A Dominant-Strategy Asset Market Mechanism

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Abstract

Asset markets—institutions that reallocate goods among agents with heterogeneous endowments, demands, and valuations—abound in the real world but have received little attention in mechanism and market design. Assuming constant marginal, private values and known endowments and maximum demands, we provide a detail-free, dominant-strategy asset market mechanism that allocates efficiently or close to efficiently, respects traders’ individual rationality constraints ex post, and never runs a deficit. If it does not allocate efficiently, it sacrifices the trades that under efficiency would involve the lowest-value trader who efficiently would be allocated a positive amount. The mechanism always allocates the quantity traded efficiently and permits clock implementation. As the market becomes large, the mechanism’s efficiency loss converges to zero under natural conditions.

Keywords: market mechanism, endogenous trading position, deficit-free, detail-free

JEL Classification: C72, D44, L13

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1 Introduction

Problems of reallocating goods among agents who differ with respect to their valuations, endowments, and maximum demands abound in the real world. Examples include reallocating airport slots, emission and pollution permits, water rights, land, spectrum licenses, liquidity among banks, and houses. A defining feature of these problems is that an agent’s trading position—buy, hold, or sell—is determined endogenously. For example, an agent who owns some emission permits may under efficiency sell these if his valuation is low, buy additional units if his valuation is high, and not trade at all in other instances. Notwithstanding their practical relevance, mechanisms for allocation problems of this form have received so little attention from market designers that there is not even a standard terminology to refer to these problems.\footnote{Lu and Robert (2001) refer to these as problems with “ex ante unidentified traders.” More recently, Johnson (2019) uses the term “one-sided” to refer to these problems. This is useful to distinguish them from the “two-sided” allocation problems that are studied in the literature on double auctions, but it collides with the usage of the terms “one-sided” and “two-sided” as referring to auctions and double auctions with private information on only one side versus on both sides of the market (Loertscher et al., 2015). Moreover, the defining feature of these allocation problems is that a priori agents have no “sides” insomuch as an agent cannot, in general, say whether he wants to register as a buyer or a seller.} For the purpose of this paper, we call them asset markets.

We provide a detail-free, dominant-strategy mechanism for asset markets that allocates efficiently or close to efficiently, respects traders’ individual rationality constraints ex post, and never runs a deficit. We assume agents have constant marginal, private values, are endowed with some units of the good, and have a maximum quantity demanded, where endowments and maximum demands are common knowledge. However, traders’ types—their willingness to pay for additional units if they buy or their opportunity costs of selling if they sell—are traders’ private information. These assumptions imply that there are natural conditions under which all agents, or all but one, are active under efficiency, even with a large number of traders.\footnote{This is, for example, the case when all agents have different types and positive endowments that are less than their maximum demands.} This contrasts with two-sided environments in which buyers with low values and sellers with high costs are typically inactive.

Our mechanism is detail free because it makes no assumptions about the process from which traders’ values originate. It allocates either efficiently or close to efficiently. The only lost trades if it does not allocate efficiently are those that would under efficiency involve the lowest-value trader who is allocated a positive amount under efficiency. The trades that do occur are efficient because they procure the units traded at minimal cost and allocate them to maximize value.

The following example illustrates how the mechanism works. Assume there are three agents, each of whom has an endowment of one and a maximum demand for two units.
Consider the direct mechanism that asks every agent to report his type, then ranks agents according to their reports, breaking ties if required according to some pre-determined rule, and has the agent with the lowest report sell to the agent with the highest report at a price equal to the intermediate report. It is immediate that, relative to reports, this mechanism allocates efficiently because the trade that is executed is the same trade that a Walrasian auctioneer would execute. Further, not only is the trade the same, but also the price at which it occurs is the same as the price that a Walrasian auctioneer would set.

To see that it is dominant-strategy incentive compatible for agents to report truthfully and that the mechanism is ex post individually rational, notice first that an agent obtains a nonnegative payoff when reporting truthfully. Observe next that the utility of every agent remains the same as long as his report does not affect his trading position. Dominant-strategy incentive compatibility then follows because every agent obtains a weakly negative payoff if his report changes his trading position from his trading position under truthful reporting, and strictly negative in the absence of ties.

Of course, this example is nongeneric in almost all dimensions. Nevertheless, it contains the central pieces of our mechanism, which is naturally called a trade-sacrifice mechanism. First, the agent with the lowest value who under efficiency is allocated a positive quantity is precluded from trading. Second, the type of this agent is equal to the (highest) Walrasian price given the reports. Last, when the quantity demanded is equal to the quantity supplied at a price equal to the report of the excluded agent, as is the case in the example, the mechanism takes no further action and all trades occur at this price. If the quantity demanded and supplied differ at this price, then the short side trades at this price, which thus determines the quantity traded, while the long side competes in a multi-unit Vickrey auction with a reserve equal to this price, yielding an efficient allocation of the quantity traded.

We show how the mechanism can be implemented dynamically using a clock auction, which has been deemed useful for practical purposes (see, e.g., McAfee, 1992; Ausubel, 2004; Milgrom and Segal, forth.). In addition, under natural assumptions on the process that generates types and restrictions on the endowments and maximum demands, we show that the efficiency loss of the mechanism goes to zero at rate $1/m$, where $m$ is the number of replicas of the market.

Because trading positions in asset markets are endogenous and whoever does not act as a net buyer typically trades as a net seller, asset markets may be particularly susceptible

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3If all reports are the same under truthful reporting, then all agents' payoffs are zero given these reports, and each agent’s payoff remains zero if he deviates unilaterally. If only two reports are the same and one of the two agents who, under truthful reporting, was part of the tie, makes a report that induces his trading position to be swapped with that of the agent with whom he tied under truthful reporting, then his payoff is zero before and after the deviation. Otherwise, his payoff is negative.

4In the example, the Walrasian price is unique, but in general this need not be the case.
to manipulation. As a case in point, consider the LIBOR, which is a benchmark interest rate calculated on the basis of reports by 16 panel banks, each of which could be, at any point in time, either a net borrower or a net lender. As noted by Abrantes-Metz et al. (2012, p. 137), “From a financial perspective, banks that are net borrowers would benefit from lower rates, while banks that are net lenders would benefit from higher rates.” Scope for manipulation arises because the reports of the banks are converted into the LIBOR through a simple arithmetic average of the middle 8 reports (BBA, 2012). Indeed, billions of dollars in penalties have been imposed on panel banks for attempted manipulation and false reporting concerning LIBOR and other benchmark interest rates (Eisl et al., 2017).

The mechanism design literature on asset markets is scant and confined to papers that recognize that the partnership model of Cramton et al. (1987) can equivalently be interpreted as an asset market in which agents’ ownership shares serve as their endowments. This includes McAfee (1991), Lu and Robert (2001), Gershkov and Schweinzer (2010), Johnson (2019), and Loertscher and Wasser (2019). The standard assumption in this literature is that the designer and agents know the distributions from which types are drawn, and the focus is typically on mechanisms that are Bayesian incentive compatible and interim individually rational. We provide a mechanism that is detail-free, endows agents with dominant strategies, and respects their individual rationality constraints ex post.\(^5\) Because partnership models relate to the literature on the possibility or impossibility of efficient trade, our paper is, of course, also related to that literature. The discussion of the relationship to this strand of literature is most insightful if we can connect it to the specifics of our model. We therefore defer it to Section 6.

Our paper is directly related to the literature on dominant-strategy auctions and double auctions initiated by Vickrey (1961), with subsequent contributions by McAfee (1992), Ausubel (2004), Milgrom and Segal (forth.), and Loertscher and Mezzetti (2018). As in double auction settings with two-sided private information, asset market problems give rise to a nontrivial price discovery problem. McAfee (1992) was the first to introduce a trade-sacrifice mechanism. He did so for a two-sided setting with buyers and sellers with single-unit demands and supplies, respectively. When his mechanism sacrifices a trade, it therefore necessarily precludes two agents from trading who would trade under efficiency. In contrast, our mechanism, as matter of principle, only precludes one agent from trading.\(^6\)

\(^5\) Assuming an infinite time horizon and the sequential arrival of traders, Loertscher and Muir (2019) study the efficient adjustment dynamics for a dynamic asset market without requiring Bayesian incentive compatibility. However, because of the dynamic nature of the problem, their efficient mechanism makes use of distributional assumptions.

\(^6\) Depending on the specifics of capacities and endowments, this may reduce the quantity that another agent trades without completely preventing him from trading, or it may induce other agents to become inactive who would be active traders under efficiency. Because McAfee (1992) assumes single-unit demands
In both our approach and that of Loertscher and Mezzetti (2018), the market maker aims to identify a Walrasian price, which is then used as a reserve in a Vickrey (or Ausubel) auction to efficiently allocate the quantity from the short side of the market to the long side. Our setup is both more complicated in that it is an asset market (and hence has no predetermined sides or trading positions) and simpler because, even though we have multi-unit traders, our traders have one-dimensional types and constant marginal values. This allows the designer to identify a Walrasian price from agents’ reported types without any uncertainty and to use it as a reserve without violating dominant-strategy incentive compatibility.

The remainder of this paper is organized as follows. Section 2 introduces the setup. Section 3 defines our mechanism and derives its properties for finite numbers of agents. Section 4 discusses clock-auction implementation, and Section 5 analyzes asymptotic properties. In Section 6, we expand on the discussion of the relation of our paper to the literature. Section 7 concludes the paper. Proofs are contained in the appendix.

2 Model

There are $N$ agents, and each agent $i \in \mathcal{N} \equiv \{1, \ldots, N\}$ has a type $\theta_i \in \Theta \subset \mathbb{R}$ that is his private, constant marginal value for units of a homogeneous good. Let $r_i \in \{0, 1, \ldots\}$ be agent $i$’s endowment of the good and $k_i \in \{r_i, r_i + 1, \ldots\}$ be his maximum demand or capacity. We assume that $k_i > r_i$ for at least one $i$ and, without loss of generality, $k_i = r_i = 0$ for no $i$. The total endowment (or maximum supply) $R$ and maximum demand $K$ in this economy are then

$$R \equiv \sum_{i \in \mathcal{N}} r_i < \sum_{i \in \mathcal{N}} k_i \equiv K.$$  

As is standard, we assume that the designer knows $r = (r_i)_{i \in \mathcal{N}}$ and $k = (k_i)_{i \in \mathcal{N}}$ but not the agents’ types $\theta = (\theta_i)_{i \in \mathcal{N}}$.

Supply $S(p)$ and demand $D(p)$ at price $p$ depend on the agents’ types and are interval-valued, with the interval being nondegenerate when the price is equal to some agent’s type, thereby making that agent indifferent about trade. We have:

$$S(p) \equiv \left[ \sum_{i \in \mathcal{N} \text{ s.t. } \theta_i < p} r_i, \sum_{i \in \mathcal{N} \text{ s.t. } \theta_i \leq p} r_i \right]$$

and supplies, sacrificing one trade makes exactly two agents inactive who would trade under efficiency.
and
\[
D(p) \equiv \left[ \sum_{i \in N \text{ s.t. } \theta_i \geq p} k_i - r_i, \sum_{i \in N \text{ s.t. } \theta_i > p} k_i - r_i \right].
\]

A Walrasian (or market clearing) price \( p^\ast \) is such that
\[
D(p^\ast) \cap S(p^\ast) \neq \emptyset.
\] (1)

Figure 1 illustrates that both uniqueness and nonuniqueness of Walrasian prices are generic in this environment. In panel (a), the set of Walrasian prices is the interval \([\theta_3, \theta_2]\), that is, \( p^\ast \) is a market clearing price if and only if \( p^\ast \in [\theta_3, \theta_2] \). In contrast, in panel (b) we have \( p^\ast = \theta_2 \).

To address ties among agents’ types, we assume the existence of a tiebreaking rule with the property that the resolution of a tie between two agents is not affected by the reports of any other agents. Specifically, we assume the existence of a strict order \( \prec \) (with complementary strict order \( \succ \)) on the set of agent-type pairs, \( \mathcal{N} \times \Theta \)\(^7\), such that for all agents \( i \neq j \): \( \theta_i < \theta_j \) implies \( (i, \theta_i) \prec (j, \theta_j) \); \( \theta_i > \theta_j \) implies \( (i, \theta_i) \succ (j, \theta_j) \); and \( \theta_i = \theta_j \) implies either \( (i, \theta_i) \prec (j, \theta_j) \) or \( (i, \theta_i) \succ (j, \theta_j) \), but not both.

Given \( \Theta \), let \( x^\ast(\Theta) = \{x^\ast_i(\Theta)\}_{i \in \mathcal{N}} \) denote the efficient allocation under the tiebreaking rule, i.e., \( x^\ast(\Theta) \) solves
\[
\max_x \sum_{i \in \mathcal{N}} x_i \theta_i \quad \text{subject to } \forall i \in \mathcal{N}, \ x_i \in \{0, ..., k_i\}, \text{ and } \sum_{i \in \mathcal{N}} x_i = R,
\]
and, if we have agents \( i \neq j \) with \( \theta_i = \theta_j \) and \( (i, \theta_i) \prec (j, \theta_j) \), then \( x^\ast_i(\Theta) > 0 \) implies

\(^7\)As strict orders, \( \prec \) and \( \succ \) are irreflexive, antisymmetric, and transitive.
Thus, any allocations to agents with the same type are directed first to agents ranked higher according to the tiebreaking rule.

Given a vector of types \( \theta \), let \( p^*(\theta) \) be the smallest and \( \overline{p}^*(\theta) \) be the largest market clearing price. If \( p^*(\theta) = \overline{p}^*(\theta) \), so that there is a single market clearing price, denoted \( p^*(\theta) \), then there exists an agent \( j \) with type \( \theta_j = p^*(\theta) \) such that for all \( i \in \mathcal{N} \),

\[
x^*_i(\theta) = \begin{cases} 
k_i & \text{if } (i, \theta_i) \succ (j, \theta_j), 
0 & \text{if } (i, \theta_i) \prec (j, \theta_j), 
R - \sum_{\ell \text{ s.t. } (\ell, \theta_\ell) \succ (j, \theta_j)} k_\ell & \text{if } i = j.
\end{cases}
\]

In contrast, if the Walrasian prices are multi-valued, that is, if \( p^*(\theta) < \overline{p}^*(\theta) \), then there is no agent \( i \) with \( \theta_i \in (p^*(\theta), \overline{p}^*(\theta)) \) and for all \( i \in \mathcal{N} \),

\[
x^*_i(\theta) = \begin{cases} 
k_i & \text{if } \theta_i \geq \overline{p}^*(\theta), 
0 & \text{if } \theta_i \leq p^*(\theta).
\end{cases}
\]

## 3 Trade-sacrifice mechanism and its properties

The \textit{trade-sacrifice mechanism (TSM)} is a direct mechanism that elicits reports from all agents about their types and then operates in four steps. First, it identifies as \textit{cutoff agent} the agent with the lowest type who under efficiency consumes a positive amount. In the TSM, this agent does not trade. This implies that the TSM “sacrifices” trades that under efficiency involve the cutoff agent. We refer to agents with reported types greater than the cutoff agent’s reported type as potential buyers and agents with reported types less than the cutoff agent’s reported type as potential sellers.

Second, the TSM compares the demand by potential buyers to the supply by potential sellers at a price equal to the cutoff agent’s reported type. This identifies the short and long sides of the market, calling, arbitrarily, the seller side the long side if demand and supply are equal: if the vector of reported types is \( \hat{\theta} \) and agent \( j \) is the cutoff agent, then the potential buyers are on the long side if

\[
\sum_{\ell \text{ s.t. } (\ell, \hat{\theta}_\ell) \succ (j, \hat{\theta}_j)} (k_\ell - r_\ell) > \sum_{\ell \text{ s.t. } (\ell, \hat{\theta}_\ell) \prec (j, \hat{\theta}_j)} r_\ell.
\]

Otherwise, the potential sellers are on the long side.
Third, the TSM has agents on the short side trade at the cutoff agent’s reported type, satisfying all of their demand or supply at that price.

Fourth, given the quantity $Q$ traded with agents on the short side, the TSM makes the balancing trades for the same quantity $Q$ with agents on the long side of the market: if the potential buyers are on the long side, then the TSM sells $Q$ units to them at the prices that would emerge from a multi-unit Vickrey auction with a reserve equal to the cutoff agent’s reported type (see, e.g., Krishna, 2002), which we refer to as dominant-strategy prices with a reserve equal to the cutoff agent’s reported type. The formal definition is in the appendix. If the potential sellers are on the long side, then the TSM procures $Q$ units from them using a multi-unit Vickrey auction with a reserve equal to the cutoff agent’s reported type, which we also refer to as dominant-strategy prices with a reserve equal to the cutoff agent’s reported type. Using the properties of Vickrey prices (see, e.g., Krishna, 2002, Chapter 12), given $Q$, price $p$, and set of agents $A$, reporting truthfully is a dominant strategy in the game that uses dominant strategy prices with reserve $p$ to sell $Q$ units to agents in $A$. Similarly, reporting truthfully is a dominant strategy in the game that uses dominant-strategy pricing with reserve $p$ to buy $Q$ units from agents in $A$.

As mentioned, the TSM depends on a cutoff agent, which we define now. Given the vector of agents’ reported types $\hat{\theta}$, the cutoff agent is the lowest ranked agent according to $\prec$ for whom $x^*_j(\hat{\theta}) > 0$. Thus, if there exists an agent $j$ with $x_j(\hat{\theta}) \in (0, k_j)$, then agent $j$ is the cutoff agent; and, if there is no such agent, then the cutoff agent is agent $j$ with $x_j^*(\hat{\theta}) = k_j$ such that $(i, \theta_i) \prec (j, \theta_j)$ implies $x_i^*(\hat{\theta}) = 0$.

It follows that there is always a cutoff agent. The cutoff type $j$ is always in the set of Walrasian prices, and that price is unique if $x_j^*(\theta) \in (0, k_j)$. In contrast, if $x_j^*(\theta) = k_j$, then the cutoff type $\theta_j$ is equal to $p^*(\theta)$. In addition, the cutoff agent has the property that $x_i^*(\hat{\theta}) = k_i$ for all agents $i$ with agent-type pairs ranked above the cutoff agent according to $\prec$, and $x_j^*(\hat{\theta}) = 0$ for all agents $i$ with agent-type pairs ranked below the cutoff agent according to $\prec$.

To illustrate, consider the examples of Figure 2 and assume truthful reporting. In Figure 2(a), there is no agent $j$ with $x_j^*(\theta) \in (0, k_j)$. Agent 3 is the lowest type agent with $x_3^*(\theta) = k_3$, so agent 3 is the cutoff agent. Net demand by agents with types above $\theta_3$ at a price of $\theta_3$ is 3, and supply by agents with types below $\theta_3$ at a price of $\theta_3$ is 3. Because net demand and supply are in balance, under the TSM, agents 1 and 2 purchase the units that they demand at a price of $\theta_3$, and those units are procured from agents 4 and 5 using dominant-strategy prices with a reserve of $\theta_3$. In Figure 2(b), agent 3 is again the cutoff agent, but in that case $x_3^*(\theta) \in (0, k_3)$. At a price equal to $\theta_3$, there is demand of 4 by agents other than the cutoff agent and supply of 3 by agents other than the cutoff agent. Because the seller side is the short side of the market, the TSM
Figure 2: Panel (a): The cutoff agent is agent 3 with \(x_3^* (\theta) = k_3\), and we have a range of market clearing prices, i.e., \(p^* (\theta) = \theta_4 < \theta_3 = \theta_5\). Demand and supply at price \(\theta_3\) by agents other than the cutoff agent are balanced. Panel (b): The cutoff agent is agent 3 with \(x_3^* (\theta) \in (0, k_3)\), and the cutoff agent’s type, \(\theta_3\), is the unique market clearing price, i.e., \(p^* (\theta) = \theta_3\). At price \(\theta_3\), there is excess demand by agents other than the cutoff agent.

purchases 3 units from agents 4 and 5 at a price of \(\theta_3\) and sells those units to agents 1 and 2 using dominant-strategies prices with a reserve equal to \(\theta_3\). Two of the 3 units are allocated to agent 1, and the remaining unit is allocated to agent 2. Agent 2 pays \(\theta_3\) for his unit, and agent 1 pays a total of \(\theta_2 + \theta_3\) for his two units.\(^9\)

**Theorem 1** The TSM endows agents with dominant strategies to report truthfully, is ex post individually rational, allocates the quantity traded efficiently, and never runs a deficit.

The proof of Theorem 1 makes use of an intuitive monotonicity property of the cutoff agent’s type: As one agent’s report increases from below to above \(\theta_j\), where \(\theta_j\) is the cutoff type before the increase, the new cutoff type will be at least as large as \(\theta_j\). This monotonicity implies that no agent can benefit from switching sides by misreporting his type. Because a cutoff type has a payoff of zero, misreporting to become the cutoff agent also cannot be profitable. This leaves as the only potentially profitable deviations those that keep an agent on the same side that he is on under truthful reporting. Whether this side is long or short does not depend on his report, provided the report is on the same side of the cutoff type as his true type. If the agent is on the short side, any report that keeps him on this side does not affect his payoff because he simply trades the same quantity at a price equal to the cutoff type. If he is on the long side, by the properties of dominant strategies prices, any report that keeps him on this side cannot make him better off than reporting his type truthfully.

Theorem 1 implies that the TSM offers a dominant-strategy mechanism for reallocation among a set of agents that never runs a deficit. It remains to explore the efficiency

\(^9\)With 3 units available, each of agents 1 and 2 is guaranteed to get at least 1 unit. So each pays the reserve, \(\theta_3\), for his “first” unit. The allocation of the remaining unit to agent 1 (rather than agent 2) imposes an externality on agent 2 in the amount of agent 2’s type, so agent 1 pays \(\theta_2\) for his “second” unit.
properties of this mechanism, but as is clear from its construction, the only trades that are lost in the TSM relative to the efficient allocation are the trades of the cutoff agent. Thus, no gains from trade are lost under the TSM if the cutoff agent consumes exactly his endowment under efficiency, which gives us the following corollary.

**Corollary 1** No gains from trade are lost under the TSM if and only if \( x_j^*(\theta) = r_j \).

For the symmetric setup, where all agents have endowment \( r > 0 \) and maximum demand \( k > r \), the total supply is \( Nr \). In this setup, we let

\[
a = \frac{Nr}{k}
\]

(2)
denote the number of agents (including potentially fractional numbers of agents) whose demand can be satisfied based on the total supply of \( Nr \), implying that \( [a] \) is the integer number of agents whose demand is fully satisfied under efficiency.

**Corollary 2** In the symmetric setup, \( x_j^*(\theta) = r \) if and only if \( k(a - [a]) = r \).

Using Corollary 2 it follows that in the symmetric setup with \( r = k/2 \), there exists a deficit-free, dominant-strategy incentive compatible, ex post individually rational mechanism that is efficient when \( N \) is odd and that, when \( N \) is even, precludes trade from only one buyer and one seller who would trade under efficiency.

### 4 Clock-auction implementation

The TSM can be implemented via a clock auction by using an increasing clock that starts at a price below the lowest type so that all agents are active initially. As the clock increases, agents can exit, where a decision to exit is permanent.

The state of the clock auction is given by the clock price and the set of remaining active agents \((p, A)\). The state is updated following each exit. If the set of active agents prior to the clock price reaching \( p \) is \( A \) and agent \( i \) exits at clock price \( p \), then following the exit, the state is updated to \((p, A\{i\})\). If multiple agents exit at the same clock price, then they are processed in sequence according to the tiebreaking rule embodied in \( \prec \), proceeding from the lowest ranked to the highest ranked. Thus, if the set of active agents prior to the clock price reaching \( p \) is \( A \) and agents \( i \) and \( j \) both exit at price \( p \), where \((i, p) \prec (j, p)\), then the state is first updated to \((p, A\{i\})\) and then to \((p, A\{i, j\})\) before the clock price continues to increase. This allows for the possibility that the clock auction could end following the exit of agent \( i \) and with agent \( j \) still remaining in the set of active agents.
When the clock auction begins, the set of active agents is equal to $\mathcal{N}$, and so the demand by active agents is greater than the total supply: $\sum_{i \in \mathcal{N}} k_i = K > R$. The clock continues to increase until all agents have exited or there is an exit by agent $j$ such that the set of active agents following agent $j$’s exit, $\mathcal{A}$, satisfies $\sum_{i \in \mathcal{A}} k_i < R$.

Given that the clock auction ends at price $p$ following agent $j$’s exit, with agents in $\mathcal{A}$ remaining active, let $\hat{p}_i$ be the price at which agent $i \notin \mathcal{A}$ exited and let $\bar{p} = (\hat{p}_i)_{i \in \mathcal{N}}$ with

$$\hat{p}_i \equiv \begin{cases} \hat{p}_i, & \text{if } i \notin \mathcal{A} \\ p, & \text{if } i \in \mathcal{A}. \end{cases}$$

Then we have $x^*_i(\bar{p}) = 0$ for all agents $i$ exiting prior to agent $j$. That is, agents exiting prior to agent $j$ would have an efficient allocation of zero under the assumptions that these agents exited at clock prices equal to their types and that the remaining active agents have types of at least $p$. Further, for the final agent to exit, we have $x^*_j(\bar{p}) > 0$, so that agent $j$ is the TSM’s cutoff agent. As this shows, one can identify the TSM’s cutoff agent while preserving the privacy of the buyer-side agents, where preserving privacy means that it does not require the revelation of the agent’s type other than that it is above some threshold (which here is the cutoff agent’s type).

When the stopping condition is satisfied, the total net demand of the remaining active agents at the final clock price $p$ is compared to the total supply of the exited agents other than the cutoff agent. If there is weak excess supply, then the auctioneer sells at price $p$ to the remaining active agents and procures the required amount from the exited agents other than the cutoff agent using dominant-strategy prices with a reserve of $p$, using agents’ exit prices as their reported types. If there is excess demand, then the auctioneer purchases at price $p$ from the exited agents other than the cutoff agent and then allocates those units to the remaining active agents using an Ausubel (2004) clinching auction that starts from price $p$. Because the Ausubel auction makes sincere bidding—in our setup “sincere bidding” means net demanding $k_i - r_i$ units until the price reaches $\theta_i$ and then reducing demand to zero—a dominant strategy, we have the following result:

**Proposition 1** The clock-auction implementation of the TSM induces agents to exit at their types as a dominant strategy.

## 5 Asymptotics

A natural question is whether the loss of gains from trade associated with the TSM (if there is any, that is) vanishes as the number of agents increases and, if so, at what rate. For

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10See the appendix for an illustration of this application of the Ausubel auction.
the purposes of considering this question, we assume the existence of \( N \) agents with types, demands, and endowments \((\theta_{1,1}, k_1, r_1), ..., (\theta_{N,1}, k_N, r_N)\), where for all \( i \in \mathcal{N}, k_i > r_i > 0 \), and where \( \theta_{i,1} \) is an iid draw from \( F \), which has a density \( f \) that is positive everywhere on the support \([\underline{\theta}, \overline{\theta}]\), where \( \underline{\theta} > 0 \). We increase the set of agents by replicating this initial set of \( N \) agents. We assume that each time we add \( N \) agents, these agents have the same capacities \( k \) and the same endowments \( r \) as the initial set of \( N \) agents. However, all types are iid draws from \( F \). So, for example, with two replicas, the environment is characterized by \( 2N \) agents with the properties: \((\theta_{1,1}, 1, k_1, r_1), ..., (\theta_{N,1}, 1, k_N, r_N)\), \((\theta_{1,2}, 1, k_1, r_1), ..., (\theta_{N,2}, 1, k_N, r_N)\), where all types are iid realizations from \( F \). With \( m \) replicas, we have \( mN \) agents characterized by \( \{(\theta_{i,t}, k_i, r_i)\}_{i \in \mathcal{N}, t \in \{1, ..., m\}} \). As above, we assume there is a tie-breaking rule defined by a strict order on the set of agent-type pairs.

We now show that the loss of gains from trade converges in probability to zero at rate \( 1/m \), where \( m \) is the number of replicas.\(^{11}\) Specifically, given \( m \in \{1, 2, ..., \} \) and \( \theta \in [\underline{\theta}, \overline{\theta}]^{mN} \), and letting \( L_m(\theta) \geq 0 \) be the efficiency loss under the TSM when there are \( m \) replicas and the type vector is \( \theta \), we show that there exists a constant \( c > 0 \) such that for all \( \epsilon > 0 \), \( \lim_{m \to \infty} \Pr \left( |m \cdot L_m(\theta) - c| > \epsilon \right) = 0. \)

Define \( \hat{q}_m(\theta) \) to be the rank of the cutoff agent with \( m \) replicas when the type realization is \( \theta \).\(^{12}\) This agent does not trade in the TSM. If the \( \hat{q}_m(\theta) \)-th indexed agent consumes exactly his endowment under efficiency, then there is no inefficiency in the TSM for that type realization. The following lemma provides a lower bound for the rank of the cutoff agent when there are \( m \) replicas and implies that the rank of the cutoff agent goes to infinity as \( m \) goes to infinity.

**Lemma 1** There exists \( d \in (0, 1) \) such that for all \( m \in \{1, 2, ...\} \) and \( \theta \in [\underline{\theta}, \overline{\theta}]^{mN} \), \( \hat{q}_m(\theta) \geq dmN \).

If the cutoff agent consumes strictly more than his endowment under efficiency, then the agent has net purchases under efficiency of at most \( \max_{i \in \mathcal{N}} \{k_i - r_i\} \). Because we assume that \( \min_{i \in \mathcal{N}} r_i > 0 \), those purchases can be supplied by \( \hat{\ell} \) agents, where

\[
\hat{\ell} = \left\lceil \frac{\max_{i \in \mathcal{N}} \{k_i - r_i\}}{\min_{i \in \mathcal{N}} r_i} \right\rceil.
\]

When the cutoff agent consumes strictly more than his endowment under efficiency, this gives us an upper bound for the expected efficiency loss in the environment that is repli-

\(^{11}\)This is consistent with the result of McAfee (1992) that inefficiency in his setup vanishes at a rate equal to 1 divided by the number of agents.

\(^{12}\)To avoid cluttered notation, we do not explicitly distinguish between realizations and random variables. It will be clear from the context what \( \theta \) stands for.
cated $m$ times of

$$\hat{B}_m \equiv \max_{i \in \mathcal{N}} \{k_i - r_i\} \mathbb{E}_\theta \left[ \theta(q_m;mN) - \theta(q_m+\ell;mN) \right],$$

where $\theta(i;n)$ denotes the $i$-th highest of $n$ independent draws from $F$.

If the cutoff agent consumes strictly less than his endowment under efficiency, then the cutoff agent has net sales under efficiency of at most $\max_{i \in \mathcal{N}} r_i$. Those sales can be absorbed by $\tilde{\ell}$ agents, where

$$\tilde{\ell} \equiv \left\lceil \max_{i \in \mathcal{N}} r_i / \min_{i \in \mathcal{N}} (k_i - r_i) \right\rceil.$$

Thus, when the cutoff agent consumes strictly less than his endowment under efficiency, an upper bound for the expected efficiency loss in the environment that is replicated $m$ times is

$$\tilde{B}_m \equiv \max_{i \in \mathcal{N}} r_i \mathbb{E}_\theta \left[ \theta(q_m - \tilde{\ell};mN) - \theta(q_m;mN) \right].$$

It follows then that $\mathbb{E}_\theta[L_m(\theta)]$ is bounded above by $\max\{\hat{B}_m, \tilde{B}_m\}$. Using this bound along with Lemma 1 from [Loertscher and Marx (2019)](Loertscher and Marx (2019)), which relates spacings between order statistics to hazard rates, we can prove the following result:

**Proposition 2** If $\min_{i \in \mathcal{N}} r_i > 0$, then the expected efficiency loss of the TSM converges in probability to zero at rate $1/m$ as the number $m$ of replicas goes to infinity.

The analysis above can be extended to allow agents’ endowments and maximum demands to vary across replicas by assuming that for agent $i$ in replica $j$, $k_{i,j} \in \{\underline{k}, \ldots, \overline{k}\}$ and $r_{i,j} \in \{\underline{r}, \ldots, \overline{r}\}$, with $\underline{r} \leq \overline{r} \leq \underline{k} \leq \overline{k}$, are drawn independently from discrete distributions with finite means $\mu_k > \mu_r$ (according to Lebesgue integration), so that the strong law of large numbers holds. Then with $m$ replicas, the share of the demand of the $mN$ agents that can be met, $Q_m \equiv \sum_{j=1}^{m} \sum_{i=1}^{N} r_{i,j} / \sum_{j=1}^{m} \sum_{i=1}^{N} k_{i,j}$, converges with probability one to $\mu_r / \mu_k$. To see this, note that by the strong law of large numbers,

$$\frac{\sum_{j=1}^{m} \sum_{i=1}^{N} r_{i,j}}{mN} \to \mu_r \text{ and } \frac{\sum_{j=1}^{m} \sum_{i=1}^{N} k_{i,j}}{mN} \to \mu_k \neq 0 \text{ a.s. as } m \to \infty.$$

Then, (it can be easily verified that),

$$Q_m = \frac{\sum_{j=1}^{m} \sum_{i=1}^{N} r_{i,j} / mN}{\sum_{j=1}^{m} \sum_{i=1}^{N} k_{i,j} / mN} \to \frac{\mu_r}{\mu_k} \text{ a.s. as } m \to \infty.$$

In the limit, $Q_m$ is the share of the $mN$ agents with $m$ replicas who are fully served.
Because that share is constant with probability one for all replicas beyond some \( m \), then as in Lemma 1, the rank of the cutoff agent with \( m \) replicas goes to infinity with \( m \), and the analysis above applies.

6 (Im)possibility of efficient trade in asset markets

As mentioned in the introduction, asset market models are variants of the partnership models introduced by Cramton et al. (1987), with subsequent generalizations by Che (2006), Figueroa and Skreta (2012), and Loertscher and Wasser (2019), among many others. As such, our model is related to the literature on the (im)possibility of ex post efficient trade initiated by Vickrey (1961) and Myerson and Satterthwaite (1983), with subsequent generalizations by, among many others, Williams (1999) and Delacrétaz et al. (2019). Much of this literature presumes commonly known distributions and focuses on Bayesian incentive compatibility and expected budget surplus (or deficit). In contrast, our focus is on dominant strategies and no deficit ex post. Nevertheless, some additional discussion is useful.

The literature just mentioned raises the interesting and relevant question of under which conditions one obtains the possibility or impossibility of efficient trade in asset markets. However, as we now show, the answer will depend on the finer details of the environment. Specifically, we provide a simple example that illustrates that whether or not ex post efficiency is possible without running a deficit, subject to incentive compatibility and individual rationality, depends on assumptions about endowments, maximum demands, and distributions. Rather than studying conditions under which fully efficient trade obtains (or is impossible), we have instead provided an incentive compatible, individually rational mechanism that always avoids a deficit and allocates efficiently or close to efficiently.

To illustrate, assume that there are three traders, each drawing a type \( \theta_i \in [\underline{\theta}, \overline{\theta}] \) independently from the commonly known distribution \( F \) with density \( f \) that is positive on \( [\underline{\theta}, \overline{\theta}] \). Each agent \( i \) has a maximum demand of \( k_i = 2 \) and an endowment \( r_i \) with \( R = \sum_{i=1}^{3} r_i = 3 \). This implies that under the efficient allocation rule, an agent of type

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\[13\]

Of course, due to the insights of d’Aspremont and Gérard-Varet (1979) and Arrow (1979), the distinction between budget balance ex post and ex ante is somewhat superficial. Nonetheless, expectations (and hence commonly known distributions) matter, whereas most of our analysis is free of such details.

\[14\]

One may also wonder how our mechanism compares to a Walrasian market. The short answer is it does not: Our mechanism gives agents incentives to provide the information necessary to set prices and determine the allocation, whereas, in general, Walrasian prices do not. That being said, when the Walrasian price gap is zero and the agent whose type is equal to the Walrasian price consumes his endowment under efficiency, our mechanism is efficient (and budget balanced). For more on the tight relationship between Walrasian prices and the deficit in a VCG double auction, see, for example, Loertscher and Mezzetti (forth.).
$\theta$ has an expected allocation of $2F(\theta)$\footnote{To see this, notice that an agent obtains 2 units with probability $F^2(\theta)$, 1 with probability $2(1 - F(\theta))F(\theta)$, and 0 otherwise. Adding up gives the result.}. Standard arguments imply that the ex ante expected transfer from agent $i$, denoted $T_i$, under the efficient allocation rule, subject to Bayesian incentive compatibility, is given by

$$T_i = \int_{\theta}^{\overline{\theta}} \Phi(\theta)2F(\theta)f(\theta)d\theta - U_i(\overline{\theta}),$$

where $\Phi(\theta) = \theta - (1 - F(\theta))/f(\theta)$ is the virtual type of an agent with type $\theta$ and $U_i(\overline{\theta})$ is the interim expected payoff of agent $i$ when of type $\theta$. Consequently, the mechanism’s ex ante expected revenue is

$$\sum_{i=1}^{3} T_i = 3 \int_{\theta}^{\overline{\theta}} \Phi(\theta)2F(\theta)f(\theta)d\theta - \sum_{i=1}^{3} U_i(\overline{\theta}).$$

Further, by payoff equivalence, the interim expected payoff of agent $i$ when of type $\theta$ is $U_i(\theta) = U_i(\overline{\theta}) + \int_{\theta}^{\overline{\theta}} 2F(x)dx$, which means that the interim expected net payoff of agent $i$ with type $\theta$ is $\overline{U_i}(\theta) - r_i \theta$. Thus, agent $i$’s worst-off type $\hat{\theta}_i$ satisfies $r_i = 2F(\hat{\theta}_i)$\footnote{As is well known, $U_i(\theta)$ is convex and therefore differentiable almost everywhere. The net payoff $\theta$, $U_i(\overline{\theta}) - r_i \theta$, is minimized at $U_i'(\overline{\theta}) = r_i$. Plugging in the derivative of $U_i$ gives the result.} Making the interim individual rationality constraints for the worst-off types bind, i.e., $U_i(\hat{\theta}_i) - r_i \hat{\theta}_i = 0$, yields $U_i(\overline{\theta}) = r_i \hat{\theta}_i - \int_{\theta}^{\overline{\theta}} 2F(\theta)d\theta$.

We can now illustrate how the possibility of efficient trade is sensitive to the agents’ endowments using a distribution $F$ that is the uniform distribution on $[0, 1]$. First, consider the case of symmetric endowments: $r_i = 1$ for all $i$. In this case, the ex ante expected revenue under the efficient allocation rule, subject to incentive compatibility and interim individual rationality, satisfies $\sum_{i=1}^{3} T_i = 1/4$, which is positive\footnote{For $F$ uniform on $[0, 1]$, we have $3 \int_{\theta}^{\overline{\theta}} \Phi(\theta)2F(\theta)f(\theta)d\theta = 1$, and further, if $r_i = 1$ for all $i$, then $\hat{\theta}_i = 1/2$ and $U_i(\overline{\theta}) = 1/4$ for all $i$. Thus, $\sum_{i=1}^{3} T_i = 1/4$.}. Thus, as one might expect from Cramton et al. (1987), ex post efficiency is possible with symmetric endowments and equal demands.

Second, and for contrast, consider the case of asymmetric endowments, with $r_1 = 2$, $r_2 = 1$, and $r_3 = 0$. This asymmetric ownership makes agent 1 a seller in the sense that his endowment equals his maximum demand, so that his only trading positions are not to trade or to sell. Likewise, agent 3 is now a buyer. This specification thus resembles a bilateral trade problem à la Myerson and Satterthwaite (1983). However, their impossibility result does not directly apply because of agent 2, from whom the mechanism can extract a positive net revenue of $T_2 = 1/12$ as before. The agents’ worst-
off types are $\hat{\theta}_1 = 1$, $\hat{\theta}_2 = 1/2$, and $\hat{\theta}_3 = 0$, and $U_1(\hat{\theta}) = 1$, $U_2(\hat{\theta}) = 1/4$, and $U_3(\hat{\theta}) = 0$. It follows then that $\sum_{i=1}^3 T_i = -1/4$, which is negative. Thus, in the spirit of Myerson and Satterthwaite (1983), ex post efficient reallocation is impossible.

7 Conclusions

The defining features of asset markets are that, in general, agents’ trading positions—buy, sell, or hold—are determined endogenously and that typically all agents, or all but one agent, are active under efficiency. This makes asset markets particularly interesting from a strategic point of view and possibly susceptible to manipulation (as in the LIBOR case). We provide a detail-free mechanism that never runs a budget deficit, endows agents with dominant strategies, respects their individual rationality constraints ex post, and either allocates efficiently or close to efficiently. We also show that this mechanism is both amenable to implementation via clock auctions and asymptotically efficient under natural conditions.

Several interesting and challenging avenues for future research emerge from our analysis. First, the analysis raises new questions about optimal ownership structures for asset markets and how those differ from optimal ownership structures in partnership models. Second, extending the model to allow agents to have multi-dimensional types and decreasing marginal values would, while adding generality and, some might argue, realism, considerably complicate the problem. The key problem will be, as far as we understand it, in identifying the cutoff agent(s). A natural first step to tackle problems with decreasing marginal values will be to impose decreasing marginal utility while maintaining the assumption of one-dimensional types. Yet, even for this simpler setup, the problem of identifying cutoff agent(s) remains far from trivial. Third, while maintaining the assumptions of one-dimensional types (and constant marginal values), the model could be extended by introducing interdependent values. The main challenge with this extension will be in ensuring that there is no deficit under ex post incentive compatibility.

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As before, $3 \int_0^\theta \Phi(\theta) [F^2(\theta)] d\theta = 1$, and so $\sum_{i=1}^3 T_i = 1 - \sum_{i=1}^3 U_i(\hat{\theta}) = -1/4$.

For partnership models, where the standard assumption is $k_i = R$ for all $i$, extreme ownership structures that assign all ownership to one agent imply impossibility of ex post efficient trade. (With extreme ownership, partnership models specialize to two-sided allocation problems, and impossibility follows from existing results.) While our second specification has the flavor of such extreme ownership, the impossibility result does not follow from known results because even though agent 1 is a seller, he does not have full ownership (because $r_2 = 1$). Put differently, in a partnership model, agent $i$ being a seller, i.e., $r_i = k_i$, is equivalent to $i$ having full ownership because $k_i = R$, whereas in an asset market, agent $i$ being a seller does not imply that agent $i$ has full ownership.
Appendix

Appendix A describes dominant-strategy prices. Appendix B provides the proofs of the formal results in the paper.

A Appendix: Dominant-strategy prices

In this appendix, we first consider dominant-strategy prices in the context of the direct TSM and then in the context of the clock implementation of the TSM.

A.1 Dominant-strategy prices in the direct TSM

Given a quantity to be traded $Q$, a reserve $p$, and a set of agents $A$, the TSM uses what we call dominant-strategy prices to allocate the quantity $Q$ efficiently—allocate it to maximize the sum of values or procure it by minimizing the costs—among agents in $A$ at prices that are bounded by $p$\footnote{If $Q$, $p$, and $A$ were fixed, standard arguments (see e.g. Krishna 2002) could be invoked to conclude that these prices endow agents with dominant strategies to bid truthfully. However, because they are endogenous in our setup, additional work is required to reach this conclusion.}. These prices are the Vickrey prices for the problem of trading quantity $Q$ with the set of agents $A$ using a reserve of $p$.

We begin by defining dominant-strategy prices for the case in which they are used to sell $Q$ units with a reserve of $p$ to a set of agents $A$ with types that are all greater than or equal to $p$.

Each agent $i \in A$ is allocated $q_i$ units, where

$$q_i = \min\{k_i - r_i, \max\{0, Q - \sum_{\ell \in A \text{ s.t. } (\ell, \theta_{\ell}) \succ (i, \theta_i)} (k_{\ell} - r_{\ell})\}\}.$$  

Thus, agent $i$ is allocated units whenever there remain units available from the total of $Q$ after the net demands of agents in $A$ with agent-type pairs ranked above $(i, \theta_i)$ according to $\succ$ have been satisfied, up to a maximum of $k_i - r_i$ units.

The amount paid by agent $i$ is based on an individualized price vector consisting of $p$ followed by the types of the other agents in $A$, in increasing order by $\prec$. Specifically, letting $\overline{\theta}_{[i]}^{-i}$ denote the type associated with the $\ell$-th lowest ranked element of $\{(j, \theta_j)\}_{j \in A \setminus \{i\}}$ according to $\prec$, we define:

$$\overline{p}_i \equiv (p, \overline{\theta}_{[1]}^{-i}, \ldots, \overline{\theta}_{[|A|-1]}^{-i}).$$

For example, in the setup of Figure 3 if $A = \{1, 2, 3\}$ and $Q = 3$, then the 3 units are allocated first to the higher-type agents so that $q_1 = 1$, $q_2 = 2$, and $q_3 = 0$. Further,
the individualized price vectors are simply the vectors consisting of \( p \) followed by the types of the other agents in \( A \), in increasing order: \( \mathbf{p}^1 = (p, \theta_3, \theta_2) \), \( \mathbf{p}^2 = (p, \theta_3, \theta_1) \), and \( \mathbf{p}^3 = (p, \theta_2, \theta_1) \).

\[
\begin{array}{cccc}
p < \theta_3 < \theta_2 < \theta_1 \\
k_i - r_i & 1 & 2 & 1 \\
\end{array}
\]

Figure 3: Example of net demand by agents with types greater than \( p \).

Agent \( i \)'s total payment is determined by applying the prices in \( \mathbf{p}^i \) to traunches of agent \( i \)'s units. Agent \( i \) pays \( p \) for the first \( b^i_0 \) units that he purchases, pays \( \theta_{\ell=[1]} \) for the next \( b^i_1 \) units, etc. The traunches of units are defined so that each agent pays an amount equal to the externality that he exerts on the other agents. That is, letting \( d_{\ell=[i]} \) be the net demand (capacity minus endowment) of the agent associated with type \( \theta_{\ell=[i]} \), we define \( b^i_\ell \) to be the maximum quantity (up to \( q \)) out of \( Q \) that can be allocated to agent \( i \) without affecting the allocation of the other agents in \( A \),

\[
b^i_0 \equiv \max \left\{ 0, \min \left\{ \bar{q}_i, Q - \sum_{\ell=1}^{\lvert A \rvert - 1} d_{\ell=[i]} \right\} \right\}, \tag{3}
\]

and for \( \ell \in \{1, \ldots, \lvert A \rvert - 1\} \), we define \( b^i_\ell \) iteratively to be the additional quantity (up to \( \bar{q}_i - \sum_{\ell=0}^{\ell-1} b^i_\ell \)) that can be allocated to agent \( i \) by imposing an externality on the \( \ell \)-th lowest type agent in \( A \),

\[
b^i_\ell \equiv \max \left\{ 0, \min \left\{ \bar{q}_i - \sum_{t=0}^{\ell-1} b^i_t, d_{\ell=[i]} \right\} \right\}. \tag{4}
\]

Applying the prices in \( \mathbf{p}^i \) to the traunches for agent \( i \), we get a total payment for agent \( i \) of

\[
\mathbf{p}^i \cdot \mathbf{b}^i. \tag{5}
\]

In the example of Figure 3, assuming \( Q = 3 \), we have \( b^1 = (0, 1, 0) \) and \( b^2 = (1, 1, 0) \). Thus, agent 1 pays \( \theta_3 \) for his unit and agent 2 pays \( p \) for his first unit and \( \theta_3 \) for his second unit. Because there are 3 units for sale and only two units demanded by agents other than agent 2, agent 2 is guaranteed to trade one unit and so pays the reserve for that unit. Agent 2’s consumption of that unit has no impact on the trades of the others. However, agent 2’s purchase of a second unit precludes agent 3 from trading, so agent 2 pays \( \theta_3 \) for his second unit. Similarly, agent 1’s purchase of a unit precludes agent 3 from trading, so agent 1 pays \( \theta_3 \) for his unit.
The supply-side version of dominant-strategy prices in which \( Q \) units are purchased with reserve \( p \) from a set of agents \( A \) with types less than or equal to \( p \) is analogous. In that case, the quantity purchased from agent \( i \) is

\[
q_i \equiv \min\{r_i, \ max\{0, Q - \sum_{\ell \in A \ s.t. \ ((\ell, \theta_\ell)) \prec (i, \theta_i)} r_\ell\}\},
\]

so that agent \( i \) sells units whenever there remain units demanded after the agents ranked below agent \( i \) have sold their endowments, up to a maximum of \( r_i \) units. Analogously to the buyer side case, let \( \theta_\ell^{-i} \) be the type associated with the \( \ell \)-highest element of \( \{(j, \theta_j)\}_{j \in A \setminus \{i\}} \) according to \( \prec \).

The individualized price vector for agent \( i \) is

\[
p^i \equiv (p, \theta_1^{-i}, ..., \theta_{|A|-1}^{-i}).
\]

The payment made to agent \( i \) is determined by applying the prices in \( p^i \) to tranches of units \( b^i \) defined as in (3)–(4), but with \( q_i \) replaced by \( q_i \) and \( d^{-1}_{[\ell]} \) replaced by the supply of the agent with type \( \theta_\ell^{-i} \). Then each agent \( i \in A \) is paid \( p^i \cdot b^i \).

### A.2 Dominant-strategy prices in the clock-auction implementation

In the clock implementation of the TSM, dominant-strategy prices are achieved when the buyer side is the long side through the use of an Ausubel auction with a starting price equal to the report of the cutoff agent.

We illustrate the Ausubel auction for the example of Figure 4. In that example, agent 3 is the cutoff agent. The supply at price \( \theta_3 \) by agents ranked below the cutoff agent is 4 units, and the demand at price \( \theta_3 \) by agents ranked above the cutoff agent is 5 units. Thus, there is excess demand. The clock implementation of the TSM purchases 4 units from agents 4 and 5 at a price of \( \theta_3 \) and then allocates those units using an Ausubel clinching auction. The clinching auction begins with a clock price equal to \( \theta_3 \), with agents 1 and 2 being the active agents. At that price, each active agent is asked how many units he would like to purchase at that price.

At a price of \( \theta_3 \), agent 2 has demand for 2 units, and agent 1 has demand for 3 units. Because the quantity supplied is 4, this means that agent 2 is guaranteed to purchase at least 1 unit and so he clinches 1 unit at a price of \( \theta_3 \). In addition, because agent 1 is guaranteed to purchase at least 2 units, agent 1 clinches 2 units at a price of \( \theta_3 \). Once these 3 units are clinched, only 1 unit remains to be allocated. At a price of \( \theta_2 \), agent 2’s demand falls to zero, and so agent 1 clinches the final unit at a price of \( \theta_2 \). This dynamic
\[
\begin{array}{cccc}
\theta_5 < \theta_4 < \theta_3 < \theta_2 < \theta_1 \\
r_i & 2 & 2 & 3 & 0 & 0 \\
k_i & 2 & 2 & 3 & 2 & 3 \\
x_i^*(\theta) & 0 & 0 & 2 & 2 & 3 \\
TSM & 0 & 0 & 3 & 1 & 3 \\
\end{array}
\]

Figure 4: Example with a cutoff agent 3 with \(x_3^*(\theta) < k_3\) and excess demand at price \(\theta_3\) by agents other than the cutoff agent.

mechanism replicates the outcome of the VCG mechanism (applied to the problem of allocating 4 units to agents 1 and 2 with a reserve of \(\theta_3\)) and makes sincere bidding a dominant strategy.

B Appendix: Proofs

Proof of Theorem 1. The TSM balances quantities bought and sold by construction. That it never runs a deficit follows from this balanced trade property and the fact that all sellers receive per-unit prices that are weakly less than the price that buyers pay. Given the reported types, the quantity traded is allocated efficiently. Under the TSM with truthful reporting, each agent has a nonnegative payoff. Thus, individual rationality constraints are satisfied ex post, and we are left to prove dominant-strategy incentive compatibility.

The proof of dominant-strategy incentive compatibility relies on the following lemma, which establishes a fundamental monotonicity property of the cutoff type. Let \(\theta\) and \(\theta'\) be two vectors satisfying \(\theta'_i = \theta_i\) for all \(i \neq \ell\) with \(\ell \neq j\) and \((\ell, \theta_\ell) \prec (j, \theta_j) \prec (\ell, \theta'_\ell)\). In words, all types are the same under \(\theta\) and \(\theta'\) except for agent \(\ell\)'s, whose type changes so that agent \(\ell\) goes from ranked below agent \(j\) to ranked above agent \(j\). Assume \(\theta_j\) is the cutoff type when the types are \(\theta\), that is, \(\bar{p}^*(\theta) = \theta_j\).

Lemma 2. \(p^*(\theta') \geq p^*(\theta)\).

Proof of Lemma 2. Net demand by agents ranked above agent \(j\) is weakly larger under \(\theta'\) than under \(\theta\). Consequently, \(x_j^*(\theta') \leq x_j^*(\theta)\). Thus, the cutoff agent under \(\theta'\) is either agent \(j\) or an agent ranked above agent \(j\), which implies that \(\bar{p}^*(\theta') \geq \bar{p}^*(\theta)\). □

Lemma 2 implies that no agent who would be ranked below the cutoff agent under truthful reporting can benefit from deviating so as to be ranked above the cutoff agent.
And similarly, no agent who would be ranked above the cutoff agent under truthful reporting can benefit from deviating so as to be ranked below the cutoff agent. To see this, note that an agent $\ell$ who is on the seller side under truthful reporting, i.e., $(\ell, \theta_{\ell}) \prec (j, \theta_j)$, would at best make a payoff of zero when reporting a higher type such that he is on the buyer side, and possibly a negative payoff, because following the deviation agent $\ell$ would only be able to buy at a price that is greater than or equal to $p^*(\theta')$, which using Lemma 2 satisfies $p^*(\theta') \geq p^*(\theta) = \theta_j \geq \theta_{\ell}$. Likewise, an agent who is on the buyer side when reporting truthfully would at best make a payoff of zero when reporting a lower type such that he is on the seller side, and possibly a negative payoff.

Because an agent makes a nonnegative payoff under truthful reporting and makes a payoff of zero when he is the cutoff type, no agent can benefit from deviating by reporting a type such that he becomes the cutoff agent.

The agent who is the cutoff agent under truthful reporting cannot benefit from deviating so as to become a buyer or a seller because if he deviates by reporting a lower type so that he is on the seller side, the new cutoff type will be weakly lower than his type, and so he can only sell at a price less than or equal to his type. Similarly, if he reports a higher type so that he is on the buyer side, the new cutoff type will be weakly higher than his type, and so he can only buy at a price that is greater than or equal to his type.

We are thus left to consider deviations from truthful reporting that do not affect an agent’s trading position. We begin with the cutoff agent. His payoff is zero as long as he is the cutoff agent, so no report is better for him than truthful reporting. A deviation by an agent on the buyer or seller side of the market that does not affect which side of the market they are on also does not affect whether their side is the short side or the long side. If it is the short side, then all agents trade their full demand or supply at a price equal to the cutoff agent’s type, and so the deviation has no effect. If it is the long side, then by the properties of dominant strategy prices, no deviation is profitable. ■

**Proof of Proposition** The proof follows from the text. ■

**Proof of Lemma** Let $m$ and $\theta$ be given. Because all agents ranked $\hat{q}_m(\theta) + 1, ..., mN$ sell all of their endowment under efficiency, the number of trades under efficiency is at least

$$(mN - \hat{q}_m(\theta)) \min_{j \in \{1, ..., N\}} r_j.$$ 

Because each of the $\hat{q}_m(\theta)$ highest ranked agents can buy at most $\max_{i \in \{1, ..., N\}} (k_i - r_i)$, it follows that the number of trades with $m$ replicas under efficiency is no more than $\hat{q}_m(\theta)$.
times this amount. Putting these together,

\[
(mN - \hat{q}_m(\theta)) \min_{j \in \{1, \ldots, N\}} r_j \leq \hat{q}_m(\theta) \max_{i \in \{1, \ldots, N\}} (k_i - r_i),
\]

which implies that

\[
mN \frac{\min_{j \in \{1, \ldots, N\}} r_j}{\max_{i \in \{1, \ldots, N\}} (k_i - r_i) + \min_{j \in \{1, \ldots, N\}} r_j} \leq \hat{q}_m(\theta).
\]

Letting

\[
d \equiv \frac{\min_{j \in \{1, \ldots, N\}} r_j}{\max_{i \in \{1, \ldots, N\}} (k_i - r_i) + \min_{j \in \{1, \ldots, N\}} r_j} \in (0, 1),
\]

we have \(mNd \leq \hat{q}_m(\theta)\), which completes the proof. \(\blacksquare\)

**Proof of Proposition 2** Lemma 1 in [Loertscher and Marx (2019)](Loertscher2019) implies that for any \(q \in \{1, \ldots, N - 1\}\),

\[
\mathbb{E}_\theta \left[ \theta(q:N) - \theta((q+1):N) \right] = \frac{1}{q} \mathbb{E}_\theta \left[ \frac{1 - F(\theta(q:N))}{f(\theta(q:N))} \right].
\]

Thus, for any \(q \in \{1, \ldots, N - 1\}\) and \(\ell \in \{1, \ldots, N - q\}\),

\[
\mathbb{E}_\theta \left[ \theta(q:N) - \theta((q+\ell):N) \right] = \mathbb{E}_\theta \left[ (\theta(q:N) - \theta((q+1):N)) + \cdots + (\theta((q+\ell-1):N) - \theta((q+\ell):N)) \right]
\]

\[
= \mathbb{E}_\theta \left[ \frac{1}{q} \frac{1 - F(\theta(q:N))}{f(\theta(q:N))} \right] + \cdots + \mathbb{E}_\theta \left[ \frac{1}{q+\ell-1} \frac{1 - F(\theta((q+\ell-1):N))}{f(\theta((q+\ell-1):N))} \right] \quad (6)
\]

and similarly, for any \(q \in \{2, \ldots, N\}\) and \(\ell \in \{1, \ldots, q - 1\}\),

\[
\mathbb{E}_\theta \left[ \theta((q-\ell):N) - \theta(q:N) \right] = \mathbb{E}_\theta \left[ \frac{1}{q-\ell} \frac{1 - F(\theta((q-\ell):N))}{f(\theta((q-\ell):N))} \right] + \cdots + \mathbb{E}_\theta \left[ \frac{1}{q-1} \frac{1 - F(\theta(q-1:N))}{f(\theta(q-1:N))} \right] \quad (7)
\]

Letting \(f \equiv \inf_{\theta \in [\underline{\theta}, \overline{\theta}]} f(\theta)\), which is positive by assumption, and noting that for all
\( m \in \{1, 2, \ldots \} \) and \( \theta \in [\underline{\theta}, \overline{\theta}]^m, \hat{\ell} \in \{1, \ldots, mN - \hat{q}_m(\theta)\} \), it follows that

\[
\hat{B}_m = \max_{i \in N} \left\{ k_i - r_i \right\} \sum_{j=0}^{\hat{\ell}-1} \mathbb{E}_\theta \left[ \frac{1}{\hat{q}_m(\theta) + j} \frac{1 - F(\theta(\hat{q}_m(\theta)+j;mN))}{f(\theta(\hat{q}_m(\theta)+j;mN))} \right]
\]

\[
\leq \max_{i \in N} \left\{ k_i - r_i \right\} \sum_{j=0}^{\hat{\ell}-1} \mathbb{E}_\theta \left[ \frac{1}{mNd \hat{\ell}} \right]
\]

\[
= \max_{i \in N} \left\{ k_i - r_i \right\} \frac{\hat{\ell}}{mNd \hat{\ell}},
\]

where the first equality uses (6), the inequality uses Lemma 1 and the definition of \( f \), and the second equality simplifies. Similarly, noting that for all \( m \in \{1, 2, \ldots \} \) and \( \theta \in [\underline{\theta}, \overline{\theta}]^m, \tilde{\ell} \in \{1, \ldots, \hat{q}_m(\theta) - 1\} \), we have

\[
\tilde{B}_m = \max_{i \in N} r_i \tilde{\ell} \sum_{j=1}^{\tilde{\ell}} \mathbb{E}_\theta \left[ \frac{1}{\hat{q}_m(\theta) - \tilde{\ell} - j} \frac{1 - F(\theta(\hat{q}_m(\theta)-j;mN))}{f(\theta(\hat{q}_m(\theta)-j;mN))} \right]
\]

\[
\leq \max_{i \in N} r_i \tilde{\ell} \sum_{j=1}^{\tilde{\ell}} \mathbb{E}_\theta \left[ \frac{1}{mNd - \tilde{\ell}} \right]
\]

\[
= \max_{i \in N} r_i \tilde{\ell} \frac{1}{mNd - \tilde{\ell}},
\]

where the first equality uses (7), the inequality uses Lemma 1 and the definition of \( f \), and the second equality simplifies.

Recalling that \( \mathcal{L}_m(\theta) \) is the efficiency loss under the TSM when there are \( m \) replicas and the type vector is \( \theta \), and so \( \mathcal{L}_m(\theta) \) is a nonnegative random variable, and recalling
that $\max\{\hat{B}_m, \tilde{B}_m\}$ is an upper bound for $\mathbb{E}_\theta [\mathcal{L}_m(\theta)]$, we have

$$
0 \leq \lim_{m \to \infty} \inf m \mathbb{E}_\theta [\mathcal{L}_m(\theta)] \\
\leq \lim_{m \to \infty} \sup m \mathbb{E}_\theta [\mathcal{L}_m(\theta)] \\
\leq \lim_{m \to \infty} m \max\{\hat{B}_m, \tilde{B}_m\} \\
\leq \lim_{m \to \infty} \max \left\{ \max_{i \in \mathcal{N}} \{k_i - r_i\} \frac{\hat{\ell}}{f \hat{N} d}, \max_{i \in \mathcal{N}} \frac{r_i}{Nd - \ell/m} \right\} \\
= \max \left\{ \max_{i \in \mathcal{N}} \{k_i - r_i\} \frac{\hat{\ell}}{f \hat{N} d}, \max_{i \in \mathcal{N}} \frac{r_i}{Nd} \frac{\ell}{\ell/m} \right\} \\
\equiv c,
$$

where $0 < c < \infty$. It follows that $(m \mathbb{E}_\theta [\mathcal{L}_m(\theta)])_{m=1}^\infty$ is a bounded sequence and so there exists $\hat{c} \in [0, c]$ such that $\hat{c} = \lim_{m \to \infty} \sup \mathbb{E}_\theta [\mathcal{L}_m(\theta)]$. Using Markov’s inequality and the definition of $\hat{c}$, for all $\varepsilon > 0$,

$$
0 \leq \lim_{m \to \infty} \inf \Pr (|m \mathcal{L}_m(\theta) - \hat{c}| \geq \varepsilon) \\
\leq \lim_{m \to \infty} \Pr (|m \mathcal{L}_m(\theta) - \hat{c}| \geq \varepsilon) \\
\leq \lim_{m \to \infty} \sup \Pr (|m \mathcal{L}_m(\theta) - \hat{c}| \geq \varepsilon) \\
\leq \lim_{m \to \infty} \sup \frac{1}{\varepsilon} \mathbb{E}_\theta [|m \mathcal{L}_m(\theta) - \hat{c}|] \\
= 0,
$$

which establishes that $\lim_{m \to \infty} \Pr (|m \mathcal{L}_m(\theta) - \hat{c}| \geq \varepsilon) = 0$, i.e., the efficiency loss $\mathcal{L}_m(\theta)$ converges to 0 in probability at the rate of $\frac{1}{m}$. ■
References


——— (forth.): “The Deficit on Each Trade in a Vickrey Double Auction Is at Least as Large as the Walrasian Price Gap,” *Journal of Mathematical Economics*.


