Efficient spot markets through forward markets*

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September 6, 2020

Abstract

This paper provides an efficiency rationale for forward markets in an independent private values model with risk neutral agents. Agents have constant marginal values up to their commonly known maximum demands. Forward markets operate before, and spot markets operate after, agents' values are realized. Appropriately designed forward markets induce an equilibrium in which the agents' holdings from the forward market guarantee that incentive compatible, individually rational spot markets can allocate efficiently without running a deficit, which may not be possible otherwise. Because forward markets eliminate the deficit problem, they reduce the market power of the agents. Extending the model to allow for uncertainty about the total supply available in the spot market, we also show that forward markets permit efficient spot markets if all agents are ex ante symmetric.

Keywords: market mechanisms, optimal ownership structure, Coase Theorem, Vickrey-Clarke-Groves mechanism
JEL Classification: C72, D44, L13

*We thank Peter Cramton and Toan Le and seminar audiences at the University of Melbourne for insightful discussions and suggestions. Financial support from the Samuel and June Hordern Endowment, a University of Melbourne Faculty of Business & Economics Eminent Research Scholar Grant, and Australian Research Council Discovery Project Grant DP200103574 is gratefully acknowledged.

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1 Introduction

Spot markets have long been recognized as a means to efficiently reallocate resources, but their efficient operation has proven difficult to reconcile with private information held by economic agents, incentive compatibility, and individual rationality. Relying less on private information, forward markets have been welcome because they complete markets and, in particular, allow risk averse agents to share risk. However, this makes forward markets difficult to reconcile with risk neutrality and common priors.

In this paper, we take a mechanism design perspective to the analysis of forward and spot markets. Specifically, we consider an independent private values model with risk neutral agents and a common prior in which the agents are endowed with the total supply of the good to be traded. We show that appropriately designed forward markets that run before private information is realized are sufficient to guarantee that efficient, incentive compatible, individually rational spot markets operate without running a deficit. In the forward market, the agents voluntarily trade their endowments until they reach holdings that permit an efficient spot market. Interestingly, no money is poured into or extracted from the forward market or transferred from the forward market into the spot market. Moreover, efficiency is obtained for an arbitrary number of agents, and without imposing any assumption of price-taking behaviour. Rather, price-taking behaviour results from the dominant-strategy mechanism in the spot market (and the discriminatory pricing rule in the forward market).

The following is a sketch of the model and mechanics at work. At date 0, before private information is realized, the agents can trade their endowments in the forward market, thereby determining their holdings for the purposes of the spot market. The spot market operates at date 1, after private information is realized. Agents' types are drawn independently from possibly different but commonly known distributions. An agent's realized type is its private information and constitutes its constant marginal value (or willingness to pay) for a homogeneous good up to its maximum demand. Like the distributions, the agents' maximum demands are allowed to differ across the agents but are commonly known. The agents in aggregate own the entire endowment of the good, and the only restriction that we impose on the agents' maximum demands is that, in aggregate, they exceed the total endowment, so that there is scarcity. We impose no restrictions on the ownership structure of the total endowment at the outset. It is possible that one agent owns everything, or that all agents have equal shares. We assume that the spot market allocates efficiently and is incentive compatible and individually rational when it operates and that it operates only if it does not run a deficit in expectation.

The holdings with which agents enter the spot market render that market an *asset* market in the sense that an agent's trading position—buy, sell, do not trade—is determined endogenously as a function of its own type, holding, and maximum demand, as well as those of the other agents. Asset markets are generalizations of the partnership model introduced by Cramton et al. (1987). Like partnership models, asset markets have the characteristics that an agent's value from consuming its own holding determines the value of its outside option. Consequently, agents' worst-off types are endogenous and in the interior of the type space when the holdings are interior. As we show, efficient, incentive compatible, and individually rational asset markets do not run a deficit if each agent's worst-off type is the same. Thus, if the agents trade in the forward market until their holdings are such that the implied worst-off types in the spot market are the same, then the forward market can be said to induce an efficient spot market to operate. We show that, with an appropriate, discriminatory pricing rule in the forward market, according to which the agents pay and receive differential prices for different units traded, the agents can be induced to do precisely that.

In an independent private values setting such as ours, the efficient allocation can always be implemented using a dominant-strategy mechanism, which makes every trader a price taker. In this setting, market power manifests itself in the values of the agents' outside options. If the sum of the values of the outside options is too large, then efficient, incentive compatible, individually rational reallocation is not possible without running a deficit. In this sense, forward markets that allow the agents to trade their endowments can be viewed as reducing overall market power because they reduce the sum of the values of these outside options.

The independent private values setting also offers a natural and parsimonious way to endogenize price uncertainty because the market clearing prices depend on type realizations. Although, in contrast to the earlier literature on forward markets in competitive environments,[1] our framework does not require aggregate uncertainty for forward market to play an efficiency inducing role, it can be amended to accommodate aggregate uncertainty, as we show in an extension. In particular, under the assumption that all agents are ex ante symmetric in the sense of having the same maximum demands and drawing their types from identical distributions, the holdings obtained in the forward market are robust to unanticipated shocks that commonly affect total supply or, equivalently, each agent's maximum demand. If aggregate uncertainty is anticipated and there is common knowledge of the distribution of shocks, still assuming symmetry, then appropriately structured forward markets continue to induce the agents to obtain holdings that, in every state of the world, allow the spot market to operate efficiently, subject to incentive compatibility and individual rationality, without

[1] See, for example, Arrow and Debreu (1954), Arrow (1964), Townsend (1978), and Holthausen (1979).

running a deficit.

Our paper contributes to the literature on forward markets and the mechanism design literature on partnership models and asset markets. The early literature on forward markets assumes competitive behaviour with risk-averse agents who face aggregate (or economy-wide) uncertainty; see, for example, Arrow and Debreu (1954), Arrow (1964), Townsend (1978) or Holthausen (1979). While more recent work, such as Allaz and Vila (1993) or Mahenc and Salanié (2004), has studied forward markets in oligopoly models, to the best of our knowledge, ours is the first paper that combines the mechanism design methodology with the ideas of market completeness and risk sharing that are prevalent in the early literature and with the ideas of mitigating market power advanced more recently (see, e.g., Ausubel and Cramton, 2010; Cramton and Doyle, 2017).[2]

Initiated by Cramton et al. (1987), the literature on partnership models has largely studied under which conditions on what we call holdings it is possible to have efficient, incentive compatible, and individually rational reallocation without running a deficit. Asset markets are generalizations of partnership models insofar as they allow agents' maximum demands to be less than aggregate endowments, and so spot markets that operate after forward markets are asset markets in that sense. For partnership models, Cramton et al. (1987), Che (2006), and Figueroa and Skreta (2012) derive holdings that are sufficient to guarantee possibility of efficient reallocation, while Loertscher and Wasser (2019) derive holdings that are optimal in the sense of maximizing the value of the designer's objective (which may be profit). Lu and Robert (2001) derive the optimal mechanism for an asset market problem without deriving optimal holdings,[3] while Loertscher and Marx (2020a), building on McAfee (1992), and Loertscher and Marx (2020b) develop, respectively, a prior-free trade sacrifice mechanism that is almost efficient and a prior-free mechanism for asset markets that is asymptotically optimal.

The remainder of this paper is organized as follows. In Section 2, we introduce the setup, and in Section 3, we provide basic results from mechanism design theory. In Section 4, we show how forward markets support efficient spot markets. In Section 5, we consider extensions: we allow for uncertainty regarding the ratio of total demand to supply; introduce groups of homogeneous agents; and analyze the holdings that maximize the expected revenue to the spot market designer. Section 6 concludes the paper.

[2]Although the agents in our setting are risk neutral, the extension to aggregate uncertainty with ex ante symmetric agents captures a feature of risk sharing in that it shows that the symmetric holdings obtained in the forward market permit efficient reallocation in the spot market, regardless of aggregate shocks.

[3]See also Chen and Li (2018) for an application of robust mechanism to an asset market problem.

2 Setup

We assume that there is set of n agents denoted by $\mathcal{N}$ and a homogeneous good whose total supply we normalize to 1. Each agent $i \in \mathcal{N}$ has a commonly known *maximum demand* for $k_i \in (0, 1]$ units of the good. To focus on the case in which there is scarcity, we assume that $\sum_{i \in \mathcal{N}} k_i > 1$. Each agent i has a constant marginal value of $\theta_i \in [\underline{\theta}, \overline{\theta}]$ for up to k_i units of the good and zero value for additional units. Agent i's type θ_i is drawn independently from a distribution F_i with positive density on $[\underline{\theta}, \overline{\theta}]$. The realization θ_i is agent i's private information. The type distributions $F_1, \ldots, F_n$ are common knowledge. The agents are the owners of the total supply. That is, denoting by $\mathbf{e}$ the vector of *endowments*, we have $e_i \geq 0$ for each agent i and $\sum_{i \in \mathcal{N}} e_i = 1$.

We consider a setup with a forward market followed by a spot market. The timing is as follows: First, prior to the realization of agents' types, there is a forward market in which the agents can trade their endowments, thereby determining the *holdings*, denoted by $\mathbf{r}$ and satisfying, for all $i \in \mathcal{N}$, $r_i \geq 0$ and $\sum_{i \in \mathcal{N}} r_i = 1$, with which they will enter the spot market. The holdings are observed by all. After the forward market closes, the agents' types are realized but remain the agents' own private information. Next, agents choose whether or not to participate in the spot market, where any agent i that chooses not to participate receives its outside option payoff of $\min\{r_i, k_i\}\theta_i$.

We model the spot market as a direct, Bayesian incentive compatible, interim individually rational, efficient, and deficit-free mechanism. Formally, the spot market is a mechanism $\langle \mathbf{Q}, \mathbf{T_r} \rangle$, where $\mathbf{Q} : [\underline{\theta}, \overline{\theta}]^n \to \mathbb{R}^n_+$ is the efficient allocation rule and $\mathbf{T_r} : [\underline{\theta}, \overline{\theta}]^n \to \mathbb{R}^n$ is a payment rule (where payments are from agents to the mechanism) such that Bayesian incentive compatibility and interim individual rationality are satisfied and such that $\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i \in \mathcal{N}} T_{\mathbf{r},i}(\boldsymbol{\theta})\right] \geq 0$. Let $\mathcal{M}_{\mathbf{r}}$ denote the set of such mechanisms. If $\mathcal{M}_{\mathbf{r}}$ is empty, then all agents simply receive their outside options. Otherwise, the spot market operates based on one of the mechanisms in $\mathcal{M}_{\mathbf{r}}$. For given holdings $\mathbf{r}$, we say that the spot market operates if $\mathcal{M}_{\mathbf{r}}$ is nonempty. In this case, the spot market delivers an efficient allocation for all possible type realizations.

This setup is a generalization of a number of existing models. First, if $k_i = 1$ for all $i \in \mathcal{N}$, then we obtain a version of the standard partnership model of Cramton et al. (1987) with heterogeneous type distributions as analyzed by Che (2006), Figueroa and Skreta (2012), and Loertscher and Wasser (2019).[4] Second, if for some $m \in \{1, \ldots, n-1\}$, we have $k_i = 1/m$ for all $i \in \mathcal{N}$ and $r_j = 1/m$ for m agents, then we obtain the standard two-sided allocation

[4]Like Che (2006) and Figueroa and Skreta (2012), Loertscher and Wasser (2019) derive optimal ownership structure for partnership models with heterogeneous type distributions, but they also allow for interdependent values and rent extraction by the designer.

problem with single-unit traders that has been studied extensively in the literature (see, e.g., Loertscher et al. (2015) for an overview). Third, for $m = 1$, our setup yields a version of the bilateral trade model of Myerson and Satterthwaite (1983) with $n-1$ buyers and a special case of Gresik and Satterthwaite (1989).[5] Fourth, for $k_i = k$ and $F_i = F$ for all $i \in \mathcal{N}$, the setup specializes to the spot market model analyzed by Lu and Robert (2001).[6] Although we do not pursue this avenue here, it seems possible to extend the setup to allow for continuous quantities and decreasing marginal utility along the lines of McAfee (1991).

3 Mechanism design results

Denoting the interim expected efficient allocation for agent i with type θ_i by $q_i(\theta_i)$ and i's interim expected payment to the mechanism by $t_{\mathbf{r},i}(\theta_i)$, we have

$$q_i(\theta_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i(\theta_i, \boldsymbol{\theta}_{-i})] \quad \text{and} \quad t_{\mathbf{r},i}(\theta_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[T_{\mathbf{r},i}(\theta_i, \boldsymbol{\theta}_{-i})].$$

The interim expected payoff from the spot market of agent i with type θ_i, net of the agent's outside option, can thus be expressed as

$$q_i(\theta_i)\theta_i - t_{\mathbf{r},i}(\theta_i) - \min\{r_i, k_i\}\theta_i. \tag{1}$$

For each agent i, there exists a worst-off type, which we denote by $\hat{\theta}_i(r_i)$, which is the type in $[\underline{\theta}, \overline{\theta}]$ that minimizes (1). Standard arguments (see, e.g., Cramton et al., 1987) imply that:

Lemma 1. *Given r_i, $\hat{\theta}_i(r_i) = \min\{\theta \mid q_i(\theta) \geq \min\{r_i, k_i\}\}$.*

As intuition for Lemma 1, an agent benefits from trade, both as a buyer or seller, so an agent's worst-off type is the one whose allocation in the spot market is equal to its holding from the forward market. Because q_i is increasing in agent i's type, $\hat{\theta}_i(r_i)$ is unique.

As described above, the spot market reallocates agents' holdings from the forward market (to an efficient allocation) only when it can do so without running a deficit. Let $\Pi(\boldsymbol{\theta}, \mathbf{r})$ denote the *ex post* budget surplus of an efficient, dominant-strategy mechanism that satisfies the agents' *interim* individual-rationality constraints.

[5] Gresik and Satterthwaite (1989) allow for multiple sellers and Myerson and Satterthwaite (1983) for non-identical supports for the buyer's and the sellers' distributions, while confining attention to a single buyer.

[6] Lu and Robert (2001) derive the optimal reallocation mechanisms without addressing the question of what are the optimal holdings. In light of the underlying symmetry and of the results of Loertscher and Wasser (2019), who find that with ex ante identical agents equal ownership shares are always optimal in a partnership model, a natural conjecture is that equal holdings will be optimal for this model. However, here we do not pursue the question of what are optimal holdings when the spot market is not efficient.

Standard arguments based on the payoff equivalence theorem imply that this is the VCG mechanism (see, e.g., Holmström, 1979; Börgers, 2015), and the budget surplus of the VCG mechanism is well known (see, e.g., Che, 2006): letting $W(\boldsymbol{\theta})$ denote maximal social welfare at the type profile $\boldsymbol{\theta}$, defined by

$$W(\boldsymbol{\theta}) \equiv \max_{\hat{\boldsymbol{Q}}} \sum_{i\in\mathcal{N}} \theta_i \hat{Q}_i(\boldsymbol{\theta})$$

subject to

$$\sum_{i\in\mathcal{N}} \hat{Q}_i(\boldsymbol{\theta}) \leq 1 \quad \text{and} \quad \hat{Q}_i(\boldsymbol{\theta}) \leq k_i \quad \text{for all} \quad i \in \mathcal{N},$$

the budget surplus of the VCG mechanism is

$$\Pi(\boldsymbol{\theta}, \mathbf{r}) = \sum_{i\in\mathcal{N}} W(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - (n-1)W(\boldsymbol{\theta}) - \sum_{i\in\mathcal{N}} \min\{r_i, k_i\}\hat{\theta}_i(r_i). \tag{2}$$

To see that (2) holds, recall that $\langle \mathbf{Q}, \mathbf{T_r} \rangle$ satisfies dominant strategies if and only if the transfer from agent i to the mechanism when the type profile is $\boldsymbol{\theta}$ satisfies

$$T_{\mathbf{r},i}(\boldsymbol{\theta}) = Q_i(\boldsymbol{\theta})\theta_i - W(\boldsymbol{\theta}) + h_i(\boldsymbol{\theta}_{-i}, \mathbf{r}),$$

where $h_i(\boldsymbol{\theta}_{-i}, \mathbf{r})$ is a constant that is independent of i's report. By choosing

$$h_i(\boldsymbol{\theta}_{-i}, \mathbf{r}) = W(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - \min\{r_i, k_i\}\hat{\theta}_i,$$

one obtains

$$T_{\mathbf{r},i}(\boldsymbol{\theta}) = Q_i(\boldsymbol{\theta})\theta_i - W(\boldsymbol{\theta}) + W(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - \min\{r_i, k_i\}\hat{\theta}_i.$$

The budget surplus of the mechanism is $\Pi(\boldsymbol{\theta}, \mathbf{r}) = \sum_{i\in\mathcal{N}} T_{\mathbf{r},i}(\boldsymbol{\theta})$, which is the same as in (2).[7] Letting $\Pi_{\mathbf{r}} \equiv \mathbb{E}_{\boldsymbol{\theta}}[\Pi(\boldsymbol{\theta}, \mathbf{r})]$, we thus have the following result:

Lemma 2. *Given holdings* $\mathbf{r}$*, the spot market operates if and only if* $\Pi_{\mathbf{r}} \geq 0$*.*

To see why the operation of the spot market is at stake, consider a parameterization such that for some $\mathcal{S} \subset \mathcal{N}$, we have $\sum_{i\in\mathcal{S}} k_i = 1$, implying that total demand by agents in $\mathcal{S}$ is equal to total supply. If $r_i = k_i$ for all $i \in \mathcal{S}$, then we have a two-sided allocation problem.

[7] To see that the mechanism satisfies interim individual rationality, notice that

$$t_{\mathbf{r},i}(\theta_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[T_{\mathbf{r},i}(\theta_i, \boldsymbol{\theta}_{-i})] = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[W(\theta_i, \boldsymbol{\theta}_{-i})] - q_i(\theta_i)\theta_i + \min\{r_i, k_i\}\hat{\theta}_i - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[W(\hat{\theta}_i, \boldsymbol{\theta}_{-i})].$$

Because $q_i(\hat{\theta}_i) = \min\{r_i, k_i\}$, we have $t_{\mathbf{r},i}(\hat{\theta}_i) = 0$ and so the payoff to agent i with type $\hat{\theta}_i$ is $\min\{r_i, k_i\}\hat{\theta}_i$, which is equal to the agent's outside option. Thus, the interim individual-rationality constraints are satisfied.

The results in Loertscher and Mezzetti (2019) then imply that under the VCG mechanism, the deficit on each unit traded is at least as large as the difference between the highest and the lowest market-clearing, Walrasian price. Hence, $\Pi(\boldsymbol{\theta}, \mathbf{r}) \leq 0$ for all $\boldsymbol{\theta}$ and $\Pi(\boldsymbol{\theta}, \mathbf{r}) < 0$ for a positive measure set of $\boldsymbol{\theta}$, implying an expected deficit for the spot market.[8]

4 Forward and spot markets

We now analyze forward and sport markets, beginning with the latter.

4.1 Efficient spot markets

A sufficient condition for the spot market to operate is that the agents' holdings from the forward market imply a common worst-off type:

Proposition 1. *The spot market operates if $\hat{\theta}_i = \hat{\theta}$ for all $i \in \mathcal{N}$.*

Proof. See Appendix A.1.

Recall that the spot market only operates if it does not run a deficit in expectation and induces participation by all types of each agent, i.e., satisfies interim individual rationality. This is achieved if and only if the participation constraint is satisfied for each agent's worst-off type, i.e., for all $i \in \mathcal{N}$, $q_i(\hat{\theta}_i)\hat{\theta}_i - t_{\mathbf{r},i}(\hat{\theta}_i) \geq \min\{r_i, k_i\}\hat{\theta}_i$. In addition, the payoff equivalence theorem and incentive compatibility imply that each agent i receives an information rent of $W(\boldsymbol{\theta}) - W(\hat{\theta}_i, \boldsymbol{\theta}_{-i})$. Given these conditions, it is possible to satisfy the no-deficit constraint if the sum of the agents' outside options, $\sum_{i\in\mathcal{N}} \min\{r_i, k_i\}\hat{\theta}_i$, plus the sum of their information rents, $\sum_{i\in\mathcal{N}} W(\boldsymbol{\theta}) - W(\hat{\theta}_i, \boldsymbol{\theta}_{-i})$, is always no more than social welfare under the efficient allocation, that is, $W(\boldsymbol{\theta})$. In the proof of Proposition 1, we establish the possibility result for the case of common worst-off types $\hat{\theta}$ by putting an upper bound on each agent's information rent by using the fact that $W(\hat{\theta}, \boldsymbol{\theta}_{-i})$ maximizes social welfare for the type profile $(\hat{\theta}, \boldsymbol{\theta}_{-i})$ and is thus always weakly higher than the social welfare that would be obtained under the efficient allocation for the type profile $\boldsymbol{\theta}$ plus $(\hat{\theta} - \theta_i)Q_i(\boldsymbol{\theta})$.

Proposition 1 provides a sufficient condition on worst-off types for an efficient spot market to operate. We now show that holdings from the forward market exist that generate such worst-off types:

[8] Moreover, under the VCG mechanism, the interim individual-rationality constraints are binding for the highest possible types of the sellers (agents in $\mathcal{S}$) and for the lowest possible types of the buyers (agents in $\mathcal{N} \setminus \mathcal{S}$). Hence, there are no fixed payments that increase revenue without violating the interim individual-rationality constraints.

Proposition 2. *There exist a unique* $\hat{\theta} \in (\underline{\theta}, \overline{\theta})$ *and unique holdings*

$$\mathbf{r}^* \equiv (q_1(\hat{\theta}), \ldots, q_n(\hat{\theta})),$$

such that when the holdings are $\mathbf{r}^*$*, all agents have the same worst-off type in the spot market, and this worst-off type is* $\hat{\theta}$*.*

Proof. See Appendix A.2.

Proposition 2 shows that there exist holdings from the forward market $\mathbf{r}^*$ that induce the same worst-off type for all agents. Proposition 1 then implies that the spot market operates given holdings $\mathbf{r}^*$ from the forward market. Indeed, with holdings $\mathbf{r}^*$ from the forward market, the no-deficit constraint for the spot market is satisfied ex post. Taken together, these two propositions provide a remarkably simple and general possibility result for the operation of an efficient spot market.

The previous mechanism design literature has focused on partnership models, which corresponds to the special case in which $k_i = 1$ for all $i \in \mathcal{N}$. Assuming identical distributions, Cramton et al. (1987) establish possibility by deriving a lower bound for ex ante expected revenue. Schweizer (2006) proves possibility using a fixed point theorem. Figueroa and Skreta (2012) derive the holdings that correspond to our $\mathbf{r}^*$ assuming heterogeneous distributions without directly proving that it permits possibility. Our proof is closest to the arguments first employed by Che (2006) (see also Figueroa and Skreta, 2012), but is simultaneously simpler and more general.

4.2 Efficiency-inducing forward markets

If there are no transaction costs in the forward market, then the Coase Theorem, via Propositions 1 and 2 and the individual rationality of the spot market mechanism, suggests that trading among agents before their types are realized would result in forward market holdings that allow efficient spot markets to operate. The proofs of Propositions 1 and 2 reveal that $\Pi_{\mathbf{r}^*} > 0$, implying that holdings exist that allow the spot market to operate. Hence, one would expect decentralized bargaining over holdings to result in $\overline{\mathbf{r}}$ such that $\Pi_{\overline{\mathbf{r}}} = 0$ if such holdings exist and in some $\hat{\mathbf{r}} \in \arg\min_{\mathbf{r}} \Pi_{\mathbf{r}}$ if $\Pi_{\mathbf{r}} > 0$ for all $\mathbf{r}$ because otherwise the agents leave money on the table. However, even without private information, various other sources of transaction costs, such as search costs, can make decentralized bargaining inefficient. We now construct a centralized forward market that has an equilibrium in which the agents obtain the holdings $\mathbf{r}^*$, which implies that subsequently the spot market operates.

Recall that we take as given the commonly known endowments $\mathbf{e}$. In our centralized forward market, each agent can demand $x_i \in [0, 1]$ units. If agent i is allocated x_i units, then x_i will be its holding for the purposes of the spot market. If $x_i < e_i$ (respectively, $x_i > e_i$), agent i is a net seller (respectively, net buyer) in the forward market. To define the pricing rule in the centralized forward market, let $P : [0, 1] \to [\underline{\theta}, \overline{\theta}]$ be a continuous and "sufficiently" increasing function satisfying $P(0) = \hat{\theta}$; see the *single-crossing* condition below for a precise statement of sufficiently increasing. For notational ease, let $X_{-i} \equiv \sum_{j \in \mathcal{N} \setminus \{i\}} (x_j - e_j)$ and $X^*_{-i} \equiv \sum_{j \in \mathcal{N} \setminus \{i\}} (r^*_j - e_j)$. The payment that agent i has to make when it demands x_i units and other agents' demands are $\mathbf{x}_{-i}$ is

$$\int_{e_i}^{x_i} P(z - e_i + X_{-i}) dz.$$

Note that the payment that agent i makes can be negative. This happens, for example, when $X_{-i} \geq e_i$, in which case agent i's payment when demanding 0 is $\int_{e_i}^{0} P(z - e_i + X_{-i}) dz \leq 0$, where the inequality follows because P is increasing, and where the inequality is strict if $e_i > 0$.

Using Lemma 1, an agent i with a holding $r_i \leq k_i$ from the forward market has an expected payoff from the spot market equal to

$$\pi_i(r_i) \equiv r_i \hat{\theta}_i(r_i) - \int_{\underline{\theta}}^{\hat{\theta}_i(r_i)} F_i(\theta) q_i(\theta) d\theta + \int_{\hat{\theta}_i(r_i)}^{\overline{\theta}} (1 - F_i(\theta)) q_i(\theta) d\theta.$$

Thus, agent i's expected payoff when buying x_i units in the centralized forward market when all the other agents demand $\mathbf{x}_{-i}$ is

$$\pi_i(x_i) - \int_{e_i}^{x_i} P(z - e_i + X_{-i}) dz. \tag{3}$$

We say that P satisfies *single-crossing* if for all $i \in \mathcal{N}$,

$$\hat{\theta}_i(x) \gtrless P(x - r^*_i) \quad \Leftrightarrow \quad x \lessgtr r^*_i.$$

Proposition 3. *If P satisfies single-crossing, then the centralized forward market has a Nash equilibrium $\mathbf{x}^*$ in which each agent i demands $x^*_i = r^*_i$.*

Proof. See Appendix A.3.

It is well known that if contracts can be written at an ex ante stage, it is possible to satisfy (Bayesian or dominant-strategy) incentive-compatibility, (interim or ex post) individual-rationality constraints without running a deficit. For example, at the ex ante stage, agents can be required to post bonds, which can then be used to guarantee that individual-rationality constraints are satisfied without running a deficit. What is remarkable about Proposition 3 and our other forward market results is then not that efficient spot markets are possible when agents can contract at the ex ante stage, but rather that a simple device that is widely used in the real world—forward markets—achieves this *without* money from the ex ante stage being used to grease the wheels of the spot market mechanism. The agents voluntarily trade their holdings in the forward market to the point where efficient reallocation in the spot market, after private information has been realized, is possible.

In addition, the forward market can be budget balanced, that is,

$$\sum_{i\in\mathcal{N}}\int_{e_i}^{r_i^*} P(z-e_i+X_{-i}^*)dz = 0.$$

We define a family $\mathcal{P}$ of functions P that support $\mathbf{r}^*$ as an equilibrium of the forward market and show that we can find functions in the family such that the forward market balances its budget in equilibrium. For $\mathbf{e}\neq\mathbf{r}^*$, we index the agents by $e_n - r_n^* \leq e_{n-1} - r_{n-1}^* \leq \cdots \leq e_1 - r_1^*$, where $e_n - r_n^* < 0$ and $e_1 - r_1^* > 0$. For $y\in[e_n - r_n^*, 0)$, let $\delta(y)\equiv\min_{j\in\mathcal{N}}\hat{\theta}_j(y+r_j^*)$ and let $\underline{\delta}(y)$ be the convex hull of $\delta(y)$. For $y\in(0, e_1 - r_1^*]$, let $\delta(y)\equiv\max_{j\in\mathcal{N}}\hat{\theta}_j(y+r_j^*)$ and $-\overline{\delta}(y)$ be the convex hull of $-\delta(y)$. Let $\mathcal{P}$ be the family of increasing functions such that $P(y)<\underline{\delta}(y)$ for all $y\in[e_n-r_n^*,0)$, $P(0)=\hat{\theta}(=\underline{\delta}(0)=\overline{\delta}(0))$ and $P(y)>\overline{\delta}(y)$ for all $y\in(0,e_1-r_1^*]$.

Proposition 4. *For any $P\in\mathcal{P}$, $\mathbf{r}^*$ is a Nash equilibrium of the centralized forward market, and there exists $P'\in\mathcal{P}$ such that the centralized forward market is budget balanced in this equilibrium.*

Proof. See Appendix A.4.

Proposition 4 establishes the existence of a budget-balanced centralized forward market that results in the operation of an efficient spot market. In addition, the spot market itself can operate with an ex ante balanced budget by Proposition 1 by using the Arrow-d'Aspremont-Gérard-Varet (AGV) mechanism (Arrow, 1979; d'Aspremont and Gérard-Varet, 1979b); see Appendix B.

The analysis above raises the question whether one might be able to support holdings $\mathbf{r}^*$ as equilibrium holdings based on a forward market with a uniform price; however, uniform pricing does not support $\mathbf{r}^*$ as equilibrium holdings. Uniform pricing implies that $P(z)=p$ for all z. For $\mathbf{r}^*$ to be an equilibrium, we need $r^*\in\arg\max\pi_i(x_i)-(x_i-e_i)p$, the first-order condition for which is $p=\hat{\theta}_i$. But the second-order derivative is $\hat{\theta}_i'(r_i^*)>0$, which implies that $r^*\notin\arg\max_{x_i}\pi_i(x_i)-(x_i-e_i)p$, and thereby that $\mathbf{r}^*$ cannot be an equilibrium of a forward market with a uniform price.

5 Extensions

In Section 5.1, we consider aggregate uncertainty about the ratio of total supply to demand. Section 5.2 contains comparative statics and allows for groups of homogeneous agents. In Section 5.3, we determine the holdings that maximize the expected revenue from an efficient, incentive compatible, individually rational spot market.

5.1 Aggregate uncertainty

The analysis above shows that aggregate uncertainty is not required in our independent private values model for forward markets to induce efficient spot markets. As shown next, the model can readily be extended to accommodate aggregate uncertainty. To that end, we focus on a symmetric setup with $F_i = F$, $k_i = k(\omega)$ for all i, where $\omega \in \Omega$ is a random variable that represents the state of the world that is realized before the spot market operates but after forward market holdings are determined.

The simplest case is when aggregate uncertainty is not anticipated. In this case, we assume that agents enter the spot market with some holdings $\mathbf{r}$ and then learn, as a shock, $k(\omega)$. To model aggregate uncertainty that is anticipated, we let $\omega \in \Omega$ have a commonly known distribution $\alpha(\omega)$, where Ω is compact and $\alpha(\omega)$ is positive and finite.

The following corollary to Proposition 1 goes to the heart of why $\mathbf{r}^*$ permits the spot market to operate even with aggregate uncertainty:

Corollary 1. *If $F_i = F$ and $k_i = k(\omega)$ for all $i \in \mathcal{N}$, the holdings $\mathbf{r}^* = (1/n, \ldots, 1/n)$ permit the spot market to operate in state $\omega \in \Omega$.*

With Corollary 1, we can establish the results that forward markets with the pricing functions $P \in \mathcal{P}$ still result in efficiency-inducing forward market holdings and that certain pricing functions can also balance the budget in the forward market:

Proposition 5. *If aggregate uncertainty is not anticipated, then the resulting forward market holdings for all $P \in \mathcal{P}$ result in an efficient spot market for all states $\omega \in \Omega$, and there exists a $P' \in \mathcal{P}$ that balances the budget in the forward market.*

Proof. See Appendix A.5.

When there is anticipated uncertainty, even if there are no additional transaction costs, decentralized bargaining among agents over holdings is unlikely to result in forward market holdings that allow the spot market to operate. As briefly discussed above, absent uncertainty about the state ω, bargaining among themselves the agents can be expected to trade until they reach holdings that are either such that $\Pi_{\bar{\mathbf{r}}} = 0$ or minimize $\Pi_{\mathbf{r}}$ if $\Pi_{\mathbf{r}} > 0$ for all $\mathbf{r}$. With uncertainty about the state, denote by $\Pi_{\mathbf{r}}(\omega)$ the expected spot market profit in state ω given $\mathbf{r}$ and assume there is a state ω such that there are holdings $\bar{\mathbf{r}}$ satisfying $\Pi_{\bar{\mathbf{r}}}(\omega) = 0$. The same holdings $\bar{\mathbf{r}}$ will now generically be such that $\Pi_{\bar{\mathbf{r}}}(\omega') > 0$ for some state ω', in which case the agents leave money on the table by increasing the revenue of the operator of the spot market, or such that $\Pi_{\bar{\mathbf{r}}}(\omega') < 0$, in which case the agents leave money on the table because the spot market does not operate. Hence, facing uncertainty about the state, the agents face a non-trivial tradeoff between when to induce the spot market to operate, possibly leaving the spot market with positive expected revenue, and preventing the spot market from operating for lack of nonnegative expected revenue.[9] In contrast, forward markets that result in the holdings $\mathbf{r}^*$ induce a positive profit in the spot market for all states. Likewise, because uniform pricing in the forward market cannot, as noted, induce the holdings $\mathbf{r}^*$ in equilibrium, one would expect uniform pricing in the forward market not to be capable of inducing the spot market to operate in all states when there is anticipated aggregate uncertainty.

With anticipated aggregate uncertainty, there exists a family of pricing functions $P^U \in \mathcal{P}^U$ that makes equal forward market holdings an equilibrium, and there also exist functions in $\mathcal{P}^U$ that balance the budget in the forward market. To see that, let $\pi_i(r_i;\omega)$ be the ex ante expected payoff of agent i in the spot market, conditional on that the state ω. Then for a pricing function $P^U(\cdot)$, agent i's expected payoff when buying x_i units when all the other agents demand $\mathbf{x}_{-i}$ is

$$\int_\Omega \pi_i(x_i;\omega)\alpha(\omega)d\omega - \int_{e_i}^{x_i} P^U(z - e_i + X_{-i})dz. \tag{4}$$

For Ω compact and $\alpha(\omega)$ finite, the first-order condition is that

$$\int_\Omega \pi_i'(x_i;\omega)\alpha(\omega)d\omega = P^U(x_i - e_i + X_{-i}).$$

Let $\hat{\theta}_i(x_i;\omega)$ be the worst-off type of agent i for a forward holding x_i and a state ω. We have $\pi_i'(x_i;\omega) = \hat{\theta}_i(x_i;\omega)$. The single-crossing condition in the context of aggregate uncertain is that

$$\int_\Omega \hat{\theta}_i(x_i;\omega)\alpha(\omega)d\omega \gtreqless P^U(x_i - r_i^*) \quad \Leftrightarrow \quad x_i \lesseqgtr r_i^*.$$

[9] This discussion raises the question of what comprises transaction costs when there is aggregate uncertainty. The short answer is that it is the restriction of bargaining to state-independent holdings. If the holdings could be made state-contingent, one would expect bargaining to result in holdings that parallel $\bar{\mathbf{r}}$ and $\hat{\mathbf{r}}$ for each state as defined at the beginning of Section 4.2. Put differently, without restrictions on what they bargain over, one would expect agents to choose something akin to Arrow-Debreu securities.

We can now define the family of functions $\mathcal{P}^U$. For $\mathbf{e} \neq \mathbf{r}^*$, we can index the agents by $e_n - r_n^* \leq e_{n-1} - r_{n-1}^* \leq \cdots \leq e_1 - r_1^*$, where $e_n - r_n^* < 0$ and $e_1 - r_1^* > 0$. For $y \in [e_n - r_n^*, 0)$, let $\delta(y) \equiv \min_{j \in \mathcal{N}} \int_\Omega \hat{\theta}_j(y + r_j^*)\alpha(\omega)d\omega$ and $\underline{\delta}(y)$ be the convex hull of $\delta(y)$. For $y \in (0, e_1 - r_1^*]$, let $\delta(y) \equiv \max_{j \in \mathcal{N}} \int_\Omega \hat{\theta}_j(y + r_j^*)\alpha(\omega)d\omega$ and $-\overline{\delta}(y)$ be the convex hull of $-\delta(y)$. Let $\mathcal{P}$ be the family of increasing functions such that $P(y) < \underline{\delta}(y)$ for all $y \in [e_n - r_n^*, 0)$, $P(0) = \int_\Omega \hat{\theta}\alpha(\omega)d\omega (= \underline{\delta}(0) = \overline{\delta}(0))$ and $P(y) > \overline{\delta}(y)$ for all $y \in (0, e_1 - r_1^*]$.

The result that $\mathcal{P}^U$ supports the equal forward holdings equilibrium and the existence of P^U that balances the budget in the forward market then follows using the same argument as in the proof of Proposition 4. Thus, we have:

Proposition 6. *With anticipated aggregate uncertainty, for any $P^U \in \mathcal{P}^{\mathcal{U}}$, equal forward market holdings are a Nash equilibrium. In addition, there exists $P^{U'} \in \mathcal{P}^{\mathcal{U}}$ such that the forward market is budget balanced in the Nash equilibrium with equal forward market holdings.*

With aggregate uncertainty, even risk neutral agents face the risk that the spot market may not operate in some states because it raises insufficient revenue. Proposition 6 shows that appropriately structured forward markets can safeguard against this risk by enabling the spot market to operate efficiently in every state of the world. The heavy lifting here is done by the designer of the forward market because that market needs to take into account the economy-wide uncertainty.

5.2 Comparative statics and groups

We now provide a brief discussion of comparative statics, including in the context of an extension to allow groups of homogeneous agents, to understand the structure of forward market holdings that equalize agents' worst-off types in different setups and to discuss market design and public policy implications.

Strength-ordered agents To formalize the relation between agents' maximum demands and $\mathbf{r}^*$, we say that we have "strength-ordered agents" if lower-indexed agents are "stronger" in the sense of having first-order stochastically dominating distributions and larger maximum demands, i.e., for all $i \in \{1, \ldots, n-1\}$ and $\theta \in [\underline{\theta}, \overline{\theta}]$,

$$F_i(\theta) \leq F_{i+1}(\theta) \quad \text{and} \quad k_i \geq k_{i+1}. \tag{5}$$

Then we have the following result:

Proposition 7. *With strength-ordered agents, for all $i, j \in \mathcal{N}$ with $i < j$, $r_i^* \geq r_j^*$.*

Proof. See Appendix A.6.

The proof of Proposition 7 makes use of the fact that, upon efficient allocation, both the allocations and the probabilities of obtaining higher shares are larger for the stronger agents. Thus, holdings that equalize agents' worst-off types imply that stronger agents have larger holdings.

Strength-ordered groups We now amend the above setup by allowing agents to belong to different ex ante groups, where agents within a group have the same type distribution and same maximum demand. For example, in the context of healthcare professionals and medical equipment distribution, hospitals of different specialities and in different districts might define separate groups, and in an emission permit context, coal-fired power plants and cement manufacturers might define separate groups. For the setup with groups, we show an "equal treatment of the equals" result: in the interior solution, agents in the same group have the same holdings from the forward market. When agents can be grouped into a strong group and a weak group, which we define below, we show that a weak agent that transforms into a strong agent will not have a smaller holding from the forward market.

Denoting the set of groups by $\mathcal{G} \equiv \{G_1, G_2, \ldots, G_m\}$,[10] we obtain the result of equal treatment of the equals stated in Corollary 2:

Corollary 2. *Ex ante identical agents have the same interior solution forward market holdings, that is, for all $G_l \in \mathcal{G}$ and for all $i \in G_l$, there exists some $r^*_{G_l}$ such that $r^*_i = r^*_{G_l}$.*

The proof uses the fact that all agents have the same worst-off type $\hat{\theta}$ and that $r^*_i = q_i(\hat{\theta})$, where the interim efficient allocation q_i's are the same for agents in the same group.

To extend the notion of strength-ordered agents to groups, we say that a setup has "strength-ordered groups" if for all $\theta \in (\underline{\theta}, \overline{\theta})$,

$$F_{G_1}(\theta) < F_{G_2}(\theta) < \cdots < F_{G_m}(\theta) \quad \text{and} \quad k_{G_1} > k_{G_2} > \cdots > k_{G_m}.$$

In this case, a lower numbered group is "stronger" in that it has a better cost distribution in the sense of first-order stochastic dominance and a larger maximum demand. A similar line of argument as in Proposition 7, together with Proposition 9 and Corollary 2, implies that agents in stronger groups have larger holdings from the forward market:

Corollary 3. *In the setup with strength-ordered groups, if group ℓ_1 is stronger than group ℓ_2, i.e., $\ell_1 < \ell_2$, then for all $i \in G_{\ell_1}$ and $j \in G_{\ell_2}$, we have $r^*_i \geq r^*_j$.*

[10] Formally, we assume $\cap_{G \in \mathcal{G}} = \emptyset$ and $\cup_{G \in \mathcal{G}} = \mathcal{N}$. The model studied thus far is a special case of this with $|\mathcal{G}| = n$ and, for each group G, $|G| = 1$.

In some cases, such as, in the medical equipment distribution context, when a district becomes a hot spot in a pandemic, or in the emission permit context when an energy provider changes from relying on fossil fuel to solar power, an agent may switch from one group to another. To study the effect of such transformations on the holdings from the forward market, suppose now that there are two groups of agents, G_1 and G_2, and suppose that one agent changes its distribution and maximum demand so that it moves from G_1 and G_2. More specifically, let $m \in \{0, \ldots, n\}$ be the number of agents in group G_1 and $n - m$ be the number of agents in group G_2. For $m \in \{0, \ldots, n\}$, let $r^*_{G_i}(m)$ be the interior solution holding from the forward market for an agent in group G_i when there are m agents in group G_1.

Proposition 8. *In the setup with two strength-ordered groups, for any $m \in \{0, \ldots, n-1\}$, we have $r^*_{G_1}(m+1) \geq r^*_{G_2}(m)$.*

Proof. See Appendix A.7.

On the one hand, Proposition 7 suggests that when forward market holdings equalize agents' worst-off types, the addition of 'stronger' agents increases forward market competition, thus tending to reduce each agent's holding. On the other hand, when an agent transforms from a weak agent to a strong agent, its holding will likely increase. Proposition 8 shows that the effect of an increase in its holding is stronger when there are two groups. Beyond two groups, how the two opposite effects interplay depends on the type distribution, scale of maximum demand, and group composition.

5.3 Boundary solutions for spot market revenue maximization

An interesting question is which holdings $\mathbf{r}$ maximize the revenue of an efficient, incentive compatible, individually rational spot market. As we show now, $\mathbf{r}^*$ is an interior maximum, but in some cases, the global maximum is a boundary solution in which one agent corners the forward market and obtains a share that exceeds its maximum demand.

First, we show that if holdings are restricted to be no more than agents' maximum demands in the spot market, then holdings of $\mathbf{r}^*$ maximize the expected revenue to the spot market designer:

Proposition 9. *If agents' holdings prior to the realization of types are restricted to be no more than their maximum demands in the spot market, then holdings $\mathbf{r}^*$ maximize the expected budget surplus in the spot market.*

Proof. See Appendix A.8.

In cases in which it is feasible for agents to have holdings that exceed their maximum demand in the spot market, the holdings that maximize the expected budget surplus in the spot market may be a boundary solution. Letting agent i^* be the agent with the minimal type integrated distance between its maximum demand and its interim expected spot market allocation, that is,

$$i^* \in \arg\min_{i \in \mathcal{N}} \int_{\underline{\theta}}^{\overline{\theta}} (k_i - q_i(x))dx,$$

Proposition 10 shows that agent i^* corners the forward market if a boundary solution is optimal:

Proposition 10. *If a boundary solution for holdings* $\mathbf{r}^c$ *is optimal, then it has the form* $r^c_{i^*} = 1 - \sum_{j \in \mathcal{N} \backslash \{i^*\}} r^c_j$ *and for all* $j \in \mathcal{N} \backslash \{i^*\}$, $r^c_j \in [0, \max\{0, 1 - \sum_{\ell \in \mathcal{N} \backslash \{j\}} k_\ell\}]$.

Proof. See Appendix A.9.

As intuition, because agent i^* values anything beyond its maximum demand k_i at zero, a cornered forward market is analogous to creating a one-sided allocation problem for the spot market where the "supplier," agent i^*, is chosen to be the "least likely" agent to trade in the spot market. Of course, the interior solution $\mathbf{r}^*$ continues to characterize a local maximum, and whether the interior or boundary solution is the global maximum depends on the details of the setup. When one agent is endowed with all of the asset, relatively many transactions would be expected in order to achieve efficiency, suggesting relatively high revenue from the spot market, but the payment required to induce that agent to participate can be large. In contrast, if the asset is spread among the agents in a way that corresponds to the likely efficient allocation, relatively few spot market transactions are expected to achieve efficiency, but each of the agents would need to be induced to participate. When the holdings are $\mathbf{r}^*$, the amount needed to induce participation of all agents is equal to the common worst of type $\hat{\theta}$. Hence, the comparison between the boundary solution and the interior solution lies in the outside option of agent i^* (it depends on the smallest maximum demand in the case of identical type distributions) and the common worst-off type.

As an illustration, consider a setup with $n = 3$, $k_2 = k_3 = 2/3$, and $F_i(\theta) = \theta$ for all $i \in \{1, 2, 3\}$ and $x \in [\underline{\theta}, \overline{\theta}]$ and let the value of k_1 go from 0 to 2/3. For this range of k_1, Proposition 10 shows that $\mathbf{r} = (1, 0, 0)$ is a boundary solution. Figure 1 plots the ex ante spot market revenue from the interior solution $\mathbf{r}^*$ as a solid line and from the boundary solution $(1, 0, 0)$ as a dashed line, for $k_1 \in [0, 2/3]$. For sufficiently small k_1, the boundary solution generates higher ex ante revenue. However, when k_1 is sufficiently large, even if $r_i > k_i$ is allowed, a profit-driven operator of the spot market prefers the interior solution.

In the illustration of Figure 1, the ex ante revenue generated by the interior solution

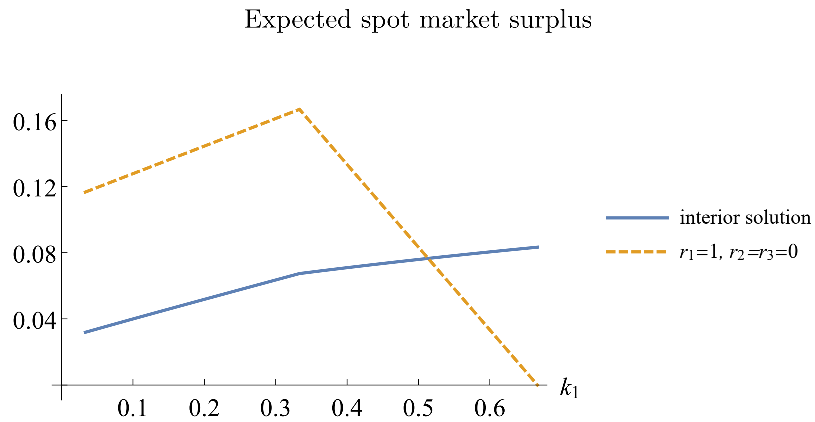

Figure 1: Ex ante spot market revenue as a function of k_1 under the interior solution for $\mathbf{r}$ and under the boundary solution in which agent 1's holding prior to the realization of types is equal to the entire supply. Assumes $n = 3$, $k_2 = k_3 = \frac{2}{3}$, k_1 varying between 0 and $\frac{2}{3}$ as indicated, and costs that are uniformly distributed on $[0, 1]$.

increases in k_1 but the rate of increase drops once k_1 is greater than 1/3. When k_1 increases, the expected spot market allocations to other agents decrease, which implies a lower common worst-off type and a higher revenue. The increase is relatively steep when $k_1 < 1/3$ because then agents expect to receive a positive quantity in the spot market even when they have the lowest type. The same explanation applies to the kink at $k_1 = 1/3$ in the expected surplus generated by the boundary solution. For the boundary solution, an increase in k_1 increases the outside option of agent 1 and, when $k_1 > 1/3$, *reduces* the spot market revenue.

6 Conclusion

In this paper, we show that with appropriately structured holdings, ex post efficient reallocation is possible in a spot market with independent private values and commonly known distributions and maximum demands, subject to incentive-compatibility, individual-rationality, and no-deficit constraints. We derive the holdings from the forward market that permit this by inducing each agent to have the same worst-off type, and we show that these holdings vary with the agents' type distributions and their maximum demands in natural and intuitive ways.

Of course, many real-world spot markets to which the analysis of this paper applies, are dynamic, with the agents in these markets being naturally thought of as drawing their valuations in every period, whereas our model is one-shot. The allocation that is ex post efficient in the spot market fully satisfies the demands of all the agents with the highest types or of all but one of these agents. Consequently, if this allocation determines the holdings in next period's spot market, efficient reallocation may then not be possible. However, if the next period's spot market is preceded by a forward market in which the current period spot market allocation plays the role of the endowments, then efficient reallocation in the spot market may again be possible. If one insists on budget balance in the forward market, then the pricing rule in the forward market may need to adjust as the endowments, or spot market allocations, change. Moreover, bidding behaviour in the forward market is then also affected by the continuation value of future payoffs, making the equilibrium analysis more complicated, which offers scope for future research.

Another avenue for future research would be to allow the spot market to operate under a second-best mechanism whenever the first-best is not possible without running a deficit. To simplify the analysis, we maintain the admittedly strong assumption that the spot market operates if and only if the first-best is possible without a deficit. While we suspect that our results generalize to a setup that assumes smoother transitions, it would, of course, be reassuring if our conjecture could be corroborated and interesting if it could not. Maybe more importantly, the model could be extended to allow for differentiated goods, decreasing marginal values, and continuous quantities.

A Appendix: Proofs

A.1 Proof of Proposition 1

Take $\boldsymbol{\theta}$ and $\mathbf{r}$ as given. Assume that for all $i \in \mathcal{N}$, $\hat{\theta}_i = \hat{\theta}$. Fix an agent $i \in \mathcal{N}$. When the type profile is $(\hat{\theta}, \boldsymbol{\theta}_{-i})$, social welfare under $\mathbf{Q}(\hat{\theta}, \boldsymbol{\theta}_{-i})$, is at least as large as the social welfare under $\mathbf{Q}(\boldsymbol{\theta})$, that is, $W(\hat{\theta}, \boldsymbol{\theta}_{-i}) \geq W(\boldsymbol{\theta}) + (\hat{\theta} - \theta_i)Q_i(\boldsymbol{\theta})$, which is equivalent to

$$W(\boldsymbol{\theta}) - W(\hat{\theta}, \boldsymbol{\theta}_{-i}) \leq (\theta_i - \hat{\theta})Q_i(\boldsymbol{\theta}). \tag{6}$$

Because $\sum_{i \in \mathcal{N}} Q_i(\boldsymbol{\theta}) = 1$, summing up yields

$$\sum_{i \in \mathcal{N}} \Big(W(\boldsymbol{\theta}) - W(\hat{\theta}, \boldsymbol{\theta}_{-i}) \Big) \leq W(\boldsymbol{\theta}) - \hat{\theta}. \tag{7}$$

Substituting $\hat{\theta}_i = \hat{\theta}$ for all $i \in \mathcal{N}$ into (2) yields the first equality in the following display:

$$\Pi(\boldsymbol{\theta}, \mathbf{r}) = W(\boldsymbol{\theta}) - \hat{\theta} - \sum_{i \in \mathcal{N}} \Big(W(\boldsymbol{\theta}) - W(\hat{\theta}, \boldsymbol{\theta}_{-i}) \Big) \geq 0,$$

and the inequality follows from (7). Because our initial choice of $\boldsymbol{\theta}$ was arbitrary, by Lemma 2, the spot market operates. ■

A.2 Proof of Proposition 2

Notice first that, because for any $\theta < \overline{\theta}$ the probability $\prod_{j \in \mathcal{N} \setminus \{i\}} (1 - F_j(\theta))$ that all competitors have a higher type is strictly positive, $q_i(\theta) < k_i$ for all $\theta < \overline{\theta}$. Let $I \subset \mathcal{N}$ be the set of agents, where I could be an empty set or $\mathcal{N}$, such that $q_i(\underline{\theta}) = 0$ for all $i \in I$ and $q_j(\underline{\theta}) = \min\{k_j, 1 - \sum_{l \in \mathcal{N} \setminus \{j\}} k_l\} > 0$ for all $j \in \mathcal{N} \setminus I$. Then,

$$\sum_{i \in \mathcal{N}} q_i(\underline{\theta}) = \sum_{j \in \mathcal{N} \setminus I} \min\{k_j, 1 - \sum_{l \in \mathcal{N} \setminus \{j\}} k_l\} < 1 - \sum_{l \in \mathcal{N} \setminus \{j\}} k_l + \sum_{j \in \mathcal{N} \setminus \{j\}} k_l = 1.$$

Moreover, $\sum_{i \in \mathcal{N}} q_i(\overline{\theta}) = \sum_{i \in \mathcal{N}} k_i > 1$. Thus, because $q_i(\theta)$ is continuous and increasing in θ, there exists a unique $\hat{\theta} \in (\underline{\theta}, \overline{\theta})$ such that

$$\sum_{i \in \mathcal{N}} q_i(\hat{\theta}) = 1. \tag{8}$$

For all $i \in \mathcal{N}$ and $\theta \in [\underline{\theta}, \overline{\theta}]$, $q_i(\theta) \in [0, k_i]$. Hence, $\mathbf{r}^* = (q_1(\hat{\theta})), \ldots, q_n(\hat{\theta}))$ satisfies $r_i^* \in [0, k_i]$ for all $i \in \mathcal{N}$ and $\sum_{i \in \mathcal{N}} r_i^* = 1$. It follows that $\mathbf{r}^*$ defines feasible holdings and results in all agents having the same worst-off type $\hat{\theta}$. Moreover, because for each $i \in \mathcal{N}$ $q_i(\theta)$ is increasing in θ, it follows that both $\hat{\theta}$ and $\mathbf{r}^*$ are unique. This completes the proof. ■

A.3 Proof of Proposition 3

We first show that given $x_j^* = r_j^*$ for all $j \neq i \in \mathcal{N}$, $x_i = r_i^*$ maximizes (3) for all $i \in \mathcal{N}$. Then we show that $\mathbf{x}^*$ with $x_i^* = r_i^*$ for all $i \in \mathcal{N}$ satisfies that for all $i \in \mathcal{N}$

$$\pi_i(r_i^*) - \int_{e_i}^{r_i^*} P(z - e_i + X_{-i}^*)dz \geq \min\{k_i, e_i\}\mathbb{E}[\theta_i], \tag{9}$$

that is, all agents are ex ante better off participating in the forward market and then the ensuing efficient spot market than holding their initial endowments $\boldsymbol{e}$.

For the first part, keeping other agents' demands fixed at r_j^*, the first-order condition for agent i is $\pi_i'(x_i) - P(x_i - e_i + X_{-i}^*) = 0$. Recalling that $\pi_i'(x_i) = \hat{\theta}_i(x_i)$, $\hat{\theta}_i(r_i^*) = \hat{\theta}$, and $P(0) = \hat{\theta}$, it follows that that the first-order condition is satisfied at $x_i = r_i^*$. The second-order condition is satisfied if P satisfies single-crossing.

For the second part,

$$\begin{aligned}
&\pi_i(r_i^*) - \int_{e_i}^{r_i^*} P(z - e_i + X_{-i}^*)dz - \min\{k_i, e_i\}\mathbb{E}[\theta_i] \\
&\geq \pi_i(r_i^*) - \int_{e_i}^{r_i^*} P(z - e_i + X_{-i}^*)dz - \left(\pi_i(e_i) - \int_{e_i}^{e_i} P(z - e_i + X_{-i}^*)dz\right) \\
&\geq 0,
\end{aligned}$$

where the first inequality follows from the individual rationality of the spot market, which implies that $\pi_i(e_i) \geq \min\{e_i, k_i\}\mathbb{E}[\theta_i]$, and from the fact that the second integral is zero, and where the second inequality follows from the fact that r_i^* is the maximizer of $\pi_i(x_i) - \int_{e_i}^{x_i} P(z - e_i + X_{-i}^*)dz$.

A.4 Proof of Proposition 4

For all $i \in \mathcal{N}$ and $x_i < r_i^*$, $P(x_i - e_i + X_{-i}^*) = P(x_i - r_i^*) \leq \hat{\theta}_i(x_i)$; for all $i \in \mathcal{N}$ and $x_i > r_i^*$, $P(x_i - e_i + X_{-i}^*) > \hat{\theta}_i(x_i)$; at $x_i = r_i^*$, $P(0) = \hat{\theta}$. It then follows that the single-crossing condition required for Proposition 3 is satisfied, which completes the proof of the first part of the proposition. Turning to the second part of the proposition, for all i and z,

$$\int_{e_i}^{r_i^*} P(z - e_i + X_{-i}^*)dz = \int_{e_i - r_i^*}^{0} P(z)dz.$$

Because $\sum_{i \in \mathcal{N}} e_i = \sum_{i \in \mathcal{N}} r_i^* = 1$, there exists at least one i such that $e_i - r_i^* > 0$, and there exists at least one j such that $e_i - r_i^* < 0$ (or $e_i = r_i^*$ for all i). For any $P \in \mathcal{P}$ with $\sum_{i \in \mathcal{N}} \int_{e_i - r_i^*}^{0} P(z)dz > 0$, there exists $P' \in \mathcal{P}$ such that for $y \in [e_n - r_n^*, 0)$, $P'(y) = P(y)$, for $y \in (0, e_1 - r_1^*]$, $P'(y) > P(y)$ and $\sum_{i \in \mathcal{N}} \int_{e_i - r_i^*}^{0} P'(z)dz = 0$. For any $P \in \mathcal{P}$ with $\sum_{i \in \mathcal{N}} \int_{e_i - r_i^*}^{0} P(z)dz < 0$, there exists $P' \in \mathcal{P}$ such that for $y \in [e_n - r_n^*, 0)$, $P'(y) < P(y)$, for $y \in (0, e_1 - r_1^*]$, $P'(y) = P(y)$ and $\sum_{i \in \mathcal{N}} \int_{e_i - r_i^*}^{0} P'(z)dz = 0$.

A.5 Proof of Proposition 5

For the symmetric setup, Corollary 1 implies that equal forward market holdings result in non-negative expected surplus in the spot market, with respect to agents' type distributions, for all $\omega \in \Omega$. By Proposition 4, for any $P \in \mathcal{P}$, equal forward market holdings are a Nash equilibrium. The first part of the proposition hence follows. The second part follows from the second part of Proposition 4.

A.6 Proof of Proposition 7

Let $j < i \leq n$. For any $M \subset \mathcal{N} \backslash \{i, j\}$, let $M^c \equiv \mathcal{N} \backslash (M \cup \{i, j\})$ and define

$$A^M(\hat{\theta}) \equiv \prod_{a \in M^c} (1 - F_a(\hat{\theta})) \prod_{\ell \in M} F_\ell(\hat{\theta}),$$

so that $A^M(\hat{\theta})$ is the probability that agents in set M have types lower than $\hat{\theta}$ and agents in set M^c have types higher than $\hat{\theta}$. Note that $A^M(\hat{\theta})$ is independent of F_i and F_j. Then we can write

$$\begin{aligned} r_i^* = \sum_{M \in \mathcal{P}(\mathcal{N} \backslash \{i,j\})} \Bigg(& F_j(\hat{\theta}) \min\{k_i, \max\{0, 1 - \sum_{a \in M^c} k_a\}\} A^M(\hat{\theta}) \\ & + (1 - F_j(\hat{\theta})) \min\{k_i, \max\{0, 1 - \sum_{a \in M^c} k_a - k_j\}\} A^M(\hat{\theta}) \Bigg) \end{aligned}$$

and

$$r_j^* = \sum_{M\in\mathcal{P}(\mathcal{N}\backslash\{i,j\})} \Bigg(F_i(\hat{\theta}) \min\{k_j, \max\{0, 1 - \sum_{a\in M^c} k_a\}\} A^M(\hat{\theta})$$
$$+ (1 - F_i(\hat{\theta})) \min\{k_j, \max\{0, 1 - \sum_{a\in M^c} k_a - k_i\}\} A^M(\hat{\theta}) \Bigg).$$

For every $M \subset \mathcal{N}\backslash\{i,j\}$, let

$$B_{i,1}^M \equiv \min\{k_i, \max\{0, 1 - \sum_{a\in M^c} k_a\}\},$$
$$B_{i,2}^M \equiv \min\{k_i, \max\{0, 1 - \sum_{a\in M^c} k_a - k_j\}\},$$
$$B_{j,1}^M \equiv \min\{k_j, \max\{0, 1 - \sum_{a\in M^c} k_a\}\},$$
$$B_{j,2}^M \equiv \min\{k_j, \max\{0, 1 - \sum_{a\in M^c} k_a - k_i\}\}.$$

Then r_i^* and r_j^* are the same sum over a linear combination of two numbers, $B_{i,1}^M$ and $B_{i,2}^M$, $B_{j,1}^M$ and $B_{j,2}^M$, respectively, with $B_{i,1}^M > B_{j,1}^M$ and $B_{i,2}^M > B_{j,2}^M$. Moreover, the two numbers have the same range, that is, $B_{i,1}^M - B_{i,2}^M = B_{j,1}^M - B_{j,2}^M$. The result that $r_i^* \geq r_j^*$ then follows from the fact $F_j(\hat{\theta}) \geq F_i(\hat{\theta})$. ■

A.7 Proof of Proposition 8

For all $m \geq 1$, $nr_{G_2}^*(m) < 1$ and $nr_{G_1}^*(m+1) > 1$ by the fact that $r_{G_2}^*(m) > r_{G_1}^*(m)$ for all $m \geq 1$. Also, $r_{G_1}^*(0) = \frac{1}{n} = r_{G_1}^*(n)$, which implies that for all m, $r_{G_1}^*(m+1) \geq \frac{1}{n} \geq r_{G_2}^*(m)$. ■

A.8 Proof of Proposition 9

In the first part of the proof, we formalize the constrained optimization problem. Let $u_i(\theta_i, \mathbf{r})$ denote agent i's interim expected payoff from the spot market, not including agent i's outside option $\theta_i r_i$:

$$u_i(\theta_i, \mathbf{r}) \equiv q_i(\theta_i)\theta_i - t_{i,\mathbf{r}}(\theta_i).$$

By the payoff equivalence theorem and the definition of $\hat{\theta}_i$ and letting individual rationality bind for type $\hat{\theta}_i$,

$$t_{\mathbf{r},i}(\theta_i) = q_i(\theta_i)\theta_i - \int_{\hat{\theta}_i(r_i)}^{\theta_i} q_i(x)dx - \hat{\theta}_i(r_i)r_i, \tag{10}$$

which implies that the ex ante expected budget surplus generated in the spot market is

$$\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}}\left(q_i(\theta_i)\theta_i-\int_{\underline{\theta}}^{\theta_i}q_i(x)dx\right)\right]+\sum_{i\in\mathcal{N}}\left(\int_{\underline{\theta}}^{\hat{\theta}_i(r_i)}q_i(x)dx-\hat{\theta}_i(r_i)r_i\right). \tag{11}$$

The first component of (11) is determined by exogenous parameters, that is, F_i and k_i, and the interim expected efficient allocation $q_i(\theta_i)$. Maximizing (11) thus requires maximizing the second component of (11). Letting

$$\Delta(\boldsymbol{k})\equiv\left\{\mathbf{r}\in\mathbb{R}^n \mid \sum_{i\in\mathcal{N}}r_i=1 \text{ and } \forall i\in\mathcal{N},\ \max\{0,1-\sum_{j\in\mathcal{N}\setminus\{i\}}k_j\}\leq r_i\leq k_i\right\},$$

the problem of maximizing the expected budget surplus generated in the spot market is

$$\max_{\mathbf{r}\in\Delta(\boldsymbol{k})}\sum_{i\in\mathcal{N}}\left[\int_{\underline{\theta}}^{\hat{\theta}_i(r_i)}q_i(x)dx-\hat{\theta}_i(r_i)r_i\right].$$

The Lagrangian for this problem is

$$\begin{aligned}L(\mathbf{r},\mu,\boldsymbol{\lambda}^o,\boldsymbol{\lambda}^N)=&\sum_{i\in\mathcal{N}}\left[\int_{\underline{\theta}}^{\hat{\theta}_i(r_i)}q_i(x)dx-\hat{\theta}_i(r_i)r_i\right]\\&+\mu(\sum_{i\in\mathcal{N}}r_i-1)+\sum_{i\in\mathcal{N}}\lambda_i^o(r_i-\max\{0,1-\sum_{j\in\mathcal{N}\setminus\{i\}}k_j\})-\sum_{i\in\mathcal{N}}\lambda_i^N(r_i-k_i).\end{aligned}$$

The Karush–Kuhn–Tucker conditions are as follows:
(a) (stationarity) $\forall i\in\mathcal{N}$, $\frac{\partial L}{\partial r_i}=0$, i.e., for all $i\in\mathcal{N}$,

$$\hat{\theta}_i(r_i)=\mu+\lambda_i^o-\lambda_i^N;$$

(b) (complementary slackness) for all $i\in\mathcal{N}$,

$$\lambda_i^o(r_i-\max\{0,1-\sum_{j\in\mathcal{N}\setminus\{i\}}k_j\})=0$$

and $\lambda_i^N(r_i-k_i)=0$;
(c) (primal feasibility) $\mathbf{r}\in\Delta(\boldsymbol{k})$;
(d) (dual feasibility) for all $i\in\mathcal{N}$,

$$\lambda_i^o,\lambda_i^N\geq 0$$

To characterize the local maxima, we examine three exhaustive cases depending on the signs of λ_i^o and λ_i^N:

Case 1. For all $i \in \mathcal{N}$, $\lambda_i^o = \lambda_i^N = 0$. Proposition 2 shows that this is a feasible solution. The concavity of the objective function in r_i ensures that $\mathbf{r}^*$ characterizes a local maximum.

Case 2. For some $i \in \mathcal{N}$, $\lambda_i^o > 0$ and for all $i \in \mathcal{N}$, $\lambda_i^N = 0$. Let $\mathcal{N}_o \equiv \{i \in \mathcal{N} \mid \lambda_i^o > 0\}$. Then stationarity implies that for all $i \in \mathcal{N}_o$, $\hat{\theta}_i(r_i) = \mu + \lambda_i^o$, and for all $i \in \mathcal{N}_o^c$, $\hat{\theta}_i(r_i) = \mu$. By the definition of $\hat{\theta}_i$ and complementary slackness, we then have for all $i \in \mathcal{N}_o$,

$$q_i(\mu + \lambda_i^o) = \max\{0, 1 - \sum_{j \in \mathcal{N} \setminus \{i\}} k_j\},$$

and for all $i \in \mathcal{N}_o^c$, $q_i(\mu) = r_i$. Then primary feasibility implies that

$$\sum_{i \in \mathcal{N}_o} q_i(\mu + \lambda_i^o) + \sum_{i \in \mathcal{N}_o^c} q_i(\mu) = 1.$$

If such μ exists, then for all $i \in \mathcal{N}_o$ and all $x \leq \mu + \lambda_i^o$, we have $q_i(x) = q_i(\mu + \lambda_i^o) = \max\{0, 1 - \sum_{j \in \mathcal{N} \setminus \{i\}} k_j\}$. Thus, for all $i \in \mathcal{N}_o$, we have $q_i(\mu) = q_i(\mu + \lambda_i^o)$, and hence,

$$\sum_{i \in \mathcal{N}_o} q_i(\mu) + \sum_{i \in \mathcal{N}_o^c} q_i(\mu) = 1.$$

The corresponding local maximum $r_i = q_i(\mu)$ for all $i \in \mathcal{N}$ is the interior solution $\mathbf{r}^*$ with $\hat{\theta} = \mu$.

Case 3. For some $i \in \mathcal{N}$, $\lambda_i^o > 0$, and for some $i \in \mathcal{N}$, $\lambda_i^N > 0$. Define $\mathcal{N}_o$ as in Case 2 and define $\mathcal{N}_N \equiv \{i \in \mathcal{N} \mid \lambda_i^N > 0\}$. Then stationarity implies that for all $i \in \mathcal{N}_N$, $\hat{\theta}_i(r_i) = \mu - \lambda_i^N$, for all $i \in \mathcal{N}_o$, $\hat{\theta}_i(r_i) = \mu + \lambda_i^o$, and for all $i \notin \mathcal{N}_o \cup \mathcal{N}_N$, $\hat{\theta}_i(r_i) = \mu$. By the definition of $\hat{\theta}_i$ and complementary slackness, we then have $\forall i \in \mathcal{N}_N$,

$$q_i(\mu - \lambda_i^N) = k_i, \tag{12}$$

for all $i \in \mathcal{N}_o$ and $j \notin \mathcal{N}_o \cup \mathcal{N}_N$,

$$q_i(\mu + \lambda_i^o) = \max\{0, 1 - \sum_{j \in \mathcal{N} \setminus \{i\}} k_j\}, \tag{13}$$

and

$$r_i = q_i(\mu). \tag{14}$$

Because $f_i(\theta) > 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$, (12) implies that $\mu > \overline{\theta}$. Then equations (13) and (14) only hold if $\mathcal{N}_N = \mathcal{N}$. However, this violates the primary feasibility that $\sum_{i \in \mathcal{N}} r_i = 1$. Thus, a local maximum with $\lambda_i^o > 0$ for some $i \in \mathcal{N}$ and $\lambda_j^N > 0$ for some $j \in \mathcal{N}$ does not exist.

Therefore, the interior solution $\mathbf{r}^*$ maximizes the expected budget surplus in the spot market. ■

A.9 Proof of Proposition 10

We first state and prove Lemma A.1, which we use in the proof of Proposition 10.

Lemma A.1. *For any* $i \in \mathcal{N}$, $1 - \sum_{j \in \mathcal{N}\setminus\{i\}} q_j(\underline{\theta}) > k_i$.

Using our assumption that $\sum_{i \in \mathcal{N}} k_i > 1$, for all $i \in \mathcal{N}$, we have

$$q_i(\underline{\theta}) = \max\{0, 1 - \sum_{j \in \mathcal{N}\setminus\{i\}} k_j\} < k_i. \tag{15}$$

Given $i \in \mathcal{N}$, if for all $j \in \mathcal{N}\setminus\{i\}$, $q_j(\underline{\theta}) = 0$, we are done. If not, choose any $j \in \mathcal{N}\setminus\{i\}$ such that $q_j(\underline{\theta}) = 1 - \sum_{\ell \in \mathcal{N}\setminus\{j\}} k_\ell > 0$. Then,

$$\begin{aligned}
1 - \sum_{\ell \in \mathcal{N}\setminus\{i\}} q_\ell(\underline{\theta}) &= 1 - q_j(\underline{\theta}) - \sum_{\ell \in \mathcal{N}\setminus\{i,j\}} q_\ell(\underline{\theta}) \\
&= 1 - (1 - \sum_{\ell \in \mathcal{N}\setminus\{j\}} k_\ell) - \sum_{\ell \in \mathcal{N}\setminus\{i,j\}} q_\ell(\underline{\theta}) \\
&> 1 - (1 - \sum_{\ell \in \mathcal{N}\setminus\{j\}} k_\ell) - \sum_{\ell \in \mathcal{N}\setminus\{i,j\}} k_\ell \\
&= k_i,
\end{aligned}$$

where the inequality uses (15). □

Proof of Proposition 10. We start the proof by formulating the constrained optimization problem. Define agent i's interim expected payoff, not including the agent's outside option, by

$$u_i(\theta_i, \mathbf{r}) \equiv q_i(\theta_i)\theta_i - t_{i,\mathbf{r}}(\theta_i).$$

For agents with $r_i \geq k_i$, the interim expected payoff of its worst-off type is $u_i(\hat{\theta}_i, \mathbf{r}) = \hat{\theta}_i k_i$. Denote this set of agents by $\mathcal{N}_P$. For all $i \in \mathcal{N}_P$, by Lemma 1 and the continuity of the density functions, $\hat{\theta}_i = \overline{\theta}$. Define $\mathcal{N}_P^c \equiv \mathcal{N}\setminus\mathcal{N}_P$. Let $\Delta_P(\boldsymbol{k}) \equiv \{\mathbf{r} : \sum_{i \in \mathcal{N}} r_i = 1, \forall i \in \mathcal{N}_p^c \ \max\{0, 1 - \sum_{j \in \mathcal{N}\setminus\{i\}} k_j\} \leq r_i \leq k_i, \forall i \in \mathcal{N}_p \ r_i \geq k_i\}$ be the set of feasible holdings from the forward market $\mathbf{r}$. The constrained optimization problem is to

$$\max_{\mathbf{r} \in \Delta_P(\boldsymbol{k})} \sum_{i \in \mathcal{N}_P^c} \left[\int_{\underline{\theta}}^{\hat{\theta}_i(r_i)} q_i(x)dx - \hat{\theta}_i(r_i) r_i \right] + \sum_{i \in \mathcal{N}_P} \left[\int_{\underline{\theta}}^{\overline{\theta}} q_i(x)dx - \overline{\theta} k_i \right]$$

The associated Lagrangian is

$$
\begin{aligned}
L(\mathbf{r},\mu,\boldsymbol{\lambda}^o,\boldsymbol{\lambda}^N,\boldsymbol{\lambda}^P) = &\sum_{i\in\mathcal{N}_P^c}\left[\int_{\underline{\theta}}^{\hat{\theta}_i(r_i)} q_i(x)dx - \hat{\theta}_i(r_i)r_i\right] + \sum_{i\in\mathcal{N}_P}\left[\int_{\underline{\theta}}^{\overline{\theta}} q_i(x)dx - \overline{\theta}k_i\right]\\
&+\mu(\sum_{i\in\mathcal{N}} r_i - 1) + \sum_{i\in\mathcal{N}_P^c}\lambda_i^o(r_i - \max\{0, 1-\sum_{j\in\mathcal{N}\setminus\{i\}} k_j\})\\
&-\sum_{i\in\mathcal{N}_P^c}\lambda_i^N(r_i-k_i) + \sum_{i\in\mathcal{N}_P}\lambda_i^P(r_i-k_i).
\end{aligned}
$$

An analysis similar to that of Case 3 in the proof for Proposition 9 shows that solution only exists when $\lambda_i^N = 0$ for all $i\in\mathcal{N}_P^c$. Stationarity implies that for all $i\in\mathcal{N}_P$, $\mu = -\lambda_i^P$. We rewrite the Lagrangian as

$$
\begin{aligned}
L(\mathbf{r},\mu,\boldsymbol{\lambda}^o,\boldsymbol{\lambda}^N) = &\sum_{i\in\mathcal{N}_P^c}\left[\int_{\underline{\theta}}^{\hat{\theta}_i(r_i)} q_i(x)dx - \hat{\theta}_i(r_i)r_i\right] + \sum_{i\in\mathcal{N}_P}\left[\int_{\underline{\theta}}^{\overline{\theta}} q_i(x)dx - \overline{\theta}k_i\right]\\
&+\mu(\sum_{i\in\mathcal{N}} r_i - 1) + \sum_{i\in\mathcal{N}_P^c}\lambda_i^o(r_i - \max\{0, 1-\sum_{j\in\mathcal{N}\setminus\{i\}} k_j\})\\
&-\mu\sum_{i\in\mathcal{N}_P}(r_i-k_i).
\end{aligned}
$$

Karush–Kuhn–Tucker conditions are as follows:
(a) (stationarity) for all $i\in\mathcal{N}$, $\frac{\partial L}{\partial r_i} = 0$, i.e., for all $i\in\mathcal{N}_P^c$,

$$\hat{\theta}_i(r_i) = \mu + \lambda_i^o;$$

(b) (complementary slackness) for all $i\in\mathcal{N}_P^c$,

$$\lambda_i^o(r_i - \max\{0, 1-\sum_{j\in\mathcal{N}\setminus\{i\}} k_j\}) = 0$$

and for all $i\in\mathcal{N}_P$, $\mu(r_i-k_i)=0$;
(c) (primal feasibility) $\mathbf{r}\in\Delta_P(\boldsymbol{k})$;
(d) (dual feasibility) for all $i\in\mathcal{N}_P^c$, $\lambda_i^o\geq 0$, and for all $i\in\mathcal{N}_P$, $\mu = -\lambda_i^P \leq 0$.

To complete the proof, we consider two exhaustive cases depending on the sign of $\lambda_i^o = 0$.
Case 1. For all i, $\lambda_i^o = 0$. The definition of the worst-off types $\hat{\theta}_i$, stationarity and primal feasibility require the existence of $\mu\in[\underline{\theta},\overline{\theta}]$ and $(r_i)_{i\in\mathcal{N}_P}$ such that

$$\sum_{i\in\mathcal{N}_P^c} q_i(\mu) + \sum_{i\in\mathcal{N}_P} r_i = 1.$$

If $\mathcal{N}_P = \emptyset$, then the solution to this case is the interior solution $\mathbf{r}^*$. If $\mathcal{N}_P\neq\emptyset$, then complementary slackness implies that $\mu = 0$. The problem is feasible if $\underline{\theta} = 0$ and for all $i\in\mathcal{N}_P^c$, $r_i = q_i(0)$. If

$\underline{\theta} > 0$, then we can expand the type space by making $q_i(x) = 0$ for all $i \in \mathcal{N}$ and $x \in [0, \underline{\theta})$. The problem becomes to

$$\max_{\mathcal{N}_P \in \mathcal{P}(\mathcal{N})} \sum_{i \in \mathcal{N}_P} \left[\int_0^{\overline{\theta}} q_i(x)dx - \overline{\theta} k_i \right].$$

Because $\int_{\underline{\theta}}^{\overline{\theta}} q_i(x)dx - \overline{\theta}k_i < 0$, the solution is to choose $\mathcal{N}_P = \{i^*\}$ where

$$i^* \in \arg\min_{i \in \mathcal{N}} \int_{\underline{\theta}}^{\overline{\theta}} (k_i - q_i(x))dx.$$

Let $\mathbf{r}^c$ satisfy that for all $j \neq i^*$, $r_j^c \in [0, q_j(\underline{\theta})]$ and $r_{i^*}^c = 1 - \sum_{j \in \mathcal{N} \setminus \{i^*\}} r_j^c$. $\mathbf{r}^c$ is feasible since $r_{i^*} \geq 1 - \sum_{j \in \mathcal{N} \setminus \{i^*\}} q_j(\underline{\theta}) > k_{i^*}$ by Lemma A.1. $\mathbf{r}^c$ is optimal since $\mathcal{N}_P = \{i^*\}$.

Case 2. For some $i \in \mathcal{N}$, $\lambda_i^o > 0$. Let $\mathcal{N}_o \equiv \{i \in \mathcal{N} \mid \lambda_i^o > 0\}$ and let $\mathcal{N}_o^c \equiv \mathcal{N}_P^c \backslash \mathcal{N}_o$. Then stationarity implies that for all $i \in \mathcal{N}_o$, $\hat{\theta}_i(r_i) = \mu + \lambda_i^o$, and for all $i \in \mathcal{N}_o^c$, $\hat{\theta}_i(r_i) = \mu$. By Lemma 1 and complementary slackness, for all $i \in \mathcal{N}_o$,

$$q_i(\mu + \lambda_i^o) = \max\{0, 1 - \sum_{j \in \mathcal{N} \setminus \{i\}} k_j\},$$

for all $i \in \mathcal{N}_o^c$, $q_i(\mu) = r_i$. Primal feasibility implies

$$\sum_{i \in \mathcal{N}_o} q_i(\mu + \lambda_i^o) + \sum_{i \in \mathcal{N}_o^c} q_i(\mu) + \sum_{i \in \mathcal{N}_P} r_i = 1$$

If $\mathcal{N}_P = \emptyset$, then the solution to this case is the interior solution $\mathbf{r}^*$ by the same argument in Case 2 in the proof of Proposition 9. If $\mathcal{N}_P \neq \emptyset$, then complementary slackness implies that $\mu = 0$, and the solution has the same form as in Case 1 of this proof. ■

B Budget balance with an AGV mechanism

One can also consider Arrow-d'Aspremont-Gérard-Varet (AGV) spot market mechanisms, which generate zero budget surplus for all type realizations.[11]

Recall that we use $\mathbf{Q}$ to denote the efficient allocation rule. Denote the expected utility of agents other than agent i, given agent i's report of θ_i, plus the outside option of agent's i's worst-off type by

$$v_i(\theta_i) \equiv \sum_{j \neq i} \mathbb{E}_{\boldsymbol{\theta}_{-i}} \left[Q_j(\theta_i, \boldsymbol{\theta}_{-i})\theta_j \right] + \min\{r_i, k_i\}\hat{\theta}_i.$$

[11]See Krishna (2002, chapter 5.3.2), with references to Arrow (1979) and d'Aspremont and Gérard-Varet (1979a), d'Aspremont and Gérard-Varet (1979b). See also Krishna and Perry (1998) and d'Aspremont et al. (2004).

Note that $v_i(\hat{\theta}_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}} W(\hat{\theta}_i, \boldsymbol{\theta}_{-i})$. Define

$$h_i(\boldsymbol{\theta}_{-i}) \equiv \frac{1}{n-1} \sum_{j \neq i} v_j(\theta_j)$$

and

$$d_i \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[h_i(\boldsymbol{\theta}_{-i})] - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[W(\hat{\theta}_i, \boldsymbol{\theta}_{-i})].$$

Proposition 11. *If* $\mathbf{r}$ *is such that*

$$\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i \in \mathcal{N}} W(\hat{\theta}_i, \boldsymbol{\theta}_{-i})\right] \geq \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i \in \mathcal{N}} v_i(\theta_i)\right], \tag{16}$$

then the mechanism $\langle \mathbf{Q}, \overline{\mathbf{T}} \rangle$*, where for* $i \in \mathcal{N} \backslash \{1\}$*,*

$$\overline{T}_i(\boldsymbol{\theta}) \equiv h_i(\boldsymbol{\theta}_{-i}) - v_i(\theta_i) - d_i$$

and

$$\overline{T}_1(\boldsymbol{\theta}) \equiv h_1(\boldsymbol{\theta}_{-1}) - v_1(\theta_1) + \sum_{j \in \mathcal{N} \backslash \{1\}} d_j,$$

is ex post efficient, Bayesian incentive compatible, interim individually rational, and ex post budget balanced.

Proof. Recall that the AGV mechanism, $\langle \mathbf{Q}, \mathbf{T}^A \rangle$, where $T_i^A(\boldsymbol{\theta}) \equiv h_i(\boldsymbol{\theta}_{-i}) - v_i(\theta_i)$, satisfies Bayesian incentive compatibility and ex post budget balance. Because $\overline{\mathbf{T}}$ is defined to be the AGV payment rule adjusted by a constant, it follows that $\langle \mathbf{Q}, \overline{\mathbf{T}} \rangle$ is Bayesian incentive compatible. Further, because $\sum_{i \in \mathcal{N}} \overline{T}_i(\boldsymbol{\theta}) = \sum_{i \in \mathcal{N}} T_i^A(\boldsymbol{\theta}) = 0$, $\langle \mathbf{Q}, \overline{\mathbf{T}} \rangle$ satisfies ex post budget balance. It remains to show that $\langle \mathbf{Q}, \overline{\mathbf{T}} \rangle$ satisfies interim individual rationality. Defining $u_i^T(\theta_i) \equiv \theta_i q_i(\theta_i) - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[T_i(\boldsymbol{\theta})] - \min\{r_i, k_i\}\theta_i$, we need to show that for all $i \in \mathcal{N}$, $u_i^{\overline{T}}(\theta_i) \geq 0$. By the definition of $\hat{\theta}_i$ as the worst-off type for agent i, for all $\theta_i \in [\underline{\theta}, \overline{\theta}]$,, $u_i^T(\theta_i) \geq u_i^T(\hat{\theta}_i)$, so it is sufficient to show that $u_i^{\overline{T}}(\hat{\theta}_i) = -\mathbb{E}_{\boldsymbol{\theta}_{-i}}[T_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i})] \geq 0$. For $i \neq 1$,

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\theta}_{-i}}[T_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i})] &= \mathbb{E}_{\boldsymbol{\theta}_{-i}}[h_i(\boldsymbol{\theta}_{-i}) - v_i(\theta_i) - d_i] \\
&= \mathbb{E}_{\boldsymbol{\theta}_{-i}}\{h_i(\boldsymbol{\theta}_{-i}) - v_i(\hat{\theta}_i) - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[h_i(\boldsymbol{\theta}_{-i})] + \mathbb{E}_{\boldsymbol{\theta}_{-i}}[W(\hat{\theta}_i, \boldsymbol{\theta}_{-i})]\} \\
&= \mathbb{E}_{\boldsymbol{\theta}_{-i}}\{-v_i(\hat{\theta}_i) + W(\hat{\theta}_i, \boldsymbol{\theta}_{-i})\} \\
&= 0.
\end{aligned}$$

Now, for agent 1,

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\theta}_{-1}}[T_1(\hat{\theta}_1, \boldsymbol{\theta}_{-1})] &= \mathbb{E}_{\boldsymbol{\theta}_{-1}}[h_1(\boldsymbol{\theta}_{-1}) - v_1(\theta_i) + \sum_{i \neq 1} d_i] \\
&= \mathbb{E}_{\boldsymbol{\theta}_{-1}}\{h_1(\boldsymbol{\theta}_{-1}) - v_1(\hat{\theta}_1) + \sum_{i \neq 1} \mathbb{E}_{\boldsymbol{\theta}_{-i}}[h_i(\boldsymbol{\theta}_{-i})] - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[W(\hat{\theta}, \boldsymbol{\theta}_{-i})]\} \\
&= \mathbb{E}_{\boldsymbol{\theta}}[\sum_{i \in \mathcal{N}} h_i(\boldsymbol{\theta}_{-i})] - \mathbb{E}_{\boldsymbol{\theta}_{-1}}[v_1(\hat{\theta}_1)] - \mathbb{E}_{\boldsymbol{\theta}}[\sum_{i \neq 1} W(\hat{\theta}_i, \boldsymbol{\theta}_{-i})] \\
&= \mathbb{E}_{\boldsymbol{\theta}}[\sum_{i \in \mathcal{N}} v_i(\theta_i)] - \mathbb{E}_{\boldsymbol{\theta}_{-1}}[W(\hat{\theta}_1, \boldsymbol{\theta}_{-i})] - \mathbb{E}_{\boldsymbol{\theta}}[\sum_{i \neq 1} W(\hat{\theta}_i, \boldsymbol{\theta}_{-i})] \\
&= \mathbb{E}_{\boldsymbol{\theta}}[\sum_{i \in \mathcal{N}} v_i(\theta_i)] - \mathbb{E}_{\boldsymbol{\theta}}[\sum_{i \in \mathcal{N}} W(\hat{\theta}_i, \boldsymbol{\theta}_{-i})] \\
&\leq 0,
\end{aligned}$$

where the last inequality follows from (16). ■

References

Allaz, B. and J.-L. Vila (1993): "Cournot Competition, Forward Markets and Efficiency," *Journal of Economic Theory*, 59, 1–16.

Arrow, K. (1979): "The Property Rights Doctrine and Demand Revelation under Incomplete Information," in *Economics and Human Welfare*, ed. by M. Boskin, New York: Academic Press, 23–39.

Arrow, K. J. (1964): "The Role of Securities in the Optimal Allocation of Risk Bearing," *Review of Economic Studies*, 31, 91–96.

Arrow, K. J. and G. Debreu (1954): "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, 265–290.

Ausubel, L. M. and P. Cramton (2010): "Using Forward Markets to Improve Electricity Market Design," *Utilities Policy*, 18, 195–200.

Börgers, T. (2015): *An Introduction to the Theory of Mechanism Design*, Oxford University Press.

Che, Y.-K. (2006): "Beyond the Coasian Irrelevance: Asymmetric Information," Unpublished Lecture Notes, Columbia University.

Chen, Y.-C. and J. Li (2018): "Revisiting the Foundations of Dominant-Strategy Mechanisms," *Journal of Economic Theory*, 178, 294–317.

Cramton, P. and L. Doyle (2017): "Open Access Wireless Markets," *Telecommunications Policy*, 41, 379–390.

Cramton, P., R. Gibbons, and P. Klemperer (1987): "Dissolving a Partnership Efficiently," *Econometrica*, 55, 615–632.

d'Aspremont, C., J. Crèmer, and L.-A. Gérard-Varet (2004): "Balanced Bayesian Mechanisms," *Journal of Economic Theory*, 115, 385–396.

d'Aspremont, C. and L.-A. Gérard-Varet (1979a): "Incentives and Incomplete Information," *Journal of Public Economics*, 11, 25–45.

——— (1979b): "On Bayesian Incentive Compatible Mechanisms," in *Aggregation and Revelation of Preferences*, ed. by J.-J. Laffont, North-Holland, 269–288.

Figueroa, N. and V. Skreta (2012): "Asymmetric Partnerships," *Economics Letters*, 115, 268–271.

Gresik, T. and M. Satterthwaite (1989): "The Rate at which a Simple Market Converges to Efficiency as the Number of Traders Increases: An Asymptotic Result for Optimal Trading Mechanisms," *Journal of Economic Theory*, 48, 304–332.

Holmström, B. (1979): "Groves' Scheme on Restricted Domains," *Econometrica*, 47, 1137–1144.

Holthausen, D. M. (1979): "Hedging and the Competitive Firm under Price Uncertainty," *American Economic Review*, 69, 989–995.

Krishna, V. (2002): *Auction Theory*, Elsevier Science, Academic Press.

Krishna, V. and M. Perry (1998): "Efficient Mechanism Design," Working Paper, Penn State University.

Loertscher, S. and L. M. Marx (2020a): "A Dominant-Strategy Asset Market Mechanism," *Games and Economic Behavior*, 120, 1–15.

——— (2020b): "A Prior-Free, Asymptotically Optimal, Dominant-Strategy Mechanism for Asset Markets," Working Paper, University of Melbourne.

Loertscher, S., L. M. Marx, and T. Wilkening (2015): "A Long Way Coming: Designing Centralized Markets with Privately Informed Buyers and Sellers," *Journal of Economic Literature*, 53, 857–897.

Loertscher, S. and C. Mezzetti (2019): "The Deficit on Each Trade in a Vickrey Double Auction Is at Least as Large as the Walrasian Price Gap," *Journal of Mathematical Economics*, 84, 101–106.

Loertscher, S. and C. Wasser (2019): "Optimal Structure and Dissolution of Partnerships," *Theoretical Economics*, 14, 1063–1114.

Lu, H. and J. Robert (2001): "Optimal Trading Mechanisms with Ex Ante Unidentified Traders," *Journal of Economic Theory*, 97, 50–80.

Mahenc, P. and F. Salanié (2004): "Softening Competition through Forward Trading," *Journal of Economic Theory*, 116, 282–293.

McAfee, R. P. (1991): "Efficient Allocation with Continuous Quantities," *Journal of Economic Theory*, 53, 51–74.

——— (1992): "A Dominant Strategy Double Auction," *Journal of Economic Theory*, 56, 434–450.

Myerson, R. and M. Satterthwaite (1983): "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, 29, 265–281.

Schweizer, U. (2006): "Universal Possibility and Impossibility Results," *Games and Economic Behavior*, 57, 73–85.

Townsend, R. M. (1978): "On the Optimality of Forward Markets," *American Economic Review*, 68, 54–66.