A Quantitative Dynamic Agency Model of Financing Constraints

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October 2012 †

Abstract
Recent theoretical research in corporate finance has highlighted the role of incentive problems between firms’ investors and insiders in determining corporations’ financial structures, dynamics and investment policies. Financial contracts are designed to mitigate these agency conflicts. However, most of the analysis is qualitative in nature. This paper develops a dynamic firm model in order to study the quantitative and empirical implications of optimal long-term contracts for firms’ policies and returns. Using a parsimonious representation of agency conflicts between firms’ outsiders and insiders, the paper embeds a dynamic contracting problem into a neoclassical model of firm dynamics. It characterizes the optimal contract using recursive techniques and then quantitatively evaluates its implications for firm financing, investment and returns. Remarkably, the empirical predictions for optimal firm behavior and return patterns under optimal long-term contracts can differ considerably from models in which financing constraints arise from commonly used reduced form specifications of costs of external finance. This

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†I have benefited from suggestions and discussions with Andrew Abel, Hal Cole, Jean-Pierre Danthine, Bernard Dumas, Antonio Falato, João Gomes, Urban Jermann, Guillaume Plantin, Vincenzo Quadrini, Adriano Rampini, Toni Whited and seminar participants at Duke University, London Business School, University of Wisconsin-Madison, Western Finance Association and American Economic Association. All errors are mine.
suggests a resolution for several controversies in the literature on quantitative implications of firms' financing constraints.
1 Introduction

A large literature in corporate finance has analyzed the impact of firms’ financial structures on firm value, firm dynamics and investment. While such effects can be dismissed in the frictionless Modigliani-Miller (MM henceforth) benchmark, they become of paramount importance in a world with capital market imperfections. Indeed, in the fifty years that have elapsed since Modigliani and Miller’s seminal contribution, corporate finance theory has studied in depth the implications of a long list of departures from this benchmark.¹ A more recent literature has started to investigate the effects of such financing frictions quantitatively.² By and large, this literature has concentrated on evaluating the quantitative implications of a limited number of very specific departures from MM, namely almost exclusively the presence of taxes and bankruptcy costs. While taxes are undoubtedly an important determinant of firms’ financial structures and the ensuing dynamic trade-off models are empirically successful, recent theoretical work in corporate finance has stressed another source of deviations from MM: incentive problems. The theoretical literature on financial contracting has documented how various agency conflicts between firms’ investors and insiders can be crucial determinants of firms’ financial structures, investment policies and dynamics.³ However, the analysis has remained mostly qualitative in nature. An important challenge therefore is to quantitatively assess the implications of such mechanisms. The aim of this paper is to take a step into this direction. To that end, it embeds a specific incentive problem, namely an enforcement friction, into a dynamic neoclassical setting and studies its quantitative implications. More specifically, it asks: Are dynamic agency models quantitatively consistent with a number of stylized facts concerning firm dynamics? Can the empirical implications of dynamic agency models be distinguished from the implications of other classes of models in quantitative corporate finance, namely tax-based trade-off models and reduced form models of costly external finance? The paper examines these questions in the context of the literature on financing constraints and documents that the

¹See e.g. the surveys of Harris and Raviv (1991) and Zingales (2000)
³Examples are asymmetric information, moral hazard or limited enforcement of contracts. See Hart (2001) for a survey.
results suggest affirmative answers to them.

In the model, entrepreneurs have valuable investment opportunities, but need to sign financing contracts with outside investors because they have limited funds. An agency conflict arises due to limited enforcement of financial contracts. More precisely, entrepreneurs have the option to divert capital for their own private benefit when it is in their best interest to do so. In this setting, optimal financial contracts provide optimal financing for the entrepreneur while assuring that diversion is never optimal. Such a contract then specifies, as a function of a firm’s history, both how profits are split between investors and entrepreneurs as well as the level of capital investment. The paper determines the properties of the optimal contract using recursive dynamic programming techniques. An important property of the optimal contract is that it jointly determines a firm’s financing, investment and dividend policies. This is in close analogy to covenants that are routinely found in both private and public loan agreements. Covenants contain often severe restrictions on firms’ financing, investment and dividend policies (Smith and Warner (1979)). In that respect, the model gives a sense of how such covenants quantitatively affect firm dynamics and shows how they naturally arise in financial contracting. The empirically relevant question is then, to what extent do these contracts have different implications for firms’ policies from trade-off models and how can we empirically distinguish them?

To shed new light on this question, the paper identifies differential empirical implications in the context of the literature on financing constraints. A broad literature has emerged that studies the implications of financing constraints for firm dynamics, asset pricing and business cycles, among others. For instance, a prominent question in corporate finance theory is whether financing constraints rationalize the observed cash-flow sensitivity of firms’ investment policies. Building on the recent insight that firms’ investment policies are a first order determinant of asset pricing patterns, the asset pricing literature investigates to what extent financing constraints are reflected in expected returns.

At an aggregate level, an influential literature in macroeconomics argues that financing constraints deliver a powerful propagation and amplification mechanism for aggregate fluctuations. However, despite the intuitive appeal of the underlying concept, no consensus has yet emerged. Conflicting evidence abounds especially in the investment and
A crucial element in all theoretical and empirical financing constraints research is translating the intuitive notion into an empirically or theoretically measurable or quantifiable concept. In other words, researchers have to provide an identification strategy, that is, an "algorithm" that identifies financially constrained firms. As the multitude of identification strategies in the literature shows, that is a far from trivial endeavor.

The paper’s first theoretical contribution is a sharp characterization of firms’ financing constraints arising from dynamic agency conflicts. This characterization allows identification of the behavior of financially constrained firms. The model then predicts that financing constraints drive a wedge between Tobin’s Q and marginal q, a wedge which is correlated with current cash flows. Constrained firms’ investment is relatively insensitive to Tobin’s Q. This suggests that cash flows should have additional explanatory power for investment beyond Q in the model. This prediction is consistent with the empirical evidence. Model simulations offer additional evidence on the financing constraints implied by the model. The model predicts that financially constrained firms in this context tend to be firms with lower leverage and high market-to-book ratios, while the indices typically used to identify constrained firms in empirical work, such as indices based on Whited and Wu (2006) and Kaplan and Zingales (1997) typically load considerably on leverage. This suggests that definitions of financing constraints may be rather sensitive to the underlying frictions.

On the asset pricing side, the model predicts that firms facing a higher degree of financing constraints should earn higher expected returns. While this appears intuitive, the literature has so far failed to produce conclusive evidence to this effect. In light of the model, a natural interpretation of these empirical findings is that they are driven by their identification of financing constraints. As suggested above, their proxy likely stands for financial distress, where financially distressed stocks have empirically been shown to exhibit lower returns (Campbell, Hilscher, Szilagy (2007)).

To sum up, this paper offers a sharp characterization of financing constraints as implied by financial contracts. Firms so identified as financially constrained are typically smaller and younger firms, with lower leverage and higher growth opportunities, in contrast to common empirical identification strategies. This casts doubt on the empirical content
of the intuitive notion of financing constraints in the absence of a model. Financing constraints in the present model restrict the growth of young firms through suboptimal access to credit, while in the extant literature financing constraints are likely often synonymous to financial distress. This suggests that a reason underlying the conflicting evidence in the literature is that these results hinge mostly on the characteristics of the firms identified as financially constrained. Moreover, it shows that the incentive problems that have been at the core of recent research in corporate finance theory have different quantitative implications than dynamic trade-off models and can be empirically distinguished. An important challenge for future empirical and theoretical work in the area is to more clearly delineate the underlying concepts of financing constraints. This should clarify a host of the recent controversies.

From a methodological point of view, this paper builds on the literature of dynamic contracting and on the literature on neoclassical models of firms’ investment and financing decisions. In formulating the contract design problem, it builds on the work of Albuquerque and Hopenhayn (2004), who study analytically a long-term financial contract between investors and entrepreneurs under limited commitment. The industry model builds on the work of Gomes (2001) and Zhang (2005), who use versions of Hopenhayn’s (1992) industry equilibrium model to quantitatively study investment and the cross-section of returns. Indeed, one contribution of the present paper is to extend and quantitatively evaluate the implications of limited commitment models in the tradition of Albuquerque and Hopenhayn for investment, financing and the cross-section of returns.

The paper is related to several strands of literature. The influential work of Fazzari, Hubbard and Petersen (1988), documenting cash flow effects in investment regressions and interpreting them as prima facie evidence for the presence of financing constraints, has started a vast theoretical and empirical literature on these effects. Recent work by Gomes (2001), Cooper and Ejarque (2003), Abel and Eberly (2005) have challenged their view and demonstrated that financing constraints are neither necessary nor sufficient to provide cash flow effects. The present paper demonstrates that suitably modelled financing frictions indeed can rationalize the empirical evidence. In this respect, the papers most closely related are the recent contributions of Lorenzoni and Walentin (2007),
DeMarzo, Fishman, He and Wang (2008) and Hennessy, Levy and Whited (2007). In contrast to the present paper, Lorenzoni and Walentin focus on aggregate investment, while DeMarzo, Fishman, He and Wang analyze qualitatively a dynamic moral hazard model with closed-form solutions in continuous time. Hennessy, Levy and Whited analytically show that collateral constraints drive a wedge between marginal q and Tobin’s Q in a Q-theoretical model with financing frictions, and empirically test its predictions. In asset pricing, the list of theoretical and empirical papers investigating the effects of financing constraints on stock returns includes Lamont, Polk and Saa-Requejo (2001), Whited and Wu (2006), Chen and Campello (2006), Gomes, Yaron and Zhang (2006), Li (2006) and Livdan, Sapriza and Zhang (2007). As briefly discussed, no consensus emerges from that literature.

The paper is also related to a growing literature using limited commitment models in the tradition of Albuquerque and Hopenhayn to study a variety of questions. Examples are industry dynamics (Monge, 2001), impact of financing constraints on changes in aggregate productivity and macroeconomic volatility (Jermann and Quadrini, 2006, 2007), amplification of aggregate shocks (Cooley, Marimon, Quadrini, 2004), investment dynamics (DeMarzo and Fishman, 2007), capital structure (Arellano, Bai, Zhang (2007)) and financial intermediation and risk management (Rampini and Viswanathan (2008)). More generally, this paper is related to the recent theoretical literature trying to link asset prices to firm characteristics (a subset is Berk, Green, Naik (1999), Gomes, Kogan, Zhang (2003), Carlson, Giammarino, Fisher (2004), Zhang (2005), Livdan, Sapriza, Zhang (2006), Gala (2006), Gomes and Schmid (2007)) and to the recent efforts in the capital structure literature that attempt to link corporate investment and leverage policies (including Hennessy and Whited (2004, 2006), Hennessy, Levy and Whited (2006) and Sundaresan and Wang (2006)) by introducing firms’ exposure to systematic risk, thus allowing for business cycle variation in financing and investment decisions.

This paper is organized as follows. Section 2 details the environment and describes a general contracting problem with limited enforcement, which is then further analyzed exploiting both a Lagrangian sequence as well as a recursive approach in section 3. Using the recursive approach the model is then calibrated and solved numerically and quantitative implications for investment, financing, returns and industry dynamics are
described in section 4. Finally, section 5 provides some concluding remarks.

2 Economic Environment

The model embeds a dynamic financial contracting problem into a discrete time neo-classical model of an industry. The industry is composed of a continuum of competitive entrepreneurial firms that produce a homogeneous product. With probability $\phi$ firms become unproductive and leave the industry. The law of large numbers implies that every period a fraction $\phi$ of randomly selected firms leave the industry. The firms leaving the industry are replaced by new firms of the same overall mass $\phi$. Therefore the total mass of firms is constant. New firms arrive with an initial capital stock $k_0$. However, production requires an initial setup cost $I_0$ with $I_0 > k_0$ which is sunk. In order to finance the setup cost new firms have to enter into a contractual relationship with an investor.

Subsection 1 describes the technology, subsection 3 describes the financial environment and investment decisions, subsection 3 details the contracting problem (which will be analyzed in more detail in section 3) for a single firm.

2.1 Technology

The stochastic production technology transforms the capital stock $k$ into output subject to two shocks, namely an aggregate shock $x$ and an idiosyncratic shock $z$. The following assumptions about the stochastic processes of the shocks are standard: The aggregate technology shock $x$ has a stationary and monotone Markov transition function $Q_x(x_{t+1}|x_t)$ as follows

$$x_{t+1} = (1 - \rho_x)x + \rho_xx_t + \sigma_xx_{t+1}$$

where $\epsilon_{t+1}^x$ is an IID standard normal random variable.

The idiosyncratic productivity shocks $z_{it}$ are uncorrelated across firms indexed by $i$ and have a common stationary and monotone Markov transition function $Q_z(z_{it+1}|z_{it})$ as follows

$$z_{it+1} = \rho_zz_{it} + \sigma_z\epsilon_{it+1}^z$$

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4 In the following the terms entrepreneur and firm are used interchangeably.
where $\epsilon_{it+1}$ is an IID standard normal random variable and $\epsilon_{it+1}$ and $\epsilon_{jt+1}$ are uncorrelated for every pair $i \neq j$, and $\epsilon_{it+1}$ is independent of $\epsilon_{jt+1}$ for every $j$.

The aggregate shock $x$ describes the overall level of technology in the economy and thereby generates the business cycle implications of the model, whereas the idiosyncratic shocks generate a non-trivial cross section of firms at every point in time.

The production function is specified as

$$y_{jt} = \exp(x_t + z_{jt}) k_{jt}^\alpha$$

where $0 \leq \alpha \leq 1$ and $y_{jt}$ and $k_{jt}$ denote output and capital stock of firm $j$ at time $t$. If $0 < \alpha < 1$ the production technology exhibits decreasing returns to scale.

Firms can sell their product at a price $P$, which is normalized to 1. Therefore firms’ profit function $\pi$ can be identified with $y$. The form of the production function can be seen as a profit function coming from a firm maximizing profits through an optimal choice of both capital and labor inputs, that is, where wage and labor input have already been “maximized out”.

For future reference denote by $s_t$ the collection of variables that the firms take as exogenous, that is $s_t \equiv (x_t, z_{it})$.

### 2.2 Financing and Investment

Since firms have to pay a setup cost $I_0$, which exceeds their initial capital stock, to start the production process, they need to obtain external funding. The firms’ shareholders do that by entering into a long-term contractual relationship with an outside investor at the time they arrive in the industry. This contract will also provide firms with external funds and liquidity over their life cycle. The contract for a firm $i$ that arrives in the industry in period $t$ specifies a sequence of transfers $\{\tau_{it+j}\}_{j=0}^\infty$ to the investor (that is, a repayment policy), a sequence of investments $\{i_{it+j}\}_{j=0}^\infty$ (that is, an investment policy) and a sequence of dividend payments $\{d_{it+j}\}_{j=0}^\infty$ to the firms shareholders (that is, a dividend policy) for all periods when the firm is productive. Transfers $\tau$ are defined to be an inflow to the investor and an outflow to the firm, such that a negative transfer means that the investor is injecting funds into the firm or, alternatively, is lending to the firm. During the life span of a contract, transfers will likely be negative in the early
stages of the agreement.

The implied law of motion for the capital stock is then as usual

\[ k_{it+1} = (1 - \delta)k_{it} + i_{it} \]

where \( \delta \) denotes the depreciation rate. All these policies are potentially fully state contingent and depend on the entire history of both aggregate and idiosyncratic shocks incurred during the life span of the firm.

A contract is feasible when \( d_{it+j} \geq 0 \) \(^5\) and

\[ d_{it+j} + \tau_{it+j} \leq \pi_{it+j} - i_{it+j} \]

for all \( j \), which is just stating the budget constraint. This expression shows that a contract essentially amounts to specifying a sharing rule that specifies how the surplus generated by the firm’s production is split between the firm’s claimants, namely its shareholders and the outside investor. Note that there is no real investment friction in the form of adjustment costs.

To model the agency conflict underlying the financing constraints in this paper parsimoniously, it is assumed that at the every period the entrepreneur can divert resources from the firm. Diversion provides firms with a private return according to the function\(^6\)

\[ D(k_t) = \lambda k_t - \xi \]

This modelling is inspired by the literature on financial contracting under limited commitment. The interpretation there is that every period the entrepreneurial firm has the option to default and doing so, generates a private return according to the above function. Financial contracts are then structured to guarantee that the entrepreneur never has incentives to default. This interpretation is also available here, however, the specification readily encompasses other agency conflicts. The interpretation entertained

\(^5\)This is clearly a crucial assumption. If the firm could obtain external funds by paying negative dividends, that is obtaining costless external finance, the contracting problem would become trivial and obsolete.

\(^6\)The implications of the model are of course to some extent sensitive to the choice of the diversion value. As long as the latter is an increasing function of the capital stock the basic properties of the model are unchanged however. Specifications of this type have been used in Cooley, Marimon, Quadrini (2004) and Jermann and Quadrini (2006, 2007)
here is that diversion stands for a broader class of agency conflicts, in which the returns to diversion exclusively accrue to the entrepreneur and therefore deprive investors of their contractual rights.

The diversion value is a function of the capital stock in the firm minus a punishment cost $\xi$. The first term is commonly interpreted as a backyard technology available to the firm which generates a present value of $\lambda k_t$. This part of the diversion value captures the idea that it should be related to the resources in the firm. When designing the contract investors anticipate that firms will divert resources whenever it is in their best interest to do so. This will be the case when the future expected discounted dividends accruing to the shareholders will be smaller than the diversion value. Therefore the firm’s equity value can be thought of as collateral for the contract.

2.3 Dynamic Problem

Assuming competition in financial markets an optimal contract maximizes the expected discounted payments (that is, the dividends) to the firm’s shareholders subject to an enforcement constraint and an investor’s participation constraint. The enforcement constraint simply asserts that at any future date, the value of continuing the contract cannot be smaller than the value of diverting. The participation constraint imposes that the investor breaks even in expectation, that is, the expected discounted value of transfers to the latter cannot be smaller than the initial setup cost. Let $m_t$ denote the possibly stochastic discount factor in the economy. Notice that the effective stochastic discount factor used to discount payments from the firms is

$$m_{t+1} = \tilde{m}_{t+1}(1 - \phi)$$

since with probability $\phi$ the firm becomes unproductive and leaves the industry.

Accordingly the contractual problem for a firm arriving in the industry at time $t$ can be stated as follows

$$\max_{\{d_{it+j}, \tau_{it+j}, \delta_{it+j}\}_{j=0}^{\infty}} E_t\{\sum_{j=0}^{\infty} m_{t+j}d_{t+j}\}$$

(1)
subject to

\[ d_{it+j} = \pi_{it+j} - i_{it+j} - \tau_{it+j} \geq 0 \]  \hspace{1cm} (2)
\[ k_{it+j+1} = (1 - \delta)k_{it+j} + i_{it+j} \]  \hspace{1cm} (3)
\[ D(k_{t+j}) \leq E_{t+j}\{\sum_{l=t+j}^{\infty} m_{t+j+l}d_{t+j+l}\} \]  \hspace{1cm} (4)
\[ I_0 \leq E_{t+j}\{\sum_{j=0}^{\infty} m_{t+j}\tau_{t+j}\} \]  \hspace{1cm} (5)

Constraint (2) is simply the budget constraint combined with the constraint that the dividend be non-negative at all times. Constraint (3) is the implied accumulation law of the capital stock.

Constraint (4) is the enforcement constraint for the firm. At the end of each period the firm’s shareholders will compare the value of diverting and thereby receiving the diversion value \( D(k_t) \) to the value of sticking to the contractual obligations, which is simply the expected value of discounted future dividend payments. Incentive compatibility then requires that the value of staying in the contract cannot be smaller than the diversion value. Notice that this constraint has to be satisfied at all future dates which implies that an infinite sequence of forward looking occasionally binding constraints need be kept track of. This is the main technical difficulty when dealing with models of this sort and will be discussed in more detail in section 3.

Constraint (5) is the participation constraint for the investor, which simply states that he has to break even in expectation. Notice that the transfers to the investor are discounted by the same stochastic discount factor as the payments to firms’ shareholders.

This can be seen as a reduced form general equilibrium effect where, ultimately, it is a representative investor’s consumption which prices all claims.

### 3 Optimal Financing Contracts

This section analyzes the financial contracting problem in some more detail and derives some basic properties of the resulting contracts. Subsection 1 uses a Lagrangian approach in the space of sequences and Section 2 deals with a recursive representation of the contracting problem.
3.1 A Lagrangian Formulation

To gain some first insights about the nature of the optimal contracts it is instructive to exploit the Lagrangian associated with the above optimization problem. Without loss of generality, assume for the moment for ease of exposition that the firm under consideration arrives in the industry in period 0. Assuming competition in financial markets, the contract maximizes the value to the entrepreneur subject to the investor’s participation constraint. Denoting by $V_{i0}$ the value of the contract to the firm, the contracting problem then becomes

$$V_{i0} = \max_{\{d_{it}, \tau_{it}, i_{it}\}_{t=0}^{\infty}} \mathbb{E}_0\left\{ \sum_{t=0}^{\infty} m_t d_t \right\}$$  \hspace{1cm} (6)

subject to

$$d_{it} = \pi_{it} - i_{it} - \tau_{it} \geq 0$$  \hspace{1cm} (7)

$$k_{i(t+1)} = (1 - \delta) k_{it} + i_{it}$$  \hspace{1cm} (8)

$$D(k_t) \leq \mathbb{E}_t\left\{ \sum_{j=1}^{\infty} m_{t+j} d_{t+j} \right\}$$  \hspace{1cm} (9)

$$I_0 \leq \mathbb{E}_0\left\{ \sum_{t=0}^{\infty} m_t \tau_t \right\}$$  \hspace{1cm} (10)

In order to be able to keep track of the firms liabilities, define a new variable $q_{it}$ to be the expected discounted value of the firms liabilities at time $t$, that is

$$q_{it} = \mathbb{E}_t\left\{ \sum_{j=0}^{\infty} m_{t+j} \tau_{t+j} \right\}$$

In view of a recursive characterization of the problem, note that

$$q_{it} = \tau_{it} + \mathbb{E}_t\left\{ \sum_{j=1}^{\infty} m_{t+j} \tau_{t+j} \right\}$$  \hspace{1cm} (11)

$$= \tau_{it} + \mathbb{E}_t\left\{ m_{t+1} q_{it+1} \right\}$$  \hspace{1cm} (12)

The participation constraint for the investor (10) then just amounts to requiring that, at the time of initiation of the contract, we have

$$q_{i0} \geq I_0$$

\footnote{Notice that no reference to any set of sufficient state variables is made at this stage. This will become crucial in the next section.}
and that the future evolution of $q$ be governed by the law of motion (12).

Attach multipliers $\tilde{\mu}_t$ to the dividend non-negativity constraint in (7), $\tilde{\alpha}_{it+1}$ to the enforcement constraint, $\tilde{q}_m^m$ to the capital accumulation constraint (8) and $\lambda_t$ to the investor’s participation constraint (9). The Lagrangian then becomes

$$L_0 = E_0\{\sum_{t=0}^{\infty} m_t[(1 + \tilde{\mu}_t)d_t + \tilde{\lambda}_t(\pi_t - i_t - d_t)$$
$$- \tilde{q}_m^m(k_{it+1} - (1 - \delta)k_{it} - i_{it}) - \tilde{\alpha}_{it+1}(D(k_{it}) - \sum_{j=1}^{\infty} m_{t+j}d_{t+j})]\}$$

Define a new set of stochastic Lagrange multipliers $\tilde{\nu}_t$ recursively by

$$\tilde{\nu}_0 = 0$$
$$\tilde{\nu}_t = \tilde{\nu}_{t-1} + m_{t-1}\tilde{\alpha}_t$$

and subsequently renormalize $\nu_t = \frac{1 + \tilde{\nu}_t}{\lambda_t}$, $\mu_t = \frac{\tilde{\mu}_t}{\lambda_t}$, $q_t^m = \frac{\tilde{q}_t^m}{\lambda_t}$ and $\alpha_{it+1} = \frac{\tilde{\alpha}_{it+1}}{\lambda_t}$. These multipliers have a straightforward interpretation. Recall that the multipliers $\alpha_{it}$ are positive whenever the enforcement constraint is binding, and zero otherwise. The multipliers $\nu_{it}$ then keep track of the histories of binding enforcement constraints. 8

The appendix shows how, using these multipliers, the Lagrangian can be written as follows

$$L_0 = E_0\{\sum_{t=0}^{\infty} m_t[\pi_t - i_t - (1 - \mu_t - \nu_t)d_t - q_t^m(k_{it+1} - (1 - \delta)k_{it} - i_{it}) - \alpha_{it+1}D(k_{it})]\}$$

Notice that this implies that $\mu_t + \nu_t \leq 1$, as otherwise the objective function would be maximized by setting $d_t$ to infinity, which by (7) would violate the investor’s participation constraint.

With the Lagrangian at hand, the first order conditions with respect to investment are readily determined. The first order conditions with respect to $i_{it}$ and $k_{it+1}$ are in that order

$$q_t^m = \mu_t + \nu_t$$
$$q_t^m = E_t\{(\mu_{it+1} + \nu_{it+1})\frac{\partial \pi_{it+1}}{\partial k_{it+1}} + (1 - \delta)q_{it+1}^m - \alpha_{it+2}\frac{\partial D(k_{it+1})}{\partial k_{it+1}}\}$$

Marcet and Marimon (1998) develop a theory of recursive contracting using such recursively defined Lagrange multipliers as state variables. Cooley, Marimon and Quadrini (2004) study a financial contracting problem similar to the one considered here using the latter approach.
The first order conditions yield a few interesting observations. Notice first that, in spite
of the absence of adjustment costs to physical investment, the shadow value of one more
unit of capital - marginal $Q$ - is in general different from one and time-varying due to
the presence of financing constraints. However, $\nu_{it}$ grows on average but cannot exceed
1. Therefore, once it reaches this value, the enforcement constraint will never be binding
in the future. In this situation, $\mu_{it}$ will be 0, and the first order condition implies that
marginal $Q$ will be equal to 1.

Next, combining the two equations yields

$$ q_{it}^m = E_t \{ m_{t+1} [ q_{it+1}^m ( \partial \pi_{it+1} + (1 - \delta)) - \alpha_{it+2} \partial D(k_{it+1}) ] \} $$

(13)

which is best interpreted when contrasting it with the corresponding equation in an
economy without financing frictions

$$ 1 = E_t \{ m_{t+1} [ \partial \pi_{it+1} + (1 - \delta) ] \} $$

The presence of the enforcement constraint leads to an additional term relative to the
frictionless case, associated namely to the default value. Since the additional term enters
negatively and the profit function exhibits decreasing returns to scale, the presence of
the enforcement friction leads to suboptimal investment relative to the frictionless case.

This has a simple intuition. An additional unit of capital not only raises expected
future dividends it also increases the firms diversion value. It is precisely that trade-off
that an optimal contract has to balance. It does so by determining optimal investment
and financing policies. Starting from the setup cost, the financing policy is such that
it provides exactly the right amount of external funds such that it can grow along a
path which optimally increases the equity value of the firm, while making sure that the
value of liabilities do not grow at a rate which would violate the enforcement constraint.

Overall, the contract then specifies the investment, dividend and financing policies in a
way that yields a growth pattern that exactly balances the friction apparent in equation
(13): Providing funding that helps increasing capital raises expected dividends within
the contract and therefore creates collateral value, but also increases the resources in

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9this not exactly the same as the definition of marginal Q in a continuous time framework because
of the time-to-build assumption implicit in the capital accumulation equation. Zhang (2006) provides
a good discussion.
the firm available for diversion.

Next define $P_t$ as follows

$$P_t = V_t - (\pi_t - i_t) + (1 - \mu_t - \nu_t)d_t + \alpha_{it+1}D(k_{it})$$

For a young firm with no history of binding enforcement constraints and a slack dividend non-negativity constraint $P_t$ is simply the equity value plus the current transfer to the investor, $\tau_t$. The following proposition establishes a link between $P_t$ and $q_{it}^m$, which in the context of this model corresponds to marginal $Q$.

**Proposition 3.1.** With the above definition we have

$$P_t + C_t = q_{it}^m k_{it+1} + D_t + P_t$$

where

$$C_t = E_t\{\sum_{j=1}^{m_t+1} (\mu_{it+j} + \nu_{it+j} - 1)\pi_{it+j}\}$$

$$D_t = E_t\{\sum_{j=1}^{m_t+1} \alpha_{it+j+1} (k_{it+j} \frac{\partial D(k_{it+j})}{\partial k_{it+j}} - D(k_{it+j}))\}$$

$$P_t = E_t\{\sum_{j=1}^{m_t+1} q_{it+j}^m (1 - \alpha)\pi_{it+j}\}$$

**Proof:** The proof is by simple manipulation of the Lagrangian. First note that by definition

$$P_t + (\pi_t - i_t) - (1 - m_t - \nu_t)d_t - \alpha_{it+1}D(k_{it}) = \pi_t - i_t - (1 - \mu_t - \nu_t)d_t$$

$$-q_t^m (k_{it+1} - (1 - \delta)k_t - i_t) - \alpha_{it+1}D(k_{it}) + E_t\{m_t+1(\pi_{it+1} - i_{it+1})$$

$$-(1 - \mu_{it+1} - \nu_{it+1})d_{it+1} - q_{it+1}^m (k_{it+2} - (1 - \delta)k_{it+1} - i_{it+1}) - \alpha_{it+2}D(k_{it+1}) + \ldots\}$$

Recursive substitution using the first order conditions and the capital accumulation yields the result.

To interpret this result, first note that

$$\frac{P_t + C_t}{k_{it+1}}$$

is closely related to a firm’s market-to-book ratio, or alternatively average $Q$. In particular for a young firm with no history of binding enforcement constraints and a slack
dividend non-negativity constraint, that is with $\mu_{it} = \nu_{it} = 0$, it is exactly average $Q$. For such a firm average and marginal $Q$ differ by two terms, namely

$$\frac{D_{it} + P_{it}}{k_{it+1}}$$

and average $Q$ is no longer a sufficient statistic for investment. Note that in case of a constant returns to scale technology $\alpha = 1$ and $P_{it}$ is identically zero for all $t$. In that case $D_{it}$ drives a time-varying wedge between average and marginal $Q$. Note that here $D_{it} \geq 0$ and therefore average $Q$ will on average be larger than marginal $Q$.\(^{10}\) Variables which carry information about this wedge should then be empirically informative for investment behavior beyond average $Q$. The wedge here is driven by expected future binding enforcement constraints through the dependence on the Lagrange multipliers associated with the enforcement constraints and the default values. In particular, conditional on $\frac{P_{it+1} + E_t \{m_{it+1} q_{it+1}\}}{k_{it+1}}$ anticipating fewer future binding constraints will decrease the wedge and thereby increase investment. Since, all else equal, higher cash flows decrease the likelihood of binding enforcement constraints through lesser dependence on external finance, one should expect cash flows to carry information about investment behavior beyond average $Q$. Additionally, one should expect cash flows to be more informative about investment for younger firms as they likely face more expected binding constraints and loose information for older firms which are less likely to face binding enforcement constraints.

The model therefore suggests that cash flows should be informative for investment beyond average $Q$ in a way that is consistent with empirical evidence. These implications will be confirmed in the numerical simulations below.

### 3.2 A Recursive Formulation

In view of a numerical solution of the model and for notational transparency’s sake it is convenient to have recursive representation of the contracting problem at hand. Following Spear and Srivastava (1987) and using - as anticipated in the last section - the value of transfers $q$ promised to the investor as an additional state variable, the contracting

\(^{10}\)This clearly depends on the specification of the default value function. Exploring the dynamic implications of alternative specification is an interesting direction for further research.
problem can be cast in recursive form. Accordingly time subscripts (and for expositional
reasons the individual firm’s index \( i \)) will be dropped in the following and next period
variables are indicated by a prime. As before, let \( s \equiv (x, z) \) denote the state variables
the firm takes as exogenous.

As before, assuming competition in financial markets an optimal contract maximizes
the value of the firm subject to an enforcement constraint, a participation constraint
for the investor and the law of motion for the value promised to the investor \( q \). In
the recursive setting control variables are next period’s capital stock \( k' \), current period
transfer \( \tau \) and the continuation values \( q(s') \). While in the general formulation of the
last section the continuation values were potentially state-contingent, in view of computa-
tional tractability, attention will from now on be restricted to an non-contingent case
with \( q' \equiv q(s') \). The contracting problem can then be stated as follows in direct analogy
with the Lagrangian formulation

\[
V(k, q, s) = \max_{\tau, i, k', q'} d + E_s\{m'V(k', q', s')\} \tag{14}
\]

subject to

\[
\begin{align*}
d &= \pi - i - \tau \geq 0 \tag{15} \\
k' &= (1 - \delta)k + i \tag{16} \\
q &= \tau + E_s\{m'q'\} \tag{17} \\
D(k) &\leq E_s\{m'V(k', q', s')\} \tag{18}
\end{align*}
\]

Contracts are initialized such that \( q_0 \geq I_0 \).

Constraint (15) is the budget constraint in conjunction with the dividend constraint
and constraint (16) is the capital accumulation equation. Constraint (17) is the promise
keeping constraint for the transfers to the investor, that is, the law of motion for the
repayment schedule. Constraint (18) then again is the enforcement constraint requiring
that the value of continuing in the contract for the firm cannot be smaller than the value
of diversion.

The above dynamic programming problem has a somewhat special structure due to the
fact that its solution - the value function - also appears in the enforcement constraint.
This leads to the difficulty that standard contraction mapping techniques for dynamic
programming are not directly applicable to the present problem. In particular, verifying Blackwell’s sufficient conditions for a contraction is impossible directly since verifying the discounting property in the present case would require knowledge of the value function to be determined. While the solution to the contracting problem will still solve a functional equation of this form, the solution may not be unique and accordingly it cannot be that iterating on any initial guess will lead to the correct value function.  

However, a slightly different approach developed by Thomas and Worrall in the context of an international trade model with a similar formal structure shows how to handle problems of this kind. The idea is to start with the value function that strictly dominates the solution to problem (14) and then to iterate using the constraints. The solution to problem (14) is then the pointwise limit of the iteration procedure. Since the limit is only pointwise a priori the limiting function need not exhibit any continuity properties. However, Thomas and Worrall show that the resulting value function will be concave almost everywhere and differentiable almost everywhere.

To make this precise let

\[ \tilde{V}(k, s) = \max_{i,k'} d + E_s \{ m' \tilde{V}(k', s') \} \]  

subject to

\[ d = \pi - i \]  
\[ k' = (1 - \delta)k + i \]  

Clearly, \( \tilde{V}(k, s) \geq V(k, q, s) \) for all \( k, q \) and \( s \). \( \tilde{V}(k, s) \) is the solution to a planner’s unrestricted problem, that is, it traces out the Pareto frontier. In this case, the standard recursive machinery directly applies and it immediately follows from Blackwell’s sufficient conditions that \( \tilde{V}(k, s) \) exists and is unique.

Let the operator \( T \) on the space of continuous functions equipped with the sup-norm be

\[ 11^{11} \text{In the context of a model studying risk sharing in a two-agent economy with limited commitment Alvarez and Jermann (2000) show that there are multiple solutions to a dynamic programming of the above kind, in particular in their case autarky without risk-sharing is always a fixed point of the mapping.} \]
defined by

\[ T(V)(k, q, s) = \max_{\tau, k', q'} (d + E_s \{ m'V(k', q', s') \}) \]

subject to constraints (15)-(18). Clearly, this is just the Bellman operator associated with the contracting problem. Finally, consider the sequence of functions defined recursively by \( f^0 = \hat{V} \) and \( f^n = T(f^{n-1}), n \geq 1 \). Then the following holds

**Proposition 3.2.** *(Thomas and Worrall)*

\( f^n \) converges to \( V \) pointwise. Additionally, \( V \) is concave almost everywhere and differentiable almost everywhere.

Operationally, this essentially gives a valid initial guess that can be applied in the iteration procedure.

Next, with this proposition at hand the first order conditions can be determined almost everywhere. Attaching multipliers \( \mu \) to the dividend non-negativity constraint, \( \gamma \) to the enforcement constraint, \( \chi \) to the promise keeping constraint and substituting out the capital accumulation equation and the budget constraint the first order conditions with respect to \( \tau, k' \) and \( q' \) are

\[
(1 + \gamma)E_s \{m'V(k', q', s')\} - (1 + \mu) = 0 \tag{23}
\]

\[
\chi E_s \{m'\} + (1 + \gamma)E_s \{m'V_q(k', q', s')\} = 0 \tag{24}
\]

\[
\chi = 1 + \mu \tag{25}
\]

and the envelope conditions become

\[
V_k = (1 + \mu)(\pi_k + 1 - \delta) - \gamma D_k(k) \tag{26}
\]

\[
V_q = -\chi \tag{27}
\]

These conditions yield a few observations. First, note from (26) that \( V \) is strictly decreasing in \( q \), which is intuitive. This observation gives the enforcement constraint a natural interpretation as an endogenous borrowing constraint. Note from (23) the unrestricted optimal next period capital stock, that is the optimal capital stock when both constraints are slack (that is, \( \gamma = \mu = 0 \)) is a function of \( s \) only. Thus in the presence of constraints a given target capital stock might only be implementable with a binding
dividend constraint, implying an increase in $q'$. However, such $q'$ might violate the enforcement constraint as $V$ is a decreasing function of the promised value of transfers. Therefore the enforcement constraint can be interpreted as endogenous debt constraint or alternatively, as an endogenous collateral constraint. More importantly, the impossibility of implementing the unconstrained optimal capital stock does not derive from the fact that it might be too expensive to obtain the necessary external funds, but from the fact that obtaining these funds would lead the firm to divert funds. This is akin restrictions in bond covenants.

To derive implications for firm dynamics under the optimal contract, combine the first order condition (24) with the envelope condition (27) to obtain

$$
\chi E_s\{m'\} = (1 + \gamma)E_s\{m'\chi(s')\}
$$

This shows that whenever the enforcement constraint binds, that is when $\gamma > 0$, $\chi$ will decrease next period on average (this could be strengthened to a state-by-state statement under risk-neutrality, that is, when $m' = \beta$). This leads to a simple pattern for the dynamics of a firm. When the firm arrives in the economy, in order to become productive it needs to sign a contract giving rise to an initial promise $q$ and an associated $\chi > 0$. With binding enforcement constraints $\chi$ will decrease on average. As $\chi \geq 1$, at some stage $\chi$ will reach the value of 1. By the first order condition at that time $\mu$ must be zero and the dividend constraint is no longer binding. Clearly, when the dividend constraint is no longer binding, the enforcement constraint cannot be binding any longer either. At this stage the first order condition (23) combined with the envelope condition (26) yields

$$
1 = E_s\{m'(\pi_k(k',s') + 1 - \delta)\}
$$

which is the familiar Euler equation for optimal investment. At this stage, the firm will employ the optimal capital input. That is, the firm will become unconstrained. Before this stage is reached however, the firm’s capital stock will be suboptimal as the enforcement constraint might be binding. Such a firm is constrained.

This suggests a straightforward measure of financing constraints in the context of this model, namely the multiplier on the enforcement constraint $\gamma$. Exploiting the envelope conditions allows to explicitly compute $\gamma$ once the value function is known. This will
be helpful in the quantitative analysis in section 4. Combining the envelope conditions it follows that

\[ \gamma = \frac{-V_k - V_q(\pi_k + 1 - \delta)}{D_k(k)} \]

In the context of this model this expression has a straightforward intuition. Recalling that \( V_q < 0 \) the second term will be positive and the first one negative. All else equal \( \gamma \) is increasing in \( q \) as this lowers firm and therefore collateral value and decreasing in \( k \) since this increases firm and collateral value.

4 Quantitative Analysis

This section presents results from simulation evidence obtained by solving the model numerically. Quantitative implications concerning financing, investment and the cross-section of stock returns are considered. Analyzing implications for stock returns requires specifying preferences towards risk. In the present partial equilibrium framework this is achieved by directly parameterizing an exogenously specified stochastic discount factor. Subsection 1 explains the specification of the stochastic discount factor, section 2 details the calibration of the model, subsection 3 discusses capital structure implications, subsection 4 considers quantitative implications for investment, subsection 5 describes implications for the cross-section of returns and subsection 6 briefly considers aggregate industry dynamics.

4.1 Stochastic Discount Factor

In the present partial equilibrium framework, discounting is done by an exogenously specifying a stochastic process for the stochastic discount factor, in line with the recent literature on cross-sectional asset pricing (e.g. Berk, Green, Naik (1999), Carlson, Fisher, Giammarino (2004), Zhang (2005)). This can be justified by the model’s emphasis on cross-sectional firm level behavior rather than aggregate pricing. Specifically, the stochastic discount factor is specified as follows:

\[ \log \hat{m}_{t+1} = \log \beta + \gamma(x_t - x_{t+1}) \]
with time-discount factor $\beta > 0$ and a constant $\gamma > 0$. Although exogenous, the pricing kernel can be justified as follows. The log pricing kernel in a consumption based model with constant relative risk aversion preferences and risk aversion parameter $\varsigma$ is given by $\log m_{t+1} = \log \beta + \varsigma (\log C_t - \log C_{t+1})$. It is then apparent that the above pricing kernel relates to a consumption based one by identifying $\log C_t \simeq \kappa x_t$ for some constant $\kappa$ and setting $\varsigma \kappa = \gamma$.

An alternative way of writing the above pricing kernel can be obtained by exploiting the properties of the log-normal distribution. Since

$$
\log \tilde{m}_{t+1} = \log \beta + \gamma(x_t - x_{t+1})
$$

$$
= \log \beta + \gamma(1 - \rho_x)x_t - \gamma(1 - \rho_x)\bar{x} - \gamma\sigma_x \epsilon_{t+1}^x
$$

and

$$
r_t = - \log E_t \{ \tilde{m}_{t+1} \}
$$

$$
= - \log \beta - \gamma(1 - \rho_x)x_t + \gamma(1 - \rho_x)\bar{x} - \frac{1}{2}\gamma^2 \sigma_x^2
$$

it follows that

$$
\log \tilde{m}_{t+1} = -r_t - \frac{1}{2}\gamma^2 \sigma_x^2 - \gamma\sigma_x \epsilon_{t+1}^x
$$

Accordingly, as one should expect from a pricing kernel similar to one derived from constant relative risk aversion preferences, it exhibits a constant market price of risk.

### 4.2 Calibration

To evaluate the quantitative implications a set of parameter values must first be picked. As usual that is achieved by letting the model match unconditional moments of investment, returns, and cash flows both in the cross-section and at the aggregate level. In order to be consistent with the empirical literature on cross-sectional asset pricing the model is calibrated on a monthly basis. The resulting parameter choices, summarized in Table 1, are for the most part close to the estimates used in other quantitative investigations in the literature after taking in account the different calibration frequency (e.g. Gomes (2001), Cooley and Quadrini).

\[12\text{Gomes, Kogan and Zhang (2003) construct a general equilibrium model with heterogeneous firms and endogenous consumption, where aggregate consumption is a function of the aggregate shock only.}\]
The persistence, $\rho_x$, and conditional volatility, $\sigma_x$, of aggregate productivity, are set equal to 0.983 and 0.003 which under time aggregation is close to the (quarterly) values reported in Cooley and Prescott (1995). For the persistence, $\rho_z$, and conditional volatility, $\sigma_z$, of firm-specific productivity, the values used correspond to those constructed by Gomes (2001) to match the cross-sectional properties of firm investment and valuation ratios in annual frequency. That is, $\rho_z$ is set to $0.96 \approx 0.62^{1/12}$ and $\sigma_z = 0.025 = 0.15/12$.

The depreciation rate of capital, $\delta$, is set equal to 0.01 which provides a good approximation to the annualized average rate of investment found in both macro and firm level studies. For the degree of decreasing returns to scale $\alpha$ is set to 0.65. Although probably low this number is almost identical to the estimates in Cooper and Ejarque (2003) as well as several other recent micro studies. On the other hand relative to the range of parameters used in literature spanning from 0.3 (Zhang) to 0.9 (Arellano, Bai and Zhang) this seems a reasonable value.

Following Zhang (2005) the pure time discount factor $\beta$ and the pricing kernel parameter $\gamma$ are set such that the model approximately matches two key moments of asset markets, namely the mean annual risk free rate and the Sharpe ratio. This pins down $\beta$ at 0.995 and $\gamma$ is set equal to 20. Note that this parameterization uniquely pins down characteristics of the aggregate stock market, whereas in the asset pricing context pursued here the emphasis is on cross-sectional risk characteristics.

The exogenous rate of exit from the economy $\phi$ is set to 0.005, in line with the value used in Cooley, Marimon, Quadrini (2004). Note that because of the exit rate the effective time discount rate is $\beta(1 - \phi)$.

Important parameters to calibrate in the model are the default value parameters $\lambda$ and $\xi$. Clearly, for such parameters not much direct empirical guidance is available. These parameters however are closely related to the tightness of the enforcement constraint and therefore to the availability of external funds for firms. In particular, they primarily drive the leverage ratios, the market-to-book values and the size distribution of the firms in the economy as they control how quickly a young firm can grow. Therefore they are chosen to match the average market leverage ratio (defined as $\frac{q}{V+q}$) and
the average market-to-book value (defined by $\frac{V+q}{k}$) and to deliver a reasonable size distribution of the firms in the economy. This yields $\lambda = 2.7$ and $\xi = 0.7$. There clearly is some discretion as to what exactly the values of these parameters should be, but beyond giving reasonable implications for the moments of some key variables, they are in line with the extant literature (Cooley, Marimon, Quadrini (2004), Jermann and Quadrini (2006, 2007), Arellano, Bai, Zhang (2006)).

To assess the fit of the calibration, table 2 reports the implied moments generated by the parameterization for some key variables. The calibration performs rather well along some dimensions crucial for the model. Apart from the asset market data mentioned above which are directly targeted, the calibration matches investment-to-asset ratio and its volatility, market-to-book ratio as well as book and market leverage ratios, indicating that the model captures average investment and financing decisions reasonably well.

Table 2 gives a first although rough pass at the nature of financing constraints arising in the model. The sample is first split into four quintiles depending on their financing constraints as measured by the value of the Lagrange multiplier of the enforcement constraint. The groups are rebalanced every period. In addition to the moments for the entire sample the moments are reported for the most constrained quintile. This suggests that the most financially constrained invest more, have higher Tobin’s Q, lower cash flows and lower leverage. The picture of a financially constrained firm emerging is then the one of a young firm that has profitable investment opportunities but which cannot yet receive all the external financing needed because it still has to build up more collateral value. Given the qualitative nature of the contract studied in the last section this should not be too surprising. On the other hand, the notion of financing constraints emerging here seems to different from the concept of financial distress, which one would intuitively rather associate with low investment, low Tobin’s Q and high leverage. This is encouraging, as empirically these two notions are often somewhat hard to separate. To gain more intuition, table 3 reports average market leverage ratios across constraint quintiles. While, not surprisingly again, the most constrained firms have relatively lower leverage, a more interesting pattern emerges: the constraint-leverage relation appears to be inverse U-shaped. The intuition is the following: while the most constrained firms are not given access external finance as their investment would increase their default
value, less constrained firms rely less on external finance as they are less inclined to invest.

Most importantly, tables 2 and 3 suggests that, maybe somewhat surprisingly, the concept of financing constraints, indeed the identification strategy, is different from those commonly used in the extant literature. Financing constraints there often - but not exclusively - appear to be more closely related to financial distress. This in turn indicates that the nature of firm to which financial constraints are attributed, is rather sensitive to the underlying model specifications and identification strategies. In particular, the concept of financing constraint emerging from this model is non linearly related to leverage.

4.3 Capital Structure Implications

This section discusses the model’s implications for capital structure and considers to what extent these are in line with the empirical evidence. In this context, (market) leverage is defined as $MLev = \frac{q}{v+q}$. In section 3, $q$ was introduced as the contractual value promised to the investor, but is interpreted as debt here. To understand this, recall that the law of motion of the promised value (the promise-keeping constraint) is $q = \tau + E_s\{m'q'\}$, where $\tau$ is the transfer to the investor. Notice that $\tau$ is fully state-contingent and can therefore be interpreted as a state-contingent coupon payment. In sum, the interpretation entertained here is that $q$ refers to a long-term bond with state-contingent coupon payment, in line with recent empirical evidence for state-contingency and renegotiation in debt contracts (Roberts and Sufi (2007, 2008)). Clearly, other interpretations are possible and may appear more natural, depending on the context. In particular, a recent literature (DeMarzo and Fishman (2007), DeMarzo and Sannikov (2007), Biais, Mariotti, Plantin, Rochet (2007)) has provided implementations of similar contracts using equity, long-term debt and either cash or credit lines.

One way to assess to what extent the model’s implications for the dynamics of capital structure are broadly in line with the empirical evidence, is to run leverage regressions akin to those performed in empirical work on simulated data. Table 4 reports the results of cross-sectional regressions of market leverage on Tobin’s Q and a measure of
profitability as in Rajan and Zingales (1995), namely

$$\text{MLe}v_{it} = \alpha_0 + \alpha_1 \frac{V_{it} + q_{it}}{k_{it}} + \alpha_2 \frac{\pi_{it}}{k_{it}} + \epsilon_{it}$$

In line with the data, both the coefficients on Tobin’s Q and on profitability are negative. While the above results on the relation between leverage and constraints suggest, that a univariate regression of leverage on Tobin’s Q would be uninformative, controlling for profitability implies a negative coefficient on both regressors, as more profitable firms need to rely less on external finance to cover investment expenditures. This suggests that the leverage dynamics implied by the model is broadly consistent with the empirical evidence. Furthermore, the correlation between aggregate output and average leverage is -0.42, rendering market leverage countercyclical, as in the data.

### 4.4 Investment Regressions

This section looks at the quantitative implications of the model for investment. Since the model delivers a concept of financing constraints, and indeed a financing constraint as implied by the optimal contract is the only friction in the model, a natural way to evaluate its performance is to see whether it can replicate empirical evidence in the area. The main empirical tool used are panel investment regressions of the form

$$\frac{i_{it}}{k_{it}} = \alpha_0 + \alpha_1 \frac{V_{it} + q_{it}}{k_{it}} + \alpha_2 \frac{\pi_{it}}{k_{it}} + \epsilon_{it}$$

that is, panel regressions of firms’ investment rates on Tobin’s Q and cash-flows. The empirical issue in this context are the persistently arising cash-flow effects. Cash-flow effects appear in two forms, namely as significant coefficients of the cash-flow term and through the fact that adding cash-flow terms to the estimating equations often adds significant explanatory power. From the perspective of a neoclassical investment theory this may appear surprising as under certain assumptions Tobin’s Q should be a sufficient statistic for investment behavior. The standard conclusion drawn from this evidence is the existence of significant financing constraints (see Fazzari, Hubbard, Petersen (1988) for an account). Intuitive as it seems this view has recently been challenged. Gomes (2001) shows that financing frictions are neither necessary nor sufficient to generate cash-flow effects and that adding cash-flows as a regressor does not add any explanatory
power beyond average Q in the simulated data. Based on these results a view has emerged that although financing frictions may be important for investment behavior, they should - as Q is a forward looking variable - already be contained in Q and cash-flows should not deliver additional information. In contrast, the present model shows that this conclusion may vitally linked to the way financing constraints are modelled.

The qualitative analysis in section 3.1. suggested that the presence of financing constraints would lead to a time-varying wedge between between average Q and marginal Q in the form of anticipated future binding enforcement constraints. Since cash-flows mitigate enforcement constraints by increasing firms’ internal funds they should carry information about the wedge and be informative about marginal Q - which determines investment - beyond average Q.

Tables 5 and 6 report results from performing empirical investment regressions as defined above on data sampled from the model. Table 5 performs two panel regressions on the entire sample of the simulated data. The first one regresses investment on Tobin’s Q only, that is α is restricted to be zero. The second adds cash-flows as an explanatory variable. Although the exact regression coefficients do - not surprisingly - not exactly match their empirical counterparts it is immediate that the cash-flow puzzle does arise naturally in the context of this model: Cash-flows enter significantly in the regressions and add explanatory power beyond average Q as measured by the $R^2$, consistent with the intuition developed here. Note that an easy way to get a better quantitative fit for the regression coefficients would be the introduction of adjustment costs to investment. Since the point here is to study financing constraints in isolation, this will not be pursued further.

If the intuition is correct, one would expect the cash-flow effects to be more important the greater the financing constraints. In the context of this model a clear cut measure of the extent a firm is subject to financing constraints is available in the form of the Lagrange multiplier attached to the enforcement constraint. Given the value function this multiplier can be numerically computed as shown in section 3.2. Table 6 reports results from regressions as above when the sample is split in four groups each period on the basis of the extent of their financing constraints (that is, the value of their multiplier) and the panel regressions are performed on those groups separately. The firms
are grouped into four quartiles based on the value of the multiplier. The first group corresponds to the highest financing constraints. While the numbers reported here do not have counterparts in the data and the regression coefficients are somewhat hard to interpret, they are nevertheless instructive. The more financially constrained firms are the more explanatory power is added by including cash-flows as regressors. While the significance levels of the regression coefficients are somewhat less clear it is nevertheless interesting that the standard deviation on $\alpha_2$ for the least constrained group is the largest.

These results seem supportive of the view that financing constraints can deliver an explanation for the empirically observed interactions between investment, average Q and cash flows. However, it seems to depend crucially on the way financing constraints are modelled. This is not to discard alternative theories put forward that can account for cash-flow effects in investment regressions as in Gomes (2001), Cooper and Ejarque (2003) and Abel and Eberly (2004), which appeal to measurement errors as linear regressions are used to proxy for highly nonlinear investment rules, decreasing returns to scale and market power.

4.5 Returns

One point to take away from the last section seems to be the potential importance of explicitly modelling financing frictions for investment. On a similar note, a similar point might apply when studying the implications of financing frictions for the cross-section of returns. This has been a field of considerable research activity recently, but, as noted in the introduction, no clear picture has emerged yet (see Lamont, Polk, Saa-Requejo (2001), Whited, Wu (2006), Gomes, Yaron, Zhang (2006), Li (2006)). This section provides quantitative evidence to this effect from data sampled from the model. The potential advantage of the framework is that it allows to study the effect of financing frictions essentially in isolation, as the only impediment to investment are those dictated by the financial contract. Furthermore as in the last section the availability of a clear measure of the extent to which a firm is subject to financing constraints in the form of a multiplier allows to track whether an effect is really due to financing constraints or not. The quantitative analysis here follows the classical portfolio sorting approach well-known
from empirical asset pricing. A first way to evaluate the cross-sectional asset pricing implications of the model is to sort stocks into portfolios on the basis of the value of their multiplier (this is also the approach implemented in the empirical work of Whited and Wu, as well as the simulation approach of Livdan, Sapriz and Zhang). This is important, as one of the difficulties encountered in empirical work is exactly that it is often hard to precisely delineate financial constraints from other frictions.

Table 7 reports the results of a one-way sort of the stocks into portfolios on the basis of their Lagrange multipliers. Stocks are sorted into ten equally-weighted portfolios at the beginning of the year and then rebalanced on a yearly basis. Qualitatively the results confirm the widespread intuition that financially constrained firms should be more risky: The returns on the portfolios are monotonically increasing in the size of the multiplier. This seems consistent with the notion of financially constrained firms obtained in the model: young firms which grow slowly and whose cash-flows have longer duration as they delay dividend payments in order to invest internal funds into profitable investment opportunities. In a way the risk that is being priced here is exactly opposite to the investment irreversibility mechanism usually exploited in the extant cross-sectional asset pricing literature. While there firms find it difficult to scale down in recessions, here there is no impediment to downsizing the capital stock, but rather firms find it difficult to scale up in good times. One should therefore expect such financing constraints risk to display strong cyclicality effects, a prediction shared with Gomes, Yaron, Zhang (2006). Quantitatively, however the effect seems rather small echoing the results obtained by Livdan, Sapriz and Zhang (2006).

Given the intuition developed above about financially constrained firms, one would expect that size should be strongly correlated with financing constraints, which in turn suggest that the well-known size effect in the cross-section might be related to a financing constraints effect. Whited and Wu (2006) and Lamont, Polk and Saá-Requejo (2001) show that after controlling for market capitalization the difference in returns between constrained and unconstrained firms becomes insignificant based on their respective financing constraints index. This is largely confirmed in the model simulations. Table 8 reports results from two-way sorts of stocks into financing constraint and size portfolios following the approach taken by Whited and Wu. The stocks are sorted into 9 portfolios
at the beginning of the year and then rebalanced on a yearly basis as follows. First sort the stocks into small-cap (S), mid-cap (M) and large-cap (B) firms defined as the lowest 40%, the middle 20% and the highest 40% firms in terms of their size (market capitalization). Similarly, sort these portfolios on the basis of their multiplier into the least 40% (L), the middle 20% (M) and the most 40% (H) financially constrained firms. This yields 9 portfolios. Whereas the returns on the portfolios do not exactly match their empirical counterparts, the empirical results quoted above are by and large confirmed. While even controlling for size financially constrained firms earn higher returns the quantitatively the effect seems small, in line with the empirical evidence in Whited and Wu.

Overall the model is consistent with the view that firms exposed more to financing constraints should exhibit higher expected returns and that this effect should be related to the size effect, although in the model the quantitative effects are modest. This reinforces the case that it is crucial to isolate the impact of financing constraints as opposed to financial distress and real frictions, which, when appearing in conjunction, might lead to misleading conclusions.

### 4.6 Aggregate Implications

Although it is not the main focus of the paper and aggregate implications of limited commitment models have been considered elsewhere in the literature (Cooley, Marimon, Quadrini (2004), Monge (2001)), it is worthwhile briefly giving a look at the aggregate implications of the model and to check whether they are in line with what has been found previously. The main implications on the aggregate (or respectively industry) level can be seen from figure 1. Two experiments are conducted there. The images in the top row consider permanent changes in the level of aggregate productivity. In the style of an event study the experiment computes aggregate industry output for a 24 year window around the time of the permanent change, the change being upwards and downwards respectively. The images in the bottom period consider a temporary change either up- or downwards, that is the aggregate productivity jumps for one period and then returns to its steady state level.

While the model is not meant to be quantitatively in line with business cycle facts one
qualitative implication nevertheless comes out clearly from both experiments, namely asymmetry for which there is considerable empirical evidence. The response to a negative shock is sharp and immediate whereas adjustment to a positive shock is slow and gradual. This almost follows mechanically from the setup of the model. While in the absence of adjustment costs to real investment firms can immediately adjust their capital stock downwards, while in order to invest in response to a persistent shock firms on average have to access capital markets. Here clearly the basic mechanism of the limited commitment model kicks in and firms slowly have to build up collateral value in order to be able to obtain the funds for investing. The idea that credit constraints can endogenously generate business cycle asymmetries is not new (Kocherlakota (2000) provides a survey), but it turns out to be surprisingly difficult to make that story tick quantitatively.

The model also delivers some amplification in the sense that the response to a one standard deviation aggregate technology shock is very big in percentage terms, likely too big. This cannot be counted as a success of the model however as it has a simple explanation. A stochastic discount factor that is calibrated to match aggregate asset market data will naturally lead to very high levels of aggregate volatility and therefore responses to shocks will automatically be quantitatively large. However, it has been shown in Cooley, Marimon, Quadrini (2004) that a limited commitment technology can provide a powerful amplification mechanism through general equilibrium forces. In turn, the latter paper will not be consistent with asset market data. This leads back to the classic quantity-price puzzle of macroeconomics and aggregate asset pricing. Since the present paper’s focus is on cross-sectional phenomena, it seeks to match basic asset price statistics.

5 Conclusion

Modern corporate finance theory has emphasized the importance of incentive and agency problems in determining firms’ financial and investment policies. While some of these mechanisms are fairly well understood qualitatively, their quantitative and empirical implications have not been widely explored. The aim of this paper is to take a step in
this direction. To that end, it develops a dynamic model of firms whose financing and investment policies are determined within long-term contracts with outside investors. The contracts are designed to address a classic agency problem, namely an enforcement friction. The model is solved using recursive contracting methodology and its empirical implications are assessed using simulations. Specifically, the results show that dynamic agency models are quantitatively consistent with a number of stylized facts concerning firm dynamics. Moreover, the empirical implications of dynamic agency models can be distinguished from the implications of other classes of models in quantitative corporate finance, namely tax-based trade-off models and reduced form models of costly external finance.

Simulations demonstrate that the model quantitatively matches a wide range of statistics and empirical regressions concerning firm’s financing and investment policies, as well as the cross-section of stock returns. Moreover, the paper suggests that in the context of financing constraints, dynamic agency models have quite distinct empirical implications. Some of the model’s predictions are that financially constrained firms typically are small and young, have low leverage and high market-to-book ratios, that their current cash flows should have additional explanatory power for investment beyond Tobin’s Q and that they should, on average, earn higher expected returns.

These implications are quite distinct from the predictions of both trade-off and reduced form costly external finance models, and similarly, from the proxies and indices for financing constraints commonly used in empirical work. While these results sheds some light on the reasons why the literature on financing constraints has largely remained inconclusive - different financial frictions lead different firm characteristics to be associated with financing constraints -, it also gives a starting point for developing further empirical hypotheses to test dynamic agency models.
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A  Lagrangian

We have

\[ L_0 = E_0 \left\{ \sum_{t=0}^{\infty} m_t [(1 + \bar{\mu}_t) d_t + \tilde{\lambda}_t (\pi_t - i_t - d_t)] \right. \\
- \tilde{q}_t^m (k_{t+1} - (1 - \delta) k_t - i_t) - \tilde{\alpha}_{t+1} (D(k_t) - \sum_{j=1}^{\infty} m_{t+j} d_{t+j}) \right\} \\
\]

\[ = E_0 \left\{ \sum_{t=0}^{\infty} m_t [\tilde{\lambda}_t (\pi_t - i_t) - (\lambda_t - 1 - \mu_t) d_t + \\
- \tilde{q}_t^m (k_{t+1} - (1 - \delta) k_t - i_t) - \tilde{\alpha}_{t+1} D(k_t) + \sum_{j=1}^{\infty} m_{t+j} \alpha_{t+j+1} d_{t+j}] \right\} \\
\]

with

\[ \hat{\nu}_{t+1} = 0 \]
\[ \hat{\nu}_t = \hat{\nu}_{t-1} + m_{t-1} \alpha_{t-1} \]

Now divide by \( \lambda_t \) and the expression in the main text follows.

B  Computation

The individual firm problem is solved using standard value function iteration on a discretized state space. Value function iteration, although rather slow, is convenient here for several reasons. First, it does not require any regularity or differentiability assumptions on the value function, which might be problematic here as a priori the latter is only differentiable almost everywhere. Second in contrast to most other solution techniques it makes occasionally binding constraints (dividend and enforcement constraint here) straightforward to handle. Moreover the method is robust and can be made very precise.

Operationally, the state spaces for \( k, q, x \) and \( z \) are all discretized using finite equally spaced grids. Because the stochastic processes specified for \( x \) and \( z \) are highly persistent,
the procedure proposed by Rouwenhorst is used to transform the continuous random variables into finite state Markov Chains. The bounds of the grids for the state variables \( k \) and \( q \) are chosen in order to make sure that they never bind. The grids for the control variables \( k' \) and \( q' \) are defined on finer grids and linear interpolation is used extensively to determine function values on non-grid points.
Table 1: **Parameter Values**

<table>
<thead>
<tr>
<th>Parameter Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>α</td>
</tr>
<tr>
<td>β</td>
</tr>
<tr>
<td>γ</td>
</tr>
<tr>
<td>δ</td>
</tr>
<tr>
<td>λ</td>
</tr>
<tr>
<td>ξ</td>
</tr>
<tr>
<td>ρ_x</td>
</tr>
<tr>
<td>σ_x</td>
</tr>
<tr>
<td>ρ_z</td>
</tr>
<tr>
<td>σ_z</td>
</tr>
</tbody>
</table>

This table reports parameter choices for the model. In order to be consistent with the empirical asset pricing literature, the model is calibrated at monthly frequency and then evaluated in its ability to match annualized data both at the macro level and in the cross-section. The ideas guiding the calibration procedure is detailed in the main text in section 4.1.
Table 2: Sample Moments

<table>
<thead>
<tr>
<th>Variable</th>
<th>Data</th>
<th>Model total</th>
<th>Model constrained</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual risk-free rate</td>
<td>0.018</td>
<td>0.023</td>
<td></td>
</tr>
<tr>
<td>Annual volatility of risk-free rate</td>
<td>0.03</td>
<td>0.015</td>
<td></td>
</tr>
<tr>
<td>Annual Sharpe ratio</td>
<td>0.430</td>
<td>0.41</td>
<td></td>
</tr>
<tr>
<td>Investment-to-asset ratio</td>
<td>0.145</td>
<td>0.136</td>
<td>0.151</td>
</tr>
<tr>
<td>Volatility of Investment-to-asset ratio</td>
<td>0.139</td>
<td>0.152</td>
<td>0.164</td>
</tr>
<tr>
<td>Market-to-book ratio</td>
<td>1.493</td>
<td>1.671</td>
<td>1.834</td>
</tr>
<tr>
<td>Market leverage</td>
<td>0.230</td>
<td>0.242</td>
<td>0.185</td>
</tr>
</tbody>
</table>

This table reports unconditional sample moments generated from the simulated data of some key variables of the model and compares them with their empirical counterparts. The column labelled model total reports moments from the entire sample and the column labelled constrained reports moments from the most constrained quintile, where sorting is done on the Lagrange multiplier which is computed numerically using the value functions in the simulated data. The first two entries are related to the specification of the pricing kernel and make sure that the model is consistent with aggregate asset market data. The corresponding data are from Campbell, Lo, and McKinlay (1997). The data moments on the investment-to-asset ratio and the market-to-book ratio are taken from Gomes (2001) and Zhang (2005) respectively. Cash flow and Leverage data are taken from Hennessy and Whited (2004) and Covas and Den Haan (2006). All data are annualized.
Table 3: **Leverage Quintiles**

<table>
<thead>
<tr>
<th>Portfolios</th>
<th>most constrained</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>least constrained</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leverage</td>
<td>0.18</td>
<td>0.29</td>
<td>0.32</td>
<td>0.26</td>
<td>0.21</td>
</tr>
</tbody>
</table>

This table reports mean market leverage ratios across constraint quintiles. Firms are sorted into quintiles based on their Lagrange multiplier which is computed numerically using the value functions in the simulated data. A higher lagrange multiplier points to more financially constrained firms.

Table 4: **Leverage Regressions**

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>0.08</td>
<td>0.6</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>-0.22</td>
<td>-0.17</td>
</tr>
</tbody>
</table>

This table reports results from cross-sectional regressions of market leverage ratios on Tobin’s Q and on Tobin’s Q and profitability in the data and on the simulated data. The data are from Rajan and Zingales (1995). The regressions are performed on annualized data to make them better comparable to the empirical literature.

Table 5: **Investment Regressions I**

<table>
<thead>
<tr>
<th>Regression</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficients</td>
<td>$\alpha_1$</td>
<td>$\alpha_2$</td>
</tr>
<tr>
<td>Investment Equation</td>
<td>0.06 (0.01)</td>
<td>0.12</td>
</tr>
<tr>
<td>Cash-Flow Augmented</td>
<td>0.06 (0.01)</td>
<td>0.25</td>
</tr>
</tbody>
</table>

This table reports results from Fama-Macbeth regressions of investment rates on Tobin’s Q (investment regression) and on Tobin’s Q and cash flow rates (cash flow augmented) in the data and on the simulated data. The data are from Gomes (2001). The regressions are performed on annualized data to make them better comparable to the empirical literature. The standard errors are given in brackets.
Table 6: **Investment Regressions II**

<table>
<thead>
<tr>
<th>Regression</th>
<th>Model</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group I</td>
<td>Investment Equation</td>
<td>0.94</td>
<td>0.39</td>
<td>0.39</td>
</tr>
<tr>
<td></td>
<td>Cash-Flow Augmented</td>
<td>0.73</td>
<td>1.46</td>
<td>0.74</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.17)</td>
<td>(0.22)</td>
<td></td>
</tr>
<tr>
<td>Group II</td>
<td>Investment Equation</td>
<td>1.58</td>
<td>0.42</td>
<td>0.42</td>
</tr>
<tr>
<td></td>
<td>Cash-Flow Augmented</td>
<td>0.67</td>
<td>1.85</td>
<td>0.69</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.31)</td>
<td>(0.18)</td>
<td></td>
</tr>
<tr>
<td>Group III</td>
<td>Investment Equation</td>
<td>1.23</td>
<td>0.49</td>
<td>0.49</td>
</tr>
<tr>
<td></td>
<td>Cash-Flow Augmented</td>
<td>0.88</td>
<td>1.06</td>
<td>0.55</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.23)</td>
<td>(0.11)</td>
<td></td>
</tr>
<tr>
<td>Group IV</td>
<td>Investment Equation</td>
<td>0.98</td>
<td>0.46</td>
<td>0.46</td>
</tr>
<tr>
<td></td>
<td>Cash-Flow Augmented</td>
<td>1.36</td>
<td>0.82</td>
<td>0.49</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.12)</td>
<td>(0.34)</td>
<td></td>
</tr>
</tbody>
</table>

This table reports results from standard investment and cash-flow augmented regressions on the simulated data. The sample is split into four quartiles on the basis of their financing constraints. The data are annualized for the regressions. The standard errors are given in brackets. Group one is the most constrained.
Table 7: Portfolio Sorting I

<table>
<thead>
<tr>
<th>Portfolios</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Returns</td>
<td>0.73</td>
<td>0.77</td>
<td>0.85</td>
<td>0.91</td>
<td>0.93</td>
<td>0.99</td>
<td>1.02</td>
<td>1.07</td>
<td>1.16</td>
<td>1.23</td>
</tr>
</tbody>
</table>

This table reports average returns from sorting stocks into then portfolios on the basis of their financing constraints as measured by the Lagrange multiplier of the enforcement constraint. The table reports equal weighted average monthly returns in excess of the risk free rate. Portfolios are rebalanced once a year.

Table 8: Portfolio Sorting II

<table>
<thead>
<tr>
<th>Category</th>
<th>Simulated Data</th>
<th>Whited and Wu</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small-Cap Firms</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low FC</td>
<td>SL</td>
<td>1.14</td>
</tr>
<tr>
<td>Middle FC</td>
<td>SM</td>
<td>1.05</td>
</tr>
<tr>
<td>High FC</td>
<td>SH</td>
<td>1.32</td>
</tr>
<tr>
<td>Mid-Cap Firms</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low FC</td>
<td>ML</td>
<td>1.06</td>
</tr>
<tr>
<td>Middle FC</td>
<td>MM</td>
<td>1.13</td>
</tr>
<tr>
<td>High FC</td>
<td>MH</td>
<td>1.09</td>
</tr>
<tr>
<td>Large-Cap Firms</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low FC</td>
<td>BL</td>
<td>0.91</td>
</tr>
<tr>
<td>Middle FC</td>
<td>BM</td>
<td>0.88</td>
</tr>
<tr>
<td>High FC</td>
<td>BH</td>
<td>0.96</td>
</tr>
</tbody>
</table>

This table reports results from double sorting stocks on the basis of their size (market capitalization) and their financing constraints into 9 portfolios. Portfolios are rebalanced once a year. The data is from Whited and Wu (2006). The returns are in average monthly percent in excess of the risk-free rate as in Whited and Wu.
Figure 1: Aggregate Responses to permanent and temporary shocks

This figure reports results from two event studies. It computes industry output in 24 year windows around permanent changes in productivity (top row) and temporary changes in productivity (bottom row).