Bilateral trade with multi-unit demand and supply*

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Abstract

We study a bilateral trade problem with multi-unit demand and supply and one-dimensional private information. Each agent geometrically discounts additional units by a constant factor. We show that when goods are complements, the incentive problem—measured as the ratio of second-best to first-best social surplus—becomes less severe as the degree of complementarity increases. In contrast, if goods are substitutes and each agent's distribution exhibits linear virtual types, then this ratio is a constant. If the bilateral trade setup arises from prior vertical integration between a buyer and supplier, with the vertically integrated firm being a buyer facing an independent supplier, the incentive problem does not vary with the degree of complementarity because the integrated firm's willingness to pay is given by its cost of production. Extensions to profit-maximization by a market maker and a discrete public good problem show that the broad insight that complementarity of goods mitigates the incentive problem generalizes to these settings.

Keywords: Substitutes, complements, multi-unit demand and supply, geometric utility, impossibility of efficient trade

JEL Classification: D44, D82, L41

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1 Introduction

Complements are typically perceived as problematic in economics. They give rise to public goods problems, an empty core, and, accordingly, the nonexistence of Walrasian prices. Further, the impossibility of efficient bilateral trade has been attributed to the fact that the buyer and the seller are complements—each agent's social marginal product is larger with the other agent present than without, which implies that the sum of the agents' Vickrey-Clarke-Groves transfers exceeds total welfare.[1] For bilateral trade settings, this suggests that the inefficiency due to incentive compatibility, individual rationality, and no-deficit constraints is exacerbated when the buyer and seller have multi-unit demand and supply and perceive the goods as complements.

In this paper, we address this question, measuring inefficiency as the ratio of second-best over first-best welfare. We study a version of the bilateral trade problem of Myerson and Satterthwaite (1983) in which the buyer's maximum demand is equal to the seller's capacity and in which each agent "discounts" the value or cost of additional units geometrically with the same nonnegative scale parameter s. That is, the buyer's value for the second and third units, respectively, are s and s^2 times its value for the first unit, and so on. The seller's cost to produce a second unit is $1/s$ times its cost for the first unit, its cost for a third unit is $1/s^2$ times its cost for the first unit, and so on. If s is less than one, then goods are substitutes, and if s is greater than one, then they are complements. The buyer's value and the seller's cost for the first unit are assumed to be independently distributed with continuous distributions with identical supports, each of which has an increasing virtual type function. The standard bilateral trade problem with single-unit demand and supply, or, equivalently, constant marginal values and costs, emerges as a special case both when s is equal to one and when s is equal to zero. When s is equal to one, marginal values and costs are constant up to agents' capacities; when s is equal to zero, units beyond the first have zero value to the buyer and infinite cost to the seller.

With this setting in place, we derive a number of results. We first show that there is a threshold greater than 1 such that for s above that threshold, the inefficiency of bilateral trade *decreases* in s. That is, the stronger is the complementarity between units, the less of a burden incentive compatibility and individual rationality constraints impose on the bilateral trade problem. As s goes to infinity, the ratio of second-best to first-best welfare goes to one, meaning that the incentive problem vanishes in the limit.

Because both the case in which s is equal to zero and the case in which s is equal to one correspond to parameterizations of the standard bilateral trade problem, it is clear a

[1] See Samuelson (1954), Milgrom (2017), and Delacrétaz et al. (2019), respectively.

priori that the ratio of second-best to first-best welfare cannot monotonically vary with s for s between zero and one. Against this backdrop, we show that, for the family of distributions that exhibit linear virtual value and virtual cost functions, which includes the uniform distribution, the ratio of second-best to first-best welfare is invariant to s for s below a threshold, which is greater than one, at which the ratio starts increasing.

With the benefit of hindsight, the intuition for why complementarity improves outcomes in the bilateral trade problem with multi-unit demand and supply is simple. If goods are complements, then the buyer and seller optimally either trade all units or none.[2] In this regard, the problem becomes akin to the standard bilateral trade problem. However, increasing s increases the buyer's willingness to pay and decreases the seller's cost of production, so the effective overlap of the two distributions decreases, which decreases the incentive problem.[3] Consequently, for the case of complements, increasing s is equivalent to decreasing the overlap of the relevant, appropriately normalized distributions. This means that the severity of the incentive problem decreases with s. However, because the impossibility result of Myerson and Satterthwaite holds as long as there is *any* overlap in the supports, it never completely vanishes as long as s is finite. The invariance result in the case of substitutes and distributions exhibiting linear virtual types derives from the fact that the distortion of the second-best mechanism relative to the first-best does not vary with s, which translates to an invariance of the ratio of second-best to first-best social surplus.

Bilateral trade is, of course, a canonical problem in economics that captures generic situations of two-person bargaining with incomplete information. One specific interpretation of the bilateral trade problem is that it emerges following vertical integration between a buyer and a supplier when prior to integration there were, say, two independent suppliers and one buyer. As noted by Loertscher and Marx (2021), with single-unit demand and supply, vertical integration is socially harmful if the suppliers' distributions have identical supports and the buyer's value is no less than the upper bound of these supports with probability one. Under these conditions the first-best is possible prior to integration and impossible after integration because, then, the vertically integrated firm's demand is given by its cost distribution, which has an identical support to the independent supplier's cost distribution.

With that and the prominence of arguments pertaining to vertical integration in concurrent antitrust thinking and practice in mind, we study a specification of the bilateral trade

[2]On scale economies as a contributor to "tipping" in multi-sided platform markets, see Bedre-Defolie and Nitsche (2020).

[3]For example, with maximum demand and capacity equal to two and a willingness to pay and cost for the first unit that are elements of the interval between zero and one, if s is equal to two, then the upper bound of support of the buyer's effective willingness to pay for two units is three while the upper bound of support of the seller's effective cost of producing tow units is half of that.

problem in which a buyer is vertically integrated with one supplier when prior to integration there were two independent suppliers. We assume that the buyer's value without integration is common knowledge and that the suppliers' cost distributions have identical supports. These assumptions imply that prior to vertical integration, the first-best is possible, while post integration it is not because the vertically integrated firm's willingness to pay for the independent supplier's good depends on the integrated supplier's cost, which is the private information of the integrated firm.

In light of the popular and influential view that vertical integration is welfare enhancing because it eliminates a double markup (see, e.g., U.S. DOJ and FTC *Vertical Merger Guidelines*, 2020), this setup is relevant because it permits an analysis of whether vertical integration becomes better or worse with multi-unit demand and supply.[4] We show that with vertical integration, the second-best mechanism does not vary with s in the case of complements, but changes with s in the case of substitutes, even with uniformly distributed types. These findings are almost the exact opposite of what we obtain in the baseline setting. The reason for the invariance result with complements is, in a nutshell, that with vertical integration and complements, it is still the case that either no or all units are traded, but the vertically integrated firm's willingness to pay is now given by its cost, which varies with s in the same way as the independent supplier's cost. Thus, other than scaling costs, s has no effect on the efficiency of the second-best mechanism that governs trade between the integrated firm and the independent supplier. In contrast, for s between zero and one, changes in s affect that mechanism. For example, for s close to zero, a small increase in s makes it possible that the integrated firm consumes two units, one of which is supplied by the independent supplier (because the cost of the second unit from either supplier is prohibitively high). Consequently, with substitutes, the model with a vertically integrated firm is fundamentally different from a single-unit setting and also from the bilateral trade setting with an independent buyer and an independent supplier.

In an extension section, we analyze profit maximization by a market-making broker in our baseline bilateral trade problem, as well as a discrete public good problem with two agents and multi-unit demands. These extensions demonstrate that inefficiencies due to rent extraction in market making and incentive issues in public goods become less problematic as the complementarity between goods, parameterized by s, increases.

This paper relates to the literature on two-sided allocation problems with private information initiated by Vickrey (1961). It extends the canonical bilateral trade problem of Myerson

[4] As argued forcefully by Choné et al. (2020), the incomplete information setting has the desirable feature that costs and benefits of vertical integration are *not* dependent on restrictions on the contracting space. In that sense, the incomplete information setting makes the analysis of vertical integration and other "policy" interventions immune to the Lucas' critique.

and Satterthwaite (1983) by allowing the buyer to have multi-unit demand and the seller to have multi-unit capacity. We do so while maintaining the assumption of one-dimensional private information, which allows us to incorporate the possibility of decreasing marginal values and increasing marginal costs without sacrificing the tractability of the mechanism design approach pioneered by Myerson (1981) and Myerson and Satterthwaite (1983).[5] Moreover, as Bolotny and Vasserman (2020) show, the setup with one-dimensional types is empirically useful to analyze problems with multi-unit demands. Interpreting the bilateral trade model as emerging from vertical integration, the paper also contributes to the nascent literature on vertical integration with incomplete information (see, for example, Choné et al., 2020; Loertscher and Marx, 2021).

Dating back to Cournot (1838), complementarities between *agents* have been recognized as a source of inefficiency. Spengler (1950) put forth the still influential argument that the elimination of this inefficiency provides an efficiency rationale for vertical integration.[6] The complementarity between buyers and sellers is at the source of impossibility results in two-sided allocation problems (see, for example, Loertscher et al., 2015; Delacrétaz et al., 2019), and, of course, complementarity between agents is what drives free-riding in public good problems (see, for example, Samuelson, 1954). Against this backdrop, the present paper provides the insight that complementarity between *goods* in an individual agent's preferences can be beneficial because it alleviates incentive problems.[7]

The remainder of the paper is organized as follows. Section 2 introduces the setup, and Section 3 derives auxiliary results, including basic mechanism concepts. In Section 4, we analyze the effects of substitutability and complementarity in the baseline bilateral trade setting. Section 5 introduces and analyzes the problem with vertical integration, while Section 6 contains extensions. Section 7 concludes the paper.

[5] With multi-dimensional private information and multiple agents, the optimal mechanism is not known (Daskalakis et al., 2017).

[6] See also Choné et al. (2020) on the prominence that this type of reasoning continues to receive.

[7] This point is related to the observation of Delacrétaz et al. (2021) that for private goods economies with privately informed agents, the Vickrey-Clarke-Groves (VCG) mechanism never runs a budget surplus if Walrasian prices exist. Because goods being perceived as substitutes is a sufficient condition for the existence of Walrasian prices, this suggests that complementarity between goods may be a good thing insofar as it may eliminate the existence of Walrasian prices and, as Delacrétaz et al. (2021) show with an example, the deficit of the VCG mechanism. Of course, in our bilateral trade setting, Walrasian prices—in the case of complements defined as market clearing prices for the entire package— always exist, and therefore, we always obtain a deficit under ex post efficiency.

2 Setup

We consider a bilateral trade setup. The buyer has demand for up to k units, and the seller has the capacity to supply up to k units, where $k \in \{1, 2, \dots\}$. The agents' utility is *geometric* in the sense that for $j \in \{1, \dots, k\}$, a buyer with type v has a willingness to pay for the j-th unit of vs^{j-1}, and a seller with type c has a cost for selling the j-th unit of $c\left(\frac{1}{s}\right)^{j-1}$, where $s \geq 0$ is the parameter that measures the strength of scale effects.

If $s = 0$, then the buyer's willingness to pay is v for the first unit and 0 for any additional units, and the seller's cost is c for the first unit and ∞ for any additional units. Thus, the case of $s = 0$ corresponds to the standard bilateral trade problem of Myerson and Satterthwaite (1983) with single-unit demand and supply. If $s = 1$, then the buyer has constant marginal values and the seller has constant marginal costs for up to k units. Thus, the case of $s = 1$ corresponds to a version of the Myerson and Satterthwaite problem that is scaled by the factor k. If $s \in (0, 1)$, then we have decreasing marginal willingness to pay and increasing marginal cost, which we refer to as the case with *substitutes*. If $s > 1$, then we have increasing marginal willingness to pay and decreasing marginal cost, which we refer to as the case with *complements*.

Under geometric utility, given that $j \in \{1, \dots, k\}$ units are traded, the value to the buyer with type v and the cost to the seller with type c are, respectively,

$$S(j, s)v \quad \text{and} \quad S(j, 1/s)c,$$

where

$$S(j, s) \equiv \sum_{i=1}^{j} s^{i-1} = \frac{1 - s^j}{1 - s} \quad \text{and} \quad S(j, 1/s) = \frac{S(j, s)}{s^{j-1}}.$$

For $s = 1$, we define $S(j, 1) \equiv \lim_{s \to 1} S(j, s) = j$.[8]

The buyer's type v is drawn from distribution F with support $[0, 1]$ and density f that is positive on the interior of the support. The seller's type c is drawn independently from distribution G with support $[0, 1]$ and density g that is positive on the interior of the support. We define weighted virtual value and weighted virtual cost functions, respectively, for weight $\alpha \in [0, 1]$ by

$$\Phi_\alpha(v) \equiv v - \alpha \frac{1 - F(v)}{f(v)} \quad \text{and} \quad \Gamma_\alpha(c) \equiv c + \alpha \frac{G(c)}{g(c)},$$

where Φ_1 and Γ_1 correspond to the usual virtual value and virtual cost functions. We assume that $\frac{1-F(v)}{f(v)}$ is decreasing in v and that $\frac{G(c)}{g(c)}$ is increasing in c, which is sufficient to guarantee

[8] For $s = 0$, it will only be relevant to evaluate $S(j, 1/s)$ for $j = 1$, in which case, using the convention that $0^0 = 1$, it is equal to 1.

that the weighted virtual type functions are increasing for all $\alpha \in [0,1]$.

The weighted virtual value for j units of the buyer with type v and the weighted virtual cost for j units of the seller with type c, respectively, are

$$\Phi_\alpha(v,j) \equiv S(j,s)\Phi_\alpha(v) \quad \text{and} \quad \Gamma_\alpha(c,j) \equiv S(j,1/s)\Gamma_\alpha(c).$$

We assume that the distributions F and G, the scale parameter s, and the maximum demand and supply k are common knowledge, while agents' types are their own private information.

We denote a direct mechanism by $\langle Q, \mathbf{M} \rangle$, with $Q : [0,1]^2 \to \{0,1,\ldots,k\}$ and $\mathbf{M} : [0,1]^2 \to \mathbb{R}^2$, where $\mathbf{M} = (M_B, M_S)$. Given reported types (v,c), $Q(v,c)$ is the number of units traded, $M_B(v,c)$ is the payment from the buyer to the mechanism, and $M_S(v,c)$ is the payment from the mechanism to the seller.

Of course, the parameter s plays no role if $k = 1$, so all our comparative static results with respect to s hold under the assumption that $k > 1$.

3 Auxiliary analysis and results

In this section, we derive mechanism design basics and other auxiliary results that will be used in the subsequent analysis.

3.1 Mechanism design basics

The buyer's expected payoff when its type is v and it reports v' to the mechanism is

$$U_B(v,v') \equiv \mathbb{E}_c \left[\sum_{j=1}^{k} \Phi_0(v,j) \cdot \mathbf{1}_{Q(v',c)=j} - M_B(v',c) \right],$$

and the seller's expected payoff when its type is c and it reports c' to the mechanism is

$$U_S(c,c') \equiv \mathbb{E}_v \left[M_S(v,c') - \sum_{j=1}^{k} \Gamma_0(c,j) \cdot \mathbf{1}_{Q(v,c')=j} \right].$$

The mechanism $\langle Q, \mathbf{M} \rangle$ is *incentive compatible* if for all $v, v' \in [0,1]$,

$$U_B(v,v') \leq U_B(v,v),$$

and for all $c, c' \in [0,1]$,

$$U_S(c, c') \leq U_S(c, c).$$

We assume that each agent's outside option if there is no trade is zero. Thus, the mechanism $\langle Q, \mathbf{M} \rangle$ is *interim individually rational* if for all $v \in [0,1]$,

$$U_B(v, v) \geq 0,$$

and for all $c \in [0,1]$,

$$U_S(c, c) \geq 0.$$

The mechanism satisfies *ex post individual rationality* if for all $(v, c) \in [0,1]^2$

$$\sum_{j=1}^{k} \Phi_0(v, j) \cdot \mathbf{1}_{Q(v,c)=j} - M_B(v, c) \geq 0$$

and

$$M_S(v, c) - \sum_{j=1}^{k} \Gamma_0(c, j) \cdot \mathbf{1}_{Q(v,c)=j} \geq 0.$$

The mechanism $\langle Q, \mathbf{M} \rangle$ satisfies the *no deficit* constraint if the expected profit of the designer is nonnegative, i.e.,

$$\mathbb{E}_{v,c}[M_B(v, c) - M_S(v, c)] \geq 0.$$

Standard mechanism design arguments can be used to characterize the worst-off types of the buyer and seller for an incentive compatible mechanism:

Lemma 1. *Given incentive compatible mechanism $\langle Q, \mathbf{M} \rangle$, the worst-off type of the buyer is $v = 0$, and the worst-off type of the seller is $c = 1$.*

Proof. See Appendix A.

If $k = 1$, then the worst-off types of buyer and seller trade with probability zero under efficiency. But in our setup with $k > 1$ and complements, i.e., $s > 1$, that is no longer necessarily the case because the buyer's total value for k units $S(k, s)v$ can exceed the cost to the seller $S(k, 1/s)c$ even when $c = 1$. Indeed, given $c = 1$, this occurs with probability $1 - F(1/s^{k-1}) > 0$, where the inequality uses $s > 1$.

3.2 Optimal mechanisms

We consider first-best and second-best mechanisms, as well as mechanisms that maximize the designer's expected profit. A unified approach to studying these mechanisms is possible by considering the incentive compatible, individually rational mechanisms that maximize the weighted sum of expected social surplus and expected designer profit, with weight $\alpha \in [0,1]$ on expected designer profit and weight $1-\alpha$ on expected social surplus (Myerson and Satterthwaite, 1983). With this in mind, we define $\mathcal{M}_\alpha$ to be the set of direct mechanisms $\langle Q, \mathbf{M} \rangle$ that maximize

$$\mathbb{E}_{v,c}\left[(1-\alpha)\sum_{j=1}^{k}(\Phi_0(v,j)-\Gamma_0(c,j))\cdot \mathbf{1}_{Q(v,c)=j}+\alpha\left(M_B(v,c)-M_S(v,c)\right)\right], \qquad (1)$$

subject to incentive compatibility and individually rationality. For $\alpha \in (0,1]$, individual rationality binds because otherwise the designer could extract additional profit from the agents. But for $\alpha = 0$, the agents' payments are not pinned down by individual rationality.[9] For continuity and to simplify notation, in what follows, we restrict $\mathcal{M}_0$ to mechanisms that satisfy individual rationality with equality.

It is immediate from the definition of $\mathcal{M}_\alpha$ that any mechanism in $\mathcal{M}_0$ delivers the first-best, and any mechanism in $\mathcal{M}_1$ maximizes the designer's profit, subject to incentive compatibility and individual rationality. As we now show, the set of second-best mechanisms is the set $\mathcal{M}_{\alpha^*}$, where α^* is the smallest value of α such that mechanisms in $\mathcal{M}_\alpha$ satisfy the no-deficit constraint, in addition to incentive compatibility and individual rationality.

To see this, note that by definition, any mechanism in $\mathcal{M}_\alpha$ with allocation rule Q generates expected social surplus of

$$\mathbb{E}_{v,c}\left[\sum_{j=1}^{k}(\Phi_0(v,j)-\Gamma_0(c,j))\cdot \mathbf{1}_{Q(v,c;s)=j}\right].$$

Using standard mechanism design results (see, e.g., Krishna, 2010) adapted to our setup, for mechanisms in $\mathcal{M}_\alpha$, we can relate the allocation rule and the designer's expected profit to the agents' weighted virtual types:

Lemma 2. *Given $\alpha \in [0,1]$ and mechanism $\langle Q_\alpha, \mathbf{M}_\alpha \rangle \in \mathcal{M}_\alpha$, Q_α solves*

$$\max_{Q}\mathbb{E}_{v,c}\left[\sum_{j=1}^{k}(\Phi_\alpha(v,j)-\Gamma_\alpha(c,j))\cdot \mathbf{1}_{Q(v,c;s)=j}\right] \qquad (2)$$

[9] As is clear from the proof of Lemma 2, for $\alpha = 0$, the expected profit of the designer is only pinned down down up to constants equal to the expected payoffs of the buyer's and seller's worst-off types.

and the expected profit to the designer is

$$\Pi_\alpha(s) \equiv \mathbb{E}_{v,c}\left[\sum_{j=1}^{k} (\Phi_1(v,j) - \Gamma_1(c,j)) \cdot \mathbf{1}_{Q_\alpha(v,c;s)=j}\right]. \tag{3}$$

Proof. See Appendix A.

Using the expression for the designer's expected profit from Lemma 2, we can now define the second-best mechanism. The second-best mechanism has the allocation rule Q_{α^*} where α^* is the smallest $\alpha \in [0,1]$ such that $\Pi_\alpha(s) \geq 0$, i.e.,

$$\alpha^*(s) \equiv \min\{\alpha \in [0,1] \mid \Pi_\alpha(s) \geq 0\}.$$

Noting that $\Pi_\alpha(s)$ varies continuously with $\alpha \in [0,1]$ and that $\Pi_1(s) \geq 0$, we see that $\alpha^*(s)$ is well defined and varies continuously with s.

Using Lemma 2 and determining the allocation rule that solves (2) pointwise (and breaking ties in favor of greater trade), we get trade if and only if $\Phi_\alpha(v) - \Gamma_\alpha(c) \geq 0$, and trade of $j \in \{1, \ldots, k-1\}$ units if and only if

$$\Gamma_\alpha(c,j) - \Gamma_\alpha(c,j-1) \leq \Phi_\alpha(v,j) - \Phi_\alpha(v,j-1) < \Gamma_\alpha(c,j+1) - \Gamma_\alpha(c,j).$$

Thus, up to indifferences, the allocation rule Q_α for any mechanism in $\mathcal{M}_\alpha$ is for $s \in [0,1]$,

$$Q_\alpha(v,c;s) \equiv \begin{cases} 0 & \text{if } \Phi_\alpha(v) < \Gamma_\alpha(c), \\ j & \text{if } \frac{1}{s^{2(j-1)}}\Gamma_\alpha(c) \leq \Phi_\alpha(v) < \frac{1}{s^{2j}}\Gamma_\alpha(c) \text{ for } j \in \{1,\ldots,k-1\}, \\ k & \text{if } \frac{1}{s^{2(k-1)}}\Gamma_\alpha(c) \leq \Phi_\alpha(v), \end{cases} \tag{4}$$

and for $s > 1$,

$$Q_\alpha(v,c;s) \equiv \begin{cases} 0 & \text{if } \Phi_\alpha(v) < \frac{1}{s^{k-1}}\Gamma_\alpha(c), \\ k & \text{otherwise.} \end{cases}$$

We summarize with the following proposition:

Proposition 1. *Given parameter s, a first-best mechanism has allocation rule $Q_0(\cdot,\cdot;s)$, a second-best mechanism has allocation rule $Q_{\alpha^*(s)}(\cdot,\cdot;s)$, and a mechanism that maximizes the designer's expected payoff has allocation rule $Q_1(\cdot,\cdot;s)$.*

In all cases, each agent's interim expected payment in an incentive compatible, individually rational mechanism is pinned down by the allocation rule, with interim expected payments $m_B(v) \equiv \mathbb{E}_c[M_B(v,c)]$ and $m_S(c) \equiv \mathbb{E}_v[M_S(v,c)]$ given by (see the proof of Lemma 2):

$$m_B(v) = \sum_{j=1}^{k} S(j,s) \left[v \Pr_c(Q_\alpha(v,c;s) = j) - \int_0^v \Pr_c(Q_\alpha(t,c;s) = j) dt \right]$$

and

$$m_S(c) = \sum_{j=1}^{k} S(j,1/s) \left[c \Pr_v(Q_\alpha(v,c;s) = j) + \int_c^1 \Pr_v(Q_\alpha(v,t;s) = j) dt \right].$$

Given an allocation rule $Q_\alpha(v,c;s)$, we denote expected social surplus by

$$SS_\alpha(s) \equiv \mathbb{E}_{v,c} \left[\sum_{j=1}^{k} (\Phi_0(v,j) - \Gamma_0(c,j)) \cdot \mathbf{1}_{Q_\alpha(v,c;s)=j} \right].$$

Accordingly, expected social surplus under the first-best and second-best, denoted $SS^{FB}(s)$ and $SS^{SB}(s)$, respectively, are given by

$$SS^{FB}(s) \equiv SS_0(s) \quad \text{and} \quad SS^{SB}(s) \equiv SS_{\alpha^*(s)}(s).$$

4 Effects of substitutability and complementarity

In this section, we consider the effects of changes in the degree of substitutability and complementarity, as parameterized by s, on outcomes. We begin by noting that first-best and second-best expected social surplus are nondecreasing in s.

Proposition 2. *First-best and second-best expected social surplus are nondecreasing in the parameter s (strictly increasing if more than one unit trades with positive probability).*

Proof. Holding fixed the allocation rule, expected social surplus per trade for units beyond the first one increases with s. This is sufficient for the result that first-best expected social surplus increases with s. In addition, the expected revenue to the designer increases with s if more than one unit trades with positive probability. Thus, when the allocation rule is optimized for the increased value of s, maximized social surplus subject to no deficit must be higher. ■

Focusing on the case of complements or neutrality, i.e., $s \geq 1$, the setup is isomorphic to a standard bilateral trade problem with single-unit demand and supply in which the buyer's distribution is F, as in our original setup, and the seller's cost distribution is derived from G by rescaling it to have support $[0, 1/s^{k-1}]$. To see this, notice that in the complements setting, trade under the first-best and under the second-best is a binary decision—the quantity traded is either 0 or k. Moreover, under the first-best, the buyer of type v trades with probability $G(v/s^{k-1})$, which is also the probability of trade under the first-best if the seller's cost were drawn from the rescaled distribution (and the buyer's value is v and the seller's cost is c). Increasing s beyond 1 is thus equivalent to decreasing the upper bound of the support of the seller's cost distribution in the single-unit setup. Stated formally, we have:

Proposition 3. *For $s \geq 1$, the bilateral trade problem with k-unit demand and supply is equivalent to the bilateral trade problem with single-unit demand and supply where the buyer draws its value from F with support $[0,1]$ and the seller draws its cost from $H(c) \equiv G(cs^{k-1})$ with support $[0, \frac{1}{s^{k-1}}]$.*

Proof. See Appendix A.

Using Proposition 3, an increase in s beyond 1 improves the seller's cost distribution in the sense of a first-order stochastically dominated shift. However, for any finite s, the supports of the buyer's and seller's distributions continue to overlap. Thus, an implication of Proposition 3, together with the Myerson-Satterthwaite result on the impossibility of efficient trade when the buyer and seller have overlapping type distributions, is that for all $s \geq 1$, $\alpha^*(s) > 0$.

Another implication of Proposition 3 is that expected social surplus increases with s for any $\alpha \in [0,1]$, so $SS^{SB}(s)$ and $SS^{FB}(s)$ are both increasing in s. But, taking the limit as s goes to infinity, Proposition 3 implies that the seller's information rent vanishes, and so in the limit, the second-best problem is equivalent to a one-sided allocation problem for a designer with cost of zero facing a privately informed buyer. Setting a reserve of zero is then efficient, and hence in the limit, no surplus is lost due to information rents (because essentially the problem is one-sided). Thus, we have:

Corollary 1. $\lim_{s\to\infty} SS^{SB}(s)/SS^{FB}(s) = 1$.

Corollary 1 leads to the natural conjecture that $SS^{SB}(s)/SS^{FB}(s)$ continues to decrease as s decreases below 1. However, as we now show, this is not necessarily the case. Specifically, we provide results for the case of distributions F and G that are parameterized by $\sigma > 0$ in such a way that

$$F(v) = 1 - (1-v)^{\sigma} \quad \text{and} \quad G(c) = c^{\sigma},$$

which we refer to as the *linear virtual type* specification because it implies that

$$\Phi_{\alpha}(v) = \frac{\alpha+\sigma}{\sigma}v - \frac{\alpha}{\sigma} \quad \text{and} \quad \Gamma_{\alpha}(c) = \frac{\alpha+\sigma}{\sigma}c.$$

For $\sigma = 1$, this specification specializes to the case of uniformly distributed types.

For the linear virtual type specification, we have the following result:

Proposition 4. *In the linear virtual type specification with distributional parameter $\sigma > 0$: for all $s \in [0, \left(\frac{2\sigma+2}{2\sigma+1}\right)^{\frac{1}{k-1}}]$, $\alpha^*(s)$ does not vary with s; for all $s \in [0,1]$, $\frac{SS^{SB}(s)}{SS^{FB}(s)}$ does not vary with s; and for all $s \in [0,1]$, $\frac{SS^{SB}(s)}{SS^{FB}(s)}$ is decreasing in σ.*

Proof. See Appendix A.

Figure 1 illustrates Proposition 4. Panels (a) and (b) focus on $\alpha^*(s)$ and the ratio of second-best to first-best surplus. As shown in panel (c), as s increases, the expected quantity traded under both the first-best and second-best approaches k. Panel (d) illustrates that the ratio of second-best to first-best expected quantities goes to 1.

As shown in Proposition 4, in the linear virtual type setting, the ratio $\frac{SS^{SB}(s)}{SS^{FB}(s)}$ is constant for substitutes, that is, for $s \in [0,1]$,[10] and increases in s if s is sufficiently larger than 1, that is, incentives become less of a problem when goods are sufficiently strong complements. This is so even though initially, that is, for $s \in [1, \left(\frac{2\sigma+2}{2\sigma+1}\right)^{\frac{1}{k-1}}]$, $\alpha^*(s)$ is constant and so the second-best mechanism only varies with s through the direct effect of s (i.e., the effect other than through the no-deficit constraint), just as does the first-best mechanism. Given a mechanism with allocation rule Q_α, as long as $\frac{1}{s^{k-1}}\Gamma_\alpha(1) \geq 1$, even the most efficient buyer type trades with probability less than 1, whereas for $\frac{1}{s^{k-1}}\Gamma_\alpha(1) < 1$, the most efficient buyer type trades with probability 1. Thus, as s increases to the point that $\frac{1}{s^{k-1}}\Gamma_\alpha(1)$ drops below 1, $\alpha^*(s)$ starts to decrease in s.

While Proposition 4 focuses on the linear type specification, we can show more generally that $\alpha^*(s)$ is constant for a range of values for s greater than 1 and that greater complementarity improves second-best outcomes.

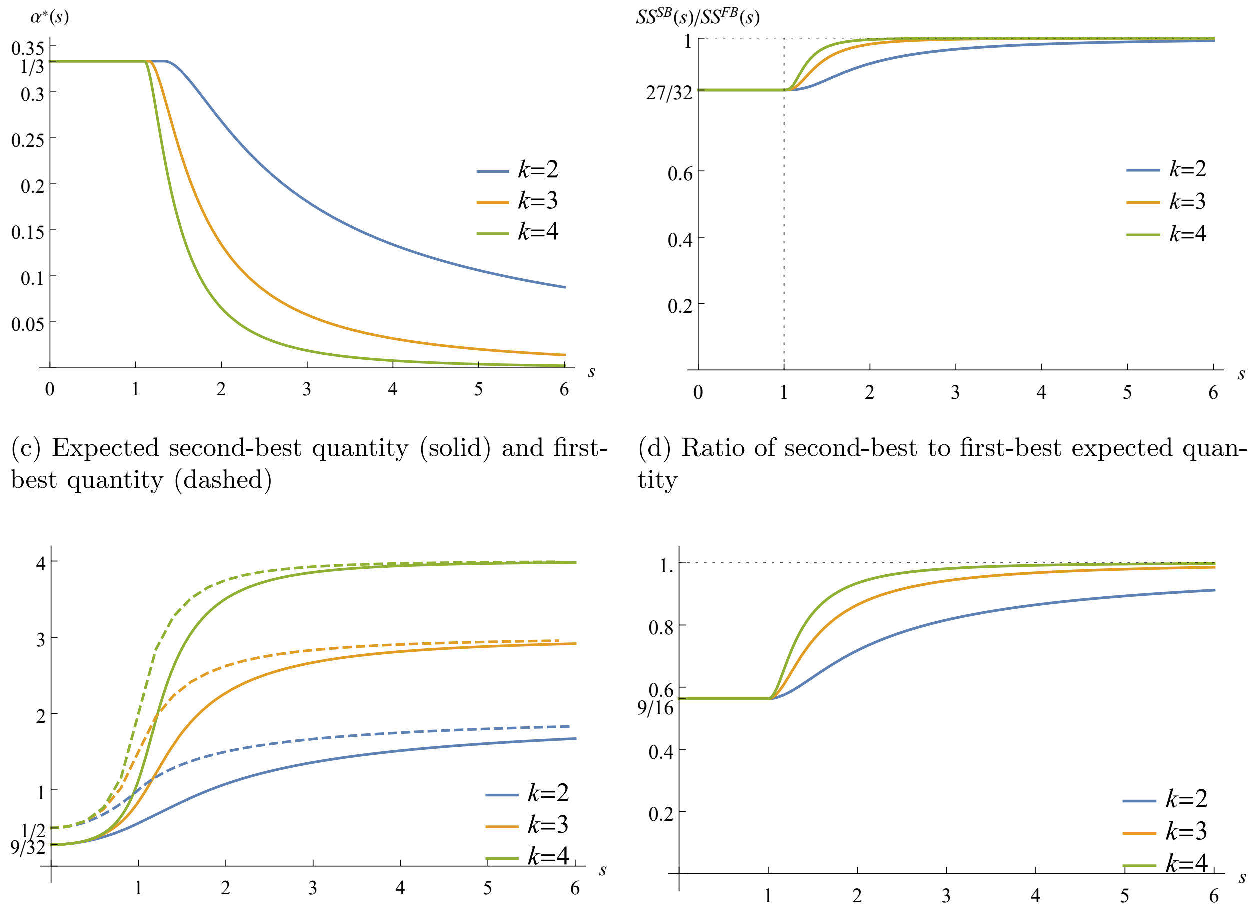
Proposition 5. *For $s' \equiv (\Gamma_{\alpha^*(1)}(1))^{\frac{1}{k-1}}$, for all $s \in [1, s']$, $\alpha^*(s) = \alpha^*(1)$, and there exists $s'' \geq s'$ such that for all $s > s''$, $\alpha^{*\prime}(s) < 0$; and for $s > 1$, $\alpha^{*\prime}(s) \leq 0$ implies that $\frac{SS^{SB}(s)}{SS^{FB}(s)}$ is increasing in s.*

Proof. See Appendix A.

Proposition 5 shows that the stronger is the complementarity between units, that is, for $s > 1$, the larger is s, the less severe is the incentive problem.[11] This is surprising in light of Delacrétaz et al. (2019), who use the fact that sellers and buyers are complements to prove the impossibility of efficient trade in two-sided setups.

[10] It remains an open question whether there are distributions such that $\frac{SS^{SB}(s)}{SS^{FB}(s)}$ varies with s for $s \in [0,1]$.

[11] If G is the uniform distribution, then $\alpha^*(1) = 1/3$, and so $s' = \left(\frac{4}{3}\right)^{\frac{1}{k-1}}$.



(a) $\alpha^*(s)$

(b) $SS^{SB}(s)/SS^{FB}(s)$

(c) Expected second-best quantity (solid) and first-best quantity (dashed)

(d) Ratio of second-best to first-best expected quantity

Figure 1: Panels show, as functions of s for different values of k: (a) $\alpha^*(s)$; (b) $SS^{SB}(s)/SS^{FB}(s)$; (c) $\mathbb{E}_{v,c}[Q_{\alpha^*(s)}(v,c;s)]$ (solid lines) and $\mathbb{E}_{v,c}[Q_0(v,c;s)]$ (dashed lines); and (d) $\mathbb{E}_{v,c}[Q_{\alpha^*(a)}(v,c;s)]/\mathbb{E}_{v,c}[Q_0(v,c;s)]$. Assumes a bilateral trade setup with k-unit demand and supply and types that are uniformly distributed on $[0,1]$.

Ex post and expected revenue under the first-best

We now consider the *ex post* revenue under the first-best. To this end, we need to introduce the revenue maximizing dominant-strategy prices that implement the efficient allocation subject to the ex post individual rationality constraints. As is well known, these dominant-strategy prices are those of the Vickrey-Clarke-Groves (VCG) mechanism.

For $s > 1$, that is, the case of complements, the dominant-strategy prices that implement the efficient allocation are simply prices for the package consisting of k units. If there is trade, then the buyer pays $p_B(v,c) = S(k,1/s)c$ and the seller receives $p_S(v,c) = S(k,s)v$, resulting in revenue of

$$p_B(v,c) - p_S(v,c) = S(k,s)\left(c/s^{k-1} - v\right),$$

which is less than or equal to zero when trade occurs, i.e., when $v \geq \frac{c}{s^{k-1}}$. As noted, for $k = 1$, s plays no role. For $k > 1$, the derivative of revenue with respect to s is

$$-\frac{ks^{k-1} - S(k,s)}{s-1}v - \frac{S(k,s) - k}{s^k(s-1)}c. \tag{5}$$

Using the observation that for $s > 1$ and $k > 1$, $k < \sum_{i=1}^{k} s^{i-1} < ks^{k-1}$, we then have $k < S(k,s) < ks^{k-1}$, which implies that the expression in (5) is negative. That is, for complements, the ex post revenue that is associated with the first-best allocation and dominant-strategy prices becomes smaller (i.e., the deficit becomes larger) as s increases.

For substitutes, that is, $s < 1$, the dominant-strategy prices are vectors $\mathbf{p}_B(v,c)$ and $\mathbf{p}_S(v,c)$ of unit prices whose j-th elements are, respectively, the price that the buyer has to pay for the j-th unit and the price that the seller receives for producing the j-th unit, with $j \in \{1, \ldots, k\}$. These vectors are

$$\mathbf{p}_B(v,c) = \left(c, \frac{c}{s}, \ldots, \frac{c}{s^{k-1}}\right) \quad \text{and} \quad \mathbf{p}_S(v,c) = \left(v, sv, \ldots, s^{k-1}v\right).$$

If there is no trade, then no payments are made. If exactly j units are traded under the first-best, then the buyer facing the price vector $\mathbf{p}_B(v,c)$ would find it optimal to buy exactly j units and the seller facing the price vector $\mathbf{p}_S(v,c)$ would find it optimal to sell exactly j units. In line with Loertscher and Mezzetti (2019), who show that the VCG mechanism runs a deficit on every unit traded, the revenue on the j-th unit traded is $c/s^{j-1} - s^{j-1}v < 0$, which is increasing in j.[12] Keeping j fixed and assuming that it is efficient to trade j units, which is equivalent to stipulating that $s^{j-1}v - c/s^{j-1} > 0$, we have

$$\frac{\partial(c/s^{j-1} - s^{j-1}v)}{\partial s} = \frac{j-1}{s}\left(s^{j-1}v - \frac{c}{s^{j-1}}\right) > 0,$$

where the inequality follows because it is efficient to trade j units. That is, for a fixed j, the revenue on the j-th efficient trade increases in s, which all else equal means that the deficit decreases in s. However, for $s < 1$, increasing s also weakly increases the efficient quantity traded. If this happens, a negative summand is added to revenue, which has a negative effect on total revenue. This suggests that for $s \in (0,1)$, the effects of increasing s on ex post revenue and on expected revenue can go either way. This is corroborated in Figure 2.

The following proposition, in which $\Pi_0(s)$ denotes expected revenue under the first-best

[12] Because of this and because $s^{h-1}v$ is the upper bound of the Walrasian price gap and c/s^{h-1} is the lower bound of the Walrasian price gap, if the efficient quantity traded is $h \in \{1, \ldots, k\}$, then it follows that the deficit of the VCG mechanism is bounded below by h times the Walrasian price gap (see also Loertscher and Mezzetti, 2019).

for given s, summarizes these findings:

Proposition 6. *Ex post revenue under the first-best is nonpositive, and expected revenue under the first-best is negative. For $s > 1$, expected revenue under the first-best is decreasing in s, i.e., $\Pi_0'(s) < 0$. For the linear virtual type specification, there exist $\underline{s}, \overline{s} \in (0,1)$ with $\underline{s} < \overline{s}$ such that expected revenue under the first-best is increasing in s for $s \in (0, \underline{s})$ and decreasing in s for $s \in (\overline{s}, 1]$, i.e., $\Pi_0'(s) > 0$ for $s \in (0, \underline{s})$ and $\Pi_0'(s) < 0$ for $s \in (\overline{s}, 1]$.*

Proof. See Appendix A.

We illustrate the results for the linear virtual type specification in Figure 2. For the case of uniform distributions ($\sigma = 1$), $\Pi_0(0) = -1/6$ and $\Pi_0(1) = -k/6$, with expected revenue under the first-best initially increasing and then decreasing as s increases from zero to one.

(a) Expected revenue under the first-best

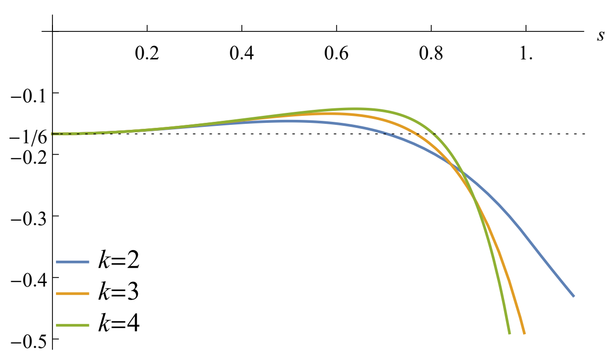


Figure 2: Expected revenue under the first-best as a function of s. Assumes a bilateral trade setup with k-unit demand and supply and types that are uniformly distributed on $[0, 1]$.

Propositions 5 and 6 show that the comparative statics for the deficit under the first-best are unreliable indicators of the severity of the incentive problem when accounting for second-best mechanisms. Indeed, as s varies in the range with $s > 1$, revenue under the first-best mechanism moves in the opposite direction from the second-best incentive problem. As noted, the incentive problem becomes less severe as s increases when goods are sufficiently strong complements.

5 Bilateral trade emerging from vertical integration

As mentioned, while the bilateral trade problem is a canonical problem in economics, a specific interpretation of the bilateral trade problem is that it emerges from vertical integration between a buyer and a supplier when, prior to integration, there were, say, two independent suppliers and one buyer.

In the case of single-unit demand and capacities, Loertscher and Marx (2021) show that vertical integration is socially harmful if the suppliers' distributions have identical supports and the lower bound of the buyer's value distribution is greater than or equal to the upper bound of the suppliers' cost distributions. In this case, the first-best is possible prior to integration and impossible after integration because the vertically integrated firm's demand for an externally supplied unit is given by its internal supplier's cost, whose support is the same as that of the external supplier's cost. This raises the question whether the harm from vertical integration is exacerbated or decreased if the agents have multi-unit demands and supplies. This is what we analyze in this section.

To this end, we now consider a setup in which, pre integration, there are two suppliers each with a capacity of k and drawing its cost independently from a distribution G with density g and support $[0, 1]$, and one buyer with value $v \in (0, 1]$, which is common knowledge. Each agent's supply or demand function is parameterized by s as in the baseline bilateral trade setup above. Vertical integration means that one supplier is vertically integrated with the buyer, so that post integration we have a setup in which there are two agents, the vertically integrated firm with cost c_0 and one independent supplier, with type c_1. Given v, if the vertically integrated firm with type c_0 purchases $q \in \{0, 1, ..., k\}$ units from the independent supplier, then it has utility $\frac{1-s^q}{1-s}v$ from the units purchased from the independent supplier and will also produce t^* additional units internally, where

$$t_q^*(c_0) \in \arg \max_{t \in \{0,...,k-q\}} \sum_{i=1}^{t} \left(s^{q+i-1}v - \frac{1}{s^{i-1}}c_0 \right). \tag{6}$$

The case of $s = 0$ corresponds to single-unit traders, and in that case, a vertically integrated firm that has $v = 1$ always consumes 1 unit, whether produced internally or purchased from the outside supplier. Moreover, if $s \geq 1$, then the buyer either consumes zero or k units, and all units consumed come from the same source, either the internal supplier or the outside supplier. It follows that $\alpha^*(s)$ is the same for $s = 0$ and for all $s \geq 1$ (e.g., for uniformly distributed types, $\alpha^*(s) = 1/3$). In a sense, $s = 0$ and $s \geq 1$ correspond to versions of Myerson and Satterthwaite (1983), so their model arises as two different limits of our model, as s goes to zero and as s goes to 1. We saw in Proposition 4, for the baseline bilateral trade setup with linear virtual types, $\alpha^*(s)$ is constant for $s \in [0, 1]$, but as we show next, $\alpha^*(s)$ is no longer constant for $s \in (0, 1)$ when bilateral trade involves a vertically integrated firm.

Define a modified weighted virtual value function as

$$\hat{\Phi}_\alpha(x, j) \equiv S(t_j^*(x), 1/s) \left(x - \alpha \frac{1 - G(x)}{g(x)} \right),$$

and define the vertically integrated firm's weighted virtual type function as

$$\Psi_\alpha(x, j) \equiv S(j + t_j^*(x), s)v - \hat{\Phi}_\alpha(x, j).$$

Then, we can use $\hat{\Phi}_\alpha$ and Ψ_α to the write the contribution of the vertically integrated firm to social surplus when j units are traded and its type is c_0 as

$$\begin{aligned} S(j + t_j^*(c_0), s)v - \sum_{i=1}^{t_j^*(c_0)} \frac{1}{s^{i-1}} c_0 &= S(j + t_j^*(c_0), s)v - \hat{\Phi}_0(c_0, t_j^*(c_0)) \\ &= \Psi_0(c_0, j), \end{aligned}$$

where the first term on the left side derives from the buyer's purchase of j units from the outside supplier and $t_j^*(c_0)$ units from the internal supplier, and the second term on the left side is the cost associated with the internal production of t^* units. In addition, we show in the proof of Proposition 7 that the buyer's expected payment in an incentive compatible, individually rational mechanism with allocation rule Q is

$$\sum_{j=0}^{k} \mathbb{E}_{c_0, c_1} \left[\Psi_1(c_0, j) \cdot \mathbf{1}_{Q(c_0, c_1) = j} \right] - S(k, s)v.$$

It then follows that one can write the α-weighted objective as:

$$\mathbb{E}_{c_0, c_1} \left[\sum_{j=0}^{k} \left(\Psi_\alpha(c_0, j) - \Gamma_\alpha(c_1, j) \right) \cdot \mathbf{1}_{Q(c_0, c_1) = j} \right] - \alpha S(k, s)v,$$

which allows us to prove the following result:[13]

Proposition 7. *The α-weighted objective for the setup with vertically integrated bilateral*

[13] If $k = 1$, so that we are in the traditional case with single-unit demand and supply, and if $v = 1$, then $t_q^*(c_0) = 1 - q$ and so $\Psi_\alpha(c_0, j) = -\hat{\Phi}_\alpha(c_0, t_j^*(x))$. Noting that $\Gamma_\alpha(c_1, 0) = \hat{\Phi}_\alpha(c_0, 0) = 0$, it follows that $Q_\alpha(c_0, c_1; s)$ is equal to 1 rather than zero if and only if $\Psi_\alpha(c_0, 1) - \Gamma_\alpha(c_1, 1) \geq 0$, i.e.,

$$c_1 + \alpha \frac{G(c_1)}{g(c_1)} < c_0 - \alpha \frac{1 - G(c_0)}{g(c_0)},$$

which is the usual result from the single-unit case.

trade is maximized with

$$Q_\alpha(c_0, c_1) \in \arg \max_{j \in \{0,\ldots,k\}} \Psi_\alpha(c_0, j) - \Gamma_\alpha(c_1, j).$$

and the associated expected designer profit is

$$\mathbb{E}_{c_0,c_1}\left[\sum_{j=0}^{k}(\Psi_1(c_0, j) - \Gamma_1(c_1, j)) \cdot \mathbf{1}_{Q_\alpha(c_0,c_1)=j}\right] - S(k, s)v.$$

Proof. See Appendix A.

We illustrate the setup with vertically integrated bilateral trade for the case with $v = 1$ and uniformly distributed types in Figure 3.

As shown in Figure 3, for $s \in (0, 1)$, variations in s affect the equilibrium quantity traded and consumed (see panel (c)); as s increases, the share of outsourcing (see panel (d)) initially increases because the vertically integrated firm sometimes buys an additional unit—with the first unit produced internally—from the external supplier, whereas producing a second unit internally is too costly when s is small. For $s = 1$, we are back in the constant marginal values setting, and from that point onward, the quantity traded is either 0 or k, so the problem becomes akin to the single-unit setting. However, in sharp contrast to the bilateral trade setting with an independent buyer and an independent supplier, further increases in s no longer decrease the overlap in the relevant parts of the support, and so no longer improve the performance of the second-best mechanism by decreasing $\alpha^*(s)$ (see panel (a)) because the integrated firm's willingness to pay is not given by $vS(k, s)$ but rather by $c_0 S(k, 1/s)$, whose distribution is the same as that for the independent supplier. This explains the invariance of $\alpha^*(s)$ in s for $s > 1$.

Comparing Figures 1(a) and 3(a), the comparative statics of $\alpha^*(s)$ with respect to s are thus almost exactly the mirror image without and with vertical integration: without it, $\alpha^*(s)$ does not vary with s (for F and G uniform, or more generally, for linear virtual types) for s below some threshold value greater than 1, and for s above the threshold, it decreases in s because of the effective decrease in overlap of the relevant supports. With vertical integration, $\alpha^*(s)$ is not monotone for $s \in (0, 1)$ and is constant for $s \geq 1$. The integrated firm's willingness to pay is given by a production cost rather than a valuation, and so varies with s in the same way and direction as the independent supplier's cost.

Figure 3(b) shows that the ratio $\frac{SS^{SB}(s)}{SS^{FB}(s)}$ is increasing for $s > 1$, despite $\alpha^*(s)$ being constant in that range. This is a general result and follows because the value of internal production, which is $S(k, s)v - S(k, 1/s)c_0$, increases, mechanically, in s:

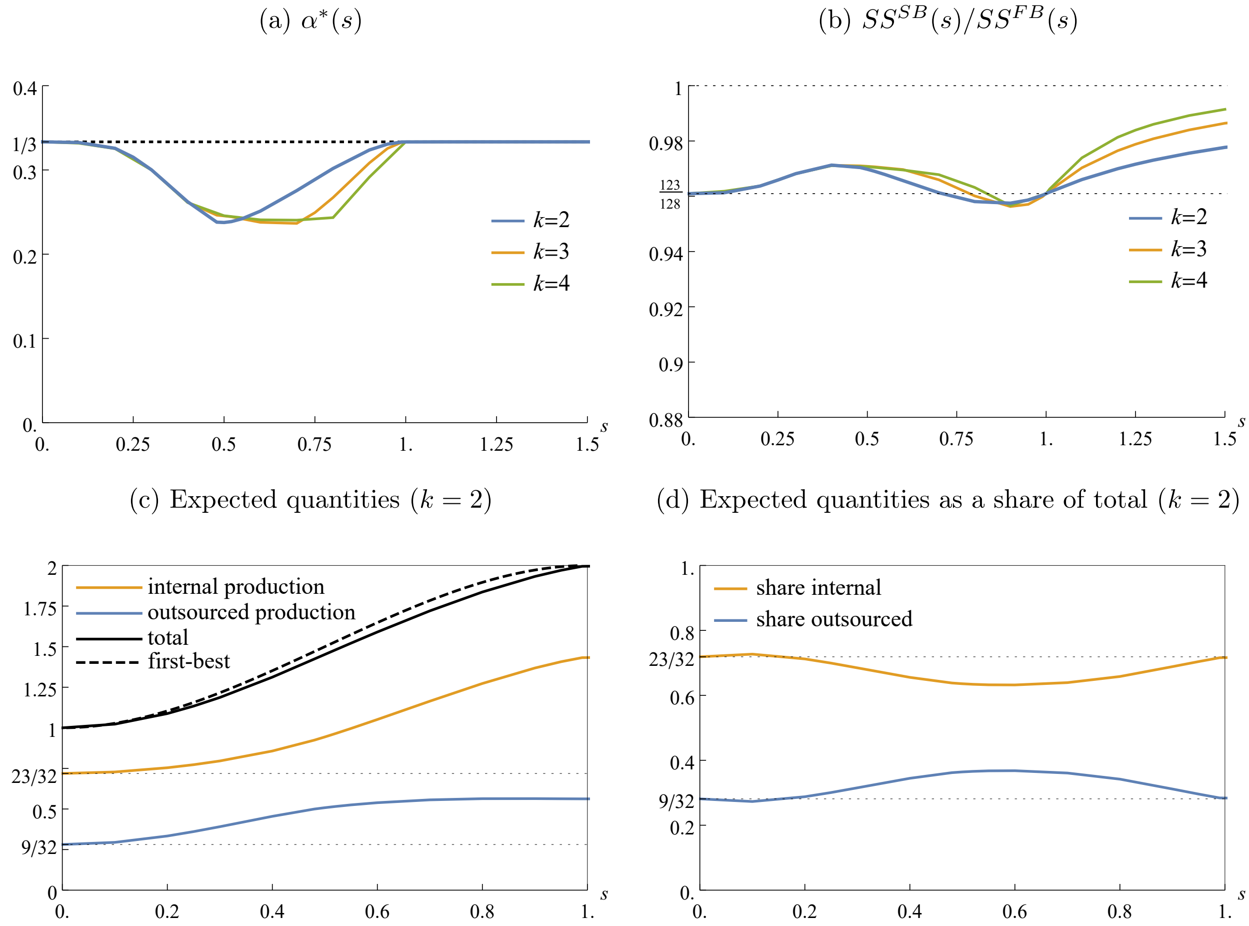


Figure 3: As a function of s and for values of k as indicated: (a) $\alpha^*(s)$; (b) $SS^{SB}(s)/SS^{FB}(s)$; (c) expected outsourced production, $\mathbb{E}_{c_0,c_1}[Q_{\alpha^*(s)}(c_0, c_1)]$, expected internal production, $\mathbb{E}_{c_0,c_1}[t^*_{Q_{\alpha^*(a)}(c_0,c_1)}(c_0)]$, total production, and first-best total production, $\mathbb{E}_{c_0,c_1}[Q_0(c_0, c_1)+t^*_{Q_0(c_0,c_1)}(c_0)]$ (where both Q_0 and $t^*_{Q_0(c_0,c_1)}$ depend on s); (d) expected outsourced and internal production as shares of total production. Assumes a vertically integrated bilateral trade setup with $v = 1$ and types that are uniformly distributed on $[0, 1]$, with calculations done for discrete values of s between 0 and 1 with increments 0.1 or less.

Proposition 8. *In the setup with vertically integrated bilateral trade, for* $s \geq 1$, $\alpha^*(s)$ *is constant and* $\frac{SS^{SB}(s)}{SS^{FB}(s)}$ *is increasing in* s, *and* $\lim_{s\to\infty} \frac{SS^{SB}(s)}{SS^{FB}(s)} = 1$.

Proof. See Appendix A.

Thus, for the case of complements, vertical integration does not change the result that $\frac{SS^{SB}(s)}{SS^{FB}(s)}$ increases with s and that incentive problems vanish in the limit.

6 Extensions

In this section, we first analyze social surplus under the designer's profit-maximizing mechanism and then extend the setup to accommodate public goods.

6.1 Social surplus under the designer's profit-maximizing mechanism

In contrast to Corollary 1, if we consider the designer's profit-maximizing mechanism, i.e., the mechanism with $\alpha = 1$, then the limit of the ratio of expected social surplus to expected first-best social surplus is bounded below 1. As intuition for this result, Proposition 3 implies that increases in s above 1 are equivalent to the seller's support shrinking to $[0, 1/s^{k-1}]$ and its distribution improving in the sense of becoming first-order stochastically dominated. In the limit, i.e., for $s \to \infty$, the seller's information rent vanishes and the problem is equivalent to a one-sided allocation problem for a designer with cost 0 facing a privately informed buyer. In the limit, the optimal reserve is $\Phi_1^{-1}(0) > 0$, which implies that the expected surplus under the profit-maximizing mechanism is $S(k,s)\mathbb{E}_v[v \mid v \geq \Phi_1^{-1}(0)] \Pr(v \geq \Phi_1^{-1}(0))$. Under the first-best, the expected surplus is $S(k,s)\mathbb{E}_v[v]$, which is larger, giving us the following result, where we use $SS^{MM}(s) \equiv SS_1(s)$ to denote the expected social surplus under the designer's profit-maximizing mechanism:

Proposition 9. $\frac{SS^{MM}(s)}{SS^{FB}(s)}$ *is nondecreasing in* s *and bounded below* 1, *i.e.,*

$$\lim_{s\to\infty} \frac{SS^{MM}(s)}{SS^{FB}(s)} = \frac{\mathbb{E}_v[v \mid v \geq \Phi_1^{-1}(0)] \Pr(v \geq \Phi_1^{-1}(0))}{\mathbb{E}_v[v]} = \frac{\int_{\Phi_1^{-1}(0)}^1 v dF(v)}{\int_0^1 v dF(v)} < 1.$$

For $s \in [0,1]$ *and the linear virtual type specification,* $\frac{SS^{MM}(s)}{SS^{FB}(s)}$ *does not vary with* s.

Proof. See Appendix A.

Thus, while the invariance result of Proposition 4 extends to the designer's optimal mechanism, the result in Corollary 1 that for the second-best mechanism, the cost imposed by private information and incentive compatibility (and individual rationality) vanishes in the limit, does not extend to the designer's optimal mechanism because, even in the limit, the buyer retains private information that affects the optimal mechanism.

6.2 Public goods

The bilateral trade problem has long been recognized as being isomorphic to a pure public good problem with two agents (see e.g. Mailath and Postlewaite, 1990). An interesting and relevant question is therefore whether this isomorphism extends to the case of multi-unit demands and supplies, which is what we explore next. As we show, the qualitative results derived above continue to hold for the provision of multiple public goods.

Consider a pure public good problem (non-rivalrous, no congestion), where the j-th unit of the public good has fixed cost K/s^{j-1}, where $K \geq 0$. Assume that there are $k \geq 1$ public goods that can be provided (e.g., roads and rail networks or an ice hockey and a soccer arena). Is the incentive problem more or less severe when the goods are substitutes or when they are complements?

Suppose that there are two agents with types v_1 and v_2, where v_1 and v_2 are both drawn from distribution F. Each agent i has value $s^{j-1}v_i$ for the j-th unit of the good.

In the public good setup, the mechanism $\langle Q, \mathbf{M} \rangle$ maps (v_1, v_2) onto the quantity provided to *both* buyers and the payments M_{B_1} and M_{B_2} provided by buyers 1 and 2, respectively, to the mechanism. Analogous to before, the designer's expected profit in an incentive compatible, individually rational mechanism with allocation rule Q is, up to a constant,

$$\mathbb{E}_{v_1,v_2}\left[M_{B_1}(v_1,v_2) + M_{B_2}(v_1,v_2)\right] = \mathbb{E}_{v_1,v_2}\left[\sum_{j=1}^{k} S(j,s)\left(\Phi_1(v_1) + \Phi_1(v_2)\right) \cdot \mathbf{1}_{Q(v_1,v_2)=j}\right],$$

which implies that the counterpart to (1) for incentive compatible, individually rational mechanisms in the public good setup is

$$\begin{aligned} & \mathbb{E}_{v_1,v_2}\left[\sum_{j=1}^{k} S(j,s)\left[(1-\alpha)\left(v_1+v_2-\frac{1}{s^{j-1}}K\right)+\alpha\left(\Phi_1(v_1)+\Phi_1(v_2)-\frac{1}{s^{j-1}}K\right)\right]\cdot \mathbf{1}_{Q(v_1,v_2)=j}\right] \\ = \; & \mathbb{E}_{v_1,v_2}\left[\sum_{j=1}^{k} S(j,s)\left[\Phi_\alpha(v_1)+\Phi_\alpha(v_2)-\frac{1}{s^{j-1}}K\right]\cdot \mathbf{1}_{Q(v_1,v_2)=j}\right]. \qquad (7) \end{aligned}$$

Thus, we can focus on the allocation rule Q_α that maximizes (7). Solving this pointwise, provision of k units is preferred to provision of zero units if

$$\Phi_\alpha(v_1) + \Phi_\alpha(v_2) \geq \frac{1}{s^{k-1}}K,$$

and for $s \in [0,1]$, provision of j units is optimal if:[14]

$$\frac{1}{s^{2(j-1)}}K \leq \Phi_\alpha(v_1) + \Phi_\alpha(v_2) < \frac{1}{s^{2j}}K.$$

It then follows that for $s \in [0,1]$,

$$Q_\alpha(v_1, v_2; s) = \begin{cases} 0 & \text{if } \Phi_\alpha(v_1) + \Phi_\alpha(v_2) < K, \\ j & \text{if } \frac{1}{s^{2(j-1)}}K \leq \Phi_\alpha(v_1) + \Phi_\alpha(v_2) < \frac{1}{s^{2j}}K \text{ for } j \in \{1, ..., k-1\}, \\ k & \text{if } \frac{1}{s^{2(k-1)}}K \leq \Phi_\alpha(v_1) + \Phi_\alpha(v_2), \end{cases}$$

and for $s > 1$,

$$Q_\alpha(v_1, v_2; s) = \begin{cases} 0 & \text{if } \Phi_\alpha(v_1) + \Phi_\alpha(v_2) < \frac{1}{s^{k-1}}K, \\ k & \text{otherwise.} \end{cases}$$

The first-best allocation is Q_0, and, analogous to before, the second-best allocation is $Q_{\alpha^*(s)}$, where $\alpha^*(s)$ is the smallest $\alpha \in [0,1]$ such that

$$\mathbb{E}_{v_1,v_2}\left[\sum_{j=1}^{k} S(j,s)\left(\Phi_1(v_1) + \Phi_1(v_2) - \frac{1}{s^{j-1}}K\right) \cdot \mathbf{1}_{Q_\alpha(v_1,v_2;s)=j}\right] \geq 0. \tag{8}$$

We then have the following result for the public good setup:

Proposition 10. *In the public good setup,* $\lim_{s\to\infty} \frac{SS^{SB}(s)}{SS^{FB}(s)} = 1$.

Proof. See Appendix A.

Thus, the public good setup is similar to our baseline setup with private goods in that the incentive problem is more severe when goods are substitutes than when they are complements and incentive problems vanish in the limit as s goes to infinity.

7 Conclusions

We analyze bilateral trade and related problems when the agents demand and supply multiple units but have one-dimensional, independently distributed private values. We show that for

[14]For $s \in [0,1]$, provision of j units is optimal if

$$(S(j-1,s) - S(j,s))(\Phi_\alpha(v_1) + \Phi_\alpha(v_2)) \leq (S(j-1,1/s) - S(j,1/s))K$$

and

$$(S(j+1,s) - S(j,s))(\Phi_\alpha(v_1) + \Phi_\alpha(v_2)) < (S(j+1,1/s) - S(j,1/s))K.$$

substitutes and the linear virtual type specification, the ratio of first-best to second-best welfare does not vary with the maximum number of units that the agents demand and supply, nor with how fast marginal values decrease and marginal costs increase. In contrast, with complements, the ratio of first-best to second-best welfare increases the stronger is the complementarity and goes to one in the limit of our specification with geometric utility.

We contribute to recent work addressing the challenges associated with modeling vertical integration in a setting with incomplete information. Our results highlight key differences between bilateral trade involving, on the one hand, a supplier and a buyer with no production capability, and, on the other hand, a supplier and a vertically integrated buyer. The integrated buyer's willingness to pay for external units of the good derives from both the integrated buyer's value and the integrated supplier's cost. As we show, relative to the case without vertical integration, a setting with vertical integration can generate distinctly different comparative statics for how the second-best allocation rule varies with the degree of substitutability and complementarity of goods. Despite these differences, under vertical integration, it continues to be the case that incentive problems vanish in the limit as the strength of the complementarity between goods increases without bound.

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A Appendix: Proofs

Proof of Lemma 1. Given incentive compatible mechanism $\langle Q, \mathbf{M}\rangle$, define the agents' expected payoffs under truthful reporting as $U_B(v) \equiv U_B(v,v)$ and $U_S(c) \equiv U_S(c,c)$. Denote the interim expected allocations for the buyer and seller, respectively, by $q_B(v) \equiv \mathbb{E}_c[Q(v,c)]$ and $q_S(c) \equiv \mathbb{E}_v[Q(v,c)]$. Standard mechanism design techniques (see, e.g., Krishna, 2010) imply that

$$U_B(v) = \int_0^v q_B(x)dx + U_B(0)$$

and

$$U_S(c) = \int_c^1 q_S(x)dx + U_S(1).$$

It then follows that for all $v \in [0,1]$, $U_B(v) \geq U_B(0)$ and for all $c \in [0,1]$, $U_S(c) \geq U_S(1)$. ■

Proof of Lemma 2. The proof follows the usual mechanism design arguments for the case of single-unit demand and supply (e.g., Krishna, 2010), which is adapted straightforwardly to our setup with multi-unit demand and supply.

Given incentive compatible, individually rational mechanism $\langle Q, \mathbf{M}\rangle$, define $\hat{q}_j(v) \equiv \Pr_c(Q(v,c) = j)$ and $m_B(v) \equiv \mathbb{E}_c[M_B(v,c)]$. Then the buyer's interim expected payoff is

$$U_B(v) \equiv \sum_{j=1}^{k} S(j,s)v\hat{q}_j(v) - m_B(v). \tag{9}$$

Incentive compatibility implies that

$$U_B(v) = \max_{z\in[0,1]} \sum_{j=1}^{k} S(j,s)v\hat{q}_j(z) - m_B(z),$$

which implies that U_B is a maximum of a family of affine functions and therefore is a convex function. In addition, incentive compatibility implies that for all $z, x \in [0,1]$,

$$\sum_{j=1}^{k} S(j,s)z\hat{q}_j(x) - m_B(x) \leq \sum_{j=1}^{k} S(j,s)z\hat{q}_j(z) - m_B(z) = U_B(z). \tag{10}$$

We can write for all $z, x \in [0,1]$,

$$\sum_{j=1}^{k} S(j,s)z\hat{q}_j(x) - m_B(x) = U_B(x) + \sum_{j=1}^{k} S(j,s)(z-x)\hat{q}_j(x),$$

which, using (10), implies that

$$U_B(z) \geq U_B(x) + (z - x) \sum_{j=1}^{k} S(j,s)\hat{q}_j(x).$$

This implies that for all $x \in [0,1]$, $\sum_{j=1}^{k} S(j,s)\hat{q}_j(x)$ is the slope of the line that supports the function U_B at the point x. A convex function is absolutely continuous and thus it is differentiable almost everywhere in the interior of its domain. Thus, at every point that U_B is differentiable,

$$U_B'(v) = \sum_{j=1}^{k} S(j,s)\hat{q}_j(v).$$

Because every absolutely continuous function is the definite integral of its derivative, we have

$$U_B(v) = U_B(0) + \int_0^v \sum_{j=1}^{k} S(j,s)\hat{q}_j(t)dt.$$

Using (9) and individual rationality, which implies that $U_B(0) = 0$,

$$m_B(v) = \sum_{j=1}^{k} S(j,s)v\hat{q}_j(v) - \int_0^v \sum_{j=1}^{k} S(j,s)\hat{q}_j(t)dt.$$

Taking expectations, we have

$$\begin{aligned}
\mathbb{E}_v\left[m_B(v)\right] &= \int_0^1 \sum_{j=1}^{k} S(j,s)v\hat{q}_j(v)f(v)dv - \int_0^1 \int_0^v \sum_{j=1}^{k} S(j,s)\hat{q}_j(t)f(v)dtdv \\
&= \int_0^1 \sum_{j=1}^{k} S(j,s)v\hat{q}_j(v)f(v)dv - \int_0^1 \sum_{j=1}^{k} S(j,s)\hat{q}_j(t)(1-F(t))dt \\
&= \int_0^1 \sum_{j=1}^{k} S(j,s)\left(t - \frac{1-F(t)}{f(t)}\right)\hat{q}_j(t)f(t)dt \\
&= \int_0^1 \sum_{j=1}^{k} \Phi_1(t,j)\hat{q}_j(t)f(t)dt \\
&= \sum_{j=1}^{k} \mathbb{E}_v\left[\Phi_1(v,j)\hat{q}_j(v)\right] \\
&= \sum_{j=1}^{k} \mathbb{E}_{v,c}\left[\Phi_1(v,j)\cdot \mathbf{1}_{Q(v,c)=j}\right],
\end{aligned}$$

where the first equality uses the definition of the expectation, the second equality reverses the order of integration for the second term and integrates, the third equality collects term, the fourth equality uses the definition of Φ_1, the fifth equality uses the definition of the expectation, and the sixth equality uses the definition of $\hat{q}_j$.

The argument for why, when $U_S(1) = 0$,

$$m_S(c) = \sum_{j=1}^{k} S(j, 1/s) \left[c \Pr_v(Q_\alpha(v, c; s) = j) + \int_c^1 \Pr_v(Q_\alpha(v, t; s) = j) dt \right]$$

and

$$\mathbb{E}_c \left[m_S(c) \right] = \sum_{j=1}^{k} \mathbb{E}_{v,c} \left[\Gamma_1(c, j) \cdot \mathbf{1}_{Q(v,c)=j} \right]$$

follows along similar lines.

One can then write the weighted objective in (1) as

$$\begin{aligned} & \mathbb{E}_{v,c} \left[(1-\alpha) \sum_{j=1}^{k} (\Phi_0(v, j) - \Gamma_0(c, j)) \cdot \mathbf{1}_{Q(v,c)=j} + \alpha \sum_{j=1}^{k} (\Phi_1(v, j) - \Gamma_1(c, j)) \cdot \mathbf{1}_{Q(v,c)=j} \right] \\ = \; & \mathbb{E}_{v,c} \left[\sum_{j=1}^{k} (\Phi_\alpha(v, j) - \Gamma_\alpha(c, j)) \cdot \mathbf{1}_{Q(v,c)=j} \right], \end{aligned}$$

where the equality uses

$$\begin{aligned} (1-\alpha)\Phi_0(v, j) + \alpha \Phi_1(v, j) &= (1-\alpha) S(j, s) v + \alpha S(j, s) \Phi_1(v) \\ &= (1-\alpha) S(j, s) v + \alpha S(j, s) \left(v - \frac{1 - F(v)}{f(v)} \right) \\ &= S(j, s) \left(v - \alpha \frac{1 - F(v)}{f(v)} \right) \\ &= S(j, s) \Phi_\alpha(v) \\ &= \Phi_\alpha(v, j), \end{aligned}$$

and similarly for the virtual costs. This proves the result that Q_α solves (2) and completes the proof. ■

Proof of Proposition 3. Letting $\hat{\Gamma}_\alpha$ be the weighted virtual cost function associated with distribution H, we have

$$\hat{\Gamma}_\alpha \left(\frac{x}{s^{k-1}} \right) \equiv \frac{1}{s^{k-1}} \Gamma_\alpha(x).$$

With single-unit demand and types v and x drawn from F and H, respectively, trade occurs if and only if $\Phi_\alpha(v) \geq \hat{\Gamma}_\alpha(x)$. With a change of variables, letting c be drawn from distribution G, we can write this as $\Phi_\alpha(v) \geq \hat{\Gamma}_\alpha(c/s^{k-1}) = \frac{1}{s^{k-1}}\Gamma_\alpha(c)$, which is the condition for trade in the bilateral trade problem with k-unit demand and supply. ■

Proof of Proposition 4. To start, focus on the case with $s \in [0,1]$, so that the allocation rule is given by (4). We can write the expected profit of the designer as

$$\Pi_\alpha(s) = \sum_{j=1}^{k} \int_0^{\Gamma_\alpha^{-1}(s^{2(j-1)})} \int_{\Phi_\alpha^{-1}(\frac{1}{s^{2(j-1)}}\Gamma_\alpha(c))}^{1} (s^{j-1}\Phi_1(v) - \frac{1}{s^{j-1}}\Gamma_1(c))dF(v)dG(c).$$

Using the assumption of linear virtual types, which implies that $\Phi_\alpha^{-1}(x) = \frac{\sigma}{\alpha+\sigma}x + \frac{\alpha}{\alpha+\sigma}$ and $\Gamma_\alpha^{-1}(x) = \frac{\sigma}{\alpha+\sigma}x$, we have

$$\Pi_\alpha(s) = \sum_{j=1}^{k} \int_0^{\frac{\sigma s^{2(j-1)}}{\alpha+\sigma}} \int_{\frac{1}{s^{2(j-1)}}c+\frac{\alpha}{\alpha+\sigma}}^{1} \left(s^{j-1}\left(\frac{1+\sigma}{\sigma}v - \frac{1}{\sigma}\right) - \frac{1}{s^{j-1}}\frac{1+\sigma}{\sigma}c\right) dF(v)dG(c).$$

Evaluating the inner integral, we have

$$\Pi_\alpha(s) = \sum_{j=1}^{k} \int_0^{\frac{\sigma s^{2(j-1)}}{\alpha+\sigma}} \frac{(s^{2(j-1)}\sigma - c(\alpha+\sigma))^\sigma (s^{2(j-1)}\alpha\sigma - c(\sigma+\alpha))}{\sigma(\alpha+\sigma)^{\sigma+1}s^{(j-1)(1+2\sigma)}} dG(c).$$

Integrating again, we get

$$\Pi_\alpha(s) = \sum_{j=1}^{k} \frac{\sigma^{2\sigma}s^{(2\sigma+1)(j-1)}\left(Gamma(\sigma+1)\right)^2}{(\alpha+\sigma)^{2\sigma+1}Gamma(2\sigma+2)}(2\alpha\sigma+\alpha-\sigma),$$

where $Gamma(\cdot)$ denotes the Euler gamma function. As is clear from this expression, $\Pi_\alpha(s) = 0$ if and only if $\alpha = \frac{\sigma}{2\sigma+1}$, which establishes that $\alpha^*(s) = \frac{\sigma}{2\sigma+1}$ independently of s and k.

Turning to the case with $s > 1$, we have

$$\begin{aligned}\Pi_\alpha(s) &= S(k,s)\int_0^{\min\{1,\Gamma_\alpha^{-1}(s^{k-1})\}} \int_{\Phi_\alpha^{-1}(\frac{1}{s^{k-1}}\Gamma_\alpha(c))}^{1} \left(\Phi_1(v) - \frac{1}{s^{k-1}}\Gamma_1(c)\right) dF(v)dG(c) \\ &= S(k,s)\,(\sigma+1)\,\sigma \int_0^{\min\{1,\Gamma_\alpha^{-1}(s^{k-1})\}} \int_{\frac{c}{s^{k-1}}+\frac{\alpha}{\sigma+\alpha}}^{1} \left(v - \frac{1}{\sigma+1} - \frac{1}{s^{k-1}}c\right)(1-v)^{\sigma-1}c^{\sigma-1}dvdc.\end{aligned}$$

Evaluating the inner integral, we have with $z(c) \equiv \frac{c}{s^{k-1}} + \frac{\alpha}{\sigma+\alpha}$,

$$\Pi_\alpha(s) = S(k,s)(\sigma+1)\int_0^{\min\{1,\frac{\sigma s^{k-1}}{\sigma+\alpha}\}} (1-z(c))^\sigma \left[\frac{1+\sigma z(c)}{1+\sigma} - \frac{1}{s^{k-1}}c - \frac{1}{\sigma+1}\right] c^{\sigma-1} dc.$$

Integrating again, we get for $\frac{\sigma s^{k-1}}{\sigma+\alpha} \leq 1$, i.e., $s \leq \left(\frac{\sigma+\alpha}{\sigma}\right)^{\frac{1}{k-1}}$,

$$\Pi_\alpha(s) = S(k,s)\left[\frac{s^{\sigma(k-1)}\sigma^{2\sigma}\left(Gamma(1+\sigma)\right)^2}{(\alpha+\sigma)^{1+2\sigma}Gamma(2\sigma+2)}\right](\alpha-\sigma+2\alpha\sigma),$$

which is zero if and only if $\alpha = \frac{\sigma}{2\sigma+1}$, which is independent of s and k. Thus, substituting $\alpha = \frac{\sigma}{2\sigma+1}$ into the upper bound for s of $\left(\frac{\sigma+\alpha}{\sigma}\right)^{\frac{1}{k-1}}$, we have that $\alpha^*(s)$ is constant at $\frac{\sigma}{2\sigma+1}$ for all $s \leq \left(\frac{2\sigma+2}{2\sigma+1}\right)^{\frac{1}{k-1}}$.

Returning to the case with $s \in [0,1]$, we now evaluate the ratio $\frac{SS^{SB}(s)}{SS^{FB}(s)}$. Writing α^* without the argument s, we have

$$\begin{aligned} SS^{SB}(s) &= \sum_{j=1}^{k}\int_0^{\frac{\sigma s^{2(j-1)}}{\alpha^*+\sigma}}\int_{\frac{1}{s^{2(j-1)}}c+\frac{\alpha^*}{\alpha^*+\sigma}}^{1}(s^{j-1}v - \frac{1}{s^{j-1}}c)dF(v)dG(c) \\ &= \frac{\sigma(1+2\sigma)^{2\sigma}\left(Gamma(\sigma)\right)^2}{(1+\sigma)^{1+2\sigma}2^{2\sigma+1}Gamma(2\sigma)}\sum_{j=1}^{k}s^{(j-1)(1+2\sigma)} \end{aligned} \tag{11}$$

and

$$\begin{aligned} SS^{FB}(s) &= \sum_{j=1}^{k}\int_0^{s^{2(j-1)}}\int_{\frac{1}{s^{2(j-1)}}c}^{1}(s^{j-1}v - \frac{1}{s^{j-1}}c)dF(v)dG(c) \\ &= \frac{\sigma(\sigma+1)\left(Gamma(\sigma)\right)^2}{Gamma(2\sigma)(2\sigma+2)(2\sigma+1)}\sum_{j=1}^{k}s^{(j-1)(1+2\sigma)}, \end{aligned} \tag{12}$$

which implies that

$$\frac{SS^{SB}(s)}{SS^{FB}(s)} = 2\left(\frac{1+2\sigma}{2+2\sigma}\right)^{2\sigma+1},$$

which is independent of s (and k).

It is then straightforward to show that $2\left(\frac{2\sigma+1}{2\sigma+2}\right)^{2\sigma+1}$ is decreasing in σ because the slope has sign equal to the sign of $1 + 2(1+\sigma)\ln\left(\frac{1+2\sigma}{2+2\sigma}\right)$, which is concave in σ and maximized and equal to zero at $\sigma = \infty$, and so it is negative for all finite $\sigma > 0$. Further, $\lim_{\sigma\to\infty} 2\left(\frac{2\sigma+1}{2\sigma+2}\right)^{2\sigma+1} = 2/e$, which is clear using $\lim_{z\to\infty}(\frac{z+1}{z})^z = e$ and letting $z = 2\sigma+1$. ■

Proof of Proposition 5. We begin the proof by proving the following lemma:

Lemma A.1. *There exists $\hat{s} > 1$ such that for all $s > \hat{s}$, $\alpha^{*\prime}(s) < 0$.*

Proof of Lemma A.1. For $s > 1$, $\Pi_\alpha(s) = S(k,s)\hat{\Pi}_\alpha(s)$, where

$$\hat{\Pi}_\alpha(s) \equiv \int_0^{\min\{1,\Gamma_\alpha^{-1}(s^{k-1})\}} \int_{\Phi_\alpha^{-1}(\frac{1}{s^{k-1}}\Gamma_\alpha(c))}^{1} \left(\Phi_1(v) - \frac{1}{s^{k-1}}\Gamma_1(c)\right) dF(v)dG(c).$$

Because $S(k,s) = \frac{1-s^k}{1-s}$ is positive and increasing in s, if $\Pi_\alpha(s) \geq 0$, then to show that $\Pi'_\alpha(s) > 0$, it is sufficient to show that $\hat{\Pi}'_\alpha(s) > 0$. Taking the derivative of $\hat{\Pi}_\alpha(s)$, we have

$$\begin{aligned}\hat{\Pi}'_\alpha(s) \;=\; & \int_0^{\min\{1,\Gamma_\alpha^{-1}(s^{k-1})\}} \frac{k-1}{s^k}\Gamma_1(c)\left(1 - F\left(\Phi_\alpha^{-1}(\frac{1}{s^{k-1}}\Gamma_\alpha(c))\right)\right) dG(c) \\ & + \int_0^{\min\{1,\Gamma_\alpha^{-1}(s^{k-1})\}} \Phi_\alpha^{-1\prime}(\frac{1}{s^{k-1}}\Gamma_\alpha(c))\frac{k-1}{s^k}\Gamma_\alpha(c) \\ & \cdot \underbrace{\left(\Phi_1(\Phi_\alpha^{-1}(\frac{1}{s^{k-1}}\Gamma_\alpha(c))) - \frac{1}{s^{k-1}}\Gamma_1(c)\right)}_{\text{negative}} f(\Phi_\alpha^{-1}(\frac{1}{s^{k-1}}\Gamma_\alpha(c)))dG(c).\end{aligned}$$

Letting $z \equiv \Phi_\alpha^{-1}(\frac{1}{s^{k-1}}\Gamma_\alpha(c))$, we can rewrite this as

$$\hat{\Pi}'_\alpha(s) = \frac{k-1}{s^k}\int_0^{\min\{1,\Gamma_\alpha^{-1}(s^{k-1})\}} \left[\Gamma_1(c)\frac{1-F(z)}{f(z)} + \frac{1}{\Phi'_\alpha(z)}\Gamma_\alpha(c)\underbrace{\left(\Phi_1(z) - \frac{1}{s^{k-1}}\Gamma_1(c)\right)}_{\text{negative}}\right] f(z)dG(c),$$

which is positive if for all $c \in (0,1)$, the term in square brackets is positive. Focusing on the term in square brackets and letting $h(x) \equiv \frac{1-F(x)}{f(x)}$, which we assume is decreasing, we can write the condition that the term in square brackets is positive as

$$\Gamma(c)h(z) + \frac{1}{1-\alpha h'(z)}\Gamma_\alpha(c)\left(z - h(z) - \frac{1}{s^{k-1}}\Gamma(c)\right) > 0.$$

Because for all $c \in (0,1)$, $\Gamma(c) \geq \Gamma_\alpha(c) > 0$, for this to hold, it is sufficient that

$$h(z) + \frac{1}{1-\alpha h'(z)}\left(z - h(z) - \frac{1}{s^{k-1}}\Gamma(c)\right) > 0$$

which we can rewrite as

$$z - \alpha h(z)h'(z) > \frac{1}{s^{k-1}}\Gamma(c). \tag{13}$$

Define $X \equiv \Phi_\alpha^{-1}(0) - \alpha h(\Phi_\alpha^{-1}(0))h'(\Phi_\alpha^{-1}(0))$ and note that by our assumption that $h'(x) < 0$, we have $X > \Phi_\alpha^{-1}(0) > 0$. Let $\varepsilon > 0$ be small. Because $\lim_{s\to\infty} \frac{1}{s^{k-1}}\Gamma(c) = 0$, there exists s' sufficiently large such that for all $s > s'$, $\frac{1}{s^{k-1}}\Gamma(c) < X - \varepsilon$. In addition, because $\lim_{s\to\infty} z - \alpha h(z)h'(z) = X$, there exists s'' sufficiently large such that for all $s > s''$, $z - \alpha h(z)h'(z) > X - \varepsilon$. Thus, for all $s > \max\{s', s''\}$, (13) holds, which implies that $\hat{\Pi}'_\alpha(s) > 0$.

Then, because $\Pi_{\alpha^*(s)}(s) \geq 0$ by the definition of α^*, it follows that for all s sufficiently large $\Pi'_{\alpha^*(s)}(s) > 0$, which implies that $\alpha^{*\prime}(s) < 0$. $\square$

Continuation of the proof of Proposition 5: Define $\tau_\alpha^s(c) \equiv \Phi_\alpha^{-1}\left(\frac{1}{s^{k-1}}\Gamma_\alpha(c)\right)$. Then

$$SS^{FB}(s) = SS^{SB}(s) + \int_0^1 \int_{\tau_0^s(c)}^{\tau^s_{\alpha^*(s)}(c)} (v - \frac{1}{s^{k-1}}c) dF(v)dG(c)$$

and

$$\frac{d}{ds}\frac{SS^{SB}(s)}{SS^{FB}(s)} = \frac{SS^{SB\prime}(s)SS^{FB}(s) - SS^{SB}(s)SS^{FB\prime}(s)}{\left(SS^{FB}(s)\right)^2},$$

which has sign equal to the sign of

$$\begin{aligned}
& SS^{SB\prime}(s)SS^{FB}(s) - SS^{SB}(s)SS^{FB\prime}(s) \\
= \; & \underbrace{\left(SS^{FB}(s) - SS^{SB}(s)\right)}_{\text{nonnegative}}\underbrace{SS^{SB\prime}(s)}_{\text{nonnegative}} + \underbrace{SS^{SB}(s)\int_0^1 \int_{\tau_0^s(c)}^{\tau^s_{\alpha^*(s)}(c)} \frac{k-1}{s^k} c dF(v)dG(c)}_{\text{nonnegative (positive if } k>1 \text{ and } \alpha^*(s)>0)} \\
& - SS^{SB}(s)\int_0^1 \frac{d\tau^s_{\alpha^*(s)}(c)}{ds} \underbrace{(\tau^s_{\alpha^*(s)}(c) - \frac{1}{s^{k-1}}c)}_{\text{positive}} f(\tau^s_{\alpha^*(s)}(c))dG(c).
\end{aligned}$$

Thus, a sufficient condition for our result is that is $\frac{d\tau^s_{\alpha^*(s)}(c)}{ds} < 0$. Let $\tilde{\Phi}_\alpha^{-1}(x)$ denote the derivative of $\Phi_\alpha^{-1}(x)$ with respect to α and note that $\tilde{\Phi}_\alpha^{-1}(x) > 0$ for $x \in [0, 1)$, and let $\tilde{\Gamma}_\alpha(x)$ denote the derivative of Γ_α with respect to α and note that $\tilde{\Gamma}_\alpha(x) > 0$ for $x \in (0, 1]$. It then follows that

$$\frac{d\tau^s_{\alpha^*(s)}(c)}{ds} = -\Phi^{-1\prime}_{\alpha^*(s)}\left(\frac{1}{s^{k-1}}\Gamma_{\alpha^*(s)}(c)\right)\frac{k-1}{s^k}\Gamma_\alpha(c) + \tilde{\Phi}^{-1}_{\alpha^*(s)}\left(\frac{1}{s^{k-1}}\Gamma_\alpha(c)\right)\frac{1}{s^{k-1}}\tilde{\Gamma}_{\alpha^*(s)}(c)\alpha^{*\prime}(s),$$

which is negative if $\alpha^{*\prime}(s) \leq 0$.

Finally, we show that $\alpha^*(s)$ is constant for a range of s greater than 1. For $s > 1$, the sign of the designer's profit is equal to the sign of

$$\hat{\Pi}_\alpha(s) = \int_0^{\min\{1,\Gamma_\alpha^{-1}(s^{k-1})\}} \int_{\Phi_\alpha^{-1}(\frac{1}{s^{k-1}}\Gamma_\alpha(c))}^1 \left(\Phi_1(v) - \frac{1}{s^{k-1}}\Gamma_1(c)\right) dF(v)dG(c).$$

Letting $H(x;s) \equiv G(xs^{k-1})$, $h(x;s) \equiv s^{k-1}g(xs^{k-1})$, and $\hat{\Gamma}_\alpha(x;s) \equiv x + \alpha\frac{H(x;s)}{h(x;s)}$, we have $\frac{1}{s^{k-1}}\Gamma_\alpha(x) = \hat{\Gamma}_\alpha(x/s^{k-1};s)$, so

$$\hat{\Pi}_\alpha(s) = \int_0^{\min\{1,\Gamma_\alpha^{-1}(s^{k-1})\}} \int_{\Phi_\alpha^{-1}(\hat{\Gamma}_\alpha\left(\frac{c}{s^{k-1}};s\right))}^1 \left(\Phi_1(v) - \hat{\Gamma}_1\left(\frac{c}{s^{k-1}};s\right)\right) f(v)\frac{1}{s^{k-1}}h\left(\frac{c}{s^{k-1}};s\right) dvdc.$$

Making a change of variables to $z = \frac{c}{s^{k-1}}$, we have:[15]

$$\hat{\Pi}_\alpha(s) = \int_0^{\min\{\frac{1}{s^{k-1}},\hat{\Gamma}_\alpha^{-1}(1;s)\}} \int_{\Phi_\alpha^{-1}(\hat{\Gamma}_\alpha(z;s))}^1 \left(\Phi_1(v) - \hat{\Gamma}_1(z;s)\right) f(v)h(z;s)\, dvdz.$$

We know from Myerson-Satterthwaite that $\alpha^*(1) > 0$, so the designer's profit at $\alpha^*(1)$ must be equal to zero, i.e., $\hat{\Pi}_{\alpha^*(1)}(1) = 0$.

Now increase s above 1. As long as it continues to be the case that $\hat{\Gamma}_{\alpha^*(1)}^{-1}(1;s) \leq \frac{1}{s^{k-1}}$, which we can also write as $\Gamma_{\alpha^*(1)}^{-1}(s^{k-1}) \leq 1$, then $\hat{\Pi}_{\alpha^*(1)}(s) = \hat{\Pi}_{\alpha^*(1)}(1) = 0$, and so $\alpha^*(s) = \alpha^*(1)$. So it suffices to show that $\Gamma_{\alpha^*(1)}^{-1}(s^{k-1}) \leq 1$ for s just above 1. This holds for all s if $k = 1$. If $k > 1$, we require that $s \leq (\Gamma_{\alpha^*(1)}(1))^{\frac{1}{k-1}}$, so we need to show that $(\Gamma_{\alpha^*(1)}(1))^{\frac{1}{k-1}} > 1$, which holds because $\Gamma_{\alpha^*(1)}(1) > 1$. ■

Proof of Proposition 6. One can show that in the case of linear virtual types with parameter σ, first-best expected revenue, $\mathbb{E}_{v,c}\left[\sum_{j=1}^k(\Phi_1(v,j) - \Gamma_1(c,j)) \cdot \mathbf{1}_{Q_0(v,c)=j}\right]$, is

$$\Pi_0(s) \equiv -\frac{(Gamma(1+\sigma))^2}{Gamma(2+2\sigma)}\left[1 + \sum_{i=1}^{k-1} s^{2i\sigma}\left(s^{2i}(1+\sigma) - \sigma\right)\right],$$

where *Gamma* is the Euler gamma function. Differentiating with respect to s, we get

$$\Pi_0'(s) \equiv -\frac{(Gamma(1+\sigma))^2}{Gamma(2+2\sigma)}\left[\sum_{i=1}^{k-1} 2is^{2i\sigma-1}\left(s^{2i}(1+\sigma)^2 - \sigma^2\right)\right], \tag{14}$$

which implies that $\Pi_0'(1) = -\frac{(Gamma(1+\sigma))^2}{Gamma(2+2\sigma)}\sum_{i=1}^{k-1} 2i\,(1+2\sigma) < 0$. Indeed for the uniform

[15] Note that $x = \Gamma_\alpha^{-1}\left(s^{k-1}\hat{\Gamma}_\alpha(x/s^{k-1};s)\right)$, so $s^{k-1}z = \Gamma_\alpha^{-1}\left(s^{k-1}\hat{\Gamma}_\alpha(z;s)\right)$, and $s^{k-1}\hat{\Gamma}_\alpha^{-1}(1;s) = \Gamma_\alpha^{-1}\left(s^{k-1}\right)$.

distribution, where $\sigma = 1$, we have $\Pi_0'(1) = -\sum_{i=1}^{k-1} i$.

The sign of $\Pi_0'(s)$ is equal to the sign of $-\sum_{i=1}^{k-1} 2is^{2i\sigma-1}\left(s^{2i}(1+\sigma)^2 - \sigma^2\right)$, which we can write as

$$\begin{aligned}&2\sigma^2 s^{2\sigma-1} - 2s^{2\sigma+1}(1+\sigma)^2\\ &+4\sigma^2 s^{4\sigma-1} - 4s^{4\sigma+3}(1+\sigma)^2\\ &+\cdots+2\sigma^2(k-1)s^{2(k-1)\sigma-1} - 2(k-1)s^{2(k-1)\sigma+2k-3}(1+\sigma)^2.\end{aligned}$$

The term in this expression for which s is raised to the lowest power is the term $2\sigma^2 s^{2\sigma-1}$. If $2\sigma - 1 < 0$, then when we evaluate this at $s = 0$, we get ∞, so $\Pi_0'(0) > 0$. If $2\sigma - 1 > 0$, then $\Pi_0^{(1)}(0) = 0$ where $\Pi_0^{(j)}$ denotes the j-th derivative of Π_0, and the term in $\Pi_0^{(j)}(s)$ for which s is raised to the lowest power is $2\sigma^2 s^{2\sigma-j}\Pi_{i=1}^{j-1}(2\sigma - i)$, which implies that $0 = \Pi_0^{(1)}(0) = \cdots = \Pi_0^{(\ell)}(0)$ where ℓ is the largest integer less than 2σ, and that $\Pi_0^{(\ell+1)}(0) > 0$ (if $\ell + 1 = 2\sigma$, then $\Pi_0^{(\ell+1)}(0) = \Pi_{i=1}^{\ell}(2\sigma - i) > 0$, and if $\ell + 1 > 2\sigma$, then $\Pi_0^{(\ell+1)}(0) = \lim_{s\to 0} s^{2\sigma-(\ell+1)}\Pi_{i=1}^{\ell}(2\sigma - i) = \infty$, implying that for some $\underline{s} > 0$, $\Pi_0(s)$ is increasing in s for $s \in (0, \underline{s})$.

This establishes that $\Pi_0(s)$ is increasing in s for $s \in (0, \underline{s})$ for some $\underline{s} \in (0, 1)$ and that $\Pi_0(s)$ is decreasing in s for $s \in (\overline{s}, 1]$ for some $\overline{s} \in (0, 1)$, which completes the proof. ■

Proof of Proposition 7. Given an incentive compatible, individually rational mechanism $\langle Q, (M_V, M_S)\rangle$, where Q is the quantity purchased from the supplier by the vertically integrated firm, define $m_V(c_0) \equiv \mathbb{E}_{c_1}[M_V(c_0, c_1)]$, and for $j \in \{0, \ldots, k\}$, define $\hat{q}_j(c_0) \equiv \Pr_{c_1}(Q(c_0, c_1) = j)$. Then the vertically integrated firm's interim expected payoff is

$$U_V(c_0) \equiv \sum_{j=0}^{k}\left(S(j + t_j^*(c_0), s)v - S(t_j^*(c_0), 1/s)c_0\right)\hat{q}_j(c_0) - m_V(c_0).$$

In what follows, it will be useful to define

$$h_j(c_0, t) \equiv S(j + t, s)v - S(t, 1/s)c_0,$$

so that we have

$$U_V(c_0) \equiv \sum_{j=0}^{k} h_j(c_0, t_j^*(c_0))\hat{q}_j(c_0) - m_V(c_0). \tag{15}$$

Incentive compatibility implies that

$$\begin{aligned}
U_V(c_0) &\equiv \max_{z\in[0,1]} \sum_{j=0}^{k} h_j(c_0, t_j^*(c_0))\hat{q}_j(z) - m_V(z) \\
&= \max_{z\in[0,1]} \sum_{j=0}^{k} \max_{t\in\{0,\ldots,k-j\}} h_j(c_0, t)\hat{q}_j(z) - m_V(z) \\
&= \max_{z\in[0,1],\ t_\ell\in\{0,\ldots,k-\ell\}\text{ for }\ell\in\{0,\ldots,k\}} \sum_{j=0}^{k} h_j(c_0, t_j)\hat{q}_j(z) - m_V(z),
\end{aligned} \tag{16}$$

which implies that U_V is a maximum of a family of affine functions (in the final line of (16), the max is taken with respect to $(z, t_1, \ldots, t_k)$, and for each choice, the resulting payoff is affine in c_0). In addition, incentive compatibility implies that for all $z, x \in [0, 1]$,

$$\sum_{j=0}^{k} h_j(z, t_j^*(z))\hat{q}_j(x) - m_V(x) \le \sum_{j=0}^{k} h_j(z, t_j^*(z))\hat{q}_j(z) - m_V(z) = U_V(z). \tag{17}$$

We can write the left size of (17) as, for all $z, x \in [0, 1]$,

$$\sum_{j=0}^{k} h_j(z, t_j^*(z))\hat{q}_j(x) - m_V(x) = U_V(x) - \sum_{j=0}^{k} \left[h_j(x, t_j^*(x)) - h_j(z, t_j^*(z))\right]\hat{q}_j(x),$$

which, using (17), implies that

$$U_V(z) \ge U_V(x) + \sum_{j=0}^{k} \left[h_j(z, t_j^*(z)) - h_j(x, t_j^*(x))\right]\hat{q}_j(x).$$

Thus, for all x such that for all $j \in \{0, \ldots, k\}$, $t_j^*(x)$ is constant in a neighborhood of x, which is all but a zero measure set of types, we have

$$\begin{aligned}
\lim_{\Delta\to 0} \frac{U_V(x+\Delta) - U_V(x)}{\Delta} &\ge \lim_{\Delta\to 0} \frac{1}{\Delta} \sum_{j=0}^{k} \left[h_j(x+\Delta, t_j^*(x+\Delta)) - h_j(x, t_j^*(x))\right]\hat{q}_j(x) \\
&= -\sum_{j=0}^{k} S(t_j^*(x), 1/s)\hat{q}_j(x),
\end{aligned}$$

and similarly,

$$\begin{aligned}\lim_{\Delta\to 0}\frac{U_V(x)-U_V(x-\Delta)}{\Delta} &\leq -\lim_{\Delta\to 0}\frac{1}{\Delta}\sum_{j=0}^{k}\left[h_j(x-\Delta,t_j^*(x-\Delta))-h_j(x,t_j^*(x))\right]\hat{q}_j(x)\\ &= -\sum_{j=0}^{k}S(t_j^*(x),1/s)\hat{q}_j(x).\end{aligned}$$

This implies that for all $x\in[0,1]$, $-\sum_{j=0}^{k}S(t_j^*(x),1/s)\hat{q}_j(x)$ is the slope of the line that supports the function U_V at the point x. A convex function is absolutely continuous and thus it is differentiable almost everywhere in the interior of its domain. Thus, at every point that U_B is differentiable, we have

$$U_V'(x)=-\sum_{j=0}^{k}S(t_j^*(x),1/s)\hat{q}_j(x),$$

which is less than or equal to zero. The monotonicity of U_V is necessary and sufficient for incentive compatibility.

Because every absolutely continuous function is the definite integral of its derivative, we have

$$U_V(c_0)=\int_0^{c_0}U_V'(t)dt+U_V(0)=U_V(0)-\int_0^{c_0}\sum_{j=0}^{k}S(t_j^*(t),1/s)\hat{q}_j(t)dt.$$

Using (15) and binding individual rationality for a vertically integrated firm with type $c_0=0$ (such a firm has no possible incremental gain from trade), which implies that $U_V(0)=S(k,s)v$, we have

$$\begin{aligned}m_V(c_0) &= \sum_{j=0}^{k}h_j(c_0,t_j^*(c_0))\hat{q}_j(c_0)-U_V(c_0)\\ &= \sum_{j=0}^{k}h_j(c_0,t_j^*(c_0))\hat{q}_j(c_0)+\int_0^{c_0}\sum_{j=0}^{k}S(t_j^*(x),1/s)\hat{q}_j(x)dx-S(k,s)v.\end{aligned}$$

Taking expectations, we have

$$
\begin{aligned}
&\mathbb{E}_{c_0}\left[m_V(c_0)\right] \\
= &\int_0^1 \sum_{j=0}^{k} h_j(t, t_j^*(t))\hat{q}_j(t)g(t)dt + \int_0^1 \int_0^t \sum_{j=0}^{k} S(t_j^*(x), 1/s)\hat{q}_j(x)dxg(t)dt - S(k,s)v \\
= &\int_0^1 \sum_{j=0}^{k} h_j(t, t_j^*(t))\hat{q}_j(t)g(t)dt + \int_0^1 \int_x^1 \sum_{j=0}^{k} S(t_j^*(x), 1/s)\hat{q}_j(x)g(t)dtdx - S(k,s)v \\
= &\int_0^1 \sum_{j=0}^{k} h_j(t, t_j^*(t))\hat{q}_j(t)g(t)dt + \int_0^1 \sum_{j=0}^{k} S(t_j^*(t), 1/s)\hat{q}_j(t)(1-G(t))dt - S(k,s)v \\
= &\int_0^1 \sum_{j=0}^{k} \left(S(j + t_j^*(t), s)v - S(t_j^*(t), 1/s)\left(t - \frac{1-G(t)}{g(t)}\right)\right)\hat{q}_j(t)g(t)dt - S(k,s)v \\
= &\int_0^1 \sum_{j=0}^{k} \Psi_1(t,j)\hat{q}_j(t)g(t)dt - S(k,s)v \\
= &\sum_{j=0}^{k} \mathbb{E}_{c_0}\left[\Psi_1(c_0,j)\hat{q}_j(c_0)\right] - S(k,s)v \\
= &\sum_{j=0}^{k} \mathbb{E}_{c_0,c_1}\left[\Psi_1(c_0,j)\cdot \mathbf{1}_{Q(c_0,c_1)=j}\right] - S(k,s)v,
\end{aligned}
$$

where the first equality uses the definition of the expectation, the second equality reverses the order of integration for the second term, the third equality integrates, the fourth equality rearranges, the fifth equality uses the definitions of Ψ_1 and Γ_1, the sixth equality uses the definition of the expectation, and the seventh equality uses the definition of $\hat{q}_j$.

Turning to the independent supplier and letting $\hat{q}_j^S(c_1) \equiv \Pr_{c_0}(Q(c_0,c_1)=j)$ and $m_S(c_1) \equiv \mathbb{E}_{c_0}\left[M_S(c_0,c_1)\right]$, we have

$$U_S(c_1) \equiv m_S(c_1) - \sum_{j=1}^{k} \frac{c_1}{s^{j-1}}\hat{q}_j^S(c_1).$$

Incentive compatibility implies that $U_S(x) = \max_{z\in[0,1]} m_S(z) - \sum_{j=1}^{k} \frac{x}{s^{j-1}}\hat{q}_j^S(z)$, which, as above, implies that U_S is convex. In addition, incentive compatibility implies that for all $z, x \in [0,1]$, $m_S(x) - \sum_{j=1}^{k} \frac{z}{s^{j-1}}\hat{q}_j^S(x) \le m_S(z) - \sum_{j=1}^{k} \frac{z}{s^{j-1}}\hat{q}_j^S(z) = U_S(z)$. So, we have $U_S(z) \ge U_S(x) + (x-z)\sum_{j=1}^{k} \frac{1}{s^{j-1}}\hat{q}_j^S(x)$, which implies that at every point that U_S is differentiable, $U_S'(x) = -\sum_{j=1}^{k} \frac{1}{s^{j-1}}\hat{q}_j^S(x)$ and so

$$U_S(c_1) = U_S(1) - \int_{c_1}^{1} U_S'(t)dt = U_S(1) + \int_{c_1}^{1} \sum_{j=1}^{k} \frac{1}{s^{j-1}}\hat{q}_j^S(t)dt.$$

Continuing as above, one can show that

$$\mathbb{E}_{c_1}\left[m_S(c_1)\right] = \sum_{j=1}^{k} \mathbb{E}_{c_0,c_1}\left[\Gamma_1(c_1,j)\cdot \mathbf{1}_{Q(c_0,c_1)=j}\right].$$

The expression for expected designer profit, $\mathbb{E}_{c_0,c_1}[m_V(c_0)-m_S(c_1)]$, then follows, completing the proof. ■

Proof of Proposition 8. For $s \geq 1$, the vertically integrated firm consumes k, either all purchased externally or all sourced internally. As a function of α, the quantity traded is k if $\Psi_\alpha(c_0,k) > \Gamma_\alpha(c_1,k)$ and zero otherwise, so by Proposition 7, expected revenue as a function of s and α is

$$\begin{aligned}\Pi(s,\alpha) \;&\equiv\; \mathbb{E}_{c_0,c_1}\left[(\Psi_1(c_0,0)-\Gamma_1(c_1,0))\cdot \mathbf{1}_{\Psi_\alpha(c_0,k)<\Gamma_\alpha(c_1,k)}\right] \\ &\quad +\mathbb{E}_{c_0,c_1}\left[(\Psi_1(c_0,k)-\Gamma_1(c_1,k))\cdot \mathbf{1}_{\Psi_\alpha(c_0,k)\geq\Gamma_\alpha(c_1,k)}\right] - S(k,s)v.\end{aligned}$$

Using $\Psi_1(c_0,0) = S(k,s)v - S(k,1/s)\left(c_0 - \frac{1-G(c_0)}{g(c_0)}\right)$ and $\Psi_1(c_0,k) = S(k,s)v$ and the definition of Γ_1, we can rewrite this as

$$\Pi(s,\alpha) \equiv S(k,1/s)\mathbb{E}_{c_0,c_1}\left[\left(\frac{1-G(c_0)}{g(c_0)} - c_0\right)\cdot \mathbf{1}_{\Psi_\alpha(c_0,k)<\Gamma_\alpha(c_1,k)} - \Gamma_1(c_1)\cdot \mathbf{1}_{\Psi_\alpha(c_0,k)\geq\Gamma_\alpha(c_1,k)}\right].$$

By the definition of $\alpha^*(1)$, which we know is positive from Myerson and Satterthwaite, $\Pi(1,\alpha^*(1)) = 0$, which implies that the term in square brackets is zero when $\alpha = \alpha^*(1)$ and so $\Pi(s,\alpha^*(1)) = 0$ for all $s \geq 1$.

Now consider the ratio $\frac{SS^{SB}(s)}{SS^{FB}(s)}$. Defining

$$C(\alpha) \equiv c_0 - \mathbb{E}_{c_0,c_1}\left[(c_0 - c_1)\cdot \mathbf{1}_{\Psi_\alpha(c_0,k)\geq\Gamma_\alpha(c_1,k)}\right],$$

the expected production cost as the cost of internal production minus the expected cost reduction due to external trade is $S(k,1/s)C(\alpha)$. Then we have

$$\frac{SS^{SB}(s)}{SS^{FB}(s)} = \frac{S(k,s)v - S(k,1/s)C(\alpha^*(1))}{S(k,s)v - S(k,1/s)C(\alpha(0))} = \frac{s^{k-1}v - C(\alpha^*(1))}{s^{k-1}v - C(\alpha(0))}, \tag{18}$$

where the second equality uses $\frac{S(k,s)}{S(k,1/s)} = s^{k-1}$. Because $\alpha^*(1) > 0$, it follows that $C(\alpha^*(1)) >$

$C(\alpha(0))$, and so

$$\frac{\partial}{\partial s}\left(\frac{SS^{SB}(s)}{SS^{FB}(s)}\right) = \frac{(k-1)s^{k-2}v\left[C(\alpha^*(1)) - C(\alpha(0))\right]}{\left[s^{k-1}v - C(\alpha(0))\right]^2} \geq 0,$$

with a strict inequality for $k > 1$. Further, it follows from the expression in (18) for $\frac{SS^{SB}(s)}{SS^{FB}(s)}$ that if $k > 1$, then $\lim_{s\to\infty} \frac{SS^{SB}(s)}{SS^{FB}(s)} = 1$, which completes the proof. ■

Proof of Proposition 9. It remains to show that $SS^{MM}(s)/SS^{FB}(s)$ is invariant to s for $s \in [0,1]$ and the linear virtual type specification. In the proof of Proposition 4, in (11) and (12), we provide expressions for $SS^{SB}(s)$ and $SS^{FB}(s)$ for $s \in [0,1]$ and linear virtual types. The designer's profit-maximizing mechanism corresponds to the second-best mechanism, but with $\alpha = 1$. Making that substitution into (11), we have

$$SS^{MM}(s) = \sum_{j=1}^{k} \int_0^{\frac{\sigma s^{2(j-1)}}{1+\sigma}} \int_{\frac{1}{s^{2(j-1)}}c + \frac{1}{1+\sigma}}^{1} (s^{j-1}v - \frac{1}{s^{j-1}}c) dF(v) dG(c).$$

Integrating, we get

$$\begin{aligned} SS^{MM}(s) \quad = \quad & \left[\frac{\sigma^{2\sigma+1}\left(\sigma^2+3\sigma+1\right) Gamma(\sigma) Gamma(\sigma+2)}{(\sigma+1)^{2(\sigma+1)} Gamma(2\sigma+2)}\right. \\ & \left. -\frac{\sigma^{3+2\sigma} Gamma(1/2) Gamma(\sigma)}{2^{2\sigma+1}\left(1+\sigma\right)^{2\sigma+1} Gamma\left(\sigma+\frac{3}{2}\right)}\right] \sum_{j=1}^{k} s^{(j-1)(1+2\sigma)}. \end{aligned}$$

Using this and the expression for $SS^{FB}(s)$ from (12) in the proof of Proposition 4, which, like the expression above for $SS^{FB}(s)$, is equal to a term that does not depend on s times $\sum_{j=1}^{k} s^{(j-1)(1+2\sigma)}$, we conclude that $\frac{SS^{MM}(s)}{SS^{FB}(s)}$ is invariant with respect to s for $s \in [0,1]$. ■

Proof of Proposition 10. Letting H_α denote the distribution of $\Phi_\alpha(V_1) + \Phi_\alpha(V_2)$, where V_1 and V_2 are independent random variables drawn from F, then H_α has an upper bound of its support of 2, and we can write for $s > 1$,

$$\frac{SS^{SB}(s)}{SS^{FB}(s)} = \frac{\int_{\frac{1}{s^{k-1}}K}^{2} (y - \frac{1}{s^{k-1}}K) dH_\alpha(y)}{\int_{\frac{1}{s^{k-1}}K}^{2} (y - \frac{1}{s^{k-1}}K) dH_0(y)} = \frac{2 - \int_{\min\{2,\frac{1}{s^{k-1}}K\}}^{2} H_\alpha(y)dy - \frac{1}{s^{k-1}}K}{2 - \int_{\min\{2,\frac{1}{s^{k-1}}K\}}^{2} H_0(y)dy - \frac{1}{s^{k-1}}K},$$

where the second equality integrates by parts and rearranges. Taking the limit as s goes to infinity, we have $\lim_{s\to\infty} \frac{SS^{SB}(s)}{SS^{FB}(s)} = 1$. ■

B Appendix: Mapping the public good problem to the bilateral trade problem

To start, take the case of $k = 1$. Suppose that there are two agents with values v_1 and v_2 for the public good, where v_1 is uniformly distributed on $[0, 1]$ and v_2 has distribution $H(x) \equiv 1 - G(K - x)$ with support $[K - 1, K]$.

It is optimal to produce the public good if and only if $v_1 + v_2 \geq K$. Letting $c = K - v_2$, it is then optimal to produce the public good if and only if $v_1 - c \geq 0$, where c is uniformly distributed on $[0, 1]$. This is the same condition for trade in the bilateral trade problem with agents that have uniformly distributed types.

Thus, we can equate "trade" in the bilateral trade problem with "provision" of the public good in the public good problem. An interesting question then is whether complements alleviate or worsen the incentive problem. To answer this question, we move to the case of $k = 2$.

For the case of $k = 2$, assume that agent 1 has value v_1 for one unit and $v_1(1 + s)$ for two units, and assume that agent 2 has value v_2 for one unit and $v_2(1 + 1/s)$ for two units. Then it is optimal to produce two units of the public good rather than zero if and only if $v_1(1 + s) + v_2(1 + 1/s) \geq (1 + 1/s)K$, which, using $c = K - v_2$ as above, is equivalent to $v_1(1 + s) \geq c(1 + 1/s)$. Again, we can equate the "provision" of the public good with "trade" in the bilateral trade problem where the buyer and seller have uniformly distributed types.

We know from the analysis of the bilateral trade problem with uniformly distributed types that for $s > 1$ a larger s alleviates incentive problems. In the public good problem, that corresponds to agent 1 having a larger value for the second unit of the good and agent 2 having a smaller value for the second unit of the good and to the second unit having a smaller cost.

Continuing, the dominant-strategy prices (for the purchase of two units) that implement the efficient allocation are

$$p_1(v_2) = \frac{1 + s}{s}(K - v_2)$$

and

$$p_2(v_1) = \frac{1 + s}{s}(K - sv_1).$$

Consequently, when there is trade, i.e., when $K < sv_1 + v_2$, we have net revenue of

$$p_1(v_2) + p_2(v_1) - \left(1 + \frac{1}{s}\right) K = \frac{1 + s}{s} (K - sv_1 - v_2) < 0.$$

The derivative of net revenue with respect to s is

$$\frac{v_2 - K - s^2 v_1}{s^2},$$

which is negative because $v_2 \in [K-1, K]$. This implies that the incentive problem becomes less severe as s increases, as in the Myerson-Satterthwaite problem.

For an analysis of the second-best, we can rely on the relation with the bilateral trade model. For example, the second-best quantity is Q_α given by

$$Q_\alpha(v, c; s) = \begin{cases} 0 & \text{if } v_1 s^{k-1} + v_2 < K + s^{k-1} \frac{\alpha}{1+\alpha}, \\ k & \text{otherwise,} \end{cases}$$

where α is the smallest value in $[0, 1]$ such that expected revenue is nonnegative.