affine term structure models

An affine term structure model hypothesizes that interest rates, at any point in time, are a time-invariant linear function of a small set of common factors. This class of models has proven to be a remarkably flexible structure for examining the dynamics of default-risk free bonds, and as a result affine modelling has become the dominant framework for term structure research since the early 1980s.

The term structure of interest rates refers to the relationship between the yields-to-maturity of a set of bonds and their times-to-maturity. It is a simple descriptive measure of the cross-section of bond prices observed at a point in time. An affine term structure model hypothesizes that the term structure of interest rates at any point in time is a time-invariant linear function of a small set of common state variables or factors. Once the dynamics of the state variables and their risk premiums are specified, the dynamics of the term structure are determined.

For the term structure of interest rates to be meaningful, the bonds being compared must have similar risk and payout characteristics. The literature we examine in this article focuses on the term structure of default-risk free nominal bonds that make a single payment at a pre-specified future date—so-called zero-coupon bonds. The models described below can be applied to other types of bonds, but zero-coupon bonds are particularly important because they represent the fundamental discount rates embedded in all financial claims that make payments through time.

The literature on term structure modelling is large and reaches back to some of the giants of early 20th century economics: Fisher, Hicks, and Keynes. The pre-eminent model of the term structure, prior to the advent of affine models, was the expectations hypothesis. While the expectation hypothesis exists in a variety of forms (see Cox, Ingersoll and Ross, 1981), most researchers today use the definition of Campbell (1986) and Campbell and Shiller (1991) that the expected returns, or so-called term premiums, on default-risk-free zero-coupon bonds are constant through time. Other commonly espoused early term structure models, namely, the liquidity preference and preferred habitat theories, can be viewed as extensions of the expectation hypothesis that make additional predictions about the size of term premiums as a function of time-to-maturity. Most empirical tests of the expectations hypothesis, including Fama and Bliss (1987) and Campbell and Shiller (1991), find strong evidence against the prediction that term premiums are constant through time. This rejection of the expectations hypothesis implies that the prices of default-risk-free zero-coupon bonds embed time-varying term premiums. Explaining the dynamics of these term premiums is an important goal of affine term structure models.

Any affine term structure model starts from the assumption that there are no arbitrage opportunities in financial markets. This assumption implies the existence of a strictly positive stochastic process, \( L \), that prices all assets. (See Duffie, 2001, for a textbook treatment of the implications of absence of arbitrage for asset pricing in general and term structure modelling in particular.) This process is typically referred to as a state price deflator in continuous-time models of asset pricing or as a stochastic discount factor in discrete-time models. We follow the more common approach in the literature and develop the affine term structure models in continuous time. The existence of a state price deflator also implies that there exists a risk-neutral
measure, \( Q \), which is distinct from the physical measure, \( P \), that generates observed variation in asset prices.

Independent of any specific model of bond prices, it is always possible to express the price at time \( t \) of a zero coupon bond that matures at time \( t + \tau \) as

\[
P_t(\tau) = E_t^Q \left[ \exp \left( - \int_0^\tau r_s \, ds \right) \right],
\]

where \( E_t^Q[\cdot] \) denotes the expected value at time \( t \) under the risk-neutral measure, and \( r \) is the instantaneous rate of interest (or short rate). The short rate can be defined as

\[
r_t = \lim_{\tau \downarrow 0} \ln P_t(\tau),
\]

but it is also related to the expected value of the instantaneous rate of change of the state price deflator because

\[
dL_t = -r_t dt + \sigma_L t dW_t^Q,
\]

where \( W_t^q \) is a Brownian motion under \( Q \), \( \sigma_L(\cdot) \) is the possibly time-and state-dependent instantaneous volatility of the state price deflator, and the second term in (3) is a common shorthand notation for an Itô stochastic integral. (See Duffie, 2001, for a textbook treatment of continuous-time stochastic processes, including the definitions of Brownian motion and the Itô integral.)

As eq. (1) clearly shows, pricing zero-coupon default-risk-free bonds boils down to specifying a model for the dynamics of the short rate under the risk-neutral measure. In choosing models for \( r_t \), there are two paramount considerations: (a) a flexible specification that does a reasonable job of capturing the dynamics of proxies for the short rate (since \( r_t \) itself is unobservable), and (b) a specification that yields a convenient form for the bond prices that are the ultimate objects of interest.

The dynamic of the short rate, when modelled in continuous time, are completely determined by the drift function, which defines the instantaneous expected value of the short rate, and the diffusion function, which determines the instantaneous volatility of the short rate. What is not clear from eq. (1) is that, in order to move from the theoretical risk-neutral measure, \( Q \), to the actual (or physical measure), \( P \), that generates the observed data, a term structure model must also specify a structure for the risk premium functions controlling the transformation between the measures \( Q \) and \( P \). While the risk-neutral measure is sufficient for pricing, researchers wanting to fit affine term structure models to observed time-series data or wanting to use these models to forecast future interest rates require also the actual measure.

We can now turn to the basic building blocks (that is, short rate dynamics and market price of risk assumptions) and the main pricing results (that is, exponentially linear bond prices) of affine term structure models. We first present the main points in the context of single-factor models and then generalize the discussion to the multifactor case. Chapman and Pearson (2001), Dai and Singleton (2003), and Piazzesi (2005) are all recent, more detailed, and more technical examinations of the material that follows.

**Single-factor models**

In a single-factor affine model, the determinant of bond prices is the short rate itself. The model is constructed by specifying a continuous-time process for the short rate and a form of the risk premium function. As Cox, Ingersoll,
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and Ross (1985) note, these choices must be mutually consistent in order to avoid accidentally introducing arbitrage opportunities into a (supposedly) arbitrage-free model. The fundamental building blocks of all affine models are the single-factor models due to Vasicek (1977) and Cox, Ingersoll, and Ross (1985) (hereafter CIR).

The Vasicek model assumes that the short rate evolves as an Ornstein–Uhlenbeck process under the risk-neutral measure

\[ dr_t = \kappa(\theta - r_t)dt + \sigma dW^Q_t, \]

where \( \kappa > 0 \) determines the speed of reversion to the constant mean, \( \theta > 0 \), and \( \sigma \) is the unconditional instantaneous volatility of the process. The conditional and unconditional distributions of interest rate changes are Gaussian in this model. Accordingly, it is possible for the short rate to be negative. The risk premium function is a constant, \( \lambda_0 \), which implies that the short rate is also Gaussian under the physical measure, \( \mathbb{P} \). Solving the conditional expectation in (1) under these assumptions generates an explicit expression for the price of a default-risk free zero coupon bond

\[ P_t(t) = \exp[a(t) + b(t)r_t], \]

where

\[ a(t) = \left( \theta - \frac{\lambda_0}{\kappa} \right) + \frac{1}{2} \sigma^2 \left( t \right) \frac{1}{\kappa} \left( 1 - \exp(-\kappa t) - t \right) - \frac{\sigma^2}{4\kappa t} \left[ 1 - \exp(-\kappa t) \right]^2 \]

and

\[ b(t) = -\frac{1}{\kappa} \left[ 1 - \exp(-\kappa t) \right]. \]

Equation (5) is the first statement of an exponential-affine pricing function. It implies a simple structure where continuously compounded yields are Gaussian with constant volatility. The term structure of forward rates implied by this simple model can assume most (but not all) of the commonly observed shapes of the term structure. In particular, the term structure of forward rates can be upward sloping, downward sloping, or humped shaped, although the model cannot generate an inverted humped shape. Since prices at all maturities are driven by a single stochastic factor, this model implies that all yield levels are perfectly correlated. In the data, yield levels are very highly, but not perfectly, correlated.

In the single-factor CIR term structure model, the short rate evolves as

\[ dr_t = \kappa(\theta - r_t)dt + \sigma r_t dW^Q_t, \]

where \( \kappa > 0 \) and \( \theta > 0 \) have the same interpretation as in the Vasicek case, but the short rate is no longer Gaussian. The parameter restriction \( 2\kappa\theta \geq \sigma^2 \) is imposed in order to ensure that the short rate process does not get trapped at zero. \( r_t \) has a conditional non-central chi-square distribution (and an unconditional Gamma distribution). The instantaneous conditional variance of the short rate is linear in the level of the rate. The risk premium specification that is consistent with no-arbitrage in the single-factor CIR specification is \( \lambda(r_t) = \lambda_1 r_t \), and the no-arbitrage bond price is, again, of the form (5) with

\[ a(t) = \frac{2\kappa\theta}{\sigma^2} \log \left[ \frac{2\gamma \exp(\frac{1}{2}(\kappa + \lambda_1 + \gamma)) \left( \kappa + \lambda_1 + \gamma \right) \left( \exp(\gamma t) - 1 \right) + 2\gamma}{\left( \kappa + \lambda_1 + \gamma \right) \left( \exp(\gamma t) - 1 \right) + 2\gamma} \right] \]

\[ b(t) = -\frac{2\gamma \left( \exp(\gamma t) - 1 \right)}{\left( \kappa + \lambda_1 + \gamma \right) \left( \exp(\gamma t) - 1 \right) + 2\gamma}, \]

where \( \gamma = \sqrt{(\kappa + \lambda_1)^2 + 2\sigma^2} \). The CIR model can generate the most common
shapes of the term structure, but it still implies that all yield levels are perfectly correlated.

The Vasicek and CIR models are the most common forms of single-factor affine models, but Duffie and Kan (1996) provide the conditions on the drift, diffusion, and risk premium functions of a short rate specification, like (4) or (8), that ensure that the bond pricing function is exponential-affine under the risk neutral measure. In particular, a pricing function of the form of (5) will follow if

\[ \mu(r_t) - \lambda(r_t) = \rho_0 + \rho_1 r_t \]  \hspace{1cm} (11)

and

\[ \sigma(r_t) = \sqrt{\beta_0 - \beta_1 r_t} \]  \hspace{1cm} (12)

hold, where \( \mu(r_t) \) is a general expression for the drift of the short rate and \( \sigma(r_t) \) is a general expression for the instantaneous volatility of the short rate. For example, in the CIR case \( \rho_0 = \kappa \theta, \rho_1 = - (\kappa + \lambda), \beta_0 = 0, \) and \( \beta_1 = \sigma^2. \)

In this more general case, the \( a(t) \) and \( b(t) \) functions do not generally have explicit closed-form expressions. Rather, they are defined as the solutions to a pair of ordinary differential equations.

The empirical evidence clearly shows that a single-factor specification is not sufficient to describe the dynamics of the default-risk-free term structure. As such, empirical analysis of simple specifications, like (4) and (8), have shifted away from attempting to completely characterize yields on all maturities and, instead, have concentrated on explaining the dynamics of a proxy for the unobservable short rate. Chan et al. (1992) pioneered this approach, using a simple generalized method of moments estimation scheme. Durham (2003) is the natural evolution of this literature using state-of-the-art approximate maximum likelihood estimation. The conclusions of this literature are: (a) the evidence of mean reversion in the short rate is weak, at best, but (b) there is little consistent evidence of nonlinear mean reversion; and (c) there are complicated volatility dynamics that are not consistent with either constant volatility (Vasicek) or instantaneous conditional variances that are linear in the short rate (CIR).

**Multifactor models**

If single-factor models are insufficient to explain the observed term structure, then how many factors are needed and what are the dynamics of these factors? The common answer to the first question is provided by the analysis of Litterman and Scheinkman (1991). Using a simple principal components approach, they argue that three factors, extracted from bond yields or returns themselves, can explain well over 95 per cent of the variation in weekly changes of US Treasury bond prices, for maturities of up to 18 years. The answer to the second question – in the most general form consistent with an exponential-affine pricing function – is provided by Dai and Singleton (2000) and extended by Duffee (2002).

The multifactor affine term structure model consists of the following components. First, there is linear relation between the short rate and the factors:

\[ r_t = \delta_0 + \delta' Y_t, \]  \hspace{1cm} (13)

where \( Y_t \) denotes the \( N \)-vector of time \( t \) factor realizations. The factor dynamics conform to an affine diffusion.
\[ dY_t = \mathcal{K}(\theta - Y_t)dt + \Sigma \sqrt{S_t} dW^Q_t, \quad (14) \]

where \( \mathcal{K} \) and \( \Sigma \) are \( N \times N \) matrices (with no general restrictions) and \( S_t \) is a diagonal matrix with the \( i \)-th diagonal element equal to

\[ [S^0_t] = \lambda_i + \beta_i' Y_t. \quad (15) \]

The \( S_t \) matrix allows for the instantaneous conditional variance of the factors to be linear functions of factor levels. If every element of \( Y \) can affect the conditional volatility of every other factor, then (14) is a multifactor generalization of the CIR model from the last section. Of course, the fact that volatility is linear in the level of \( Y \) requires strong restrictions on the parameters of the model in order to ensure that variances are non-negative.

If no elements of \( Y \) affect the conditional volatility, then (14) is a multifactor generalization of the Vasicek model. If \( m < N \) factors affect the conditional volatility, then the multifactor affine model is a mixture of the CIR and Vasicek forms. Dai and Singleton (2000) define different classes of affine models by the number of factors that affect the conditional factor volatilities, with \( A_m(N) \) being the general notation for an \( N \)-factor model with \( m \)-factors driving conditional volatilities.

Under these assumptions, bond prices satisfy a multivariate generalization of (5) given by

\[ P_t = \exp[A(\tau) + B(\tau)' Y_t]. \quad (16) \]

The functions \( A(\tau) \) and \( B(\tau) \) are the solutions to the ordinary differential equations

\[ \frac{dA(\tau)}{d\tau} = -\theta' B(\tau) + \frac{1}{2} \sum_{i=1}^{N} [\Sigma' B(\tau)]^2 \lambda_i - \delta_0 \quad (17) \]

and

\[ \frac{dB(\tau)}{d\tau} = -\lambda' \beta(\tau) + \frac{1}{2} \sum_{i=1}^{N} [\Sigma' B(\tau)]^2 \beta_i - \delta. \quad (18) \]

The final component of the general multifactor affine model is the specification of the market prices of risk, which connects pricing under the risk-neutral measure to pricing under the physical measure:

\[ L_t = \sqrt{S_t} \lambda_0 + \sqrt{S_t} \lambda Y_t, \quad (19) \]

where \( \lambda_0 \) is an \( N \)-vector of constants, \( \lambda \) is an \( N \times N \) matrix of constants, and \( S_t^{-1} \) is an \( N \)-dimensional diagonal matrix with diagonal elements equal to

\[ S_t^{-1}(ii) = \begin{cases} (\lambda_i + \beta_i' Y_t)^{-1/2}, & \text{if } \inf (\lambda_i + \beta_i' Y_t) > 0; \\ 0, & \text{otherwise} \end{cases} \quad (20) \]

The first term in (19) is a straightforward generalization of the single-factor risk premium specifications: risk premiums are proportional to factor volatilities. The second component is an important source of additional flexibility in multifactor affine models. It allows these models to provide a better fit to the distribution of bond excess returns, and it is also useful in rationalizing the observed violations of the expectations hypothesis discussed above.

The general multifactor affine model can be viewed as a blending of the Vasicek and CIR forms. These extreme specifications also reveal a critical trade-off in multifactor term structure modelling. The CIR form offers the greatest flexibility in specifying the volatility dynamics of bond prices. How-
ever, this flexibility comes at a cost. The parameter restrictions that are required to ensure that (15) provides a valid description of factor variances impose substantial restrictions on the permissible correlations between the factors. In the extreme case of the pure multifactor CIR model, the factors must be uncorrelated to ensure an admissible volatility specification.

Dai and Singleton (2002), Duffee (2002) and Brandt and Chapman (2005) fit multifactor affine term structure models to more than 25 years of monthly US bond data. Each paper considers the ability of different versions of $A_m$ (3) models to both explain the rejections of the expectations hypothesis and to provide accurate forecasts of future yields. Both Dai and Singleton (2002) and Brandt and Chapman (2005) find that a Gaussian version (an $A_0$ (3) model) can rationalize the risk premiums dynamics revealed by expectations hypothesis tests. Duffee (2002) demonstrates that an $A_0$ (3) model with the expanded risk premium specification of (19) can produce more accurate yield forecasts than a random walk benchmark model.

Although the ability to explain risk premiums and yield movements is an important success for multifactor affine models, their biggest failing to date is that the favoured Gaussian specifications require that conditional yield volatilities are constant. Essentially, the flexibility in factor correlations that are required to explain these features of the data require a stochastic structure that precludes the volatility dynamics that are an equally important feature of interest rate data.

Concluding remarks

Affine models have two important strengths compared with the earlier theories of the term structure. They explicitly rule out arbitrage opportunities in the cross-section of bond prices, and they simultaneously allow for flexible specifications of term premiums and their dynamics. Weaknesses of affine models include the fact that they are typically not easy to estimate, that model specifications which can explain the rejection of the expectations hypothesis are inconsistent with observed volatility dynamics, and that there is generally limited intuition as to the economic interpretation of the factors. Ang and Piazzesi (2003) and Ang, Dong, and Piazzesi (2005) are recent attempts to combine affine term structure modelling with elements of the macroeconomy. This line of research holds out the promise of greater intuition behind the factors as well as a greater understanding of how capital markets perceive the actions of monetary authorities.

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See also

- arbitrage;
- continuous and discrete time models;
- finance;
- finance (recent developments);
- linear models, Markov processes;
- term structure of interest rates;
- Wiener process.
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Bibliography


Index terms

affine term structure models
arbitrage opportunities
bonds
Brownian motion
continuous-time models
expectations hypothesis
generalized method of moments
Itô integral
liquidity preference
maximum likelihood
multifactor models
Ornstein–Uhlenbeck processes
preferred habitat theory
principal components
single-factor models
state price deflators
term premiums
term structure of interest rates
zero-coupon bonds

Index terms not found:
preferred habitat theory