>45.37</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>27.59</td>
<td>21.69</td>
<td>31.42</td>
<td>29.14</td>
<td>46.99</td>
<td>38.97</td>
<td>49.05</td>
<td>47.61</td>
</tr>
<tr>
<td>Soln error</td>
<td>-</td>
<td>22.8%</td>
<td>7.7%</td>
<td>4.8%</td>
<td>-</td>
<td>8.1%</td>
<td>2.0%</td>
<td>0.5%</td>
<td></td>
</tr>
<tr>
<td>Cost error</td>
<td>27.28</td>
<td>-</td>
<td>3.2%</td>
<td>1.8%</td>
<td>34.77</td>
<td>-</td>
<td>0.5%</td>
<td>0.1%</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>48.69</td>
<td>48.69</td>
<td>48.69</td>
<td>48.69</td>
<td>54.60</td>
<td>53.49</td>
<td>54.72</td>
<td>54.62</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>48.69</td>
<td>48.69</td>
<td>48.69</td>
<td>48.69</td>
<td>52.79</td>
<td>52.79</td>
<td>52.79</td>
<td>52.79</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>48.69</td>
<td>48.69</td>
<td>48.69</td>
<td>48.69</td>
<td>54.10</td>
<td>53.32</td>
<td>54.18</td>
<td>54.08</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>48.69</td>
<td>48.69</td>
<td>48.69</td>
<td>48.69</td>
<td>61.86</td>
<td>55.76</td>
<td>63.27</td>
<td>62.09</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>48.69</td>
<td>48.69</td>
<td>48.69</td>
<td>48.69</td>
<td>63.73</td>
<td>56.22</td>
<td>65.52</td>
<td>64.18</td>
</tr>
<tr>
<td>Soln error</td>
<td>-</td>
<td>0.0%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>-</td>
<td>5.4%</td>
<td>1.2%</td>
<td>0.3%</td>
<td></td>
</tr>
<tr>
<td>Cost error</td>
<td>32.51</td>
<td>-</td>
<td>0.0%</td>
<td>0.0%</td>
<td>41.40</td>
<td>-</td>
<td>0.8%</td>
<td>0.1%</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Comparison of policies in a single-stage system under the five-state demand cases.

From Figure 1, we observe that as the transition probability $q$ increases (i.e., as the probability of jumping from a high demand state to a low demand state increases), the optimal base-stock level for the high demand state decreases, and our proposed bounds and heuristic follow the same decreasing trend as do the optimal solutions in both cases with $L = 0$ and 1. These numerical observations further suggest that the insight obtained in the two-state case in Proposition 3 is robust for MMD processes with more than two states and positive leadtimes. Moreover, when leadtime increases just one period from zero, our heuristic becomes a much closer approximation to the optimal solution.
5.2 Two-Stage Systems

In this section, we present numerical results for two-stage systems. We set the unit penalty cost again to be $b = 50$, and the unit echelon holding costs to be $h_1 = 2$ and $h_2 = 1$ for Stages 1 and 2, respectively. This echelon holding cost structure corresponds to unit installation holding costs of $H_1 = 3$ for Stage 1 and $H_2 = 1$ for Stage 2, representing the value-adding process of moving product from Stage 2 to Stage 1. We further set the leadtimes to be $(L_1, L_2) = (1, 1), (1, 3), \text{ and } (3, 1)$. Thus, we again have a total of 12 scenarios. For brevity, we will only present the results of the three-state demand cases below. The results of the five-state demand cases are included in the Appendix D, which offers essentially the same numerical insights as in the three-state demand cases.

For each scenario, we compute five solutions: the optimal solution $s^*_n(k)$, the lower bound $\underline{s}_n(k)$, the upper bound $\bar{s}_n(k)$, the newsvendor heuristic $s^v_n(k)$ defined by (26), and the enhanced heuristic $s^a_n(k)$ defined by (27). We further compare the latter four policies with the optimal solution using the following weighted relative solution error percentage metric. Specifically, for a given policy $s_n(k)$, the weighted relative solution error percentage is defined as

$$\text{Weighted relative solution error percentage} = \frac{\sum_{k,n} |s_n(k) - s^*_n(k)|}{\sum_{k,n} s^*_n(k)} \times 100\%.$$ 

We evaluate the system cost performance under the policies of $s^*_n(k)$, $s^v_n(k)$, and $s^a_n(k)$. For the heuristic policies $s^v_n(k)$ and $s^a_n(k)$, we compute the relative cost error percentage with respect to the cost under the optimal policy $s^*_n(k)$. The results for the three-state demand cases are summarized in Table 3.
From Table 3, we make the following observations. First, Dong and Lee (2003) and Shang (2012) proposed the solution $s^*_{kn}(k)$ as a lower bound for the optimal solution $s^*_{kn}(k)$ for a system with time-correlated demand and independent but nonidentical demand, respectively. However, from the above table, we can spot many cases in which $s^*_{kn}(k)$ is greater than $s^*_{kn}(k)$. Thus, this numerical observation confirms our earlier discussion that the lower bound proposed by Dong and Lee (2003) and Shang (2012) may not hold in general. However, based on the numerical results, the newsvendor heuristic $s^*_{kn}(k)$ does seem to provide a reasonable approximation to the optimal solution under MMD. This is a new finding that was not previously known in the literature. Second, we observe that the enhanced heuristic $s^a_{kn}(k)$ provides better approximation to the optimal solution than the newsvendor heuristic $s^v_{kn}(k)$, especially when the leadtime is short and the demand is cyclic.

<table>
<thead>
<tr>
<th>$(L_1, L_2)$</th>
<th>$n$</th>
<th>$k$</th>
<th>$s^v_{kn}(k)$</th>
<th>$g_n(k)$</th>
<th>$s^*_{kn}(k)$</th>
<th>$s^a_{kn}(k)$</th>
<th>$s^v_{kn}(k)$</th>
<th>$g_n(k)$</th>
<th>$s^*_{kn}(k)$</th>
<th>$s^a_{kn}(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>1</td>
<td>1</td>
<td>4.63</td>
<td>4.63</td>
<td>4.63</td>
<td>4.63</td>
<td>23.13</td>
<td>21.12</td>
<td>23.40</td>
<td>23.40</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
<td>24.79</td>
<td>19.70</td>
<td>26.45</td>
<td>26.45</td>
<td>19.03</td>
<td>19.03</td>
<td>19.03</td>
<td>19.03</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
<td>22.75</td>
<td>19.74</td>
<td>26.50</td>
<td>26.50</td>
<td>29.74</td>
<td>25.49</td>
<td>31.57</td>
<td>31.57</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>23.99</td>
<td>19.79</td>
<td>31.42</td>
<td>25.09</td>
<td>27.60</td>
<td>24.44</td>
<td>36.52</td>
<td>36.52</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>22.75</td>
<td>20.11</td>
<td>31.42</td>
<td>25.09</td>
<td>25.22</td>
<td>23.56</td>
<td>33.83</td>
<td>33.83</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>29.24</td>
<td>20.11</td>
<td>31.42</td>
<td>25.09</td>
<td>36.48</td>
<td>27.08</td>
<td>45.10</td>
<td>45.10</td>
</tr>
<tr>
<td>Soln error</td>
<td></td>
<td></td>
<td>-</td>
<td>-</td>
<td>18.6%</td>
<td>10.5%</td>
<td>-</td>
<td>12.7%</td>
<td>41.4%</td>
<td>2.4%</td>
</tr>
<tr>
<td>Cost error</td>
<td></td>
<td></td>
<td>59.43</td>
<td>-</td>
<td>-</td>
<td>2.7%</td>
<td>73.83</td>
<td>-</td>
<td>-</td>
<td>0.7%</td>
</tr>
<tr>
<td>(1, 3)</td>
<td>1</td>
<td>1</td>
<td>4.63</td>
<td>4.63</td>
<td>4.63</td>
<td>4.63</td>
<td>23.13</td>
<td>21.12</td>
<td>23.40</td>
<td>23.40</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
<td>24.79</td>
<td>19.70</td>
<td>26.45</td>
<td>26.45</td>
<td>19.03</td>
<td>19.03</td>
<td>19.03</td>
<td>19.03</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
<td>33.35</td>
<td>22.82</td>
<td>33.55</td>
<td>27.22</td>
<td>45.06</td>
<td>35.21</td>
<td>52.76</td>
<td>43.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>43.76</td>
<td>27.88</td>
<td>48.70</td>
<td>40.96</td>
<td>42.21</td>
<td>34.30</td>
<td>50.09</td>
<td>40.40</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>42.31</td>
<td>28.39</td>
<td>48.74</td>
<td>40.99</td>
<td>52.36</td>
<td>37.86</td>
<td>60.70</td>
<td>50.43</td>
</tr>
<tr>
<td>Soln error</td>
<td></td>
<td></td>
<td>-</td>
<td>-</td>
<td>28.2%</td>
<td>9.1%</td>
<td>-</td>
<td>18.2%</td>
<td>30.3%</td>
<td>3.7%</td>
</tr>
<tr>
<td>Cost error</td>
<td></td>
<td></td>
<td>67.46</td>
<td>-</td>
<td>-</td>
<td>2.7%</td>
<td>84.79</td>
<td>-</td>
<td>-</td>
<td>0.7%</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>1</td>
<td>1</td>
<td>28.63</td>
<td>28.60</td>
<td>28.63</td>
<td>28.63</td>
<td>39.32</td>
<td>37.95</td>
<td>39.48</td>
<td>39.48</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
<td>39.37</td>
<td>34.67</td>
<td>42.95</td>
<td>42.95</td>
<td>45.91</td>
<td>41.53</td>
<td>47.48</td>
<td>47.48</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>27.21</td>
<td>27.22</td>
<td>33.55</td>
<td>27.22</td>
<td>42.56</td>
<td>40.18</td>
<td>52.76</td>
<td>43.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>38.87</td>
<td>32.05</td>
<td>48.70</td>
<td>40.96</td>
<td>39.72</td>
<td>39.47</td>
<td>50.09</td>
<td>40.40</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>41.29</td>
<td>32.05</td>
<td>48.74</td>
<td>40.99</td>
<td>50.13</td>
<td>42.38</td>
<td>60.70</td>
<td>50.43</td>
</tr>
<tr>
<td>Soln error</td>
<td></td>
<td></td>
<td>-</td>
<td>-</td>
<td>10.2%</td>
<td>2.9%</td>
<td>-</td>
<td>6.3%</td>
<td>41.9%</td>
<td>1.2%</td>
</tr>
<tr>
<td>Cost error</td>
<td></td>
<td></td>
<td>82.35</td>
<td>-</td>
<td>-</td>
<td>0.6%</td>
<td>101.79</td>
<td>-</td>
<td>-</td>
<td>0.5%</td>
</tr>
</tbody>
</table>

Table 3: Comparison of policies in a two-stage system under the three-state demand cases.
This is consistent with the numerical observations in the single-stage system (see §5). Third, as the Stage 1 leadtime $L_1$ increases from one to three periods, the echelon base-stock levels for different states become close to each other, and our heuristic $s_n^a(k)$ provides a much better approximation to the optimal solution, resulting in less than 1% system cost errors. This confirms the insight derived from the convergence result in Proposition 5 that $s_n^a(k)$ can be a good approximation to the optimal solution under relatively short leadtime. Finally, we also observe some limitations on the enhanced heuristic. When the Stage 1 leadtime $L_1$ is shorter than the Stage 2 leadtime $L_2$, both the enhanced heuristic $s_n^a(k)$ and the newsvendor heuristic $s_n^v(k)$ have a cost deviation greater than 5% under the cyclic demand case. This observation suggests that, under the cyclic demand, when the leadtime at the downstream stage shortens, simple heuristics may not suffice to achieve near optimal system performance. Being able to compute the optimal policy can make a difference in this case.

To further illustrate the performance of our bounds and heuristic solution, we consider the three-state demand case with transition matrix given by (28). We vary the transition probability $q$ from 0.1 to 1. When $q = 1$, the transition matrix corresponds to the cyclic demand case. We set the leadtimes to be $(L_1, L_2) = (2, 1)$. For demand state 3, we plot the optimal solution $s_n^*(3)$, the lower bound $s_n(3)$, the upper bound $\bar{s}_n(3)$, the newsvendor heuristic $s_n^v(3)$, and the enhanced heuristic $s_n^a(3)$ for Stages 1 and 2 in Figure 2 (a) and (b), respectively.

![Figure 2: Impact of transition probability on the solutions in a two-stage system.](image)

From Figure 2, we observe that as the transition probability $q$ increases (i.e., the probability of jumping from a high demand state to a low demand state increases), the optimal base-stock level for the high demand state decreases, and our proposed bounds and heuristic follow the same
decreasing trend as the optimal solution. These numerical observations further suggest that the insight obtained in the two-state case in Proposition 3 is robust for serial systems. Moreover, we observe that the enhanced heuristic closely tracks the optimal solution as transition probability $q$ increases.

5.2.1 Internal Fill Rate

In serial systems with i.i.d. demand, under optimal inventory policy, the internal fill rate from an upstream stage to the immediate downstream stage is usually substantially smaller than the system fill rate (to the external demand). See, for example, Choi et al. (2004) and Shang and Song (2006) and the references therein. However, to our knowledge, there has been no discussion in the literature of this same issue for systems with MMD.

Since we are able to compute the optimal solutions for serial systems with MMD under the gamma demand distribution, it is interesting to design our numerical study to shed light on the internal fill rate under the optimal and heuristic policies (i.e., $s^*_a(k)$, $s^*_n(k)$, and $s^*_v(k)$). For this purpose, we simulate the two-stage inventory system for 50,000 periods to measure the realized fill rates at both stages. In each period, we measure the fill rate (= fulfilled order quantity/placed order quantity) at each stage and then average the fill rates across 50,000 periods. We present the results for three-state demand cases in Table 4. The numerical insights from the five-state demand cases are essentially the same as those of the three-state demand cases.

<table>
<thead>
<tr>
<th>$(L_1, L_2)$</th>
<th>Cyclic demand</th>
<th>Noncyclic demand</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$s^*_a(k)$</td>
<td>$s^*_n(k)$</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>1 88% 89% 89%</td>
<td>89% 90% 89%</td>
</tr>
<tr>
<td></td>
<td>2 88% 60% 77%</td>
<td>88% 85% 86%</td>
</tr>
<tr>
<td>(1, 3)</td>
<td>1 90% 90% 90%</td>
<td>89% 89% 88%</td>
</tr>
<tr>
<td></td>
<td>2 98% 87% 96%</td>
<td>93% 90% 89%</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>1 91% 92% 91%</td>
<td>92% 92% 92%</td>
</tr>
<tr>
<td></td>
<td>2 80% 77% 81%</td>
<td>78% 74% 75%</td>
</tr>
</tbody>
</table>

Table 4: Fill rates in a two-stage system under the three-state demand cases.

From Table 4, we observe that the fill rates under the enhanced heuristic $s^*_a(k)$ are close to those under the optimal policy $s^*_a(k)$, except for the cyclic demand case with leadtimes $(L_1, L_2) = (1, 1)$. In this case, as can be seen from Table 3, the Stage 2 heuristic policy is the same across different states (because the leadtime demand distributions are the same across different states), whereas the optimal policy varies depending on the state. This explains why the Stage 2 fill rate under the heuristic policies is significantly lower than that under the optimal policy.
We observe that, under the optimal policy, the Stage 2 fill rate need not be high, especially when the Stage 1 leadtime is relatively long. However, the difference between the internal and external fill rates is not as significant as what is observed under the i.i.d. demand. The behavior of the system also differs depending on whether \( W \) is cyclic. In the noncyclic demand case with leadtimes \((L_1, L_2) = (3, 1)\), the realized fill rate at Stage 2 is only 78%. However, when Stage 2 has longer leadtime than Stage 1, i.e., \((L_1, L_2) = (1, 3)\), the realized fill rate at Stage 2 is 98% under cyclic demand and 93% under noncyclic demand. This observation suggests that one may safely assume a high fill rate at Stage 2 to decouple the two-stage system and solve the inventory control problem for each stage separately when the downstream leadtime is shorter than the upstream leadtime (e.g., Abhyankar and Graves 2001). However, this assumption may be problematic when the downstream leadtime is longer than the upstream leadtime.

### 5.2.2 The Bullwhip Effect

The bullwhip effect, or the amplification of demand variability propagating from the downstream to the upstream stages of a supply chain, has been extensively studied in the literature (see Chen and Lee 2009, 2012 and the references therein). Most studies of the bullwhip effect have focused on either a single-stage system or a two-stage system with decentralized decision making. In a serial supply chain with centralized decision making, it is known that the optimal policy is a stationary echelon base-stock policy when demand is i.i.d. As a result, the replenishment order at each stage in each period is simply the replacement of the demand in the previous period, so there is no amplification of demand variability in such a system. However, when demand follows an MMD process, the optimal inventory policy is state-dependent, and it is not clear whether demand variability would be amplified or dampened in such a system. Our paper is the first to report numerical observations for this kind of system.

Specifically, we simulate the two-stage inventory system for 50,000 periods to measure the order variability amplification ratio (i.e., the bullwhip effect ratio) at both stages. We first calculate the sample variances of external demand, orders placed by Stage 1, and orders placed by Stage 2. Then, the bullwhip effect ratio for Stage 1 is determined by the ratio of Stage 1 order variance over the external demand variance, and the bullwhip effect ratio for Stage 2 (or more precisely echelon 2) is determined by the ratio of Stage 2 order variance over the external demand variance. We present the results of three-state demand cases in Table 5. The numerical insights of the five-state demand cases are essentially the same as those of the three-state demand cases.

From Table 5, we observe that the bullwhip effect ratios under both heuristics closely track
those under the optimal policy for the noncyclic demand. When the demand is cyclic, the bullwhip effect ratios under both heuristics tend to be higher than those under the optimal policy. This observation accords with our earlier observations that the heuristics tend to perform better when demand is noncyclic.

An interesting observation from the above table is that the bullwhip effect ratio under the optimal policy is less than one in most cases, indicating a sweeping demand variability dampening effect. Specifically, for noncyclic demand, the bullwhip effect ratios for Stages 1 and 2 are consistently below 0.70. To further investigate this phenomenon, we compute the autocorrelation of sequential demand under the noncyclic demand. We find that \( \rho(1) = -0.122, \rho(2) = -0.002, \rho(3) = 0.010, \rho(4) = -0.003, \) and \( \rho(5) = 0.000, \) where \( \rho(i) = \text{Corr}\{D(t), D(t+i)\}. \) Thus, there exists only a weak negative correlation among sequential demands, which cannot fully account for the large magnitude of the variability dampening effect. Chen and Lee (2012) showed that the existence of system capacity constraints can help dampen the bullwhip effect. In our system with MMD, there is no capacity constraint, yet we still observe that the bullwhip effect can be significantly dampened under the optimal policy, suggesting that the state-dependent inventory policy under MMD may have an inherent smoothing effect on the order variability. Finding the root cause for this intriguing observation is beyond the scope of this paper, and we shall leave it for future research.

6. Conclusion

We have addressed in this paper supply chain inventory management in a volatile demand environment driven by uncertain external factors. Our goal is to develop simple analytical tools and insights to cope effectively with dynamic demand uncertainties. To this end, we focused on simplifying the inventory decision tools for single-location and serial inventory systems with MMD.
This demand model is particularly attractive because the flexibility of Markov chains enables us to account for environmental uncertainties. Advanced information technology, such as hand-held devices, GPS, and social media, as well as abundant data associated with these technologies, makes it easier for such constructions. Real-time data also makes it easier to observe the demand state and, hence, to facilitate the implementation of MMD.

We make several contributions to the literature. First, we have developed simple solution bounds and approximations for these systems. Our derivative analysis approach enables us to obtain a general solution lower bound that does not require the restrictive assumptions used in previous works.

Second, we have demonstrated through numerical examples that our approximations are quite effective. These results greatly simplify computation and, hence, facilitate implementation. For instance, the bounds provide a convenient benchmark for the maximum and minimum inventory needed for each possible demand state, which is useful for budgeting purposes. The heuristics can serve as quick decision tools. Perhaps even more importantly, these simple solutions can be obtained by solving equations that depend on the primitive model data only, making them much more transparent than the exact optimal solution. In particular, they shed light on how the transition probability among the states (and thus the environmental uncertainty) affects the optimal inventory position.

Third, we have performed the first asymptotic analysis for these systems as the leadtime goes to infinity. We prove the central limit theorem under MMD and characterize the asymptotic mean and variance. We also show the convergence rate of our bounds and approximations to the optimal policy. These results help reveal the effect of leadtime on the optimal policy and provide analytical assessment of the effectiveness of our bounds and heuristic policies (i.e., the relative errors).

Fourth, to evaluate the performance of our solution bounds and heuristic policies, we use the gamma demand distribution family and show that, under this family, both the optimal solutions and the solution bounds can be computed easily by leveraging the Laplace transformation. To our knowledge, existing algorithms in the literature for serial inventory systems with MMD only address discrete demand (such as a Poisson distribution). With the gamma demand distribution, we have presented the first set of numerical studies of serial systems under MMD.

Finally, we observe numerically that the internal fill rate under MMD may not differ from the external fill rate as much as was observed in previous studies under stationary demand. We also observe that the optimal policy can dampen the bullwhip effect significantly. Both of these intriguing observations merit further investigation. It would be interesting to establish analytical
conditions under which a serial inventory system can be decoupled into separate individual single-stage systems, and to provide analytical explanations for why demand variability propagation in the system dampens under MMD. We hope our analysis in this paper will inspire more future research efforts to further our understanding of such a complex inventory system.

References


Appendices for Online Companion

Appendix A

Proofs of All Theoretical Results

Proof (Lemma 1) Prove by induction. For case \( i = 1 \), because \( G^1(\cdot, y) \) is convex in \( y \), it is straightforward that \( G^1(1^*, y) \) is increasing convex in \( y \). Thus, by definition (4), we have

\[
\partial_y R^1(1^*, y) = \partial_y G^1(1^*, y) + p_{1^*1} E \{ \partial_y R^1(1^*, y - D_1) \},
\]

where the derivative passes through expectation because \( \partial_y R^1(\cdot, y) \) is continuous in \( y \). Because the first term on the right-hand side is nonnegative, by a standard result of renewal equations (see Sgibnev 2001 Theorem 3, p. 1032), it follows that \( \partial_y R^1(1^*, y) \geq 0 \). Taking derivative on both sides of the equation, by the same argument, we obtain \( \partial_y^2 R^1(1^*, y) \geq 0 \). Therefore, \( R^1(1^*, y) \) is increasing convex in \( y \). Next, by the definition of \( G^2(\cdot, y) \), we have \( G^2(\cdot, y) \) is convex in \( y \). Hence, it follows that \( G^2(2^*, y) \) is increasing convex in \( y \). By repeating the above induction proof step, we obtain the desired result.

\[ \square \]

Proof (Lemma 2) We first prove

\[
\partial_y R^i(y) \leq \left( I^{(i)} - P^{(i)}_{m^*} \right)^{-1} \cdot e^i \cdot \partial_y G^i(i^*, y).
\]

For the ease of exposition, we only prove for the case of \( i = 2 \); the general case can be shown similarly by following exactly the same procedure. For \( i = 2 \), (8) can be expanded into

\[
(1 - p_{1^*1^*}) \partial_y R^2(1^*, y) - p_{1^*2^*} \partial_y R^2(2^*, y) \leq 0,
\]

\[
-p_{2^*1^*} \partial_y R^2(1^*, y) + (1 - p_{2^*2^*}) \partial_y R^2(2^*, y) \leq \partial_y G^2(2^*, y).
\]

Recall that \( \partial_y R^i(j^*, y) \) and \( \partial_y G^i(i^*, y) \) are always nonnegative, then it is easy to derive that

\[
\partial_y R^2(1^*, y) \leq \frac{p_{1^*2^*}}{(1 - p_{1^*1^*})(1 - p_{2^*2^*}) - p_{1^*2^*}p_{2^*1^*}} \partial_y G^2(2^*, y) = \frac{p_{1^*2^*}}{I^{(2)} - P^{(2)}_{m^*}} \partial_y G^2(2^*, y);
\]

\[
\partial_y R^2(2^*, y) \leq \frac{1 - p_{1^*1^*}}{(1 - p_{1^*1^*})(1 - p_{2^*2^*}) - p_{1^*2^*}p_{2^*1^*}} \partial_y G^2(2^*, y) = \frac{1 - p_{1^*1^*}}{I^{(2)} - P^{(2)}_{m^*}} \partial_y G^2(2^*, y).
\]

On the other hand, it’s easy to verify

\[
\left( I^{(2)} - P^{(2)}_{m^*} \right)^{-1} e^2 = \left( I^{(2)} - P^{(2)}_{m^*} \right)^{-1} \left( \begin{array}{c} p_{1^*2^*} \\ 1 - p_{1^*1^*} \end{array} \right),
\]

therefore we have shown

\[
\partial_y R^2(y) \leq \left( I^{(2)} - P^{(2)}_{m^*} \right)^{-1} \cdot e^2 \cdot \partial_y G^2(2^*, y),
\]

which completes the proof of (8). By lemma 1, \( \partial_y R^i(y) \) is always nonnegative, which immediately implies that \( \left( I^{(i)} - P^{(i)}_{m^*} \right)^{-1} \cdot e^i \geq 0. \)
Now note that
\[
(I^{(i)} - P^{(i)}_{m^*})^{-1} = \begin{pmatrix}
1 - p_{1^*1^*} & -p_{1^*2^*} & \cdots & -p_{1^*i^*} \\
-p_{2^*1^*} & 1 - p_{2^*2^*} & \cdots & -p_{2^*i^*} \\
\vdots & \ddots & \ddots & \vdots \\
-p_{i^*1^*} & \cdots & \cdots & 1 - p_{i^*i^*}
\end{pmatrix}^{-1}.
\]

Let \((a)^c\) denote the minor of an element \(a\) in \(I^{(i)} - P^{(i)}_{m^*}\). From the standard result of matrix inverse, we have
\[
(I^{(i)} - P^{(i)}_{m^*})^{-1} \cdot e^i = \left|I^{(i)} - P^{(i)}_{m^*}\right|^{-1} \begin{pmatrix}
(-1)^i+1(-p_{i^*1^*})^c \\
(-1)^i+2(-p_{i^*2^*})^c \\
\vdots \\
(1 - p_{i^*i^*})^c
\end{pmatrix},
\]

Now note that \(I^{(i)} - P^{(i)}_{m^*}\) is a diagonally dominant matrix (because \(1 - p_{i^*i^*} \geq \sum_{j=1}^{i-1} p_{i^*j^*}\)). By Brenner (1959, Corollary 1), it follows that
\[
\begin{pmatrix}
(-1)^i+1(-p_{i^*1^*})^c \\
(-1)^i+2(-p_{i^*2^*})^c \\
\vdots \\
(1 - p_{i^*i^*})^c
\end{pmatrix} \leq \begin{pmatrix}
(1 - p_{i^*1^*})^c \\
(1 - p_{i^*2^*})^c \\
\vdots \\
(1 - p_{i^*i^*})^c
\end{pmatrix},
\]

where we know \((1 - p_{i^*i^*})^c = \left|I^{(i-1)} - P^{(i-1)}_{m^*}\right|\). Therefore, we have \( (I^{(i)} - P^{(i)}_{m^*})^{-1} e^i \leq e \cdot \beta_i(m^*) \), and the desired result follows.

**Proof (Proposition 1)**

By applying (3) repeatedly and taking derivative with respect to \(y\), we have
\[
\partial_y G^{[k]}(k, y) = \partial_y G^1(k, y) + \sum_{l=2}^k \sum_{u \in U^l} p_{ku} \partial_y \left\{ R^{l-1}(u, y - D_k) \right\} 
\leq \partial_y G^1(k, y) + \sum_{l=2}^k \sum_{u \in U^l} p_{ku} \beta_{l-1}(m^*) \partial_y G^{l-1}((l - 1)^*, y) 
\leq \partial_y G^1(k, y) + \sum_{l=2}^K (1 - p_{kk}) \beta_{l-1}(m^*) \partial_y G^{l-1}((l - 1)^*, y),
\]

where the first inequality follows from Lemmas 1 and 2. Therefore we have
\[
\partial_y G^{[k]}(k, y) \leq \partial_y G(k, y) + M(k, y), \tag{A1}
\]

where
\[
M(k, y) = \sum_{l=2}^K (1 - p_{kk}) \beta_{l-1}(m^*) \partial_y G^{l-1}((l - 1)^*, y) = (1 - p_{kk}) \sum_{i=1}^{K-1} \beta_i(m^*) \partial_y G^i(i^*, y) \tag{A2}
\]
where \( \mathbf{m}^* \) denotes the optimal sequence. Furthermore, for \( i = 1, \ldots, K - 1 \),
\[
\partial_y G^i(i^*, y) = \max \{0, \partial_y G^i(i^*, y)\}
\]
\[
= \max \left\{ 0, \partial_y G^1(i^*, y) + \sum_{l=1}^{i-1} \sum_{k=1}^l \partial y E \left( R^l(k^*, y - D_{l^*}) \right) \right\}
\]
\[
\leq \left( \partial_y G^1(i^*, y) \right)^+ + \sum_{l=1}^{i-1} \sum_{k=1}^l \partial y R^l(k^*, y)
\]
\[
\leq \left( \partial_y G^1(i^*, y) \right)^+ + \sum_{l=1}^{i-1} \phi_{l,i}(\mathbf{m}^*) \partial_y G^l(l^*, y),
\] (A3)
where the last inequality follows from Lemma 2.

Define
\[
\partial_y \mathbf{G} = \left[ \partial_y G^1(1^*, y), \partial_y G^2(2^*, y), \ldots, \partial_y G^{K-1}(K-1^*, y) \right]^T,
\]
\[
(\partial_y \mathbf{G}^1)^+ = \left[ (\partial_y G^1(1^*, y))^+, (\partial_y G^1(2^*, y))^+, \ldots, (\partial_y G^1(K-1^*, y))^+ \right]^T,
\]
and
\[
\Psi = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\phi_{2,1}(\mathbf{m}^*) & 0 & \cdots & 0 \\
\phi_{3,1}(\mathbf{m}^*) & \phi_{3,2}(\mathbf{m}^*) & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots \\
\phi_{K-1,1}(\mathbf{m}^*) & \phi_{K-1,2}(\mathbf{m}^*) & \cdots & \phi_{K-1,K-2}(\mathbf{m}^*) & 0
\end{pmatrix}.
\]

Then (A3) can be rewritten in matrix form as:
\[
\partial_y \mathbf{G} \leq (\partial_y \mathbf{G}^1)^+ + \Psi \partial_y \mathbf{G},
\]
or equivalently,
\[
\left( \mathbf{I}^{(K-1)} - \Psi \right) \partial_y \mathbf{G} \leq (\partial_y \mathbf{G}^1)^+.
\] (A4)

Note that the matrix in (A4) is a lower triangular matrix with one on the diagonal, and hence its inverse must also be a lower triangular matrix with one on the diagonal. From the fact that \( \phi_{ij} \)'s are nonnegative, it is easy to verify that the matrix in (A4) is invertible and its inverse has nonnegative entries. Therefore we have
\[
\partial_y \mathbf{G} \leq \left( \mathbf{I}^{(K-1)} - \Psi \right)^{-1} (\partial_y \mathbf{G}^1)^+.
\]

Define \( \beta(\mathbf{m}^*) = [\beta_1(\mathbf{m}^*), \ldots, \beta_{K-1}(\mathbf{m}^*)] \). Then we have
\[
\sum_{i=1}^{K-1} \beta_i(\mathbf{m}^*) \partial_y G^i(i^*, y) = \beta(\mathbf{m}^*) \partial_y \mathbf{G} \leq \beta(\mathbf{m}^*) \left( \mathbf{I}^{(K-1)} - \Psi \right)^{-1} (\partial_y \mathbf{G}^1)^+ \\
\leq \alpha(\mathbf{m}^*)(\partial_y \mathbf{G}^1)^+ = \sum_{i=1}^{K-1} \alpha_i(\mathbf{m}^*) \left( \partial_y G^1(i^*, y) \right)^+,
\] (A5)
where \( \alpha(\mathbf{m}^*) = [\alpha_1(\mathbf{m}^*), \ldots, \alpha_{K-1}(\mathbf{m}^*)] \) with \( \alpha_i(\mathbf{m}^*) \) given by
\[
\alpha_i(\mathbf{m}^*) = \sum_{l=i+1}^{K-1} \alpha_l(\mathbf{m}^*) \phi_{l,i}(\mathbf{m}^*) + \beta_l(\mathbf{m}^*).
\]

40
Combining (A2) and (A5) yields
\[ M(k, y) \leq (1 - p_{kk}) \sum_{i=1}^{K-1} \alpha_i(m^*) (\partial_y G^1(i^*, y))^+ \]
\[ \leq (1 - p_{kk}) \max_{m \in \mathcal{S}} \sum_{i=1}^{K-1} \alpha_i(m) (\partial_y G(m, y))^+. \]  \hspace{1cm} (A6)

By (A1) and (A6), we have
\[ \partial_y G^i(k, y) \leq \partial_y G(k, y) + (1 - p_{kk}) \Delta(y), \]
where \( \Delta(y) \) is defined by (10). Thus, the proof is completed. \( \square \)

Proof (Corollary 1) \( \bar{s}(k) \leq s^*(k) \leq \bar{s}(k) \) is a direct result from Proposition 1. By the definition of \( s^{\text{min}} \), and noting that \( \partial_y G(k, y) \) and \( (\partial_y G(k, y))^+ \) are both increasing in \( y \), it is easy to verify that \( \partial_y G(k, s^{\text{min}}) + (1 - p_{kk}) \Delta(s^{\text{min}}) \leq 0 \), implying \( s^{\text{min}} \leq \bar{s}(k) \). \( \square \)

Proof (Corollary 2) By the definition of \( \alpha_i(m) \), we have \( \alpha_i(m) \geq \beta_i(m) \). Therefore,
\[ \partial_y G(k, y) \leq \partial_y G(k, y) + \frac{1}{2} \sum_{i=1}^{j-1} p_{k \bar{m}_i} \cdot \beta_i(\bar{m}) \cdot F_k(y - \bar{s}(\bar{m}_i)) (\partial_y G(\bar{m}_i, y))^+ \]
\[ \leq \partial_y G(k, y) + (1 - p_{kk}) \cdot \alpha_i(\bar{m}) (\partial_y G(\bar{m}_i, y))^+ \leq \partial_y G(k, y) + (1 - p_{kk}) \Delta(y), \]
which implies the desired result. \( \square \)

Proof (Lemma 3)
Let \( \pi^t_k \) denote the distribution of Markov chain states at period \( t \) given any initial state \( k \in \{1, ..., K\} \), with \( \pi^t_k = [\pi^t_k(1), ..., \pi^t_k(K)] \). Then by definition, \( \lim_{t \to \infty} \pi^t_k = \pi \). Let \( X^t_k \) denote the demand in period \( t \) given initial state \( k \), then
\[ X^t_k = \begin{cases} D^{(t)}_{1} & \text{with prob. } \pi^t_k(1), \\ D^{(t)}_{2} & \text{with prob. } \pi^t_k(2), \\ \vdots & \\ D^{(t)}_{K} & \text{with prob. } \pi^t_k(K), \end{cases} \]  \hspace{1cm} (A7)
where \( D^{(t)}_{j} \) is identically distributed with \( D_{i} \), and \( D^{(t_1)}_{i_1}, ..., D^{(t_n)}_{i_n} \) are independent of each other for any positive integer \( n \) and for any given set of \( \{i_1, ..., i_n\} \) and \( \{t_1, ..., t_n\} \), as long as \( t_j \neq t_{j'} \) for \( j \neq j' \).

Let us now consider another Markov chain with the same transition matrix but with stationary distribution from the very first period. Denote the MMD under this Markov chain at each period to be \( Y^t \). Then \( Y^1 = D^{(1)}_{i} \) with probability \( \pi(i) \). By the property of stationary distribution, we have \( Y^t = D^{(t)}_{i} \) with probability \( \pi(i) \). Therefore,
\[ E \{ Y^t \} = \sum_{i=1}^{K} E \{ Y^t | Y^t = D^{(t)}_{i} \} \pi(i) = \sum_{i=1}^{K} \mu_i \pi(i) = \mu. \]  \hspace{1cm} (A8)
Let $\tilde{Y}^t = Y^t - \mu$. Then we have $E\left\{\tilde{Y}^t\right\} = 0$, and it is easy to check that

$$E\left\{(\tilde{Y}^t)^2\right\} = \sum_{k=1}^{K} (\sigma_k^2 + \mu_k^2 - \mu^2) \pi(k) \text{ and } E\left\{\tilde{Y}^t \tilde{Y}^{t+s}\right\} = \sum_{k=1}^{K} \sum_{l=1}^{K} (\mu_k - \mu) \mu_k p_k^{(s)} \pi(k).$$

By construction, $\{\tilde{Y}^t\}$ is stationary. It is easy to verify that $\{\tilde{Y}^t\}$ is $\alpha$-mixing with $\alpha_s = K \rho^s$, where $A \geq 0$, $0 \leq \rho < 1$ (see Billingsley 1995, p. 363 for the definition of $\alpha$-mixing). Note that $\alpha_s$ is apparently $O(s^{-5})$. Define $\tilde{S}^L = \tilde{Y}^1 + \cdots + \tilde{Y}^{L+1}$, by Theorem 27.4 in Billingsley (1995, p. 364), we have, as $L \to \infty$,

$$(L + 1)^{-1} \text{var}\left\{\tilde{S}^L\right\} \to E\left\{(\tilde{Y}^1)^2\right\} + 2 \sum_{s=1}^{\infty} E\left\{\tilde{Y}^1\tilde{Y}^{1+s}\right\} = \sum_{k=1}^{K} [\sigma_k^2 + \mu_k^2 - \mu^2] \pi(k) + 2 \sum_{s=1}^{\infty} \sum_{k=1}^{K} \sum_{l=1}^{K} (\mu_k - \mu) \mu_k p_k^{(s)} \pi(k) = \sigma^2, \quad (A9)$$

where the series converge absolutely. If $\sigma > 0$, we further have, as $L \to \infty$,

$$\frac{\tilde{S}^L}{\sigma \sqrt{L + 1}} \xrightarrow{\text{dist.}} N(0, 1). \quad (A10)$$

Let us go back to the original Markov chain. Define $\tilde{X}_k^L = \tilde{X}_k^1 - \mu$ and $S_k^L = \tilde{X}_k^1 + \cdots + \tilde{X}_k^{L+1}$. For any $\epsilon > 0$, there exists $N_\epsilon$ such that $K \rho^N \leq \epsilon/3$ for $N > N_\epsilon$. Fix $N_\epsilon$, it is easy to check that the following results hold:

$$\lim_{L \to \infty} P\left(\frac{\tilde{S}^L}{\sigma \sqrt{L + 1}} \leq y\right) = \lim_{L \to \infty} P\left(\frac{\left(\sum_{i=1}^{N_\epsilon} \tilde{Y}^i\right) + \tilde{Y}^{N_\epsilon+1} + \cdots + \tilde{Y}^{L+1}}{\sigma \sqrt{L + 1}} \leq y\right)$$

$$= \lim_{L \to \infty} P\left(\frac{\tilde{Y}^{N_\epsilon+1} + \cdots + \tilde{Y}^{L+1}}{\sigma \sqrt{L + 1}} \leq y\right),$$

and similarly,

$$\lim_{L \to \infty} P\left(\frac{S_k^L}{\sigma \sqrt{L + 1}} \leq y\right) = \lim_{L \to \infty} P\left(\frac{\tilde{X}_k^{N_\epsilon+1} + \tilde{X}_k^{L+1}}{\sigma \sqrt{L + 1}} \leq y\right).$$

Therefore, there exists $L(\epsilon, N_\epsilon)$ such that for $L > L(\epsilon, N_\epsilon)$,

$$P\left(\frac{\tilde{S}^L}{\sigma \sqrt{L + 1}} \leq y\right) - P\left(\frac{\tilde{Y}^{N_\epsilon+1} + \cdots + \tilde{Y}^{L+1}}{\sigma \sqrt{L + 1}} \leq y\right) < \epsilon/3,$$

and

$$P\left(\frac{S_k^L}{\sigma \sqrt{L + 1}} \leq y\right) - P\left(\frac{\tilde{X}_k^{N_\epsilon+1} + \cdots + \tilde{X}_k^{L+1}}{\sigma \sqrt{L + 1}} \leq y\right) < \epsilon/3.$$
Note that
\[ P \left( \frac{\tilde{Y}_{N+1}^{L} + \tilde{Y}_{L+1}^{L}}{\sigma \sqrt{L + 1}} \leq y \right) \]
\[ = \sum_{i_{N+1}=1}^{K} \cdots \sum_{i_{L+1}=1}^{K} P \left( \frac{\tilde{Y}_{N+1}^{L} + \tilde{Y}_{L+1}^{L}}{\sigma \sqrt{L + 1}} \leq y \middle| Y_{N+1}^{L} = D_{i_{N+1}}, \ldots, Y_{L+1}^{L} = D_{i_{L+1}} \right) \]
\[ \times P \left( Y_{N+1}^{L} = D_{i_{N+1}}, \ldots, Y_{L+1}^{L} = D_{i_{L+1}} \right) \]
\[ = \sum_{i_{N+1}=1}^{K} \cdots \sum_{i_{L+1}=1}^{K} P \left( \frac{D_{i_{N+1}}^{(N+1)} + D_{i_{L+1}}^{(L+1)} - (L - N + 1)\mu}{\sigma \sqrt{L + 1}} \leq y \right) \cdot \pi(i_{N+1})p_{i_{N+1}i_{N+2}\cdots i_{L+L+1}} \]
and similarly,
\[ P \left( \frac{\tilde{X}_{k}^{N+1} + \tilde{X}_{k}^{L+1}}{\sigma \sqrt{L + 1}} \leq y \right) \]
\[ = \sum_{i_{N+1}=1}^{K} \cdots \sum_{i_{L+1}=1}^{K} P \left( \frac{D_{i_{N+1}}^{(N+1)} + D_{i_{L+1}}^{(L+1)} - (L - N + 1)\mu}{\sigma \sqrt{L + 1}} \leq y \right) \cdot p_{kN_{N+1}i_{N+1}i_{N+2}\cdots i_{L+L+1}} \]
Therefore,
\[ \left| P \left( \frac{\tilde{X}_{k}^{N+1} + \tilde{X}_{k}^{L+1}}{\sigma \sqrt{L + 1}} \leq y \right) - P \left( \frac{\tilde{Y}_{N+1}^{L} + \tilde{Y}_{L+1}^{L}}{\sigma \sqrt{L + 1}} \leq y \right) \right| \]
\[ \leq \sum_{i_{N+1}=1}^{K} \cdots \sum_{i_{L+1}=1}^{K} P \left( \frac{D_{i_{N+1}}^{(N+1)} + D_{i_{L+1}}^{(L+1)} - (L - N + 1)\mu}{\sigma \sqrt{L + 1}} \leq y \right) \]
\[ \times \left| p_{kN_{N+1}}^{(N+1)} - \pi(i_{N+1}) \right| p_{i_{N+1}i_{N+2}\cdots i_{L+L+1}} \]
\[ \leq \psi\psi^{N+1} \sum_{i_{N+1}=1}^{K} \cdots \sum_{i_{L+1}=1}^{K} P \left( \frac{D_{i_{N+1}}^{(N+1)} + D_{i_{L+1}}^{(L+1)} - (L - N + 1)\mu}{\sigma \sqrt{L + 1}} \leq y \right) \cdot p_{i_{N+1}i_{N+2}\cdots i_{L+L+1}} \]
\[ \leq K\psi^{N+1}, \]
where $0 < \psi < 1$, and the second to last inequality follows from Theorem 8.9 in Billingsley (1995, p. 131). Therefore, for $L > L(\epsilon, N_{\epsilon})$,
Since ε is chosen arbitrarily, we must have, \( \frac{S^L_k}{\sigma \sqrt{L+1}} \) and \( \frac{\tilde{s}^L_k}{\sigma \sqrt{L+1}} \) have the same asymptotic distribution. From (A10), the desired result follows. \( \square \)

**Proof (Proposition 2)** Let us first examine the case when the Markov chain has a stationary distribution. Let \( \eta = \frac{b}{b+h} \). Then \( \tilde{s}(k) \) is the solution of

\[
P \left( X^1_k + X^2_k + \ldots + X^{L+1}_k \leq \tilde{s}(k) \right) = \eta.
\]

The above equation can be further written as

\[
P \left( \frac{X^1_k + X^2_k + \ldots + X^{L+1}_k - (L+1)\mu}{\sqrt{L+1}\sigma} \leq \frac{\tilde{s}(k) - (L+1)\mu}{\sqrt{L+1}\sigma} \right) = \eta.
\]

(A11)

Let \( L \) goes to infinity, since the left-hand side of the inequality inside the parenthesis converges to \( N(0,1) \) in distribution, we must have, \( \lim_{L \to \infty} \frac{\tilde{s}(k) - (L+1)\mu}{\sqrt{L+1}\sigma} = \Phi^{-1}(\eta) \), where \( \Phi(\cdot) \) is the standard normal distribution function.

Then for different initial states \( k \) and \( k' \), we have:

\[
\sqrt{L+1} \frac{\tilde{s}(k) - \tilde{s}(k')}{\tilde{s}(k)} = \frac{(\tilde{s}(k) - \tilde{s}(k'))/(L+1)\sigma}{\tilde{s}(k)/(L+1)\sigma} = \frac{(\tilde{s}(k) - (L+1)\mu)/(L+1)\sigma}{\tilde{s}(k)/(L+1)\sigma}.
\]

(A13)

Note that \( \lim_{L \to \infty} \frac{\tilde{s}(k) - (L+1)\mu}{(L+1)\sigma} = 0 \). Therefore, we have

\[
\lim_{L \to \infty} \frac{\tilde{s}(k)}{(L+1)\sigma} = \frac{\mu}{\sigma}.
\]

Combining the above result with (A13), we conclude that, for any \( k \) and \( k' \),

\[
\lim_{L \to \infty} \sqrt{L+1} \cdot \frac{\tilde{s}(k) - \tilde{s}(k')}{\tilde{s}(k)} = 0,
\]

which implies \( \lim_{L \to \infty} \sqrt{L+1} \cdot \frac{\tilde{s}(k) - s^{\min}}{s^{\min}} = 0 \).

When the Markov chain is cyclic, for any given lead time \( L \), we can write \( L = mK + l \), where \( l = L \mod K \). The first \( mK \) periods of aggregated demand of \( D_k(L) \) and \( D_{k'}(L) \) are exactly the same. As \( L \) increases to infinity, the residual part is negligible compared with the first \( mK \) periods of aggregated demand. Therefore, it is easy to establish central limit theory for cyclic leadtime demand. With similar but much simpler analysis, the same convergence rate result can be obtained. We omit the detailed proof here. \( \square \)

**Proof (Proposition 3)** We shall follow the notations illustrated above Proposition 3. Denote \( s^* = s^*(1) \) for simplification. Because \( s^*(2) \geq s^* \), we can introduce the following change of variable: \( y = s^* + x \) with \( x \geq 0 \). Substituting this into (17), we obtain

\[
\frac{\partial_x G^2(2, s^* + x)}{p} = \frac{p + q_h}{p} \left( h + b + h \frac{q_{\theta_1}}{p \theta_1 - \theta_2} e^{-\theta_2(x+s^*)} + h \frac{q_{\theta_2}}{p^2 \theta_1 - p \theta_2} - e^{-\theta_1(x+s^*)} \right) e^{-\theta_2(x+s^*)} + h \frac{q_{\theta_2}}{p^2 \theta_1 - p \theta_2} e^{-\theta_1 x}
\]

= \frac{p + q_h}{p} \left( h + b \right) e^{-\theta_2(x+s^* - h \frac{q_{\theta_1}}{p \theta_1 - \theta_2} e^{-\theta_2 x} + h \frac{q_{\theta_2}}{p^2 \theta_1 - p \theta_2} e^{-\theta_1 x}}
\]

44
By substituting \( s^* = \ln(1 + b/h)/\theta_1 \) into the above equation we have
\[
\partial_x G^2(2, s^* + x) = \frac{p + q}{p} h - (h + b)e^{-\theta_2 x} \left( \frac{h}{h + b} \right)^{\theta_2/\theta_1} - h \frac{q \theta_1}{p \theta_1 - \theta_2} e^{-\theta_2 x} + h \frac{q \theta_2}{p^2 \theta_1 - p \theta_2} e^{-p \theta_1 x}.
\]
Further take derivative with respect to \( q \) of the above expression. We obtain
\[
\partial_q \left( \partial_x G^2(2, s^* + x) \right) = \frac{h}{p} - h \frac{\theta_1}{p \theta_1 - \theta_2} e^{-\theta_2 x} + h \frac{\theta_2}{p^2 \theta_1 - p \theta_2} e^{-p \theta_1 x}.
\]
Note that \( \partial_q \left( \partial_x G^2(2, s^* + x) \right) \mid_{x=0} = 0 \), \( \partial_q \left( \partial_x G^2(2, s^* + x) \right) \mid_{x=\infty} = h/p > 0 \), and
\[
\partial_x \left( \partial_q \left( \partial_x G^2(2, s^* + x) \right) \right) = \frac{h}{p} \frac{\theta_1 \theta_2}{p \theta_1 - \theta_2} \left( e^{-\theta_2 x} - e^{-p \theta_1 x} \right) \geq 0 \text{ for } x \geq 0.
\]
Therefore we must have \( \partial_q \left( \partial_x G^2(2, s^* + x) \right) \geq 0 \) for \( x \geq 0 \), which implies that \( s^2(2^*) \) decreases as \( q \) increases.

Next we examine our heuristic. Since \( s^a(2) \geq s^* \), we can also apply the above change of variable to (18). Therefore, \( s^a(2) \) is given by the solution to
\[
h - (h + b)e^{-\theta_2 (x+s^*)} + \frac{qh}{p} \cdot \frac{1 - e^{-\theta_2 x}}{2} = 0.
\]
Clearly the left-hand side of the above equation is increasing in \( q \), which implies that our heuristic solution for state 2 also decreases as \( q \) increases.

\( \square \)

**Proof (Proposition 4)** By repeatedly applying the procedure we used to obtain (23), we can prove by induction that,
\[
\partial_y G^{[k]}_n(k,y) \geq \partial_y G^1_n(k,y) \geq -(b + H_{n+1}) + (b + H_n)F_{k,L_n}(y).
\]
By applying (3) repeatedly, taking derivative with respect to \( y \), and applying Lemma 2, we have
\[
\partial_y G^{[k]}_n(k,y) = \partial_y G^1_n(k,y) + \sum_{i=2}^{[k]} \sum_{u \in U_i^y} \frac{p_{ku}}{\beta_i} E \left\{ R_{i-1}^n(u, y-D_k) \right\}
\leq \partial_y G^1_n(k,y) + \sum_{i=2}^{[k]} \sum_{u \in U_i^y} \frac{p_{ku}}{\beta_i} \beta_{i-1}(m^*) \partial_y G^{[i-1]}_n((i-1)^*, y)
\leq \partial_y G^1_n(k,y) + \sum_{i=2}^{K} \frac{1-p_{kk}}{\beta_{i-1}(m^*)} \partial_y G^{[i-1]}_n((i-1)^*, y), \tag{A14}
\]
where the last inequality utilizes the facts that \([k] \leq K\), transition probabilities \( p_{ku} \) sum up to 1, and \( \partial_y R_{n}^i(u,y) \) is increasing and nonnegative. For ease of exposition, denote \( v_j = W_k(t+L_{n-L_j-1}) \). By using the derivatives of (20) and (21), we have
\[
\partial_y G^1_n(k,y) = h_n + E \left\{ \partial_y G^1_{n-1,n} \left( v_n, y-D_k[t,t+L_n] \right) \right\}
= h_n + E \left\{ \partial_y G^1_{n-1} \left( v_n, \min \left\{ y-D_k[t,t+L_n], s_{n-1}^{[m]}(v_n) \right\} \right) \right\}
= h_n + E \left\{ \partial_y G^1_{n-1} \left( v_n, \min \left\{ y-D_k[t,t+L_n], s_{n-1}^{[m]}(v_n) \right\} \right) \right\}
\leq h_n + E \left\{ \partial_y G^1_{n-1} \left( v_n, y-D_k[t,t+L_n] \right) \right\}, \tag{A15}
\]
45
where the inequality utilizes the fact that the derivative of \( \partial_y G_n^{[v_n]}(v_n, y) \) is increasing in \( y \) (see Chen and Song 2001). By applying (3) to (A15), we further have

\[
\partial_y G_n^1(k, y) = h_n + E \left\{ \partial_y G_n^{1} (v_n, y - D_k[t, t + L_n]) \right\} \\
+ E \left\{ \sum_{i=2}^{[v_n]} \sum_{u \in U_{n-1}^i} p_{vn} E_{dn} \left[ \partial_y R_{n-1}^{i-1} (u, y - D_k[t, t + L_n] - D_{vn}) \right] \right\} \\
\leq h_n + E \left\{ \partial_y G_n^{1} (v_n, y - D_k[t, t + L_n]) \right\} \\
+ E \left\{ \sum_{i=2}^{[v_n]} \sum_{u \in U_{n-1}^i} p_{vn} \partial_y R_{n-1}^{i-1} (u, y - D_k[t, t + L_n]) \right\}.
\]  

(A16)

Define \( g_n(k, y) = -(b + H_{n+1}) + (b + H_{1}) F_{k, L_n}(y) \). By repeatedly applying the above techniques and using Lemma 2, we have

\[
\partial_y G_n^1(k, y) \\
\leq h_n + h_{n-1} + \ldots + h_2 + E \left\{ \partial_y G_n^{1} (v_2, y - D_k[t, t + L_n' - L_1]) \right\} \\
+ E \left\{ \sum_{j=2}^{[v_2]} \sum_{i=2}^{n} \sum_{u \in U_{j-1}^i} p_{v_2 u} \partial_y R_{j-1}^{i-1} (u, y - D_k[t, t + L_{n}' - L_{j-1}']) \right\} \\
= g_n(k, y) + E \left\{ \sum_{j=2}^{[v_2]} \sum_{i=2}^{n} \sum_{u \in U_{j-1}^i} p_{v_2 u} \partial_y R_{j-1}^{i-1} (u, y - D_k[t, t + L_{n}' - L_{j-1}']) \right\} \\
\leq g_n(k, y) + E \left\{ \sum_{j=2}^{[v_2]} \sum_{i=2}^{n} \sum_{u \in U_{j-1}^i} p_{v_2 u} \beta_{i-1} (m_{j-1}^*) \partial_y G_{j-1}^{i-1} ((i - 1)^*, y - D_k[t, t + L_{n}' - L_{j-1}']) \right\} \\
\leq g_n(k, y) + E \left\{ \sum_{j=2}^{[v_2]} \sum_{i=2}^{n} \beta_{i-1} (m_{j-1}^*) \partial_y G_{j-1}^{i-1} ((i - 1)^*, y - D_k[t, t + L_{n}' - L_{j-1}']) \right\},
\]  

(A17)

where the equality follows from \( v_2 = W_k(t+L_n'-L_1) \) and \( F_{v_2, L_1} (y - D_k[t, t + L_{n}' - L_1]) = F_{k, L_n}(y) \).

Combining (A14) and (A17) leads to

\[
\partial_y G_n^{[k]}(k, y) \leq g_n(k, y) + M_n(k, y),
\]

where

\[
M_n(k, y) = \sum_{j=1}^{n-1-K} \sum_{i=1}^{K-1} \beta_i (m_j^*) E \left[ \partial_y G_j^i (i^*, y - D_k[t, t + L_{n}' - L_j']) \right] + \sum_{i=1}^{K-1} (1 - p_{kk}) \beta_i (m_n^*) \partial_y G_n^i (i^*, y),
\]  

(A18)
with \( m_j^* \) being the optimal sequence in the \( j \)th stage. We will further relax (A18) as follows.

\[
\partial_y G_j^i(i^*, y) = \max \{ 0, \partial_y G_j^i(i^*, y) \}
\]

\[
= \max \left\{ 0, \partial_y G_j^1(i^*, y) + \sum_{l=1}^{i-1} \sum_{k=1}^l p_{i^*k} \partial_y E \left\{ R_j^l(k^*, y - D_{it}) \right\} \right\}
\]

\[
\leq \left( \partial_y G_j^1(i^*, y) \right)^+ + \sum_{l=1}^{i-1} \sum_{k=1}^l p_{i^*k} \partial_y R_j^l(k^*, y)
\]

\[
\leq \left( \partial_y G_j^1(i^*, y) \right)^+ + \sum_{l=1}^{i-1} \phi_{i,l}(m_j^*) \partial_y G_j^l(i^*, y).
\]  

(A19)

Define

\[
\partial_y G_j = \left[ \partial_y G_j^1(1^*, y), \partial_y G_j^2(2^*, y), ..., \partial_y G_j^K((K-1)^*, y) \right]^T,
\]

\[
(\partial_y G_j^1)^+ = \left[ (\partial_y G_j^1(1^*, y))^+, (\partial_y G_j^1(2^*, y))^+, ..., (\partial_y G_j^1((K-1)^*, y))^+ \right]^T,
\]

and

\[
\Psi_j = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\phi_{2,1}(m_j^*) & 0 & \cdots & 0 \\
\phi_{3,1}(m_j^*) & \phi_{3,2}(m_j^*) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{K-1,1}(m_j^*) & \phi_{K-1,2}(m_j^*) & \cdots & \phi_{K-1,K-2}(m_j^*) & 0
\end{pmatrix}.
\]

Then (A19) can be rewritten in matrix form as:

\[
\partial_y G_j \leq (\partial_y G_j^1)^+ + \Psi_j \partial_y G_j.
\]

With the same analysis as in the proof of Proposition 1, we can show

\[
\sum_{i=1}^{K-1} \beta_i(m_j^*) \partial_y G_j^i(i^*, y) = \beta(m_j^*) \cdot \partial_y G_j \leq \beta(m_j^*) \left( I^{(K-1)} - \Psi_j \right)^{-1} (\partial_y G_j^1)^+ \\
\triangleq \alpha(m_j^*) \cdot (\partial_y G_j^1)^+ = \sum_{i=1}^{K-1} \alpha_i(m_j^*) (\partial_y G_j^1(i^*, y))^+,
\]  

(A20)

where \( \alpha(m_j^*) = [\alpha_1(m_j^*), ..., \alpha_{K-1}(m_j^*)] \), with \( \alpha_i(m_j^*) \) given by

\[
\alpha_i(m_j^*) = \sum_{l=i+1}^{K-1} \alpha_l(m_j^*) \phi_{l,i}(m_j^*) + \beta_i(m_j^*).
\]

Combining (A18) and (A20) yields

\[
M_n(k, y) \leq \sum_{j=1}^{n-1} E \left\{ \sum_{i=1}^{K-1} \alpha_i(m_n^*) \left( \partial_y G_j^1(i^*, y - D_{it}, t + L_n' - L_j') \right)^+ \right\}
\]

\[
+ (1 - p_{kk}) \sum_{i=1}^{K-1} \alpha_i(m_n^*) \left( \partial_y G_n^1(i^*, y) \right)^+.
\]  

(A21)
By (A17) and Lemma 2, we have
\[
(\partial_y G_j^*(i^*, y))^+ \leq \max \left\{ 0, g_j(i^*, y) + \sum_{s=1}^{j-1} E \left\{ \sum_{l=1}^{K-1} \beta_l(m_s^*) \partial_y G_j^*(l^*, y - D_l^*[t, t + L_j^* - L_s^*]) \right\} \right\} \\
\leq \left( g_j(i^*, y) + \sum_{s=1}^{j-1} E \left\{ \sum_{l=1}^{K-1} \alpha_l(m_s^*) (\partial_y G_j^*(l^*, y - D_l^*[t, t + L_j^* - L_s^*]))^+ \right\} \right)^+.
\]

(A22)

Combining (A21) and (A22), we can verify that
\[
M_n(k, y) \leq (1 - p_{kk}) \Delta_n(y) + \sum_{j=1}^{n-1} \tilde{\Delta}_{j,n}(k, y)
\]
where \( \Delta_n(y) \) and \( \tilde{\Delta}_{j,n}(k, y) \) are defined in Proposition 4. Thus, we arrive at the desired result. □

**Proof (Corollary 3)** The proof follows the same logic as the proof for Corollary 1. □

**Proof (Corollary 4)** The proof follows the same logic as the proof for Corollary 2. □

**Proof (Proposition 5)** For ease of exposition, we provide the proof for the case of \( N = 2 \). The general \( N > 2 \) case can be proved following the same logic. Let us first consider the case when the Markov chain has a stationary distribution.

When \( L_1^* \) goes to infinity, we can directly apply Proposition 2 to obtain the result for Stage 1. So we only need to look at Stage 2. When \( L_2^* \) going to infinity, at least one of \( L_1 \) and \( L_2 \) goes to infinity. We shall prove the case of \( L_1 \) going to infinity while \( L_2 \) staying finite, and the other cases can be proved similarly.

By Proposition 4, \( \bar{s}_2(k) \) is the solution to \(- (b + H_3) + (b + H_2) F_{k,L_2^*}(y) = 0\), or equivalently,
\[
F_{k,L_2^*}(\bar{s}_2(k)) = \frac{b + H_3}{b + H_2}.
\]

(A24)

We next establish a further relaxation of the upper bound derivative (corresponding to the solution lower bound) in Proposition 4. Define the following functions recursively:
\[
\hat{\Delta}_n(y) = \bar{\alpha} \sum_{k=1}^{K} \left( (b + H_1) F_{k,L_n^*}(y) + \sum_{j=1}^{n-1} \tilde{\Delta}_{j,n}(k, y) \right)
\]
\[
\tilde{\Delta}_{j,n}(k, y) = E \left\{ \hat{\Delta}_j \left( y - D_k[t, t + L_n^* - L_j^*] \right) \right\},
\]
where \( \bar{\alpha} = \max_{i \in \{1, \ldots, K-1\}, m \in S} \alpha_l(m) \). It is easy to verify that \( \Delta_2(y) \leq \hat{\Delta}_2(y) \) and \( \tilde{\Delta}_{1,2}(k, y) \leq \hat{\Delta}_{1,2}(k, y) \). Then the upper bound derivative provided in Proposition 4 can be relaxed into
\[
\partial_y G_{2}^{[k]}(k, y) \leq -(b + H_3) + (b + H_1) F_{k,L_2^*}(y) + (1 - p_{kk}) \hat{\Delta}_2(y) + \tilde{\Delta}_{1,2}(k, y).
\]

Therefore, we obtain a relaxed lower bound solution, denoted by \( g_2^*(k) \), from solving the equation:
\[
-(b + H_3) + (b + H_1) F_{k,L_2^*}(y) + (1 - p_{kk}) \hat{\Delta}_2(y) + \tilde{\Delta}_{1,2}(k, y) = 0.
\]

(A26)
Now expand the terms $\hat{\Delta}_2(y)$ and $\hat{\Delta}_{1,2}(k, y)$ based on their definitions. We have

$$
\hat{\Delta}_1(y) = \hat{\alpha} \sum_{k=1}^{K} (b + H_1)F_{k, L_1}(y),
$$

$$
\hat{\Delta}_{1,2}(k, y) = E \left\{ \hat{\Delta}_1 \left( y - D_k[t, t + L_2' - L_1'] \right) \right\}
= \hat{\alpha}(b + H_1)E \left\{ \sum_{l=1}^{K} F_{l, L_1} \left( y - D_k[t, t + L_2] \right) \right\}
= \hat{\alpha}(b + H_1)\sum_{l=1}^{K} P \left( D_l[t, t + L_1] + D_k[t, t + L_2] \leq y \right),
$$

$$
\hat{\Delta}_2(y) = \hat{\alpha} \sum_{k=1}^{K} \left( (b + H_1)F_{k, L_2'}(y) + \hat{\Delta}_{1,2}(k, y) \right)
= \hat{\alpha}(b + H_1)\sum_{k=1}^{K} F_{k, L_2'}(y) + \hat{\alpha}^2(b + H_1)\sum_{k=1}^{K} \sum_{l=1}^{K} P \left( D_l[t, t + L_1] + D_k[t, t + L_2] \leq y \right).
$$

Since $L'_2$ goes to infinity, by the same argument as we used for Lemma 3, we have

$$
\frac{D_k[t, t + L_2'] - (L_2' + 1)\mu}{\sqrt{L'_2 + 1}\sigma} \xrightarrow{d} N(0, 1),
$$

where $\mu$ and $\sigma$ are given by (A8) and (A9), respectively. We also have

$$
\frac{D_l[t, t + L_1] + D_k[t, t + L_2] - (L_2' + 1)\mu}{\sqrt{L'_2 + 1}\sigma} = \frac{D_l[t, t + L_1] - (L_1 + 1)\mu + D_k[t, t + L_2] - L_2\mu}{\sqrt{L'_2 + 1}\sigma},
$$

(A27)

Recall that $L_1$ goes to infinity and $L_2$ stays finite, as $L'_2$ goes to infinity. We have

$$
\frac{D_l[t, t + L_1] - (L_1 + 1)\mu}{\sqrt{L'_2 + 1}\sigma} \xrightarrow{d} N(0, 1),
$$

(A28)

and

$$
\lim_{L'_2 \to \infty} \frac{D_k[t, t + L_2] - L_2\mu}{\sqrt{L'_2 + 1}\sigma} = 0.
$$

Combining (A27) and (A28) leads to

$$
\frac{D_l[t, t + L_1] + D_k[t, t + L_2] - (L_2' + 1)\mu}{\sqrt{L'_2 + 1}\sigma} \xrightarrow{d} N(0, 1).
$$

With similar arguments as we used in Proposition 2, we can obtain that the relaxed lower bound satisfies:

$$
\lim_{L'_2 \to \infty} \frac{s_2(k) - (L_2' + 1)\mu}{\sqrt{L'_2 + 1}\sigma} = \Phi^{-1}(\eta),
$$

where $\eta$ is a constant determined by $\hat{\alpha}, b, H_1, H_3$. Similarly, we can obtain from (A24) that,

$$
\lim_{L'_2 \to \infty} \frac{s_2(k) - (L_2' + 1)\mu}{\sqrt{L'_2 + 1}\sigma} = \Phi^{-1}(\psi),
$$

49
where $\psi = \frac{b + H_0}{\sigma + H_0}$. It is easy to verify that $\eta < \psi$. Therefore, we have, for any $k$ and $k'$,

$$\sqrt{L_2' + 1} \cdot \frac{s_2(k) - s_2'(k')}{s_2'(k')} = \sqrt{L_2' + 1} \cdot \frac{(s_2(k) - s_2'(k')) / \sqrt{L_2' + 1}}{s_2'(k') / \sqrt{L_2' + 1}}$$

$$= \frac{(s_2(k) - (L_2' + 1)\mu) / \sqrt{L_2' + 1} - (s_2'(k') - (L_2' + 1)\mu) / \sqrt{L_2' + 1}}{(L_2' + 1)^{-1/2} [(s_2'(k') - (L_2' + 1)\mu) / \sqrt{L_2' + 1}] + \mu / \sigma} \xrightarrow{L_2' \to \infty} \frac{\Phi^{-1}(\psi) - \Phi^{-1}(\eta)}{\mu / \sigma},$$

which implies that

$$\lim_{L_2' \to \infty} \sqrt{L_2' + 1} \frac{s_2(k) - s_2'_{\text{min}}}{s_2'_{\text{min}}} = \frac{\Phi^{-1}(\psi) - \Phi^{-1}(\eta)}{\mu / \sigma}.$$  

Since $s_2(k) \geq s_2'(k)$, by the definition of $s_2'_{\text{min}}$, we have

$$0 \leq \frac{s_2(k) - s_2'_{\text{min}}}{s_2'_{\text{min}}} \leq \frac{s_2(k) - \min_{k'} s_2'(k')}{\min_{k'} s_2'(k')}.$$  

Hence,

$$0 \leq \limsup_{L_2' \to \infty} \sqrt{L_2' + 1} \frac{s_2(k) - s_2'_{\text{min}}}{s_2'_{\text{min}}} \leq \limsup_{L_2' \to \infty} \sqrt{L_2' + 1} \frac{s_2(k) - \min_{k'} s_2'(k')}{\min_{k'} s_2'(k')} = \frac{\Phi^{-1}(\psi) - \Phi^{-1}(\eta)}{\mu / \sigma}.$$  

Therefore we can claim that

$$\limsup_{L_2' \to \infty} \sqrt{L_2' + 1} \frac{s_2(k) - s_2'_{\text{min}}}{s_2'_{\text{min}}} = c,$$

where $c$ is a nonnegative constant.

When the Markov chain is cyclic, we can apply the same argument as in Proposition 2 to obtain the convergence result. We omit the detailed proof here.  

\[\square\]
Appendix B

Gamma Demand Distribution and Laplace Transform

Recall from (3) and (4) that the exact algorithm requires convolutions involving the demand distributions $F_k$. In this section, we assume the demand distributions under different demand states all belong to the gamma distribution family. We show that, in this case, the optimal policy can be computed relatively easily by leveraging the Laplace transform technique.

Specifically, assume that the single-period demand $D_k$ under demand state $k$ follows a gamma density

$$f(x|n_k, \theta_k) = \frac{(\theta_k)^{n_k}x^{n_k-1}e^{-\theta_k x}}{\Gamma(n_k)}, \text{ with } x \geq 0,$$

where $n_k$ is the shape parameter and $\theta_k$ the rate parameter ($1/\theta_k$ is the scale parameter). It is easy to verify that $\mu_k = n_k/\theta_k$ and $\sigma_k = n_k/\theta_k^2$.

Denote the Laplace transform of $\partial_y R_i^j(k,y)$ by

$$r^i(k,\lambda) = \int_0^\infty e^{-\lambda y}dR_i^j(k,y),$$

and denote the Laplace transform of $\partial_y G_i^j(i^*,y)$ by

$$g^i(i^*,\lambda) = \int_0^\infty e^{-\lambda y}dG_i^j(i^*,y).$$

Given the optimal demand state sequence $m^* = (1^*, ..., K^*)$, define $r^i(\lambda) = [r^i(1^*,\lambda), ..., r^i(K^*,\lambda)]^T$, $g^i(\lambda) = [0, ..., 0, g^i(i^*,\lambda)]^T$. Because the Laplace transform of a gamma density $f(x|n_k, \theta_k)$ is given by $\frac{\theta_k^{n_k}/(\lambda + \theta_k)^{n_k}}{\Gamma(n_k)}$, the Laplace transform of $D_m^i(x)$ defined in (6) is given by

$$d_m^i(\lambda) = \left(\begin{array}{cccc}
\frac{\theta_1^{i^*}}{(\lambda + \theta_1^{i^*})^{n_1}} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \frac{\theta_K^{i^*}}{(\lambda + \theta_K^{i^*})^{n_K}}
\end{array}\right).$$

Under the Laplace transform, and noting that $I^i - d_m^i(\lambda)P_m^{i*}$ is strictly diagonal dominant, (7) can be simplified to

$$r^i(\lambda) = \left(I^i - d_m^i(\lambda)P_m^{i*}\right)^{-1}g^i(\lambda).$$

Here, because $d_m^i(\lambda)$ has a simple form, the inverse Laplace transform of $r^i(\lambda)$ can be fairly easily determined by the Laplace transform table or by standard software packages such as Matlab. Thus, we can obtain $\partial_y R_i^j(k,y)$ from the inverse Laplace transform, and substitute it into (3) to compute the optimal policy.

It is worth reiterating here that the computation of the solution bounds and heuristic in Corollary 2 does not require the function of $\partial_y R_i^j(k,y)$; it only involves evaluation of simple newsvendor derivative functions. Thus, unlike the optimal policy, the solution bounds and heuristic proposed in this paper can be computed for demand distributions other than the gamma distribution family.
Appendix C

Single-Stage Approximation under i.i.d. Demand

Shang and Song (2003) established solution bounds for a serial system with i.i.d. demand, which is a special case of our system. Our results relate to theirs as follows. Let $\Omega_N = (N, (h_i, L_i)_{i=1}^N, b, D)$ denote the original $N$-stage serial system. Shang and Song (2003) showed that under i.i.d. demand, the echelon problem is equivalent to an $n$-stage serial system $\Omega_n = (n, (h_i, L_i)_{i=1}^n, b + \sum_{i=n+1}^N h_i, D)$. That is, the echelon problem is the same as the original $N$-stage system truncated at stage $n$, except that the backorder cost rate is modified to be $b + \sum_{i=n+1}^N h_i$. They also showed that (the cost of) $\Omega_n$ is bounded by a lower bound system $\Omega_l_n = (n, (h_i, L_i)_{i=1}^n, b + \sum_{i=n+1}^N h_i, D)$, with $\bar{h}_n = h_n$ and $\bar{h}_i = 0$ for $i < n$, and an upper bound system $\Omega_u_n = (n, (\bar{h}_i, L_i)_{i=1}^n, b + \sum_{i=n+1}^N h_i, D)$, with $\bar{h}_n = \sum_{i=1}^n h_i$ and $\bar{h}_i = 0$ for $i < n$. They further showed that the optimal solution to the lower (upper) bound system is an upper (lower) bound for the optimal echelon base-stock solution of the original system. Because the two bounding systems are equivalent to single-stage systems, where the myopic solution is optimal, their solution bounds are simple single-stage newsvendor solutions, and one can regard these solutions as single-stage approximations.

We can adapt the same two bounding systems to our problem with MMD. Specifically, the derivative of the single-period cost function for demand state $k$ in the lower bound system $\Omega^l_n$ is given by

$$\partial_y G_n(k, y|\Omega^l_n) = -(b + \sum_{i=n+1}^N h_i) + \left(b + \sum_{i=n+1}^N h_i + \bar{h}_n \right) F_{k,L_i'}(y) = -(b + H_{n+1}) + (b + H_n) F_{k,L_i'}(y). \quad (A29)$$

And its counterpart in the upper bound system $\Omega^u_n$ is given by

$$\partial_y G_n(k, y|\Omega^u_n) = -(b + \sum_{i=n+1}^N h_i) + \left(b + \sum_{i=n+1}^N h_i + \bar{h}_n \right) F_{k,L_i'}(y) = -(b + H_{n+1}) + (b + H_1) F_{k,L_i'}(y). \quad (A30)$$

Observe that (A29) is exactly the same as the derivative lower bound in Proposition 4 (a). On the other hand, (A30) is only the first term in the derivative upper bound in Proposition 4 (b). Under the i.i.d. demand process, i.e., $K = 1$, the $\Delta$ terms in Proposition 4(b) vanish, and we recover the solution bound results in Shang and Song (2003). However, these solution bounds cannot be fully extended to the general MMD process: While the myopic solution to the lower bound system remains an upper bound for the optimal solution of the original system, the myopic solution to the upper bound system is no longer guaranteed to be a lower bound.
Appendix D

<table>
<thead>
<tr>
<th>$(L_1, L_2)$</th>
<th>n</th>
<th>k</th>
<th>$s_n^*(k)$</th>
<th>$s_{n+1}^*(k)$</th>
<th>$s_n^*(k)$</th>
<th>$s_{n+1}^*(k)$</th>
<th>$s_n^*(k)$</th>
<th>$s_{n+1}^*(k)$</th>
<th>$s_n^*(k)$</th>
<th>$s_{n+1}^*(k)$</th>
<th>$s_n^*(k)$</th>
<th>$s_{n+1}^*(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1)$</td>
<td></td>
<td></td>
<td>4.47</td>
<td>4.03</td>
<td>4.63</td>
<td>4.63</td>
<td>4.49</td>
<td>4.49</td>
<td>23.12</td>
<td>21.39</td>
<td>23.40</td>
<td>23.40</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>3.88</td>
<td>3.88</td>
<td>3.88</td>
<td>3.88</td>
<td>20.01</td>
<td>20.01</td>
<td>20.01</td>
<td>20.01</td>
<td>20.01</td>
<td>20.01</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3</td>
<td>26.44</td>
<td>13.17</td>
<td>26.44</td>
<td>26.44</td>
<td>22.03</td>
<td>20.92</td>
<td>22.22</td>
<td>22.22</td>
<td>22.05</td>
<td>22.05</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3</td>
<td>34.21</td>
<td>23.74</td>
<td>40.85</td>
<td>40.85</td>
<td>38.84</td>
<td>30.15</td>
<td>31.58</td>
<td>31.58</td>
<td>30.48</td>
<td>30.48</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>2</td>
<td>20.65</td>
<td>13.21</td>
<td>26.50</td>
<td>26.50</td>
<td>24.25</td>
<td>32.91</td>
<td>24.98</td>
<td>35.04</td>
<td>35.04</td>
<td>33.61</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3.14</td>
<td>5.09</td>
<td>6.50</td>
<td>5.40</td>
<td>5.40</td>
<td>28.97</td>
<td>24.83</td>
<td>36.92</td>
<td>28.98</td>
<td>28.91</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>25.02</td>
<td>6.50</td>
<td>31.35</td>
<td>25.03</td>
<td>25.03</td>
<td>28.18</td>
<td>24.47</td>
<td>35.25</td>
<td>27.31</td>
<td>27.31</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>34.15</td>
<td>7.37</td>
<td>46.58</td>
<td>38.85</td>
<td>38.85</td>
<td>29.63</td>
<td>24.75</td>
<td>36.57</td>
<td>28.60</td>
<td>28.51</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td>37.45</td>
<td>7.41</td>
<td>46.63</td>
<td>38.89</td>
<td>36.90</td>
<td>38.27</td>
<td>25.96</td>
<td>46.71</td>
<td>38.02</td>
<td>36.76</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>5</td>
<td>27.34</td>
<td>6.72</td>
<td>31.42</td>
<td>25.00</td>
<td>22.88</td>
<td>39.98</td>
<td>26.09</td>
<td>49.05</td>
<td>40.13</td>
<td>38.75</td>
</tr>
<tr>
<td>Soln error</td>
<td></td>
<td></td>
<td>1  - 58.5%</td>
<td>20.5%</td>
<td>9.9%</td>
<td>8.5%</td>
<td>-</td>
<td>19.0%</td>
<td>14.8%</td>
<td>2.2%</td>
<td>2.0%</td>
<td>0.6%</td>
</tr>
<tr>
<td>Cost error</td>
<td></td>
<td></td>
<td>66.46</td>
<td>-</td>
<td>-</td>
<td>4.3%</td>
<td>2.2%</td>
<td>79.23</td>
<td>-</td>
<td>-</td>
<td>0.6%</td>
<td>0.6%</td>
</tr>
<tr>
<td>$(L_1, L_2)$</td>
<td>n</td>
<td>k</td>
<td>$s_n^*(k)$</td>
<td>$s_{n+1}^*(k)$</td>
<td>$s_n^*(k)$</td>
<td>$s_{n+1}^*(k)$</td>
<td>$s_n^*(k)$</td>
<td>$s_{n+1}^*(k)$</td>
<td>$s_n^*(k)$</td>
<td>$s_{n+1}^*(k)$</td>
<td>$s_n^*(k)$</td>
<td>$s_{n+1}^*(k)$</td>
</tr>
<tr>
<td>$(1, 3)$</td>
<td></td>
<td></td>
<td>4.47</td>
<td>4.03</td>
<td>4.63</td>
<td>4.63</td>
<td>4.49</td>
<td>4.49</td>
<td>23.12</td>
<td>21.39</td>
<td>23.40</td>
<td>23.40</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3.88</td>
<td>3.88</td>
<td>3.88</td>
<td>3.88</td>
<td>20.01</td>
<td>20.01</td>
<td>20.01</td>
<td>20.01</td>
<td>20.01</td>
<td>20.01</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4</td>
<td>26.44</td>
<td>13.17</td>
<td>26.44</td>
<td>26.44</td>
<td>22.03</td>
<td>20.92</td>
<td>22.22</td>
<td>22.22</td>
<td>22.05</td>
<td>22.05</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3</td>
<td>34.21</td>
<td>23.74</td>
<td>40.85</td>
<td>40.85</td>
<td>38.84</td>
<td>30.15</td>
<td>31.58</td>
<td>31.58</td>
<td>30.48</td>
<td>30.48</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>2</td>
<td>20.65</td>
<td>13.21</td>
<td>26.50</td>
<td>26.50</td>
<td>24.25</td>
<td>32.91</td>
<td>24.98</td>
<td>35.04</td>
<td>35.04</td>
<td>33.61</td>
</tr>
<tr>
<td>Soln error</td>
<td></td>
<td></td>
<td>1  - 36.8%</td>
<td>12.2%</td>
<td>11.6%</td>
<td>10.1%</td>
<td>-</td>
<td>25.8%</td>
<td>10.3%</td>
<td>4.9%</td>
<td>4.9%</td>
<td>5.1%</td>
</tr>
<tr>
<td>Cost error</td>
<td></td>
<td></td>
<td>72.90</td>
<td>-</td>
<td>-</td>
<td>7.7%</td>
<td>6.2%</td>
<td>89.63</td>
<td>-</td>
<td>-</td>
<td>1.0%</td>
<td>1.2%</td>
</tr>
<tr>
<td>$(3, 1)$</td>
<td></td>
<td></td>
<td>28.57</td>
<td>28.57</td>
<td>28.57</td>
<td>28.57</td>
<td>40.79</td>
<td>39.58</td>
<td>40.93</td>
<td>40.93</td>
<td>40.82</td>
<td>40.82</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>42.53</td>
<td>31.96</td>
<td>42.90</td>
<td>42.90</td>
<td>38.81</td>
<td>38.81</td>
<td>38.81</td>
<td>38.81</td>
<td>38.81</td>
<td>38.81</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4</td>
<td>41.77</td>
<td>31.97</td>
<td>42.94</td>
<td>42.94</td>
<td>40.22</td>
<td>39.39</td>
<td>40.32</td>
<td>40.32</td>
<td>40.20</td>
<td>40.20</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3</td>
<td>40.17</td>
<td>31.97</td>
<td>42.95</td>
<td>42.95</td>
<td>47.86</td>
<td>41.83</td>
<td>49.22</td>
<td>49.22</td>
<td>48.08</td>
<td>48.08</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>2</td>
<td>28.57</td>
<td>28.57</td>
<td>28.57</td>
<td>28.57</td>
<td>49.79</td>
<td>42.30</td>
<td>51.57</td>
<td>51.57</td>
<td>50.24</td>
<td>50.24</td>
</tr>
<tr>
<td>Soln error</td>
<td></td>
<td></td>
<td>1  - 18.2%</td>
<td>9.9%</td>
<td>3.3%</td>
<td>2.8%</td>
<td>-</td>
<td>9.8%</td>
<td>11.5%</td>
<td>1.1%</td>
<td>1.0%</td>
<td>0.4%</td>
</tr>
<tr>
<td>Cost error</td>
<td></td>
<td></td>
<td>87.90</td>
<td>-</td>
<td>-</td>
<td>0.6%</td>
<td>0.6%</td>
<td>108.59</td>
<td>-</td>
<td>-</td>
<td>0.4%</td>
<td>0.1%</td>
</tr>
</tbody>
</table>

Table 6: Comparison of policies in a two-stage system under the five-state demand cases.