Bidder Collusion

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Abstract

Within the heterogeneous independent private values model, we analyze bidder collusion at first and second price single-object auctions, allowing for within-cartel transfers. Our primary focus is on coalitions that contain a strict subset of all bidders. To analyze collusion, a richer environment is required than in the analysis of non-cooperative behavior. We must account for the possibility of shill bidders as well as mechanism payment rules that may depend on the reports of cartel members, their bids at the main auction, and auction outcomes. We show there are environments in which a coalition at a first price auction can produce no gain for the coalition members beyond what is attainable from non-cooperative play. In contrast, a coalition at a second price auction produces a gain for coalition members in all but a few cases. For the environments for which we have contrasting results, a coalition does at least as well at a second price auction as at a first price auction. If a coalition can accomplish anything at a first price auction, then some of the surplus must “leak out” to non-coalition bidders.

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1 Introduction

Auctions are a prevalent mechanism of exchange.\(^1\) It is natural for bidders to attempt to suppress rivalry and thus capture some of the rents that would be transferred to the seller if their bidding had been non-cooperative. Case law is replete with examples of Section 1 violations of the Sherman Act for bid rigging—and these cases are just the bidders who were apprehended. As a casual observation, whenever new auction mechanisms are proposed or designed, there seems to be remarkably little attention paid to the issue of bidder collusion. Nevertheless, in terms of foregone revenue, bidder collusion is probably the most serious practical threat to revenue. For many bidders, a potential Section 1 violation is just the cost of doing business.

Numerous arguments regarding the comparative robustness of auction schemes to collusion have been offered over the years.\(^2\) However, formal analysis, particularly for cases where the coalitions contain fewer than all bidders,\(^3\) has been lacking.

Within the heterogeneous independent private values model, we analyze bidder collusion at first and second price single-object auctions, allowing for within-cartel transfers. Our primary focus is on coalitions that contain a strict subset of all bidders. To analyze collusion, we require a richer environment, as compared to non-cooperative behavior, to account for shill bidders as well as mechanism payment rules that may depend on the reports of cartel members, their bids at the main auction, and auction outcomes. We show there are environments in which a coalition at a first price auction can produce no gain for coalition members beyond what is attainable from non-cooperative play. In contrast, we show that coalitions at second price auctions produce a gain for coalition members in all but a few cases. For the environments for which we have contrasting results, a coalition does at least as well at a second price auction.

\(^1\)So are procurements. Our results apply to procurements, but we refer to auctions throughout the paper.

\(^2\)In 1977, Congress conducted hearings to assess how the Forest Service could avert collusive bidding in timber sales. In these hearings, it was suggested that first price auctions would be more robust to collusion than second price auctions.

\(^3\)Bidder coalitions are often referred to as “rings” (see Cassady 1967, Chapter 13).
auction as at a first price auction. If a coalition can accomplish anything at a first price auction then some of the surplus must “leak out” to non-coalition bidders.

What is the intuition for these findings? There has been some intuition offered for years that goes as follows. At a second price auction a bidder cartel must suppress the bids of all members except the bidder with highest value. The cartel bidder with highest value goes to the auction and bids as he would were he acting non-cooperatively. Any cartel member who thinks of breaking ranks and competing at the main auction faces the highest cartel bidder and the highest non-cartel bidder, each submitting bids that are the same as if all were acting non-cooperatively. Thus, there is no gain to deviant behavior.

The first price auction is quite different. In order to secure a collusive gain the ring member with highest value must lower his bid below what he would have bid acting non-cooperatively, and other ring members must suppress their bids. But when the highest-valuing ring member lowers his bid, the non-coalition bidders optimally lower theirs in response, and the opportunity is created for a non-highest-valuing coalition member to enter a serious bid at the main auction, either on his own or through a shill, and secure an item that he may not have been able to win acting non-cooperatively. This possibility jeopardizes the feasibility of the coalition. In addition, the optimal reduction in bids by non-cartel bidders implies that some of the collusive gain leaks out to them. This inability of the cartel to keep all of the collusive gain, which the cartel can do at a second price auction, further jeopardizes the feasibility of the ring at a first price auction.

In this paper we provide exact environments where these intuitions are borne out.

The paper proceeds as follows. The literature review is in Section 2, the model is in Section 3, and the results are in Section 4. A discussion of the results is in Section 5. Proofs and results for all-inclusive coalitions are in appendices.
2 Literature review

Perhaps the starting point for auction theory is the work of Vickrey (1961, 1962). Three papers in the early 1980’s—Riley and Samuelson (1981), Myerson (1981), and Milgrom and Weber (1982)—resolve major conundrums and provide benchmark results from which much progress has been made in the ensuing two decades. The modeling framework of both Riley and Samuelson and Myerson is called the “independent private value” model (IPV). In Riley and Samuelson it is assumed that bidders independently draw values from the same distribution $F$ and that each bidder knows his value but not the value of any other bidder (values are private information). It is important to note that $F$ is not bidder specific. We refer to this particular variant of the IPV model as “symmetric IPV” or simply “IPV”. In Myerson, the IPV framework allows for the possibility that bidders independently draw private values from different distributions, $F_i$. We refer to this variant of the IPV model as “heterogeneous IPV” or “asymmetric IPV”.\(^4\) For these environments, the authors were able to establish a revenue equivalence theorem—a broad class of auction mechanisms, including those most commonly used in practice, produce identical revenue for the seller when bidders are risk neutral and act non-cooperatively. Much ensuing research in auction theory addressed the relaxation of the underlying modeling assumptions to determine the impact on expected revenue for different auction schemes. For example, when bidders are risk averse the first price auction outperforms the second price auction (Matthews 1983, 1987). One vein of work in this regard focuses on the relaxation of the non-cooperative assumption.\(^5\)

\(^4\)We focus on a variant that requires all value distributions to have the same lower and upper support.

\(^5\)For repeated auctions, collusion by an all-inclusive ring can be sustained in some environments. Fudenberg, Levine, and Maskin (1994) prove a folk theorem for the case in which bidders can communicate prior to each auction and can observe each others' bids but cannot make transfers. They show that as the discount factor increases to one, the optimal collusive scheme is efficient. Even without communication or the ability to observe bids, Blume and Heidhues (2001) and Skrzypacz and Hopenhayn (2001) show that for discount factors sufficiently large, the ring can do better than noncooperative play or a bid rotation scheme by using implicit transfers of equilibrium continuation payoffs, although efficiency cannot be achieved.
Within an IPV, single-object framework, Graham and Marshall (1987) provide a profitable mechanism for a coalition of any size at a second price or English auction.⁶ In their framework, \( k \) of \( n \) bidders are in the ring where \( k \leq n \). Prior to the auction the \( k \) ring members each receive a fixed ex ante non-contingent payment from a “center”.⁷ Each ring member makes a report \( r_i \) to the center. The center recommends that the \( k - 1 \) ring members with lowest reports bid below the reserve price at the auction,⁸ while the ring member with highest report bids up to his report at the auction. If the ring member wins the auction, he pays the center nothing if the auction price is greater than the second-highest report from the ring. If the second-highest ring report exceeds the price paid at the auction then the winning ring bidder pays the center the difference between the second-highest report and the price at the auction. The ex ante expectation of this payment to the center, divided by \( k \), is the fixed ex ante non-contingent payment made by the center to each ring member (thus the mechanism is ex ante balanced budget). Graham and Marshall show that this mechanism is incentive compatible (ring members report truthfully to the center and follow her recommendations). Also, each bidder wants to join the coalition and each ring member wants a potential new member to join. The mechanism is efficient in that the winner is always the bidder with the highest value. Finally, there is no alternative mechanism that all ring members would prefer. A critical implicit assumption of Graham and Marshall is that the designated ring bidder cannot circumvent payment to the center when he wins and the second-highest report is greater than the price paid at the auction.

McAfee and McMillan (1992) provide an analysis of collusion within an IPV frame-

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⁶For heterogeneous IPV bidders at a second price auction, Mailath and Zemsky (1991) find an optimal mechanism. Graham, Marshall, and Richard (1990) show that a side payment scheme that is commonly employed by practicing rings when its members are heterogeneous allocates each ring member his Shapley value.

⁷The notion of a center, an incentiveless agent who facilitates implementation of the mechanism, was introduced by Myerson (1983).

⁸In a later section of the paper, Graham and Marshall describe optimal “disguised” bids by the \( k - 1 \) lowest-valuing ring members. These meaningless “competitive” bids are submitted by the ring so that the auctioneer cannot infer whether bids are coming from a ring or non-ring bidder.
work for a first price auction, where emphasis is on the surplus division game for an all-inclusive cartel. When the cartel members cannot make internal transfers (weak cartel), McAfee and McMillan show that the outcome of the auction is potentially inefficient in that a cartel member is selected at random (from those willing to pay in excess of the reserve price) to be the sole bidder at the auction. When side payments are possible (strong cartel), then the members conduct an ex ante first price auction, where the winning bid is equally distributed to all losers and the winner is the sole bidder at the main auction.\(^9\) Strong cartels produce efficient allocations, provided the highest value exceeds the reserve price.\(^10\) Individual rationality is not addressed in the McAfee and McMillan paper until their Section 5. In that section some general characterizations are offered, but because of the analytic intractability that emerges from the heterogeneity implied by collusion within an IPV model, results are only provided for a special discrete case.

The existence of equilibrium in a heterogeneous IPV setting has been demonstrated by a number of authors including Athey (2001), Maskin and Riley (2000b), and Lebrun (1996).\(^11\) Bidding behavior and expected revenue within an asymmetric IPV framework has been analyzed by Maskin and Riley (2000a).\(^12\) A remarkable non-result emerges from this work—it is extremely difficult to provide any meaningful general analytic characterization as to the conditions under which one auction scheme will outperform another in terms of expected revenue.

Marshall et al. (1994) provide numerical methods for obtaining solutions to the

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\(^9\)Lyk-Jensen (1997a) shows there exist several efficient, ex post budget balanced, pre-auction mechanisms for an all-inclusive ring.

\(^10\)Relaxing the IPV assumption, Lyk-Jensen (1996) shows that an all-inclusive ring can sustain collusion using the second price pre-auction knock-out of Graham and Marshall (1987) or the first price pre-auction knock-out and McAfee and McMillan (1992) in the general symmetric model with affiliated values (see Milgrom and Weber (1982)). In this case, efficiency is not achieved with a first or second price pre-auction knock-out, but can be achieved with a pre-auction knock-out that allows information sharing and is ex ante budget balanced (Lyk-Jensen (1996)) or ex post budget balanced (Lyk-Jensen (1997b)).

\(^11\)The sweepingly comprehensive contribution of Athey (2001) covers heterogeneous IPV as a special case.

\(^12\)The working paper circulated for nearly a decade prior to publication and thus influenced work published much earlier.
differential equations that implicitly define bids when \( k \) of \( n \) IPV bidders (\( k \leq n \)) collude and the remaining bidders act non-cooperatively (a specific kind of asymmetric IPV). Because of numerical instabilities at the origin the solutions involve “backward shooting” methods. The appendix of their paper provides an exact analytic solution for the terminal point of the bid functions for a special case. Unfortunately, for most situations the terminal condition must also be numerically obtained.

Maskin and Riley (1996a), Bajari (1997, 2001), and Lebrun (1999) analyze a heterogeneous IPV model in which each bidder’s distribution has common lower and upper support and shows that there is a unique equilibrium. This implies that the bid functions in Marshall et al. (1994) are unique. Further, Bajari (2001) implies that, whatever mechanism is used by a cartel at a first price auction, if the designated cartel bidders and non-cartel bidders arrive at the main auction with values consistent with a heterogeneous IPV model, then the equilibrium is unique. This result will be used frequently in this paper.

3 Model

We first provide the ingredients of the heterogeneous IPV model and restate known results. We then discuss the taxonomy needed to analyze collusion within this framework.

3.1 Heterogeneous IPV Model\(^\text{13}\)

We consider a single object auction with a non-strategic seller within a heterogeneous IPV framework.\(^\text{14}\) There are \( n \) risk neutral bidders where bidder \( i \) independently draws a value \( v_i \) from a distribution \( F_i \).

\(^{13}\)The initial model characterization in this section borrows from Bajari (2001), with straightforward conversions from costs to values.

\(^{14}\)In the case of a tie, we assume the object is randomly allocated to one of the bidders with the high bid.
Assumption 1 For all $i$, $F_i(v_i)$ has support $[\underline{v}, \bar{v}]$. The probability density function $f_i(v_i)$ is continuously differentiable and, for all $i$, $f_i(v_i)$ is bounded away from zero on $[\underline{v}, \bar{v}]$.

Lemma 1 Under Assumption 1, an equilibrium exists in pure strategies, the bid function is strictly increasing and differentiable, and the equilibrium is unique.$^{15}$

3.2 General features of a collusive mechanism

We are interested in the existence of collusive mechanisms that generate an expected surplus for each ring member that strictly exceeds the expected surplus each ring member could attain acting non-cooperatively. For the remainder of the paper this is what we mean by “profitable collusion”. In particular, we are interested in the case in which there are $n \geq 3$ bidders, and $k$ of those bidders are eligible to participate in a ring, where $2 \leq k \leq n - 1$ (see Appendix A for results for an all-inclusive ring, i.e., $k = n$). We use indices $1, \ldots, k$ to denote ring members and $k + 1, \ldots, n$ to denote outside bidders.

A potential ring member first decides if they want to be in the ring. This could be an ex ante (before learning his own value) or interim (after learning his own value) decision. If the bidder decides to join the ring, then a “center”, the standard Myerson (1983) incentiveless mechanism agent,$^{16}$ makes a payment to all ring members (could be zero to all).$^{17}$ Then each ring member makes a report to the center. Based on these reports the center either recommends a bid to be made by each ring member or submits a bid for each ring member at the main auction. Ring members then decide on what action to take at the main auction, which may include the use of shill bidders to bid on their behalf. Incentive compatibility involves (i) making honest reports to the center, (ii) following the center’s recommendation regarding the bid to submit at

$^{15}$See citations in Section 2.
$^{16}$The center in this paper is also a banker when ex ante budget balance is required.
$^{17}$Given that bidders are ex ante heterogeneous, the ex ante non-contingent payments could be different.
the main auction, and (iii) not using a shill to submit a second bid. Payments are made back to the center based on some subset of initial reports, auction outcomes, and bids submitted by ring members at the main auction. The center’s budget is balanced either ex ante or ex post.

Compared to non-cooperative behavior, a ring member has a richer set of questions to confront. Can he increase his expected payoff by misrepresenting his report to the center? Can he increase his expected payoff by submitting a second bid at the main auction through a shill either to (i) avoid making a payment to the ring or, alternatively, (ii) take advantage of suppressed ring bids to win an item that he would not win if he only followed the center’s recommendation?

In practice, the answers to these questions depend on the restrictions implied by the environment within which a bidder functions. It seems unrealistic that a defense contractor for a major project could use a shill bidder. However, it seems quite possible at an antique auction. At a Forest Service timber sale, the bids of every bidder are revealed after the auction; thus, the payment to the center can be conditioned on reports, the auction outcome, and the bids submitted by every ring member. At auctions of precious gems, it is sometimes the case that neither the identity of the winner nor the bids submitted are publicly revealed. In this case, the payments to the center can only be a function of the initial ring reports.

There are several possible assumptions one could make about the center’s ability to collect payments from the ring members. In particular, we let \( p_i(\cdot) \) be the center’s required payment from ring member \( i \), and we consider various assumptions on the arguments of \( p_i \). For example, the payment required from ring member \( i \) might depend on the ring members’ reports to the center, bidder \( i \)’s bid at the auction, and/or the outcome of the auction, such as the identity of the winner and the amount paid by the winner.\(^{18}\) We also consider restrictions on when a payment can be required of an

\(^{18}\)When payments to the center depend only on the reports made by the ring members, the payment could be required either before or after the auction. This detail is potentially important if a ring member can infer something about the values of the other ring members based on the payment he is required to make to the center. Our results hold regardless of which assumption is
individual ring member. For example, we consider the restriction that \( p_i \) be positive only if ring member \( i \) wins the auction.\(^{19}\)

We consider two possible assumptions about the ring members’ individual rationality constraints.\(^{20}\) Under ex ante individual rationality, a ring member decides whether to participate in the ring before learning his value, and under interim individual rationality, a ring member decides whether to participate in the ring after learning his value. We assume that if one or more potential ring members chooses not to join the ring, the ring does not operate, and all bidders participate in the auction non-cooperatively.\(^{21}\)

As has been noted in the literature, the ability of ring members to use shills to place bids on their behalf can affect the profitability of a collusive mechanism.\(^{22}\) We contrast two assumptions about the possibility of shill bidding. Under the assumption of no shills, each bidder can bid only once and only under his true identity. Under the assumption that shill bidding is possible, each bidder can bid as many times as he would like and under as many different identities as he would like, although bidders (and the center) may be able to infer the use of a shill from the observed outcome of

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\(^{19}\)Descriptions of such mechanisms can be found in Graham and Marshall (1987) and Lyk-Jensen (1996).

\(^{20}\)There are two features of ring membership. One is individual rationality—does a given bidder want to participate in the ring? The converse concerns whether \( k - 1 \) ring members want to include the \( k^{th} \) bidder as a ring member. This latter question, given that we are emphasizing a heterogeneous IPV model, draws attention to the ex ante fixed non-contingent payments that are equal for all bidders. Namely, do these payments somehow reflect the marginal value of a ring member? Our results for first price auctions are largely negative. These negative results are obtained, through backward induction, before ever reaching the issue of membership. The few positive results we obtain for first price auctions are within a symmetric IPV framework, where equal ex ante payments are sensible. Our second price results are often positive. The individual rationality constraint is satisfied trivially in these cases by any positive payment to a ring member. With regard to the issue of whether existing ring members want to include the \( k^{th} \) bidder as a co-conspirator, the size of the \( k^{th} \) ring member’s fixed payment could be problematic; however, Graham, Marshall, and Richard (1990, Theorem 7) have addressed the issue of how to construct these fixed ex ante payments. Thus, we do not address the issue in the paper.

\(^{21}\)This is a common, but not innocent, simplifying assumption in the auction literature.

\(^{22}\)Some literature uses “shill bidding” to mean bids submitted by the auctioneer (or seller) under the guise of being a regular bidder (see Chakraborty and Kosmopoulou (2001) and Hidvégi, Wang, and Whinston (2001)). We assume a non-strategic auctioneer and use “shill bidding” to mean bids submitted by ring members under a different name, which cannot be traced to them.
the auction.

We show that the profitability of collusion depends on whether the auction is first price or second price and on features of the environment, such as the feasible payment rules for the ring, whether ring members face an ex ante or interim individual rationality constraint, and whether the use of shills is possible.

We say that a ring captures the entire collusive gain if, when the ring operates, the payoff to the auctioneer and bidders outside the ring weakly decreases relative to non-cooperative play for all value realizations. In this case, payments to the auctioneer fall because of collusion, and an outside bidder wins the object only if he would have won it under non-cooperative play and at the same price. We say a collusive mechanism is ex post efficient if the highest valuing bidder, whether a ring member or outside bidder, always wins the object.

3.3 Taxonomy for studying collusion

Let \( \mathbf{r} = (r_1, ..., r_k) \) be the reports to the center by each of the \( k \) ring members. Let \( \beta_i(\mathbf{r}) \) be the recommendation by the center to bidder \( i \). Let \( b_i \) be bidder \( i \)'s bid (in his own name) at the auction. Let \( A \) be the announcements by the auctioneer, which we assume includes at least the identity of the winner. We assume the ring center can make ex ante non-contingent payments to ring members. The payment by ring member \( i \) back to the ring center is denoted by \( p_i(\mathbf{r}, b_i, A) \).\(^{23}\) (Separately, winning bidders are assumed to pay the auctioneer according to the auction scheme under consideration.)

An enumeration of some interesting cases is below. Within each case, shills can be allowed or excluded, and individual rationality can be ex ante or interim. Also,

\[^{23}\text{It is without loss of generality that we restrict attention to payment rules for bidder } i \text{ that depend only on } b_i \text{ and not on the entire vector } \mathbf{b} = (b_1, ..., b_k). \text{ To see this, note that the outcome of any incentive compatible mechanism involving a payment rules } p_i(\mathbf{r}, \mathbf{b}, ...) \text{ that depend on } \mathbf{b} \text{ can be replicated by instead requiring a payment from } i \text{ of } \bar{v} \text{ if } b_i \neq \beta_i(\mathbf{r}) \text{ and a payment of } p_i(\mathbf{r}, \beta_i(\mathbf{r}), ..., \beta_k(\mathbf{r}), ...) \text{ if } b_i = \beta_i(\mathbf{r}). \text{ This new payment rule replaces other ring members’ actual bids with their recommended bids in the payment rule and depends only on } \mathbf{r} \text{ and } b_i.\]
for each case, one can assume that $\beta_i(r) \equiv \beta_i(r_i)$ or that $\beta_i(r)$ depends on the entire vector $r$. This latter distinction is critical in terms of the information set that each ring bidder possesses when they bid at the auction. In fact, for cases 2 through 10 below, we could consider the special case in which $r$ is replaced with $r_i$. If we did this, when paired with $\beta_i(r) \equiv \beta_i(r_i)$, no ring bidder can infer anything about the values of the other ring members from his required payment to the center or his recommended bid, and the uniqueness result of Lemma 1 applies.

1. $p_i(r, b_i, A) \equiv 0$ for all $r, b_i, A, i$. Trivially, the center can make no ex ante non-contingent payments and bidding at the auction is the same as under non-cooperative play.

2. $p_i(r, b_i, A) \equiv p_i(r)$. This implies that the payment rule does not depend on $b_i$ or $A$. In practice, this fits the environment of the aforementioned gemstone auctions.

3. $p_i(r, b_i, A) \equiv p_i(r, b_i) = \begin{cases} \infty, & \text{if } b_i \neq \beta_i(r) \\ p_i(r), & \text{otherwise}. \end{cases}$

In this case, one can envision a bidding machine that enters bids for the ring members as a function of their reports, preventing the possibility of deviations from the center’s recommended bids. Alternatively, the center might prevent ring members whose recommended bids are not highest from attending the auction. Or, each ring member might post a performance bond that is forfeited if any bid appears at the main auction under his name that is different from the center’s recommendation.

4. $p_i(r, b_i, A) \equiv p_i(r, b_i)$. This case extends case 3 by allowing for the payment of a compliant ring member to be a function of $b_i$.

5. $p_i(r, b_i, A) \equiv p_i(r, A)$, where $p_i(r, A) = 0$ if $i$ does not win the auction and $A$ reveals only the identity of the winner. Note that this case is more restrictive
than case 2. The payment rule does not depend on \( b_i \), but it does depend on the auction outcome for a ring bidder who wins. Because \( A \) does not reveal \( i \)'s bid, nothing can be done to ensure that a ring bidder follows the recommendation of the center. Equivalently put, payment bonds are allowed but performance bonds are not possible.

6. \( p_i(r, b_i, A) \equiv p_i(r, A) \), where \( p_i(r, A) = 0 \) if \( i \) does not win the auction. This case extends case 5 by removing the restriction that \( A \) only reveal the identity of the winner. For example, \( A \) might reveal the identity of the winner and the price paid, something that is true for numerous auctions in the private sector.

7. \( p_i(r, b_i, A) \equiv p_i(r, A) \). This case extends case 6 by allowing for payments by non-winners.

8. \( p_i(r, b_i, A) \equiv p_i(r, b_i, A) \), where \( p_i(r, b_i, A) = 0 \) if \( i \) does not win the auction and \( A \) reveals only the identity of the winner. This case extends case 5 by allowing the payment of a winning ring member to be contingent on the ring member’s bid. Thus, the winning ring member can be penalized for not complying with the center’s recommendation.

9. \( p_i(r, b_i, A) \equiv p_i(r, b_i, A) \), where \( p_i(r, b_i, A) = 0 \) if \( i \) does not win the auction. This case extends case 6 by allowing the payment of a winning ring member to be contingent on the ring member’s bid, and it extends case 8 by removing the restriction that \( A \) only reveal the identity of the winner.

10. \( p_i(r, b_i, A) \equiv p_i(r, b_i, A) \). This case extends case 8 by allowing for payments by non-winners.
4 Results

4.1 Contrasting first and second price auctions

We begin by considering mechanisms that require payments only as a function of reports made to the ring, which corresponds to case 2 in Section 3.3. In particular, the mechanisms considered here do not rely on any information from the auction itself. We focus on mechanisms that result in the highest-valuing ring member’s bidding at the auction, but that suppress the bids of the other ring members. As we show, in this environment there is a stark difference between profitability of collusion at a second price versus a first price auction. Proposition 1 shows that the ring can capture the entire collusive gain if the auction is second price, but Proposition 2 shows that there is no profitable collusive mechanism if the auction is first price. Thus, Propositions 1 and 2 formalize the intuition that the likelihood of collusion can be reduced by using a first price rather than a second price auction.

**Proposition 1** Assuming ex ante or interim individual rationality, with or without the possibility of shills, when payments to the center depend only on reports (case 2 of Section 3.3), there exists a profitable, ex post efficient collusive mechanism for a second price auction in which non-highest-valuing ring members bid $v$ and the ring captures the entire collusive gain.

*Proof.* See the Appendix.

**Proposition 2** Assuming ex ante or interim individual rationality, with or without the possibility of shills, when payments to the center depend only on reports (case 2 of Section 3.3), there does not exist a profitable collusive mechanism for a first price auction in which non-highest-valuing ring members bid $v$.

*Proof.* See the Appendix.
Proposition 1 shows that at a second price auction, a center with no ability to observe or use either bids or auction outcomes can still support a profitable collusive mechanism. Furthermore, the ring can capture the maximum possible collusive gain subject to the outside bidders following their weakly dominant strategy of bidding their values. In contrast, Proposition 2 shows that at a first price auction there is no mechanism based on reports that enables the ring to suppress the bids of all but the highest-valuing ring member.

The proof of Proposition 1 is by construction. The collusive mechanism proposed specifies that the highest-reporting ring member pay the center an amount equal to the expected surplus for a bidder with value equal to the second-highest report from bidding at the auction against the outside bidders. For example, if bidders are symmetric, the ring member with the highest report pays the center \( \tilde{p}(r_2) \), where \( r_2 \) is the second-highest report and

\[
\tilde{p}(r_2) = \mathbb{E}_{v_{k+1}, \ldots, v_n} \left( r_2 - \max_{j \in \{k+1, \ldots, n\}} v_j \mid r_2 \geq \max_{j \in \{k+1, \ldots, n\}} v_j \right) \Pr \left( r_2 \geq \max_{j \in \{k+1, \ldots, n\}} v_j \right)
\]

\[
= \int_{r_2}^{\infty} F^n_{-k}(x) dx.
\]

Ring members with lower reports pay nothing. The center recommends that the bidder with the highest report bid his report at the auction and that all other ring members bid \( v \). In equilibrium, ring members truthfully report their values and follow the recommendations of the center. Integrating over the possible second-highest values in the ring, the ex ante non-contingent payment to each ring member is

\[
\frac{1}{k} \int_0^v \tilde{p}(x) k(k-1) F^{k-2}(x) (1 - F(x)) f(x) dx,
\]

which is positive and satisfies ex ante budget balance for the center.

To see that interim individual rationality is satisfied (and therefore also ex ante individual rationality), note that if a bidder \( i \in \{1, \ldots, k\} \) with value \( v \) joins the ring, he has expected payoff equal to the ex ante non-contingent payment plus

\[
\tilde{p}(v) - \frac{\int_0^v \tilde{p}(x) (k-1) F^{k-2}(x) f(x) dx}{F^{k-1}(v)} F^{k-1}(v),
\]

(1)
where the first term is the ring member’s expected payoff from competing at the auction and the second term is the ring member’s expected payment to the center. Rearranging (1) and integrating by parts, it can be shown to be equal to \( \int_{v}^{F^{-1}(x)}dx \), which is the expected payoff to a bidder with value \( v \) under non-cooperative play. Thus, a potential ring member bidder strictly prefers to join the ring by the amount of the ex ante non-contingent payment.\(^{24}\)

Proposition 1 establishes the profitability of collusion when the auction is second price for all cases in our taxonomy except those that restrict the mechanism from collecting payments from a ring member who does not win the auction (Proposition 12 shows that collusion continues to be profitable in that case as long as ring members cannot use shills).

**Corollary 1** Assuming ex ante or interim individual rationality, with or without the possibility of shills, in cases 2, 3, 4, 7, and 10 of Section 3.3, there exists a profitable, ex post efficient collusive mechanism for a second price auction in which non-highest-valuing ring members have zero probability of winning the object and the ring captures the entire collusive gain.

In contrast, when the auction is first price, Proposition 2 shows that, when the ring is restricted to use a payment rule that depends only on the reports to the center, there is no profitable collusive mechanism that suppresses the bids of all but the highest-valuing ring member. Because the center cannot penalize deviations from the recommended bids, in any profitable collusive mechanism that suppresses the bids of non-highest-valuing ring member, the highest-valuing ring member bids

\(^{24}\)For a numerical example, consider the case with \( n = 3 \) and \( k = 2 \), so there are two bidders in the ring and one outside bidder, and assume values are uniformly distributed on \([0,1]\). In this case, the ex ante non-contingent payment is \( \frac{1}{2} \). Let bidders 1 and 2 be the ring members, and let \( v_1 \geq v_2 \). In equilibrium, bidder 1 (or a randomly selected ring member if \( v_1 = v_2 \)) pays \( \frac{v_2^3}{3} \) to the center and competes against bidder 3 at the auction. Bidder 1 expects payoff \( \frac{v_2^3}{3} \) from non-cooperative play if he does not join the ring. If he does join the ring, he gets ex ante payment \( \frac{1}{3} \), expects to pay \( \frac{v_2^3}{3} \) to the center, and expects surplus \( \frac{v_2^3}{3} \) from the auction, for an expected payoff of \( \frac{1}{3} + \frac{v_2^3}{3} \), which is greater than his non-cooperative expected payoff by the amount of the ex ante payment.
optimally against the outside bidders, implying that he bids strictly less than his value (for all values above $v$) and that his bid does not depend on the values of the other ring members. But then with positive probability there exists a ring member who is supposed to suppress his bid, but who can profitably deviate by competing at the auction against the highest-valuing ring member and the outside bidders, a contradiction.

Because Proposition 2 focuses on mechanisms that suppress the bids of all but the highest-valuing ring member, the result does not rule out the possibility that there exists some other kind of profitable collusive mechanism at a first price auction when payments to the center can depend only the reports of the ring members; however, it does suggest that in this environment, collusion has limited benefit. In particular, the proposition implies that in this environment no collusive mechanism can suppress all competition among the ring members. In fact, as we now show, for symmetric bidders, even if the ring could suppress competition among all ring members other than the first and second-highest-valuing ring members, that would not be sufficient to secure a collusive gain. The proof relies on Assumption 1 and the uniqueness result of Lemma 1.

**Proposition 3** Assume bidders are symmetric, ex ante or interim individual rationality, no shills, and that payments to the center depend only on reports (case 2 of Section 3.3). If all ring members other than the first and second-highest-valuing ring members have zero probability of winning the object, and if at the time of the auction, the first and second-highest-valuing ring members know only that their values are either highest or second highest, but not which, then the unique equilibrium of the auction subgame for a first price auction is for the two highest-valuing ring members and all outside bidders to bid according to their non-cooperative bid functions.

**Proof.** See the Appendix.

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25 One way in which profitable collusion may be achieved in a first price auction is for the ring to require that ring members who do not have the highest value submit bids directly under the bid of the highest-valuing ring member. See Proposition ??.
Proposition 3 applies to first price auctions, but a similar result holds for second price auctions if we restrict attention to the equilibrium in weakly dominant strategies in which bidders submit bids equal to their values. Proposition 3 further highlights the difficulty of finding a profitable collusive mechanism that relies only on information provided by ring members to the center. It says that as long as the two highest-valuing ring members participate in the auction, then the equilibrium of the auction subgame involves non-cooperative bidding. So for there to be any gain relative to non-cooperative play, it is essential that the ring not simply release the two ring members with highest values, each knowing only that they have one of the two highest values from the ring, to compete at the auction.

The next proposition has interesting empirical implications. It says that an environment in which payments to the center can depend only on reports, in order for bidders to profitably collude at a first price auction (where only the highest-valuing ring member has positive probability of winning the object), the mechanism must require that ring members other than the highest-valuing ring member bid at the auction. In particular, the mechanism must require that a ring member other than the highest-valuing ring member submit a bid that is slightly less than the bid of the highest-valuing ring member.

**Proposition 4** Assuming ex ante or interim individual rationality and small, discrete bid increments, with or without the possibility of shills, when payments to the center depend only on reports (case 2 of Section 3.3), in any profitable collusive mechanism for a first price auction in which non-highest-valuing ring members have zero probability of winning the object, for a positive measure set of ring members’ values, one of the non-highest-valuing ring members bids one bid increment below the bid of the highest-valuing ring member.

*Proof.* See the Appendix.
To see the intuition for Proposition 4, first note that the center must recommend that the highest-valuing ring member bid above the value of the second-highest-valuing ring member at least some of the time in order to prevent non-highest-valuing ring members from having an incentive to submit a bid and attempt to win the object. But given the bid strategies of the outside bidders, it will not always be optimal for the highest-valuing ring member to submit a bid that is greater than the second-highest value in the ring. In these cases, it is critical that the ring require that some other ring member submit a bid slightly less than the highest-valuing ring member’s recommended bid to prevent the highest-valuing ring member from having an incentive to deviate.

4.2 Collusion at a first price auction

As Proposition 2 shows, it is not possible to suppress the bids of all but the highest-valuing ring member without giving greater power to the center to penalize ring members who do not bid according to the center’s recommendation. For example, if the center can condition the payment required from ring member \( i \) on ring member \( i \)’s bid, then the center can penalize a ring member who bids differently from the recommendation of the center. Thus, in this section, we allow the payments to the center to depend on both the reports of the ring members and an individual ring member’s bid at the auction.

Our previous results suggest that profitable collusion may not be possible at first price auctions when payments to the center can depend only on ring members’ reports. In this section we show that there do exist environments in which collusion can be sustained at a first price auction. In particular, we consider environments in which the payment from ring member \( i \) to the center is a function \( p_i(r, b_i) \), where \( r \) is the vector of reports and \( b_i \) is ring member \( i \)’s bid at the auction (cases 3 and 4 of Section 3.3).\(^{26}\)

\(^{26}\)In this environment, for a second price auction, Corollary 1 implies that there exists a profitable,
In this environment, the ring mechanism can fix ring members’ bids (as a function of the reports) by requiring a large payment from ring members who bid anything other than the center’s recommended bids. One way a mechanism like this might be implemented is by requiring that certain ring members not attend the auction or by having the center submit bids on behalf of the ring members.

We construct a mechanism in which ring members report their values and then only the highest-valuing ring member bids at the auction against the outside bidders. The center recommends that the highest-valuing ring member bid according to the equilibrium bid function for an auction in which the highest-valuing ring member bids against the $n - k$ outside bidders. Note that an equilibrium of this auction subgame exists if, for example, all bidders draw their values from the same distribution.\(^{27}\) Let $\beta^\text{in}(v)$ be the equilibrium first price bid for ring member whose value $v$ is the highest in the ring when facing the $n - k$ outside bidders, and let $\beta^\text{out}_i(v)$ be the equilibrium bid for outside bidder $i$ with value $v$. Equilibrium bid functions $\beta^\text{in}$ and $\beta^\text{out}_i$, which are unique by Lemma 1, are defined by the conditions that for all $v$,

$$
\beta^\text{in}(v) \in \arg \max_b E_{v_{k+1}, \ldots, v_n} \left( (v - b) 1_{b \geq \max_{j \in \{k+1, \ldots, n\}} \beta^\text{out}_j(v_j)} \right)
$$

and

$$
\beta^\text{out}_i(v) \in \arg \max_b E_{v_{-i}} \left( (v - b) 1_{b \geq \max\{\beta^\text{in}(\max_{j \in \{1, \ldots, k\}} v_j), \max_{j \in \{k+1, \ldots, n\} \setminus \{i\}} \beta^\text{out}_j(v_j)\}} \right).
$$

Note that the equilibrium bid function for the ring member, $\beta^\text{in}$, does not depend on which ring member has the highest value. Note also that $\beta^\text{in}$ and $\beta^\text{out}_i$ define the equilibrium of the auction subgame for a mechanism that prevents all but the highest-valuing ring member from bidding at the auction, but does not place any restriction on the bid of the highest-valuing ring member.

\(^{27}\)In this case, the auction involves asymmetric bidders with one bidder drawing his value from $F^k$ and $n - k$ bidders drawing their values from $F$. See Bajari (2001) for a proof of the existence and uniqueness of equilibrium bid functions in this case, and see Marshall et al. (1994) on the numerical calculation of the bid functions.
Consider a payment rule that requires that ring members whose reports are less than the highest report pay \( \hat{v} \) to the center if they bid an amount greater than \( v \), and zero otherwise, and that the ring member with the highest report pay \( \hat{p}(r) \) to the center, where \( r \) is the second-highest report and

\[
\hat{p}(r) \equiv E_{v_{k+1},\ldots,v_n} \left( \left( r - \beta^{in}(r) \right) 1_{\beta^{in}(r) \geq \max_{j \in \{k+1,\ldots,n\}} \beta^{out}(v_j)} \right),
\]

which can be implemented by having the ring members compete in a second price ex ante auction for the right to be the sole ring member who bids at the auction (see Graham and Marshall (1987)). The mechanism recommends bids of \( v \) for ring members who do not have the highest report and recommends a bid of \( \beta^{in}(r_i) \) for the ring member \( i \) with the highest report. This mechanism induces truthful revelation, and it is a best reply for bidders to follow the recommendations of the ring (see the proof of Proposition 5).

Given this payment rule, ex ante budget balance for the center implies an ex ante non-contingent payment to each ring member of:

\[
X \equiv \frac{1}{k} E_{v_1,\ldots,v_k} \left( \hat{p}(v_j) 1_{v_i \geq v_j} \geq \max_{i \in \{1,\ldots,k\} \setminus \{i,j\}} v_j \right) \cdot
\]

Letting \( \beta_i^{nc}(v) \) be the non-cooperative equilibrium first price bid for bidder \( i \) with value \( v \) facing \( n - 1 \) other bidders, the interim individual rationality constraint for ring member \( i \) with value \( v_i \) can be written as the requirement that \( g_i(v_i \mid \hat{p}) \geq 0 \), where \( g_i \) is defined by

\[
g_i(v_i \mid \hat{p}) \equiv X + E_{v_{-i}} \left( \left( v_i - \beta^{in}(v_i) \right) 1_{v_i \geq \max_{j \in \{1,\ldots,k\} \setminus \{i\} v_j \text{ and } \beta^{in}(v_i) \geq \max_{j \in \{k+1,\ldots,n\}} \beta^{out}(v_j)} \right)
\]

\[
- E_{v_{-i}} \left( \hat{p} \left( \max_{j \in \{1,\ldots,k\} \setminus \{i\} v_j \right) 1_{v_i \geq \max_{j \in \{1,\ldots,k\} \setminus \{i\} v_j} \right)
\]

\[
- E_{v_{-i}} \left( (v_i - \beta_i^{nc}(v_i)) 1_{\beta_i^{nc}(v_i) \geq \max_{j \neq i} \beta_j^{nc}(v_j)} \right),
\]

where the first term is the ex ante non-contingent payment, the second term is the expected surplus from the auction, the third term is the expected payment to the center, and the fourth term is the expected surplus from non-cooperative play.

\[28\text{Non-symmetric payments could also be made.}\]
Proposition 5. Assuming no shills, when the payment by ring member \(i\) to the center depends on both reports and ring member \(i\)'s bid, if \(i\)'s bid differs from the center’s recommendation (case 3 of Section 3.3), then there exists a profitable collusive mechanism for a first price auction under interim individual rationality if, for all \(i \in \{1, ..., k\}\) and \(v_i \in [v, \bar{v}]\), \(g_i(v_i | \hat{p}) \geq 0\), and under ex ante individual rationality if, for all \(i \in \{1, ..., k\}\), \(E_{v_i}(g_i(v_i | \hat{p})) \geq 0\).

Proof. See the Appendix.

Noting that in the proof of Proposition 5, the dependence of \(p_i\) on \(b_i\) is only used to prevent a ring member who does not have the highest value from having an incentive to bid at the auction, it is clear that the same result can be achieved when \(p_i\) depends not on \(b_i\), but on the identity of the winner of the auction. This gives us the following corollary.

Corollary 2. Assuming no shills, in cases 3, 4, 7, and 10 of Section 3.3, there exists a profitable collusive mechanism for a first price auction under interim individual rationality if, for all \(i \in \{1, ..., k\}\) and \(v_i \in [v, \bar{v}]\), \(g_i(v_i | \hat{p}) \geq 0\), and under ex ante individual rationality if, for all \(i \in \{1, ..., k\}\), \(E_{v_i}(g_i(v_i | \hat{p})) \geq 0\).

Proposition 5 and Corollary 2 provide conditions under which a profitable collusive mechanism exists for a first price auction, but they leave unanswered the question whether these conditions can be satisfied. Even assuming symmetric bidders, the bid functions \(\beta^{in}\) and \(\beta^{out}\) cannot be represented analytically, so we rely on a numerical calculation to show that the conditions of Proposition 5 can be satisfied.

Proposition 6. Assuming no shills and symmetric bidders, in cases 3, 4, 7, and 10 of Section 3.3, there exists a profitable collusive mechanism for a first price auction under ex ante individual rationality when \(n = 3, k = 2\), and values are drawn from the uniform distribution on \([0, 1]\).
Proof. See the Appendix.

The proof of Proposition 6 involves the numerical calculation of the ex ante individual rationality constraint using the methods of Marshall et al. (1994) to solve for the equilibrium bid functions when the highest-valuing ring member bids against the outside bidder at the auction. It is interesting that in the example of Proposition 6, although ex ante individual rationality is satisfied, interim individual rationality is not satisfied for ring members with values above approximately 0.8. Thus, bidders with sufficiently high values are not willing to join the ring. A potential ring member must weigh whether he captures enough of the collusive gain to justify deviating from non-cooperative play, and bidder with a high value typically gains less from collusive play because his required payment to the center is larger than the ex ante non-contingent payment. In effect, ring members with high values subsidize ring members with low values in satisfying the center’s ex ante balanced-budget constraint.

One difficulty in proving general results about the profitability of collusion at first price auctions stems from the fact that at a first price auction, the ring cannot capture the entire collusive gain. Because a profitable ring at a first price auction must reduce the bids submitted by the ring members relative to their non-cooperative bids, some of the collusive gain must go to the bidders outside the ring. The following proposition shows that if a collusive mechanism is profitable in the sense of strictly increasing the expected joint payoff of the ring members and weakly reducing the auctioneer’s revenue relative to non-cooperative play, then the collusive mechanism increases the expected payoff to bidders outside the ring. Thus, a portion of the collusive gain is captured by bidders outside the ring. To state the result, it is useful to let $R^{nc}(v)$ be the joint payoff of the ring members and $A^{nc}(v)$ be the auctioneer’s payoff when the vector of bidders’ values is $v$ and play is non-cooperative, and let $R^{c}(v)$ be the joint payoff of the ring members and $A^{c}(v)$ by the auctioneer’s payoff when the vector of

\footnote{In fact, we have been unable to find any example within the context implicitly defined by Proposition 4 in which interim individual rationality is satisfied for all feasible value realizations.}
bidders’ values is \( v \) and bidders 1, \( \ldots, k \) participate in collusive mechanism \( \mu \). Also let \( \beta_1^\mu(v_1, \ldots, v_k), \ldots, \beta_k^\mu(v_1, \ldots, v_k), \beta_{k+1}^\mu(v_{k+1}), \ldots, \beta_n^\mu(v_n) \) be the equilibrium bids under collusive mechanism \( \mu \), and let

\[
\hat{\beta}_i^\mu(v) \equiv \begin{cases} 
\beta_i^\mu(v_1, \ldots, v_k), & \text{if } i \in \{1, \ldots, k\} \\
\beta_i^\mu(v_i), & \text{otherwise.}
\end{cases}
\]

Note that for \( i \in \{1, \ldots, k\} \), \( \beta_i^\mu \) is a function of \( v_1, \ldots, v_k \) rather than just \( v_i \) because the center’s bid recommendations are a function of the entire vector of reports.

**Proposition 7** Given collusive mechanism \( \mu \) for a first price auction, if \( E_v(R_\mu^\mu(v)) > E_v(R_{nc}^\mu(v)) \) and \( \forall v, A_\mu(v) \leq A_{nc}(v) \), then \( \forall i \in \{k+1, \ldots, n\} \),

\[
E_v\left((v_i - \beta_i^{nc}(v_i)) \mathbf{1}_{\beta_i^{nc}(v_i) \geq \max_{j \neq i} \beta_j^{nc}(v_j)}\right) < E_v\left((v_i - \hat{\beta}_i^\mu(v_i)) \mathbf{1}_{\beta_i^\mu(v_i) \geq \max_{j \neq i} \hat{\beta}_j^\mu(v)}\right).
\]

**Proof.** See the Appendix.

In words, Proposition 7 says that if collusion strictly increases the ring’s expected payoff and weakly reduces the auctioneer’s revenue, then collusion strictly increases the expected payoff of every outside bidder. Proposition 7 identifies a reason why collusion may not be sustainable at a first price auction—some of the gains from collusion necessarily spill over to the bidders outside the ring. This occurs because in order for there to be a collusive gain at a first price auction, the ring members must reduce their bids relative to their non-cooperative bids. The outside bidders profit from this—for example, even if the outside bidders do not change from their non-cooperative bid functions, their expected payoff is strictly higher when the ring members bid collusively, i.e., reduce their bids, than when they bid non-cooperatively.

Returning to Proposition 6, the result that collusion at a first price auction can be profitable requires the assumption that the use of shills is not possible. This is because a ring member with less than the highest value, although prevented from bidding at the auction himself because the center can base payments on the ring member’s bid,
may have an incentive to compete against the high-valuing ring member using a shill, which reduces the profitability of the ring.

**Proposition 8** Assuming interim or ex ante individual rationality, when the payment by ring member $i$ to the center depends on reports and ring member $i$’s bid, if $i$’s bid differs from the center’s recommendation (case 3 of Section 3.3), and there exists a profitable collusive mechanism without shills, then the ability to use shills reduces the expected joint payoff of the ring members from any incentive compatible collusive mechanism for a first price auction in which non-highest-valuing ring members have zero probability of winning the object.

*Proof.* See the Appendix.

Although Proposition 6 shows that collusion can be sustained in some first price environments when the center can base payments on reports and ring members’ bids, Proposition 8 implies that this may not be possible when ring members can use shills to submit bids at the auction. If the ring mechanism operates by essentially sending only one ring member to bid at the auction and suppressing the bids of the other ring members, then any ring member not sent to the auction can profitably use a shill as long as there is some probability that the ring member officially sent to the auction submits a bid strictly below his value—the latter being a necessity for collusion to be profitable.

Proposition 8 together with Corollary 1 implies that the presence of shills has a larger impact on the profitability of a ring when the auction is first price than when it is second price.

### 4.3 Mechanisms requiring payments only from winners

A natural restriction one might consider on the feasible transfers within a ring is that a ring member should not have to make any payment to the center unless he wins the
object. We now consider the restriction that a payment can only be required from a ring member if he wins the auction. As we show, this restriction severely limits the ability of ring members to profitably collude when the use of shills is possible, and, even in the absence of shills, can prevent efficient collusion from being possible.

**Proposition 9** Assuming ex ante or interim individual rationality and assuming no shills, when payments to the center depend on reports, but can only be required from a ring member who wins the auction (case 5 of Section 3.3), there does not exist a profitable, ex post efficient collusive mechanism for a second price auction in which non-highest-valuing ring members bid $v$.

*Proof.* See the Appendix.

**Proposition 10** Assuming ex ante or interim individual rationality, with or without the possibility of shills, when payments to the center depend on reports, but can only be required from a ring member who wins the auction (case 5 of Section 3.3), there does not exist a profitable collusive mechanism for a first price auction in which non-highest-valuing ring members bid $v$.

*Proof.* See the Appendix.

Case 5 of Section 3.3 differs from case 2 only in that the center cannot require a payment from a ring member unless that ring member wins the object, but Proposition 9 shows that for a second price auction, a ring’s expected payoff is strictly lower in case 5. Proposition 10 shows that for a first price auction the additional information of the identity of the winner is still not sufficient for the collusive mechanism to suppress all but the highest-valuing ring member’s bid.

Focusing on Proposition 9, what goes wrong in case 5 is that the center must require payments from at least some ring members when they win.\(^{30}\) Thus, even

\(^{30}\)Otherwise, the ring cannot achieve a collusive gain without causing at least some ring members to have an incentive to distort their reports.
though the center recommends that the highest-valuing ring member bid his value (required for ex post efficiency), the ring member expects to have to make a payment to the center if he wins. So instead of bidding his value, the ring member submits a bid equal to his value minus the expected payment to the center, preventing the mechanism from being ex post efficient.

We now expand the center’s ability to collect payments, while continuing to focus on cases in which the center cannot require payments from ring members that do not win the object. There are two important ways in which the center’s ability to collect payments might be increased. First, the center might be able to condition payments on the ring member’s observed bids. This prevents the type of problem that arises in Proposition 9, where ring members have an incentive to deviate from the recommended bids, and as Proposition 11 below shows, collusion is efficient in this case, at least for some symmetric value distributions. Second, the center might be able to condition payments on the amount a winning ring member pays at the auction. In that case, Proposition 12 below shows that the ring can achieve the maximum collusive gain by using a payment rule that makes use of the amount paid by the winner.

**Proposition 11** Assuming ex ante or interim individual rationality and no shills, when the payment by ring member $i$ to the center depends on reports and ring member $i$’s bid, if $i$’s bid differs from the center’s recommendation, but payments can only be required from a ring member who wins the auction (case 8 of Section 3.3), there exists a profitable, ex post efficient collusive mechanism for a second price auction in which the ring captures the entire collusive gain if bidders are symmetric and draw their values from the uniform distribution on $[0, 1]$.

**Proof.** See the Appendix.

Constructing a mechanism that satisfies the conditions of Proposition 11 requires care. To see this, suppose the center recommends that the highest-reporting ring
member bid his report and that all others bid $\underline{v}$, and suppose the center imposes a large penalty on any ring member who wins the object with a bid different from his recommended bid. Then when a ring member decides what to report, he weighs two things. If a ring member increases his report above his value, it increases the chance he will be the ring member selected to bid at the auction. This is the benefit of over reporting. But if a ring member submits a report greater than his value and it is the highest report in the ring, then he will have to submit a bid equal to his report in order to avoid the possibility of the large penalty. But bidding above his value means he may have to pay the auctioneer an amount greater than his value for the object. This is the cost of over reporting. The payment rule used by the center must require a payment from a ring member who wins the object that is an increasing function of the ring member’s report in order to provide incentives for truthful reporting. For example, as shown in the proof of Proposition 11, when all bidders draw their values from the uniform distribution on $[0, 1]$, the ring can use the following payment rule:

$$p_i(r_1, ..., r_k, b_i, A) = \begin{cases} 
0, & \text{if } i \text{ does not win} \\
\bar{v}, & \text{if } i \text{ wins and } b_i \neq \beta_i(r_1, ..., r_k) \\
\frac{(k-1)r_i}{(n-k+1)n}, & \text{if } i \text{ wins and } b_i = \beta_i(r_1, ..., r_k),
\end{cases}$$

where $\beta_i(r_1, ..., r_k) = \begin{cases} 
r_i, & \text{if } r_i = \max_{j \in \{1, ..., k\}} v_j \\
\underline{v}, & \text{otherwise.}
\end{cases}$

Given this payment rule, ring member $i$ maximizes its expected payoff with respect to its report by reporting truthfully, i.e.,

$$v_i \in \arg\max_{r_i} E_{v-i} \left( \left( v_i - \max_{j \in \{k+1, ..., n\}} v_j - \frac{(k-1)r_i}{(n-k+1)n} \right) 1_{r_i \geq \max_{j \in \{k+1, ..., n\}} v_j} \right).$$

We now return to the assumption that the center cannot condition on bids at the auction, but assume that the center can observe the amount paid by a winning ring member. In this case, efficient collusion is possible for a second price auction.

**Proposition 12** Assuming ex ante or interim individual rationality and assuming no shills, when payments to the center depend on reports and the outcome of the
auction, including the identity of the winner and amount paid, but payments can only be required from a ring member who wins the auction (case 6 of Section 3.3), there exists a profitable, ex post efficient collusive mechanism for a second price auction in which the ring captures the entire collusive gain.

Proof. See the Appendix.

**Corollary 3** Assuming ex ante or interim individual rationality and assuming no shills, in cases 6 and 9 of Section 3.3, there exists a profitable, ex post efficient collusive mechanism for a second price auction in which the ring captures the entire collusive gain.

Proposition 12 shows that a restriction that the ring only collect a payment from a ring member who wins does not affect the profitability of collusion at a second price auction if there are no shills and the center can observe the identity of the winner and the amount paid. In this case, the ring can use a payment rule that specifies a payment from ring member \( i \) of \( \max \left\{ 0, \max_{j \in \{1, \ldots, k\} \setminus \{i\}} r_j - x \right\} \) if he wins the auction and pays \( x \), and zero otherwise. Given this payment rule, a ring member has no incentive to over report because if doing so makes the difference between the ring member’s report being highest and not, then it means that the second-highest report is greater than the ring member’s value, and then the payment rule guarantees that the ring member will have to pay an amount greater than his value if he wins the object. Thus, the potential benefit to over reporting is eliminated.

Finally, we consider the effect of shills on the profitability of collusion when payments cannot be required from ring members who do not win the object.

**Proposition 13** Assuming ex ante or interim individual rationality and the possibility of shills, if a payment can only be required from a ring member who wins the auction (cases 5, 6, 8, and 9 of Section 3.3), then there does not exist a profitable collusive mechanism for a second price or first price auction.
Proof. See the Appendix.

When the use of shills is possible, a bidder always prefers to use a shill to avoid having to make any payment to the center, so collusion cannot be sustained.

5 Discussion

We have identified several things that can affect the ability of bidders to collude at a first price auction. First, the ring may not be able to suppress competition among members. Specifically, it may not be possible to dissuade ring members who do not have the highest value from competing at the auction. Second, it may not be possible to induce bidders with high values to join the ring. This is true in part because the gain to the ring is reduced by leakage to the outside bidders. Third, if bidders can use shills, the collusive gain that can be achieved by the ring is reduced.

The use of shills can affect the profitability of collusion in two ways. First, they can reduce the profitability of collusion by allowing non-highest-valuing ring members to compete at a first price auction, contrary to the recommendation of the center. Second, as shown in Proposition 13, when payments can only be collected from the winner of the auction, shills make collusion infeasible at both first price and second price auctions—the high-valuing ring member uses a shill to avoid having to make a payment to the center.

For all the cases in our taxonomy, if there exists a profitable collusive mechanism for a second price auction, there also exists one that allows the ring to capture the entire collusive gain. In contrast, for first price auctions, it appears that the profitability of a ring varies by degrees depending on the environment.

The policy implication of our results seem clear—if collusion is a major concern for auction designers, then use a first price auction.
Appendix: All-inclusive Ring

Our results assume that the ring is not all inclusive. Thus, any ring member selected to attend the auction must compete against at least one outside bidder. When the ring is all inclusive, collusion can be profitable in a wider set of environments. Our positive results for collusion at a second price auction (Propositions 1 and 12) continue to hold for an all-inclusive ring, and our negative result for collusion at a first price auction when payments to the center depend only on reports (Proposition 2) continues to hold, but when the ring is all inclusive, collusion is profitable at a first price auction for some environments in which it is not profitable when the ring is not all inclusive.

Consider the first price environment in which payments to the center depend on reports and a ring member’s bid, and where the use of shills is not possible. Suppose ring members report their values and the center recommends that the ring member with the highest report bid zero and all others bid less than zero or not at all. Ring members whose reports are less than the highest report pay $v_i$ to the center if they bid an amount greater than or equal to zero, and the ring member with the highest report pays $r_2$ to the center, where $r_2$ is the second-highest report. This mechanism induces truthful revelation and it is a best reply for bidders to follow the recommendation of the ring.

To show that collusion is profitable, we need only show that individual rationality is satisfied. Ex ante budget balance for the center implies an ex ante non-contingent payment to each ring member of:

$$X^a = \frac{1}{n} E_{v_1, \ldots, v_n} \left(\left(\left(v_j \right)_{1 \leq j \leq n} \geq \max_{i \in \{1, \ldots, n\}} \min_{j \neq i} v_j \right) v_i \right).$$

31 The proof follows as before letting $k = n$ and assuming the center recommends that ring members with less than the highest value bid an amount less than $v$ or do not bid at all.

32 Our negative result that collusion is not profitable when shills are possible and the center can only require a payment from a ring member who wins the object (Proposition 13) continues to hold when the ring is all inclusive. In this case, if a ring member does not win the object, it is common knowledge that some ring member used a shill, but it is not possible to penalize any ring member because none won the object.

33 Differential payments, reflecting bidders different ex ante marginal contributions to the ring, are also possible (see Graham, Marshall, and Richard (1990), especially Theorem 7).
The interim individual rationality for ring member $i$ with value $v_i$ can be written as the requirement that $g^a_i(v_i) \geq 0$, where $g^a_i$ is defined by

$$g^a_i(v_i) \equiv X^a + E_{v\sim_i} \left( (v_i) 1_{v_i \geq \max_{j \neq i} v_j} \right) - E_{v\sim_i} \left( \left( \max_{j \neq i} v_j \right) 1_{v_i \geq \max_{j \neq i} v_j} \right) - E_{v\sim_i} \left( (v_i - \beta^{ac}_i(v_i)) 1_{\beta^{nc}_i(v_i) \geq \max_{j \neq i} \beta^{nc}_j(v_j)} \right),$$

where the first term is the ex ante non-contingent payment, the second term is the expected surplus from the auction, the third term is the expected payment to the center, and the fourth term is the expected surplus from non-cooperative play. For symmetric bidders, $\beta^{nc}(v_i) = E_{v\sim_i} \left( \left( \max_{j \neq i} v_j \right) 1_{v_i \geq \max_{j \neq i} v_j} \right)$. Thus, a bidder’s expected payment to the center is equal to his non-cooperative bid, so $g^a_i(v_i) = X^a$ for all $v_i$, implying that interim individual rationality is satisfied.\(^{34}\) This proves the following proposition.

**Proposition A.1** Assuming interim or ex ante individual rationality and no shills, when the payment by ring member $i$ to the center depends on reports and ring member $i$’s bid, if $i$’s bid differs from the center’s recommendation (case 3 of Section 3.3), there exists a profitable collusive mechanism for a first price auction if the ring is all inclusive and bidders are symmetric.

Clearly the result of Proposition A.1 also holds in cases 4 and 10 of Section 3.3. If we modify the payment to require a payment $\bar{v}$ from any bidder who wins the auction but did not have the highest report, then the result also holds in case 7. Modifying the payment rule further to require a payment only from the ring member who wins (payment of $r_2$ if that ring member had the highest report and payment of $\bar{v}$ if that
ring member did not have the highest report), the result also holds in cases 5, 6, 8, and 9.

**Corollary 4** *Assuming interim or ex ante individual rationality and no shills, in cases 3, 4, 5, 6, 7, 8, 9, and 10 of Section 3.3, there exists a profitable collusive mechanism for a first price auction if the ring is all inclusive and bidders are symmetric.*

Proposition A.1 contrasts with the result in the text for the case of symmetric bidders with two ring members, one bidder outside the ring, and values drawn from the uniform distribution on $[0, 1]$. In that case, interim individual rationality is not satisfied for ring members with values greater than approximately 0.8. Proposition A.1 shows that when the ring is all inclusive and bidders are symmetric, interim individual rationality is satisfied for any number of bidders and any distribution of values.

Proposition A.1 raises an issue that we did not address in the body of the paper. Another reason why collusion may not be profitable at a first price auction is because of asymmetry among ring members. Our results in the body of the paper hint at this because we are only able to show existence of a profitable collusive mechanism when payments to the center depend on values and bids for the case of symmetry (Proposition 6), but Proposition A.1 makes the problem associated with asymmetric ring members more clear.
B   Appendix: Proofs

Proof of Proposition 1. Consider the following bidding rule: if bidder $i$’s report is not highest, the center recommends a bid of $v_i$, and if bidder $i$’s report is highest, the center recommends a bid equal to the report. Note that if the bidders report truthfully, there is no incentive for any bidder to deviate from the center’s recommendation, even with a shill. Consider the following payment rule: if bidder $i$’s report is not highest, bidder $i$ pays zero, but if bidder $i$’s report is highest and $r_2$ is the second-highest report, then bidder $i$ pays the center $\hat{p}(r_2) = E_{v_{k+1},\ldots,v_n} \left( (r_2 - \max_{j \in \{k+1,\ldots,n\}} v_j) 1_{r_2 \geq \max_{j \in \{k+1,\ldots,n\}} v_j} \right)$. Note that under this payment rule, the center has positive expected revenue and so can make positive ex ante non-contingent payments to the ring members.

Suppose the other $k-1$ ring members report truthfully. If a ring member with value $v_1$ reports $v_1 + \varepsilon$ (where $\varepsilon > 0$) rather than $v_1$, his payoff differs only if the highest other value in the ring is $v_2 \in (v_1, v_1 + \varepsilon)$. In this case, if the ring member reports truthfully his payoff is zero, and if he reports $v_1 + \varepsilon$ and bids $v_1$ at the auction (his weakly dominant strategy in the continuation game), his expected payoff is

$$\hat{p}(v_1) - E_{v_2} (\hat{p}(v_2) \mid v_2 \in (v_1, v_1 + \varepsilon)) < \hat{p}(v_1) - \hat{p}(v_1) = 0.$$ 

Thus, the ring member has no incentive to deviate in this way. If a ring member with value $v_1$ reports $v_1 - \varepsilon$ (where $\varepsilon > 0$) rather than $v_1$, his payoff differs only if the highest other value in the ring is $v_2 \in (v_1 - \varepsilon, v_1)$. In this case, if the ring member reports truthfully, his expected payoff is $\hat{p}(v_1) - E_{v_2} (\hat{p}_1(v_2) \mid v_2 \in (v_1 - \varepsilon, v_1))$, and if he reports $v_1 - \varepsilon$ and bids $v_1$ at the auction, he makes no payment to the center and has expected payoff

$$E_{v_{k+1},\ldots,v_n} \left( \left( v_1 - \max_{j \in \{k+1,\ldots,n\}} v_j \right) 1_{v_1 \geq \max_{j \in \{k+1,\ldots,n\}} v_j} \mid v_2 \in (v_1 - \varepsilon, v_1) \right)$$

$$= E_{v_{k+1},\ldots,v_n} \left( \left( v_1 - \max_{j \in \{k+1,\ldots,n\}} v_j \right) 1_{v_1 \geq \max_{j \in \{k+1,\ldots,n\}} v_j} \mid v_2 \geq v_2, \quad (v_1 - v_2) 1_{v_1 \geq v_2 \geq \max_{j \in \{k+1,\ldots,n\}} v_j} \mid v_2 \in (v_1 - \varepsilon, v_1) \right)$$

$$= E_{v_{k+1},\ldots,v_n} \left( \left( v_1 - \max_{j \in \{k+1,\ldots,n\}} v_j \right) 1_{v_1 \geq \max_{j \in \{k+1,\ldots,n\}} v_j} \mid v_2 \in (v_1 - \varepsilon, v_1) \right).$$
which is the same as his payoff from reporting truthfully. Thus, there is no incentive to deviate in this way.

We have shown that bidders report truthfully. It remains to show that individual rationality is satisfied. If bidder $i$ does not join the ring, play is non-cooperative, and bidder $i$ with value $v_i$ expects payoff $E_{v_{-i}} \left( (v_i - \max_{j \neq i} v_j) 1_{v_i \geq \max_{j \neq i} v_j} \right)$. If bidder $i \in \{1, \ldots, k\}$ has value $v_i$ and joins the ring, he expects payoff equal to the ex ante non-contingent payment plus

$$E_{v_{-i}} \left( (v_i - \max_{j \in \{k+1, \ldots, n\}} v_j) 1_{v_i \geq \max_{j \in \{k+1, \ldots, n\}} v_j} \right),$$

where the first term is his expected payoff from the auction and the second term is his expected payment to the center. Note that the payoff from the auction is positive only if $i$’s value is the highest among all $n$ bidders, i.e., $v_i \geq \max_{j \neq i} v_j$, but the payment to the center must be made if $i$’s value is the highest among the $k$ ring members, i.e., $v_i \geq \max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j$. Substituting the definition of $\hat{p}$ and rearranging, (B.1) is equal to

$$E_{v_{-i}} \left( \begin{pmatrix} (v_i - \max_{j \in \{k+1, \ldots, n\}} v_j) 1_{v_i \geq \max_{j \neq i} v_j} \\ (\max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j - \max_{j \in \{k+1, \ldots, n\}} v_j) 1_{v_i \geq \max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j \geq \max_{j \in \{k+1, \ldots, n\}} v_j} \\ (v_i - \max_{j \in \{1, \ldots, n\}} v_j) 1_{v_i \geq \max_{j \in \{1, \ldots, n\}} v_j \geq \max_{j \in \{k+1, \ldots, n\}} v_j} \end{pmatrix} \right),$$

$$= E_{v_{-i}} \left( \begin{pmatrix} (v_i - \max_{j \in \{k+1, \ldots, n\}} v_j) 1_{v_i \geq \max_{j \neq i} v_j} \\ (v_i - \max_{j \in \{1, \ldots, n\}} v_j) 1_{v_i \geq \max_{j \in \{1, \ldots, n\}} v_j \geq \max_{j \in \{k+1, \ldots, n\}} v_j} \end{pmatrix} \right),$$

$$= E_{v_{-i}} \left( \begin{pmatrix} (v_i - \max_{j \neq i} v_j) 1_{v_i \geq \max_{j \neq i} v_j} \end{pmatrix} \right),$$

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which is equal to bidder $i$’s expected payoff if he does not join the ring. Because we have excluded the ex ante payments, interim individual rationality is satisfied strictly. Furthermore, ex ante individual rationality is also satisfied strictly. Because the mechanism does not rely on the bids submitted at the auction or the identity of the winner, it is not affected by the possibility of shills. Q.E.D.

Proof of Proposition 2. Suppose there is an incentive compatible, profitable collusive mechanism in which non-highest-valuing ring members bid $v$. Then ring members truthfully report their values and bid according to the recommendations of the center. Let $\ell$ be the index of the ring member (randomly selected in the case of a tie) with the highest report. Because ring member $\ell$’s payment to the ring does not depend upon his bid at the auction, his recommended bid must be optimal in the auction subgame. In particular, it must be that the center’s recommended bid to bidder $\ell$, $\beta_\ell(v_1,\ldots,v_k)$, and the bids of the outside bidders, $\beta_i(v_i)$ for $i \in \{k+1,\ldots,n\}$, satisfy

$$\beta_\ell(v_1,\ldots,v_k) \in \arg \max_b E_{v_{\neq \ell}} \left( (v_\ell - b) 1_{b \geq \max_{j \in \{k+1,\ldots,n\}} \beta_j(v_j) \mid v_\ell = \max_{j \in \{1,\ldots,k\}} v_j \right)$$

$$= \arg \max_b E_{v_{\neq \ell}} \left( (v_\ell - b) 1_{b \geq \max_{j \in \{k+1,\ldots,n\}} \beta_j(v_j) \right)$$

and for $i \in \{k+1,\ldots,n\}$,

$$\beta_i(v_i) \in \arg \max_b E_{v_{\neq i}} \left( (v_i - b) 1_{b \geq \max \{\beta_i(v_1,\ldots,v_k), \max_{j \in \{k+1,\ldots,n\} \backslash \{i\}} \beta_j(v_j) \} \mid v_i = \max_{j \in \{1,\ldots,k\}} v_j \right)$$

Notice that the recommended bid to bidder $\ell$ depends only on bidder $\ell$’s value and not on the other ring members’ values or the identity of the highest-valuing ring member, i.e., $\beta_\ell(v_1,\ldots,v_k) = \hat{\beta}(v_\ell)$. Note also that for all $v > v_\ell$, $\hat{\beta}(v) < v$.

Suppose ring member $i$ has the second-highest value in the ring, where $v_i \in (\hat{\beta}(v_\ell), v_\ell)$, which is a positive probability event. Let $I_i$ be bidder $i$’s information, if any, about the values of the other ring members as a result of learning his required payment to the center and his recommended bid. Given this information, bidder $i$ forms beliefs (correct in equilibrium) about the values of the other ring members.
If the center recommends that ring member $i$ bid an amount not equal to $\hat{\beta}(v_i)$ or requires a payment from ring member $i$ not consistent with ring member $i$’s having the highest value in the ring, then ring member $i$’s belief must be that the probability that his recommended bid is highest is zero, i.e., $E_{v-i} \left( 1_{\hat{\beta}(v_i) \geq \hat{\beta}(v_i)} \mid I_i \right) = 0$. Furthermore, since $v_i \in (\hat{\beta}(v_i), v_i)$ and beliefs are correct in equilibrium, ring member $i$ must believe that there is positive probability that the center’s highest recommended bid is less than $i$’s value, i.e., $E_{v-i} \left( 1_{\hat{\beta}(v_i) < v_i} \mid I_i \right) > 0$. Thus, in this case ring member $i$ believes he has zero probability of winning the auction with a bid of $\hat{\beta}(v_i)$, but believes there exists some bid less than his value with positive probability of winning. Thus, ring member $i$ can profitably deviate from his recommended bid, a contradiction.

Thus, it must be that the center recommends that ring member $i$ bid $\hat{\beta}(v_i)$ and requires a payment from ring member $i$ that is not inconsistent with ring member $i$’s having the highest value in the ring. In this case, ring member $i$ believes that there is positive probability that his value is not highest, i.e., $E_{v-i} \left( 1_{v_i < \max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j} \mid I_i \right) > 0$ and that there is positive probability that the highest recommended value is less than his value, i.e., $E_{v-i} \left( 1_{\hat{\beta}(v_i) < v_i} \mid I_i \right) > 0$. In particular, for all $b \in (\hat{\beta}(v_i), v_i)$, ring member $i$ believes that there is positive probability that such a bid would win the auction, i.e., $b \in (\hat{\beta}(v_i), v_i)$,

$$E_{v-i} \left( 1_{b > \max \{\max_{j \in \{k+1, \ldots, n\}} \beta_j(v_j)\}} \mid I_i \right) > 0.$$  \hfill (B.2)

In this case, bidder $i$ chooses a bid to solve

$$\max_{b} E_{v-i} \left( (v_i - b)1_{v_i < v_i \text{ and } b \geq \max_{j \in \{k+1, \ldots, n\}} \beta_j(v_j)} \mid I_i \right)$$

$$+ (v_i - b)1_{v_i = v_i \text{ and } b \geq \max_{j \in \{k+1, \ldots, n\}} \beta_j(v_j)} \mid I_i \right),$$  \hfill (B.3)

where the first term in the expectation is ring member $i$’s payoff from bidding $b$ if $i$’s value is not highest in the ring, and the second term is his payoff if his value is highest in the ring. Note that by the definition of $\hat{\beta}$,

$$\hat{\beta}(v_i) \in \arg \max_b E_{v-i} \left( (v_i - b)1_{v_i = v_i \text{ and } b \geq \max_{j \in \{k+1, \ldots, n\}} \beta_j(v_j)} \mid I_i \right)$$
and because \( \hat{\beta} \) is nondecreasing,
\[
E_{v_i} \left( 1_{v_i < v} \text{ and } \hat{\beta}(v_i) \geq \max \left\{ \max \hat{\beta}(v_k), \max_{j \in \{k+1, \ldots, n\}} \hat{\beta}_j(v_j) \right\} | I_i \right) = 0.
\]
Thus, second term in the expectation in (B.3) is maximized at \( \hat{\beta}(v_i) \) and the first term is zero at \( \hat{\beta}(v_i) \). By (B.2), the first term is increasing in \( b \) at \( b = \hat{\beta}(v_i) \), implying that \( \hat{\beta}(v_i) \) does not solve (B.3), a contradiction. Q.E.D.

**Proof of Proposition 3.** Let bidders 1 and 2 be the highest and second-highest-valuing ring members, in no particular order. Let \( \hat{\beta}^{\text{in}} \) be the bid function used by the ring members, and let \( \hat{\beta}^{\text{out}} \) be the bid function used by the outside bidders. Let
\[
\hat{\beta}_i \equiv \begin{cases} 
\hat{\beta}^{\text{in}}, & \text{if } i \in \{1, \ldots, k\} \\
\hat{\beta}^{\text{out}}, & \text{if } i \in \{k + 1, \ldots, n\}.
\end{cases}
\]
Functions \( \hat{\beta}^{\text{in}} \) and \( \hat{\beta}^{\text{out}} \) are defined by the conditions that for all \( v \),
\[
\hat{\beta}^{\text{in}}(v) \in \arg \max_b E_{v-i} \left( (v - b) 1_{b \geq \max_{j \in \{2, k+1, \ldots, n\}} \hat{\beta}_j(v_j)} | \min \{v_1, v_2\} \geq \max_{j \in \{3, \ldots, k\}} v_j \right)
\]
\[
= \arg \max_b E_{v-i} \left( (v - b) 1_{b \geq \max_{j \neq 1} \hat{\beta}_j(v_j)} \right)
\]
and
\[
\hat{\beta}^{\text{out}}(v) \in \arg \max_b E_{v-n} \left( (v - b) 1_{b \geq \max_{j \in \{1, 2, k+1, \ldots, n-1\}} \hat{\beta}_j(v_j)} | \min \{v_1, v_2\} \geq \max_{j \in \{3, \ldots, k\}} v_j \right)
\]
\[
= \arg \max_b E_{v-n} \left( (v - b) 1_{b \geq \max_{j \neq n} \hat{\beta}_j(v_j)} \right).
\]
Thus, for all \( i \in \{1, \ldots, n\} \), \( \hat{\beta}_i(v) \in \arg \max_b E_{v-i} \left( (v - b) 1_{b \geq \max_{j \neq i} \hat{\beta}_j(v_j)} \right) \), which, using the uniqueness result of Lemma 1, implies \( \hat{\beta}_i(v) = \bar{\beta}_i^{\text{nc}}(v) \). Q.E.D.

**Proof of Proposition 4.** Suppose there is an incentive compatible, profitable collusive mechanism in which non-highest-valuing ring members have zero probability of winning the object. Then ring members truthfully report their values and bid according to the recommendations of the center. Let \( i \) be the index of the ring member (randomly selected in the case of a tie) with the highest report. Because ring member
\(\iota\)'s payment to the ring does not depend upon his bid at the auction, his recommended bid must be optimal in the auction subgame. Because non-highest-valuing ring members have zero probability of winning the object, it must be that for all \(i \in \{1, \ldots, k\}\backslash\{\iota\}\), \(\beta_i(v_1, \ldots, v_k) < \beta_i(v_1, \ldots, v_k)\). For \(i \in \{k + 1, \ldots, n\}\), let \(\bar{\beta}_i(v_i)\) be the equilibrium bid of outside bidder \(i\). Define

\[
\hat{\beta}(v_i) \in \operatorname{arg\,max}_b E_{v_{-i}} \left( (v_i - b) 1_{b \geq \max_{j \in \{k+1, \ldots, n\}} \bar{\beta}_j(v_j)} \right).
\]

As shown in the proof of Proposition 2, if the center recommends that ring member \(\iota\) bid \(\hat{\beta}(v_i)\) with probability one, i.e., \(\Pr(\beta_\iota(v_1, \ldots, v_k) = \hat{\beta}(v_i)) = 1\), then for a positive measure set of values for the ring members, a non-highest-valuing ring member has an incentive to deviate from his recommended bid, contradicting the incentive compatibility of the mechanism. Thus,

\[
\Pr(\beta_\iota(v_1, \ldots, v_k) = \hat{\beta}(v_i)) < 1. \quad \text{(B.4)}
\]

If the center recommends that the highest-valuing ring member bid less than \(\beta^{'\text{in}}(v_i)\), i.e., \(\beta_\iota(v_1, \ldots, v_k) < \beta^{'\text{in}}(v_i)\), then that ring member can profitably deviate by bidding \(\beta^{'\text{in}}(v_i)\). If the center recommends that the highest-valuing ring member bid an amount greater than or equal to his value, he can profitably deviate by submitting a slightly lower bid (even if that means a tie with another ring member). Thus, incentive compatibility of the mechanism implies that the center recommends that the highest-valuing ring member bid an amount less than \(v_i\) and greater than or equal to \(\beta^{'\text{in}}(v_i)\), i.e., \(\beta_\iota(v_1, \ldots, v_k) \in [\beta^{'\text{in}}(v_i), v_i)\). Using (B.4), it follows that \(\Pr(\beta_\iota(v_1, \ldots, v_k) > \hat{\beta}(v_i)) > 0\). However, if the center recommends that the highest-valuing ring member bid more than \(\beta^{'\text{in}}(v_i)\), we can show that the highest-valuing ring member can profitably deviate by lowering his bid unless the center recommends that some other ring member bid one increment below the recommended bid of the highest-valuing ring member. To see this, suppose that \(\beta_\iota(v_1, \ldots, v_k) > \hat{\beta}(v_i)\) and that

\[
\Pr\left(\max_{j \in \{1, \ldots, k\}\backslash\{\iota\}} \beta_j(v_1, \ldots, v_k) < \beta_\iota(v_1, \ldots, v_k) - \Delta \mid I_\iota\right) = 1, \quad \text{(B.5)}
\]

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where \( I_i \) is ring member \( i \)'s information about the values of the other ring members given his required payment to the center and his recommended bid. Then ring member \( i \) chooses his bid \( b_i \) to solve

\[
\max_b E_{v_i} \left( (v_i - b)I_{b \geq \max \{\max_{j \in \{1, \ldots, k\} \setminus \{i\}} \beta_j(v_1, \ldots, v_k), \max_{j \in \{k+1, \ldots, n\}} \beta_j(v_j)\} | I_i \right).
\]

Letting \( \Delta > 0 \) be the bid increment and using (B.5), we can rewrite the maximand as

\[
E_{v_i} \left( (v_i - b)I_{b \geq \max \{\max_{j \in \{1, \ldots, k\} \setminus \{i\}} \beta_j(v_1, \ldots, v_k), \max_{j \in \{k+1, \ldots, n\}} \beta_j(v_j)\} | I_i, \max_{j \in \{1, \ldots, k\} \setminus \{i\}} \beta_j(v_1, \ldots, v_k) < \hat{\beta}_i(v_1, \ldots, v_k) - \Delta \right).
\]

(B.6)

Because we suppose \( \beta_i(v_1, \ldots, v_k) > \hat{\beta}(v_i) \), the (B.6) is decreasing in \( b \) at \( b = \beta_i(v_1, \ldots, v_k) \). Thus, there exists \( \Delta' > 0 \) such that for all \( \Delta \in (0, \Delta') \), the (B.6) is greater at \( b = \beta_i(v_1, \ldots, v_k) - \Delta \) than at \( b = \beta_i(v_1, \ldots, v_k) \), implying that ring member \( i \) can profitably deviate by bidding \( \beta_i(v_1, \ldots, v_k) - \Delta \) rather than the center’s recommendation of \( \beta_i(v_1, \ldots, v_k) \), a contradiction. Thus, it must be that whenever \( \beta_i(v_1, \ldots, v_k) > \hat{\beta}(v_i) \), which occurs with positive probability,

\[
\Pr \left( \max_{j \in \{1, \ldots, k\} \setminus \{i\}} \beta_j(v_1, \ldots, v_k) = \beta_i(v_1, \ldots, v_k) - \Delta | I_i \right) > 0.
\]

Thus, for a positive measure set of ring values, the center recommends that the highest-valuing ring member bid above \( \hat{\beta}(v_i) \) and that some other ring member bid \( \beta_i(v_1, \ldots, v_k) - \Delta \). Q.E.D.

**Proof of Proposition 5.** Let \( r^1 \) be the highest report and \( r^2 \) the second-highest report in the ring. Let \( v^1 \) be the highest value and \( v^2 \) the second-highest value in the ring. Consider the following bidding rule: the center recommends that the ring member with the highest report bid \( \beta_{\text{in}}(r^1) \) and that all others bid \( v \). Consider the following payment rule: the bidder with the highest report pays the center \( \hat{p}(r^2) \), and all others pay zero if their bid is \( v \) and \( \bar{v} \) if their bid is greater than \( v \). Suppose the bidders join the ring and report truthfully. It is a best reply for bidders with less than the
highest value to bid \( v \) at the auction rather than bid anything else and pay \( \bar{v} \) to the center. Because the payment rule faced by the highest-valuing ring member is constant with respect to his bid, the payment rule does not distort the highest-valuing ring member’s choice of bid. Thus, in equilibrium the highest-valuing ring member bids \( \beta^{in}(v^1) \). Consider whether bidders report truthfully. If all other bidders report truthfully and a bidder with value \( \hat{v} < v^1 \) reports \( \hat{r} > \hat{v} \), causing him to have the highest report, i.e., \( \hat{v} < v^1 < \hat{r} \), then his expected payoff from participating in the auction is \( p(\hat{v}) \), but his payment to the center is \( \hat{p}(v^1) > p(\hat{v}) \), giving him negative expected payoff. If a bidder with value \( v^1 \) reports \( \hat{r} < v^1 \), causing him not to have the highest report, i.e., \( \hat{r} < v^2 < v^1 \), then his expected payoff from participating in the auction is negative because the payment rule specifies a payment of \( \hat{v} \), but if he reports truthfully his expected payoff from participating in the auction is positive. Because all other deviations have zero expected payoff, this establishes that no ring member has an incentive to misrepresent his value to the center. Given the conditions in the Proposition, individual rationality, either interim or ex ante, is satisfied. Q.E.D.

**Proof of Proposition 6.** Using the assumption of symmetry,

\[
\hat{p}(r_2) = (r_2 - \beta^{in}(r_2)) \int_{\bar{v}}^{\beta^{out-1}(\beta^{in}(r_2))} (n - k) F^{n-k-1}(x) f(x) dx,
\]

where \( X = \frac{1}{k} \int_{\bar{v}}^{0} \hat{p}(x) k (k-1) F^{k-2}(x) (1 - F(x)) f(x) dx \), and

\[
g(v_i \mid \hat{p}) = X + \int_{\bar{v}}^{\beta^{out-1}(\beta^{in}(v_i))} \left( \int_{\bar{v}}^{v_i} \left( v_i - \beta^{in}(v_i) \right) (k-1) F^{k-2}(x) f(x) dx \right) (n - k) F^{n-k-1}(y) f(y) dy
\]

\[- \int_{\bar{v}}^{v_i} \hat{p}(x) (k-1) F^{k-2}(x) f(x) dx - \int_{\bar{v}}^{v_i} (v_i - \beta^{nc}(v_i)) (n - 1) F^{n-2}(x) f(x) dx.\]

The ex ante individual rationality constraint is \( \int_{\bar{v}}^{\hat{v}} g(x \mid \hat{p}) f(x) dx \geq 0 \). Substituting in \( \bar{v} = 0, \hat{v} = 1, F(v) = v, f(v) = 1, n = 3, k = 2, \) and \( \beta^{nc}(v) = \frac{(n-1)v}{n} \), and calculating \( \beta^{out} \) and \( \beta^{in} \) numerically (see Marshall et al. (1994)), we can calculate \( g \) numerically as shown in Figure 1.
An additional numerical calculation gives $\int_0^1 g(x \mid \hat{p}) \, dx \approx .017 > 0$. (The code used to calculate this result is available from the authors on request.) Thus, ex ante individual rationality is satisfied. Q.E.D.

**Proof of Proposition 7.** Let $V \equiv \times_{j=1}^n V_j$, where $V_j \equiv [v, \bar{v}]$. Because we assume $\forall v \in V, A^\mu(v) \leq A^{nc}(v)$, it follows that $\forall v \in V$,

$$\max_j \beta^\mu_j(v) \leq \max_j \beta^{nc}_j(v_j),$$  \hfill (B.7)

i.e., the high bid under collusive play is always less than or equal to the high bid under noncooperative play. Noting that for all $j \in \{k+1, \ldots, n\}$, $\beta^\mu_j(v) = \beta^{nc}_j(v) \leq \bar{v}$, (B.7) implies that $\forall (v_1, \ldots, v_k) \in \times_{j=1}^k V_j$,

$$\max_{j \in \{1, \ldots, k\}} \beta^\mu_j(v_1, \ldots, v_k) \leq \max_{j \in \{1, \ldots, k\}} \beta^{nc}_j(v_1, \ldots, v_k),$$  \hfill (B.8)

which in turn implies that for all $j \in \{k+1, \ldots, n\}$ and $v_j \in V_j$,

$$\beta^\mu_j(v_j) \leq \beta^{nc}_j(v_j).$$  \hfill (B.9)

Let $\ell \in \{k+1, \ldots, n\}$ and suppose bidder $\ell$, who is outside the ring, bids according to his non-cooperative bid function $\beta^{nc}_\ell$ while the other bidders bid according to $\beta^\mu$. Because $\beta^\mu_\ell(v) = \beta^{nc}_\ell(v) \leq \bar{v}$, (B.7) implies that $\forall v_\ell \in V_{-\ell}$, $\max_{j \neq \ell} \beta^\mu_j(v) \leq \max_{j \neq \ell} \beta^{nc}_j(v_j)$, which implies that $\forall v_\ell \in V_{-\ell}$,

$$E_{v_{-\ell}} \left( (v_\ell - \beta^{nc}_\ell(v_\ell)) \mathbf{1}_{\beta^{nc}_\ell(v_\ell) \geq \max_{j \neq \ell} \beta^{nc}_j(v_j)} \right) \leq E_{v_{-\ell}} \left( (v_\ell - \beta^{nc}_\ell(v_\ell)) \mathbf{1}_{\beta^{nc}_\ell(v_\ell) \geq \max_{j \neq \ell} \beta^\mu_j(v)} \right).$$  \hfill (B.10)
Thus, a bidder outside the ring who bids non-cooperatively is weakly better off if the other bidders bid collusively than if they bid non-cooperatively. The surplus in the event the bidder wins is the same, but the probability of winning with a given bid is increased.

Because we assume \( E_\mathcal{V}(R^u(v)) > E_\mathcal{V}(R^{nc}(v)) \), it follows that for all \( j \in \{1, \ldots, k\} \), there exists positive measure set \( \hat{V}_j \subseteq V_j \) such that \( \forall \{v_1, \ldots, v_k\} \in \times_{j=1}^k \hat{V}_j \),

\[
\max_{j \in \{1, \ldots, k\}} \beta^u_j(v_1, \ldots, v_k) < \max_{j \in \{1, \ldots, k\}} \beta^{nc}_j(v_j),
\]

i.e., it is a positive probability event that the maximum ring bid under collusion is strictly less than the maximum ring bid under non-cooperative play. Thus, there exists positive measure set \( \hat{V}_\ell \subseteq V_\ell \) such that for all \( v_\ell \in \hat{V}_\ell \),

\[
E_{v_{-\ell}} \left( 1_{\max_{j \in \{1, \ldots, k\}} \beta^0_j(v_1, \ldots, v_k) < \beta^{nc}_\ell(v_\ell) < \max_{j \in \{1, \ldots, k\}} \beta^{nc}_j(v_j)} \right) > 0.
\]

Thus, using (B.8) and (B.9), for all \( v_\ell \in \hat{V}_\ell \),

\[
E_{v_{-\ell}} \left( 1_{\beta^{nc}_\ell(v_\ell) \geq \max_{j \neq \ell} \beta^{nc}_j(v_j)} \right) < E_{v_{-\ell}} \left( 1_{\beta^{nc}_\ell(v_\ell) \geq \max_{j \neq \ell} \beta^0_j(v)} \right). \tag{B.11}
\]

It follows that for all \( v_\ell \in \hat{V}_\ell \),

\[
E_{v_{-\ell}} \left( (v_\ell - \beta^{nc}_\ell(v_\ell)) 1_{\beta^{nc}_\ell(v_\ell) \geq \max_{j \neq \ell} \beta^{nc}_j(v_j)} \right) < E_{v_{-\ell}} \left( (v_\ell - \beta^{nc}_\ell(v_\ell)) 1_{\beta^{nc}_\ell(v_\ell) \geq \max_{j \neq \ell} \beta^0_j(v)} \right).
\]

Together with (B.10), this implies that

\[
E_\mathcal{V} \left( (v_\ell - \beta^{nc}_\ell(v_\ell)) 1_{\beta^{nc}_\ell(v_\ell) \geq \max_{j \neq \ell} \beta^{nc}_j(v_j)} \right) < E_\mathcal{V} \left( (v_\ell - \beta^{nc}_\ell(v_\ell)) 1_{\beta^{nc}_\ell(v_\ell) \geq \max_{j \neq \ell} \beta^0_j(v)} \right). \tag{B.12}
\]

Thus, a bidder outside the ring who bids non-cooperatively has strictly greater expected payoff (taking the expectation over all bidders’ values) if the other bidders bid collusively than if they bid non-cooperatively. Because \( \beta^u_\ell \) is a best reply to \( \beta^u_{-\ell} \), it follows that

\[
E_{v_{-\ell}} \left( (v_\ell - \beta^{nc}_\ell(v_\ell)) 1_{\beta^{nc}_\ell(v_\ell) \geq \max_{j \neq \ell} \beta^0_j(v)} \right) \leq E_{v_{-\ell}} \left( (v_\ell - \beta^{nc}_\ell(v_\ell)) 1_{\beta^u_\ell(v_\ell) \geq \max_{j \neq \ell} \beta^0_j(v)} \right). \tag{B.13}
\]

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Combining (B.12) and (B.13),
\[
E_v \left( (v_\ell - \beta^\text{nc}_\ell (v_\ell)) 1_{\beta^\text{nc}_\ell (v_\ell) \geq \max_{j \neq \ell} \beta^\text{nc}_j (v_\ell)} \right) < E_v \left( (v_\ell - \beta^\ell (v_\ell)) 1_{\beta^\ell (v_\ell) \geq \max_{j \neq \ell} \beta^\ell_j (v)} \right),
\]

implying that when then ring operates every outside bidder has strictly greater ex ante expected payoff than under non-cooperative play. Q.E.D.

**Proof of Proposition 8.** Because the use of a shill does not affect a ring member’s payment to the center, the proof of Proposition 2 implies that for any collusive mechanism for a first price auction in which non-highest-valuing ring members have zero probability of winning the object, there is positive probability that a ring member can increase his expected payoff by using a shill to submit a bid greater than the bid recommended by the center. In particular, a ring member \(i\) with value \(v_i \in (\mathcal{M}_{\max} \setminus \{i\})\), whose recommended bid is \(\bar{v}_i\) can profitably deviate by using a shill to bid \(v_i - \varepsilon\) for some \(\varepsilon \in (0, v_i - \bar{v}_i)\). We must show that the use of a shill by ring member \(i\) to bid \(v_i - \varepsilon\) reduces the expected payoff to the ring. Suppose not. The expected joint payoff to the \(k\) ring members if ring member \(i\) does not use a shill is (payments to and from the center net to zero in expectation):
\[
E_{v_{k+1}, \ldots, v_n} \left( \left( \max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j - \beta^\text{in} \left( \max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j \right) \right) \cdot \right) \cdot 1_{\beta^\text{in} (\max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j) \geq \max_{j \in \{k+1, \ldots, n\}} \beta^\text{out}_j (v_j)}.
\]
(B.14)
The expected joint payoff to the \(k\) ring members if ring member \(i\) uses a shill to bid \(v_i - \varepsilon\) must take into account the probability that the ring member with the highest value wins and the probability that ring member \(i\) wins with his bid of \(v_i - \varepsilon\):
\[
E_{v_{k+1}, \ldots, v_n} \left( \left( \max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j - \beta^\text{in} \left( \max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j \right) \right) \cdot \right) \cdot 1_{\beta^\text{in} (\max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j) \geq \max\{v_i - \varepsilon, \max_{j \in \{k+1, \ldots, n\}} \beta^\text{out}_j (v_j)\}}
\]
\[
+ E_{v_{k+1}, \ldots, v_n} \left( \left( v_i - (v_i - \varepsilon) \right) \cdot \right) \cdot 1_{v_i - \varepsilon \geq \max\{\beta^\text{in} (\max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j), \max_{j \in \{k+1, \ldots, n\}} \beta^\text{out}_j (v_j)\}}
\]
(B.15)
By our supposition, (B.14) is less than or equal to (B.15). Because \( \max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j \geq v_i \), (B.15) is less than

\[
\mathbb{E}_{v_{k+1}, \ldots, v_n} \left( \max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j - \max \left\{ \beta^{\text{in}} \left( \max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j \right), v_i - \varepsilon \right\} \right). \tag{B.16}
\]

But then (B.14) is less than (B.16), which violates the definition of \( \beta^{\text{in}} \) because it implies that the ring can increase its expected payoff by having the highest-valuing ring member bid \( \max \left\{ \beta^{\text{in}} \left( \max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j \right), v_i - \varepsilon \right\} \) rather than \( \beta^{\text{in}} \left( \max_{j \in \{1, \ldots, k\} \setminus \{i\}} v_j \right) \).

Q.E.D.

Proof of Proposition 9. We begin with a lemma (the lemma is stated in more generality than necessary because we also use it in the proof of Proposition 11).

**Lemma B.1** Assuming ex ante or interim individual rationality and no shills, in case 5 (alternatively case 8) of Section 3.3, in any ex post efficient, incentive compatible collusive mechanism for a second price auction in which all but the highest-valuing ring member bid \( v \), there exists \( i \in \{1, \ldots, k\} \) and \( v_i > v \) such that

\[
\mathbb{E}_{v-i} \left( p_i (v_1, \ldots, v_k, \beta_i(v_1, \ldots, v_k), i \text{ wins}) 1_{v_i \geq \max_{j \neq i} v_j} \right) \neq 0
\]

(alternatively \( \mathbb{E}_{v-i} \left( p_i (v_1, \ldots, v_k, i \text{ wins}) 1_{v_i \geq \max_{j \neq i} v_j} \right) \neq 0 \)).

Proof of Lemma B.1. Suppose the conditions of the lemma hold and that for all \( i \) and \( v_i > v \),

\[
\mathbb{E}_{v-i} \left( p_i (v_1, \ldots, v_k, i \text{ wins}) 1_{v_i \geq \max_{j \neq i} v_j} \right) = 0 \tag{B.17}
\]

or, alternatively,

\[
\mathbb{E}_{v-i} \left( p_i (v_1, \ldots, v_k, \beta_i(v_1, \ldots, v_k), i \text{ wins}) 1_{v_i \geq \max_{j \neq i} v_j} \right) = 0, \tag{B.18}
\]

depending on the case being considered. Given that outside bidders bid their values (their weakly dominant strategy), ex post efficiency requires that \( \beta_i (v_1, \ldots, v_k) = v_i \).
if \( v_i = \max_{j \in \{1,\ldots,k\}} v_j \), and the conditions of the lemma require that \( \beta_i(v_1, \ldots, v_k) = v \) otherwise (in the case of a tie for the highest value in the ring, the mechanism recommends that a randomly chosen one of the highest-valuing ring members bids his value and that the others bid \( v \)). Thus, because no payment can be required from a ring member who does not win the auction, the expression on the left side of (B.17) (alternatively (B.18)) is the expected payment to the center for a ring member who reports \( v_i \) and follows the recommendation of the center. Let \( i \in \{1, \ldots, k\} \), and suppose \( v_i > v \). Let \( G^{\text{out}} \) be the cdf for the highest value among the \( n - k \) outside bidders, and let \( G^{\text{in}} \) be the cdf for the highest value among the \( k - 1 \) ring members other than \( i \). If ring member \( i \) reports truthfully and follows the recommendation of the center, then (B.17) (alternatively (B.18)) implies that his expected payment to the center is zero, and so his expected payoff is

\[
E_{v-i} \left( \left( v_i - \max_{j \in \{k+1,\ldots,n\}} v_j \right) 1_{v_i \geq \max_{j \neq i} v_j} \right) = G^{\text{in}}(v_i) \int_{v}^{v_i} (v_i - x) dG^{\text{out}}(x). \tag{B.19}
\]

If ring member \( i \) reports \( r_i > v_i \) and follows the recommendation of the center, his expected payment to the center is also zero—to see this, let \( v_i = r_i \) in (B.17) (alternatively (B.18))—so his expected payoff is

\[
E_{v-i} \left( \left( v_i - \max_{j \in \{k+1,\ldots,n\}} v_j \right) 1_{r_i \geq \max_{j \neq i} v_j} \right) = G^{\text{in}}(r_i) \int_{v}^{r_i} (v_i - x) dG^{\text{out}}(x). \tag{B.20}
\]

Note that (B.19) and (B.20) are equal when \( r_i = v_i \). The derivative of (B.20) with respect to \( r_i \) and evaluated at \( r_i = v_i \) is \( dG^{\text{in}}(v_i) \int_{v}^{v_i} (v_i - x) dG^{\text{out}}(x) \), which is positive. Thus, there exists \( r_i > v_i \) such that (B.19) is less than (B.20), implying that the mechanism is not incentive compatible, a contradiction. Q.E.D.

Continuation of the Proof of Proposition 9. Suppose the conditions of the proposition hold. Given that outside bidders bid their values (their weakly dominant strategy), ex post efficiency requires that the highest-valuing ring member bid his value at the auction. By Lemma B.1, there exists ring member \( i \) and value \( v_i > v \) such that ring member \( i \)'s expected payment to the center if he wins the auction, \( x \), is positive. In
case 5 of Section 3.3, if ring members report truthfully and ring member \( i \) has the highest value in the ring, then ring member \( i \) strictly prefers to bid \( v_i - x \) rather than \( v_i \) at the auction (in order to eliminate the possibility that \( i \) wins the object for a price in \( (v_i - x, v_i) \), in which case ring member \( i \)'s payoff is negative), a contradiction. Q.E.D.

**Proof of Proposition 10.** Suppose there is an incentive compatible, profitable collusive mechanism in which non-highest-valuing ring members bid \( \bar{v} \). Then ring members truthfully report their values and bid according to the recommendations of the center. Let \( \iota \) be the index of the ring member (randomly selected in the case of a tie) with the highest report. Because ring member \( \iota \)'s payment to the ring does not depend upon his bid at the auction, his recommended bid must be optimal in the auction subgame. In particular, as in the proof of Proposition 2, if for \( i \in \{k+1, \ldots, n\} \), \( \beta_i(v_i) \) is the equilibrium bid of outside bidder \( i \), it must be that the center recommends that bidder \( \iota \) bid \( \hat{\beta}(v_i) \), where

\[
\hat{\beta}(v_i) \in \arg \max_b E_{v_{-i}} \left( (v_i - b)1_{b \geq \max_{j \in \{k+1, \ldots, n\}} \beta_j(v_j)} \right).
\]

It follows from the proof of Proposition 2 that if the center does not use the identity of the winner to penalize ring members who do not bid according to the recommendation of the center, then a profitable collusive mechanism in which non-highest-valuing ring members bid \( \bar{v} \) does not exist. Thus, we can assume that the center chooses a penalty for any ring member who wins, but who did not have the highest report, large enough to deter bids different from the recommended bids. Suppose ring members other than \( i \) report truthfully. If ring member \( i \) reports truthfully, his expected payoff is

\[
E_{v_{-i}} \left( (v_i - \hat{\beta}(v_i))1_{v_i = \max_{j \in \{1, \ldots, k\}} v_j \text{ and } \hat{\beta}(v_i) \geq \max_{j \in \{k+1, \ldots, n\}} \beta_j(v_j)} \right).
\]

If ring member \( i \) with value \( v_i > v \) reports \( \bar{v} \) and then, if the center recommends that he bid above \( \bar{v} \), submits a bid of \( \hat{\beta}(v_i) \), his expected payoff is

\[
E_{v_{-i}} \left( (v_i - \hat{\beta}(v_i))1_{\hat{\beta}(v_i) \geq \max_{j \in \{k+1, \ldots, n\}} \beta_j(v_j)} \right).
\]

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which is strictly greater, contradicting incentive compatibility. Q.E.D.

Proof of Proposition 11. Assume the conditions of the proposition. By Lemma B.1, if the payment rule does not penalize a bid that differs from the recommendation of the center, then we are in the same situation as in the proof of Proposition 9, and ring member \( i \) strictly prefers to bid \( v_i - x \) rather than \( v_i \) at the auction, a contradiction. Furthermore, as shown in Lemma B.1, because \( E_{v_{i-1}} \left( \left( v_i - \max_{j \in \{k+1, \ldots, n\}} v_j \right) 1_{r_i \geq \max_{j \neq i} v_j} \right) \) is strictly increasing in \( r_i \) at \( r_i = v_i \), if the payment rule does not depend on a ring member’s report, then ring members have an incentive to report an amount greater than their values. We construct a mechanism that addresses these issues.

Let \( G^\text{out} \) be the cdf and \( g^\text{out} \) the pdf for the highest value among the \( n - k \) outside bidders, and let \( G^\text{in}_i \) be the cdf and \( g^\text{in}_i \) the pdf for the highest value among the \( k - 1 \) ring members other than \( i \). Consider the following mechanism. The center accepts reports and recommends that the bidder with the highest report bid his report and that all other ring members bid \( v_i \).

The payment rule is

\[
p_i(r_1, \ldots, r_k, b_i, A) = \begin{cases} 
0, & \text{if } i \text{ does not win} \\
\infty, & \text{if } i \text{ wins and } b_i \neq \beta_i(r_1, \ldots, r_k) \\
\gamma_i(r_i), & \text{if } i \text{ wins and } b_i = \beta_i(r_1, \ldots, r_k),
\end{cases}
\]

where \( \gamma_i \) is defined by the differential equation

\[
\frac{\partial E_{v_{i-1}} \left( \left( r_i - \max_{j \in \{k+1, \ldots, n\}} v_j - \gamma_i(x) \right) 1_{x \geq \max_{j \neq i} v_j} \right)}{\partial x} \bigg|_{x=r_i} = 0 \tag{B.21}
\]

and the initial condition that \( \gamma_i(v) = 0 \). Note that \( \gamma_i \) is defined so that a marginal increase in ring member \( i \)’s report above his value increases his expected payoff from the auction and his expected payment to the center by equal amounts. Because the expectation in (B.21) is equal to \( G^\text{in}_i(x) \int_v^x (r_i - y - \gamma_i(x)) g^\text{out}(y) dy \), we can write (B.21) as

\[
\gamma_i'(r_i) = \frac{g^\text{in}_i(r_i)}{G^\text{in}_i(r_i)G^\text{out}_i(r_i)} \int_v^{r_i} (r_i - y) g^\text{out}(y) dy - \frac{g^\text{in}_i(r_i)G^\text{out}(r_i)_i + G^\text{in}_i(r_i)g^\text{out}(r_i)}{G^\text{in}_i(r_i)G^\text{out}_i(r_i)} \gamma_i(r_i).
\]
Clearly, for any vector of reports, it is optimal for all ring members to bid according to the recommendations of the center. To see that it is optimal for ring member \(i\) to report truthfully, suppose that all ring members other than \(i\) report truthfully and follow the recommendation of the center. Assuming outside bidders follow their weakly dominant strategy of bidding their values, ring member \(i\)'s expected payoff (not including his ex ante non-contingent payment) as a function of his report is
\[
E_{v_{-i}} \left( \left( v_i - \max_{j \in \{k+1, \ldots, n\}} v_j - \gamma_i(r_i) \right) 1_{r_i \geq \max_{j \neq i} v_j} \right).
\]
Differentiating this expression with respect to \(r_i\) and evaluating at \(r_i = v_i\), we get zero by the definition of \(\gamma_i\). Thus, a truthful report satisfies the ring member's first-order condition. We must check that the second-order condition is satisfied.

Now specialize to the case of symmetry. Then \(G_{i}^{in}(x) = F^{k-1}(x)\), \(G_{i}^{out}(x) = F^{n-k}(x)\), and
\[
\gamma'_i(r_i) = f(r_i) \frac{(k-1)}{F^{n-k+1}(r_i)} \int_{v_i}^{r_i} F^{n-k}(y)dy - f(r_i) \frac{n-1}{F(r_i)} \gamma_i(r_i).
\]
Further specializing to the case of values drawn from \(U[0,1]\), \(\gamma'_i(r_i) = \frac{k-1}{n-k+1} - \frac{n-1}{r_i} \gamma_i(r_i)\), which implies \(\gamma_i(r_i) = \frac{(k-1)r_i}{(n-k+1)n}\). Substituting in for \(\gamma_i\), ring member \(i\)'s expected payoff when he reports \(r_i\) is
\[
E_{v_{-i}} \left( \left( v_i - \max_{j \in \{v_{k+1}, \ldots, v_n\}} v_j - \gamma_i(r_i) \right) 1_{r_i \geq \max_{j \neq i} v_j} \right) = v_i r_i^{n-1} - \frac{n-1}{n} r_i^n.
\]
Differentiating with respect to \(r_i\), we get \((n-1)r_i^{n-2}(v_i - r_i)\), which is zero at \(r_i = v_i\). Differentiating again with respect to \(r_i\), we get \((n-1)r_i^{n-3}((n-2)v_i - (n-1)r_i)\), which is negative for all \(r_i > \frac{n^2}{n-1} v_i\), and positive otherwise. Thus, to see that \(r_i\) is a global optimum, we need only check that ring member \(i\)'s expected payoff is greater when he reports \(v_i\) than when he reports zero. The expected payoff from a report of \(v_i\) is \(\frac{v_i^n}{n}\), and the expected payoff with a report of zero is zero. Q.E.D.
Proof of Proposition 12. Consider the following bidding rule: if ring member $i$’s report is not highest, the center recommends a bid of $v_i$. and if ring member $i$’s report is highest, the center recommends a bid equal to the report. Note that if the ring members report truthfully, there is no incentive for any ring member to deviate from the center’s recommendation. Consider the following payment rule: if ring member $i$ does not win the auction, he pays zero, but if ring member $i$ wins the auction and pays $x$, he pays the center $p_i(r, A) = \max \{0, \max_{j \in \{1, \ldots, k\} \setminus \{i\}} r_j - x\}$. Note that if ring members report truthfully and follow the recommendations of the center, under this payment rule, the center has positive expected revenue and so can make positive ex ante non-contingent payments to the ring members.

Suppose the other $k - 1$ ring members report truthfully. If a ring member with value $v'$ reports $v' + \varepsilon$ where $\varepsilon > 0$ rather than $v'$, his payoff differs only if the highest other value in the ring is $v'' \in (v', v' + \varepsilon)$. In this case, if the ring member reports truthfully, his payoff is zero. Suppose the ring member reports $v' + \varepsilon$ and bids $v'$ at the auction (his weakly dominant strategy). If his bid loses, his payoff is zero. If his bid wins and he pays $x$, his payoff is $v' - (v'' - x) - x < 0$. Thus, the ring member has no incentive to deviate in this way.

If a ring member with value $v'$ reports $v' - \varepsilon$ where $\varepsilon > 0$ rather than $v'$, his payoff differs only if the highest other value in the ring is $v'' \in (v' - \varepsilon, v')$. Suppose this is the case and let $x$ be the highest value among bidders outside the ring. Suppose the ring member reports truthfully and bids $v'$ at the auction. Then no other ring members bid at the auction. If he does not win the auction, his payoff is zero, and if he wins the auction, his payoff is $v' - \max \{v'', x\}$. Suppose the ring member reports $v' - \varepsilon$ and bids $v'$ at the auction. Then another ring member bids $v''$ at the auction. If he does not win the auction, his payoff is zero. If he wins the auction, his payoff is $v' - \max \{v'', x\}$. All other deviations give weakly lower payoff. Thus, the ring member has no incentive to deviate.

We have shown that ring members report truthfully. It remains to show that
individual rationality is satisfied. If a bidder does not join the ring, play is non-cooperative, and a bidder expects to win if and only if his value is highest, and in that case he expects to pay the second-highest value. Given the payment rule, if the bidder joins the ring, he also expects to win if and only if his value is highest, and in that case he expects to pay a total amount equal to the second-highest value (either through payments at the auction or a combination of payments at the auction and payments to the center). Thus, taking into account the ex ante non-contingent payments, the ring members strictly prefer to join the ring. Q.E.D.

Proof of Proposition 13. Assume the use of shills is possible. Suppose there exists a profitable collusive mechanism. Then by the revelation principle, there exists a profitable collusive mechanism in which ring members truthfully report their values and follow the recommendations of the ring. Let \( \bar{\beta}(r) \) be the mechanism’s recommended bids. Suppose the mechanism requires the winner at the auction to make a payment to the center. Then any ring bidder with a positive probability of winning with the recommended bid strictly prefers to bid \( v \) and have a shill bid \( \bar{\beta}(r) \). In this case, the bidder’s expected payoff from the auction is the same, but his expected payment to the center is reduced to zero. Thus, no payment can be required of the winner at the auction. Because no payment can be required of a non-winner, no payments can be required at all. By ex ante budget balance, the center can make no payments, and so no bidder has strict incentive to join the ring. Q.E.D.
References


Cassady, Ralph (1967), Auctions and Auctioneering, Berkeley: UC Press.


