Diagnostic Accuracy Under Congestion

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Abstract

In diagnostic services, agents typically need to weigh the benefit of running an additional test and improving the accuracy of diagnosis against the cost of congestion, i.e., delaying the provision of services to others. Our paper analyzes how to dynamically manage this accuracy/congestion tradeoff. To that end, we study an elementary congested system facing an arriving stream of customers. The diagnostic process consists of a search problem in which the agent providing the service conducts a sequence of imperfect tests to determine whether a customer is of a given type. We find that the agent should continue to perform the diagnosis as long as her current belief that the customer is of the searched-for type falls in a congestion dependent interval. As congestion intensifies, the search interval should shrink. Due to congestion effects, the agent should sometimes diagnose the customer as being of a given type, even when all preformed tests are indicating otherwise. The optimal structure also implies that, when false negatives are negligible, the agent should first let the maximum number of customers allowed in the system increase with the number of performed tests. Finally, we show numerically that improving the validity of tests can sometimes decrease accuracy while faster tests can increase congestion.

Keywords: Service Operations, Queueing Theory, Dynamic Programming, Decision Making, Information Search, Bayes Rule.

1. Introduction

Diagnostic services focus on determining customer needs, but do not themselves perform any subsequent treatments that may be indicated by the diagnosis. Accumulating information and running additional tests on a customer is likely to improve the diagnosis, the accuracy of which obviously affects the value of the service offered. Accumulating and processing information, however, takes time and therefore, increases congestion in the system. The service provider thus needs to weigh the benefit of running an additional test against the cost of delaying the provision of services to others.
Triage nursing systems provide a typical example of such tradeoffs. The nurse elicits different pieces of information to assess the severity of the patient’s symptoms (Gerdtz and Bucknall 2001). On the other hand, long triage processes can result in adverse patient outcomes (Travers 1999). Another example of an accuracy/congestion tradeoff occurs at MTU Aero Engines, which is Germany’s leading provider of engine maintenance. One key decision the company needs to make is whether to keep or replace expensive parts of an engine. The diagnosis is performed by a dedicated team of workers, who may have many parts awaiting inspection. This can yield high costs of delay in an industry that is subject to intensive time-based competition. A similar task confronts those who carry out remanufacturing processes, which typically require determining whether returned parts are obsolete or not (Guide and Wassenhove 2001). Finally, the need to accurately determine a customer type under congestion also frequently occurs in some non-diagnostic systems, such as support centers and help desks (de Véricourt and Zhou 2005).

In this paper, we attempt to gain insight into this problem of dynamically balancing accuracy against delays in the process of rendering a diagnosis. To that end, we study an elementary diagnostic system facing a random stream of customers. The service consists in identifying each customer to be one of two types, $\bar{\tau}$ or $\tau$. Accuracy is defined as the probability that the customer type is correctly diagnosed. We represent the diagnostic process as a sequential test problem in which the agent performs imperfect tests one by one. Running a test consumes time, and service demands may therefore accumulate. The test result may be either “positive” or “negative”. Given a test result, the prior probability regarding the customer type is updated according to Bayes’ rule.

We focus our analysis on two classes of diagnostic processes. The first one corresponds to a search problem (Bertsekas 2007a) for which a negative test perfectly reveals type $\tau$ and terminates the process. This models practical situations where false negatives are negligible. The second class, on the other hand, yields identical and symmetrical two sided tests. A positive (negative) result imperfectly reveals type $\bar{\tau}$ (resp. $\tau$) and the probability of false positive is equal to the probability of false negative. This assumption greatly simplifies the analysis of the two sided case, while retaining the Bayesian updating mechanism.

In practice, diagnostic tests are often based on procedures and devices producing continuous measurements. Those measurements, however, often result in binary characterizations. For example, results of medical tests or quality control procedures in manufacturing are often framed as “positive/negative”, “high/low risk” or “pass/fail”, depending on whether a certain measurement is above or below a given threshold. As a result, many diagnosis procedures take the form of simple checklists or protocols (see Hales and Pronovost 2006 and references therein, for examples in the health care, manufacturing and aviation sectors). Further, tests are often treated as one sided (see the protocol designed by Breiman.
et al. (1984), which classifies heart attack patients according to risks\(^1\); see also HP Renew Program (2009) for examples in a remanufacturing setting\(^2\). More generally, stress testing of products is commonly used in many industries (such as maintenance, production and IT industries) and typically consists of sequentially checking whether a part conforms to different tolerance levels. A part fails the stress test as soon as it fails to satisfy one of these specifications. In our framework, type \(\tau\) customers correspond to non-conforming parts, while the one sided error tests correspond to the different tolerance levels.

The higher the number of tests performed, the more accurate the diagnosis becomes but the longer are the resulting service and waiting times. In order to limit delays in the system, the agent may stop the diagnostic process at any time and move to the next customer. We formulate this problem of balancing diagnostic accuracy against delays as a stochastic dynamic program. For both classes of diagnostic processes, we fully characterize the structure of the optimal decision rule that maximizes the long run average value to the service provider, which includes rewards for identifying customer type as well as costs associated with misidentification and delays.

Such decision policies yield insights into managing diagnostic services. First, we show that, under the optimal rule, the agent performs additional tests as long as the current probability of the customer being type \(\tau\) falls in a congestion dependent interval. The length of this interval decreases as more customers are waiting. Indeed, when a large number of customers are present in the system, the waiting cost is high. In this case, the requirement for an additional test becomes more restrictive. The identification accuracy is sacrificed in exchange for less waiting. Consequently, the agent should have less opportunities to run additional tests, increasing the expected cost of misidentification.

Further, we show that the agent should sometimes stop the diagnostic process and identify the customer as type \(\tau\) even when the results of the tests performed thus far have decreased the probability of being type \(\tau\). This means, for example, that a customer may be identified as type \(\tau\) even though the test results are all positive. Such a situation never occurs for search problems without congestion.

Finally, our numerical study reveals that insights from the existing queueing and sequential testing literatures do not always hold for diagnostic services subject to congestion. In particular, we show that, due to congestion effects, improving the validity of tests can decrease accuracy. Better tests can also increase congestion. Similarly, increasing the test

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\(^1\)The protocol tests in the following order, whether the blood pressure is above a pre-specified threshold, whether the patient is older than 62.5 years, or whether sinus tachycardia is present. The diagnosis process stops and classifies the patient as low risk as soon as a test result is negative. The process moves to next test otherwise (see also Goldstein and Gigerenzer 1999.). In our framework, type \(\tau\) customers correspond to the low risk patients and the diagnosis process falls into the one sided test case.

\(^2\)In particular, an incoming product needs to be replaced if it fails one of four (binary) cosmetic test.
elicitation rate can actually intensify congestion or decrease accuracy.

Research in the field of operations management has also addressed the problem of balancing congestion against the value offered to the customer. Hopp et al. (2007) propose a queueing model with Poisson demand and deterministic service time, in which the value offered to the customer is an increasing concave function of the service time. The objective is to balance the congestion related costs against the generated value. The authors show that, under the optimal policy, the service time should decrease with the level of congestion, or, equivalently, that the maximum number of customers allowed in the system should decrease with service time. Bouns (2003) study the same model, except that the service time has an Erlang distribution. The agent can then dynamically adjust the number of stages of the distribution to maximize profit. The optimal policy possesses a similar structure in that the maximum number of customers allowed in the system should decrease with the current stage of the service time. A related problem is the speed-congestion tradeoff studied by George and Harrison (2001), where the agent continuously adjust the service rate in order to minimize congestion-related and service rate costs\(^3\) (see also Crabill 1972, Crabill et al. 1977, and Stidham and Weber 1993 for earlier work on this problem). The optimal policy is shown to increase the maximum number of customers allowed in the system with the service rate.

Our diagnostic system retains most of the elements of the previous models: we assume Poisson demand, exponentially distributed service times and a waiting cost function. Our model only departs from the literature in its representation of the service process, which we describe as a sequential testing problem. This constitutes a simple framework which allows one to represent diagnostic accuracy as a conditional probability. To the best of our knowledge, we are the first to explore the dynamic value/congestion tradeoff in a context where value corresponds to the accuracy of a judgement. The structure of the optimal policy for this problem also differs from the existing findings of the previous models. In particular, when false negatives are negligible, our result implies that the maximum number of customers allowed in the system first increases with the number of performed tests. Only after enough tests have been run should the number of customers in the system decrease.

Recent research on customer behavior in congested systems has also focused on issues related to the value/congestion tradeoff. In particular, Wang et al. (2008) study patient behaviors in a call center of triage nurses, where the service corresponds to performing diagnoses (see also Anand et al. (2008) for the study of speed-quality tradeoffs with strategic customers). The service corresponds to continuous search problem that does not dynami-

\(^3\)Service rate costs can in our context be interpreted as customer disutility; the faster the service, the lower the generated value. With this interpretation, minimizing service rate cost is equivalent to maximizing value.
cally depend on congestion as it does in our study. More precisely, the class of policy they focus on corresponds in our context to static search intervals, which we also numerically study. On the other hand, their model allows exploring the impact of different system parameters on demand, an issue we do not address.

Finally, if we control for congestion and assume ample service capacity, the corresponding diagnostic system reduces to a single diagnostic task problem. The optimal decision rules for these systems without congestion are well established. Specifically, the one sided search problem is treated in Bertsekas (2007a) while the (symmetrical) two sided test problem was first solved by Edwards (1965). More generally, these two systems are special cases of sequential hypothesis testing problems (see, for instance, Wald 1947, and DeGroot 1970), which have not been studied under congestion, to the best of our knowledge.

We describe the models in the next section. The optimal decision rules for systems with congestion are characterized in Section 3, where we also contrast our findings with the search theory and queueing literatures. In Section 4 we numerically demonstrate that using more precise and faster tests can decrease accuracy or increase congestion. Section 5 concludes the paper.

2. Models of Diagnostic Services

Consider a service provider serving customers arriving according to a Poisson process with rate $\lambda$. The service consists of identifying a customer’s type, which can be either type $\bar{\tau}$ or type $\tau$. To that end, the server performs tests one by one. Each test takes an exponentially distributed time with rate $\mu$. We denote $\rho \equiv \lambda/\mu$ to be the test utilization rate, that is, the average number of arriving customers while a test is performed. The system is preemptive in the sense that a test can be stopped at any time. The number of available tests may be infinite.

A test result is positive (resp. negative) when it signals type $\bar{\tau}$ (resp. $\tau$). Tests, however, are not perfect. A positive result of the $k + 1^{st}$ test correctly identifies a type $\bar{\tau}$ customer with probability $\alpha_k$. Similarly a negative result of the $k + 1^{st}$ test correctly identifies a type $\bar{\tau}$ customer with probability $\beta_k$. Given a test result, the agent updates her prior probability on the customer’s type. We denote $p_0$ to present the base rate of type $\bar{\tau}$ customers in the population, and $p_k$ the probability of a customer being type $\bar{\tau}$ after completing the first $k$ tests. The subjective probability evolves according to the Bayes’ rule. If the next test result is positive, the posterior probability becomes

$$p_{k+1} = p_+^+(p_k) \equiv \frac{\alpha_k p_k}{\alpha_k p_k + (1 - \beta_k)(1 - p_k)}.$$  \hspace{1cm} (1)
If the result is negative, the posterior probability becomes

$$p_{k+1} = p^{-}(p_k) \equiv \frac{(1 - \alpha_k)p_k}{(1 - \alpha_k)p_k + \beta_k(1 - p_k)}.$$  \hspace{1cm} (2)

At any time, the agent needs to decide whether to run a new test or terminate the diagnosis process and proceed to the next customer. Specifically, the agent can take one of the following three actions: stop testing and identify the customer to be type $\bar{\tau}$; stop testing and announce the customer to be type $\tau$; or continue testing. Correctly identifying a type $\bar{\tau}$ customer brings value $\bar{v}$ to the system. This reward may also include the benefit of subsequent services that $\bar{\tau}$ customers may receive. Examples may include the value of repairing a part in maintenance services, or treating a patient, etc. On the other hand, missing and releasing a type $\bar{\tau}$ customer as $\tau$ incurs a misidentification cost $\bar{c}$. This corresponds to the disutility of not providing required health care or the expected cost of potential failures when the part is not repaired. On the flip side, we denote by $v$ and $c$ the reward and misidentification cost associated with a type $\tau$ customer, respectively. If the agent identifies the customer as type $\bar{\tau}$ given her current subjective probability $p$, the expected reward is equal to $\bar{r}(p) \equiv \bar{v} - (1 - p)\bar{c}$. The corresponding expected reward for type $\tau$ is $r(p) \equiv (1 - p)v - p\bar{c}$. Both $\bar{r}(p)$ and $r(p)$ are linear functions of $p$, with $\bar{r}(p)$ increasing and $r(p)$ decreasing in $p$. (See Figure 9 in Appendix B.) We can then define $\theta$ as the unique value of $p$ which satisfies $\bar{r}(p) = r(p)$. That is,

$$\theta = \frac{v + c}{\bar{v} + \bar{c} + v + c}.$$  

Critical fraction $\theta$ captures the relative value of correctly identifying type $\bar{\tau}$ customers. In particular, $\theta$ decreases with $(\bar{v} + \bar{c})/(v + c)$. We also consider a waiting cost $c_w(x)$ per unit of time which is incurred when $x$ customers are present in the system. We impose no restrictions on $c_w(x)$ other than it being strictly increasing in $x \geq 0$ with $c_w(0) = 0$ and unbounded (i.e., $\lim_{x \to \infty} c_w(x) = +\infty$).

A control policy determines the agent’s action at any point in time. The performance of a policy is then measured as the long-run average profit. For control policy $u$, we define the corresponding long run average profit $g^u$ as,

$$g^u = \lim_{T \to \infty} \frac{1}{T} E \left[ \bar{v} \tilde{N}^u(T) + v \tilde{N}^u(T) - \bar{c} \tilde{M}^u(T) - c M^u(T) - \int_0^T c_w(X^u(t)) dt \right]$$  \hspace{1cm} (3)

where $\tilde{N}^u(t)$ (resp. $\tilde{N}^u(t)$) is the random cumulative number of identified type $\bar{\tau}$ (resp. $\tau$) customers up to time $t$; similarly, $\tilde{M}^u(t)$ (resp. $M^u(t)$) is the random cumulative number of misidentified type $\bar{\tau}$ (resp. $\tau$) customers up to time $t$; and $X^u(t)$ is a ran-
dom process representing the number of customers in the system at time $t$. Therefore $E[\bar{M}^u(t) + \bar{N}^u(t)] = p_0 \lambda t$, and $E[\overline{M}^u(t) + \overline{N}^u(t)] = (1 - p_0) \lambda t$. Later we will show that the optimal policy which maximizes the average long-run profit is stationary, and we define $g^*$ to be the best long-run average profit.

Equation (3) captures an accuracy/congestion tradeoff. Indeed, given control policy $u$, accuracy $\delta^u$ is defined as the probability that the system identifies a type $\bar{\tau}$ customer given that she is of type $\bar{\tau}$. Accuracy $\bar{\delta}^u$ is the equivalent conditional probability for type $\bar{\tau}$. It follows that,

$$\bar{\delta}^u = \liminf_{T \to \infty} \frac{E[\bar{N}^u(T)]}{p_0 \lambda T} \quad \text{and} \quad \delta^u = \liminf_{T \to \infty} \frac{E[N^u(T)]}{(1 - p_0) \lambda T}.$$  
Congestion is measured by $L^u$, the average number of customers in the system, which is also proportional to the average waiting time by Little’s Law. We have,

$$L^u = \liminf_{T \to \infty} \frac{E[\int_0^T X^u(t)dt]}{T}.$$  

The long run average profit is then equal to,

$$g^u = \bar{a} \bar{\delta}^u + a \delta^u - \liminf_{T \to \infty} \frac{1}{T} E\left[\int_0^T c_w(X^u(t))dt\right] + b \quad (4)$$

where $\bar{a} \equiv \lambda p_0 (\bar{v} + \bar{c})$, $a \equiv \lambda (1 - p_0) (v + c)$ and $b \equiv \lambda p_0 (c - \bar{c}) - \lambda \bar{c}$, which holds from the conservation of flow of $\bar{\tau}$ and $\tau$ customers. In particular, the impact of policy $u$ on the system performance changes with $\bar{v}$, $v$, $\bar{c}$ and $c$ only through the total rewards $\bar{v} + \bar{c}$ and $v + c$. Note, finally, that for linear waiting cost with $c_w(x) = c_w \times x$, Equation (4) is linearly increasing in $\bar{\delta}^u$ and $\delta^u$, and linearly decreasing in $L^u$.

We can model the problem as a Markov Decision Process (MDP), with state space $(x, k, p)$, in which $x$ is the number of customers in the system, $k$ the number of completed tests, and $p$ the subjective probability of the customer in service being type $\bar{\tau}$. Such a system, however, is generally hard to analyze or compute, due to the fact that the state space is three dimensional, and includes a continuous dimension $p$. Our main analysis in the next section focuses on two important special cases. In the “one sided case” (1S), false negatives are absent. That is, $\alpha_k = 1$ for all $k$. The “two sided case” (2S), on the other hand, represents the other extreme situation, in which probabilities of false positives and false negatives are equal. That is, $\alpha_k = \beta_k = \beta$. In both cases, the state space is reduced to two dimensions of integers, from which we obtain insights.
3. Optimal Rules for Diagnostic Services with Congestion

3.1 The One Sided Test Case

In specification (1S), test validity \( \alpha_k = 1 \) for all \( k \). That is, if a customer is of type \( \tilde{\tau} \), the test result can only be positive. Therefore, a negative test always definitely signals type \( \tau \), in which case the diagnosis process stops. On the other hand, if the customer is indeed of type \( \tau \), after \( k \) positive tests, the next test result will be negative with probability \( \beta_k \).

The probability that the customer in service is of type \( \tilde{\tau} \) after \( k \) positive tests, \( p_k \), can be obtained recursively following the Bayes’ updating rule (1):

\[
p_{k+1} = \frac{p_k}{1 - \beta_k + \beta_k p_k},
\]

such that \( p_k \) is increasing in \( k \). Since \( k \) uniquely determines \( p_k \), the system state reduces to \((x, k)\), the number of customers in the system and the number of tests completed. The optimality equation is (see Appendix A for its derivation)

\[
\begin{align*}
g + J(x, k) &= \max \left\{ \mu(1 - p_k)\beta_k v - c_w(x) + \lambda J(x+1, k) + \mu(1 - p_k)\beta_k J(x-1, 0) + \\
& \quad \mu(1 - \beta_k + \beta_k p_k)J(x, k+1), \\
& \quad g + \bar{r}(p_k) + J(x-1, 0), \\
& \quad g + \bar{r}(p_k) + J(x-1, 0) \right\}, \text{ for } x \geq 1 \text{ and } k \geq 0, \\
g + J(0, 0) &= \lambda J(1, 0) + \mu J(0, 0), \text{ for } x = k = 0.
\end{align*}
\]

The first alternative on the right hand side of the Bellman equation represents the value of continue testing. In particular, the next event is a new customer arrival with probability \( \lambda \), in which case the system evolves to state \( x+1, k \). With probability \( \mu(1 - p_k)\beta_k \), the next event is a negative test result, when search stops and the customer leaves while generates value \( v \) with certainty. The new system state becomes \( x-1, 0 \). The remaining possibility for the next event is a positive test result with probability \( \mu(1 - \beta_k + \beta_k p_k) \), under which the system moves to state \( x, k+1 \). The second (third, resp.) alternative on the right hand side of the Bellman equation is to stop and identify the customer as type \( \tilde{\tau} \) \((\tau, \text{ resp.})\) with profit \( \bar{r}(p) \) \((r(p), \text{ resp.})\) defined in Section 2. Because \( p_k \) is increasing in \( k \), we can also define \( k_\theta \) as the smallest \( k \) such that \( p_k \geq \theta \). Hence, for all \( k \geq k_\theta \), we have \( \bar{r}(p_k) \geq r(p_k) \), and diagnosing the customer as \( \bar{\tau} \) is optimal, should the diagnosis process stop.

We further assume that tests are ordered such that \( \beta_k(1 - p_k) \) is nonincreasing in \( k \). The assumption holds when \( \beta_k \) is nonincreasing in \( k \). Such an assumption is not too restrictive
because, when the firm can choose the sequence in which tests are performed, the sequence that maximizes total profit can be shown to have nonincreasing $\beta_k$, which satisfies the assumption. The assumption also allows $\beta_k$ to increase in $k$, as long as not too fast.

The following theorem, which presents our first theoretical result, describes the optimal policy for the one sided case.\footnote{Theorem 1 for the one sided case can be generalized to test-dependent completion rate $\mu_k$, under the mild assumption that $\mu_k\beta_k(1-p_k)$ is nonincreasing.} Its proof is quite intricate, and is presented in Appendices B and C.

**Theorem 1.** (1S) For diagnostic systems with congestion, the optimal rule can be characterized by two queue length dependent thresholds $k(x)$ and $k(x)$, such that it is optimal to continue testing when $k(x) < k < k(x)$. Otherwise it is optimal to stop testing and identify the customer as type $\tau$ when $k \leq k(x)$, or as type $\bar{\tau}$ when $k \geq k(x)$. Furthermore, $k(x) < k_\theta \leq k(x)$ for all $x$, and $k(x)$ is nondecreasing in $x$ while $k(x)$ nonincreasing in $x$.

In other words, the agent runs the diagnosis as long as the number of performed tests belongs to a congestion-dependent interval, the length of which decreases as congestion intensifies.

Figure 1 depicts an example of such optimal policy. The dots on the grid represent states of the system in which the agent performs an additional test. The agent performs up to 10 tests when a single customer is present in the system (i.e. $x = 1$ with $k(1) = 10$ and

![Figure 1: Optimal Policy for a One Sided Diagnostic System with Congestion; $p_0 = 0.3, \beta_k = 0.5$ for $k \geq 0, \rho = 0.1, \bar{v} = 100, v = 500, \bar{c} = c = 0$ and $c_w(x) = x$.](image-url)
$\tilde{k}(1) < 0)$. The maximum number of tests, $\tilde{k}(\cdot)$, decreases then with $x$ to reach $k_0 = 4$. At the same time, lower threshold $k(\cdot)$ increases with congestion and meets $\tilde{k}(\cdot)$ at $x = 26$, such that any state for which $x > 26$ is a transient state. More generally, Theorem 1 implies that the overall maximum number of customers allowed in the system is achieved when $k = k_0 - 1$ or $k = k_0$. Note also that, in steady state, threshold $k(\cdot)$ is always crossed from the left-hand side, and never from above or below. That is, threshold $\tilde{k}(\cdot)$ is always reached with a customer arrival.

An alternative way to describe the optimal policy presented in Theorem 1 is in terms of thresholds on the number of customers in queue. Under the optimal policy, the search continues as long as the congestion level is below a given threshold. As soon as the number of customers exceed this maximum level, the search stops and the agent identifies the customer as type $\bar{\tau}$ when $k \geq k_0$ and $\tau$ otherwise. The maximum number of customers allowed in the system is then nondecreasing for $k < k_0$ and nonincreasing otherwise. This differs from the decreasing structure of optimal policies found in the existing queueing literature (Bouns 2003 and Hopp et al. 2007). On the other hand, the following corollary presents the condition under which the optimal policy retains a simple monotone structure.

**Corollary 1.** (1S) The optimal policy is fully characterized by nonincreasing thresholds $\bar{k}(x)$ if and only if $p_0 \geq \theta$. In this case, the agent performs an additional test if $k < \bar{k}(x)$ and declares the customer as type $\bar{\tau}$ otherwise.

Figure 2 illustrates Corollary 1 and describes the impact of $\theta$ on the optimal policy. As $\theta$ decreases from 0.8 to 0.2, $k_0$ decreases from 4 to 0. At the same time, the shape of the optimal policy moves towards a single decreasing threshold $\bar{k}(x)$. When $\theta = 0.2$, $k_0 = 0$, thresholds $k(\cdot)$ are degenerated and the optimal policy is only characterized by a nonincreasing threshold $\tilde{k}(\cdot)$.

When $\theta$ gets sufficiently close to zero or one, or the waiting cost increases high enough, it may become optimal to directly identify customers without performing any test. This happens when $\bar{k}(1) = \bar{k}(1) + 1 = k_0$. In this case, no diagnostic service is required and every customer is identified as type $\bar{\tau}$ if $k_0 = 0$ or $\tau$ otherwise.

Finally, Theorem 1 reveals that the agent may stop testing and identify the customer as type $\bar{\tau}$ even though all tests thus far are positive, i.e. indicate $\bar{\tau}$. This happens when after $k$ tests ($k < k_0$), the queue length $x$ increases to the extend that $k = \bar{k}(x)$. This observation, however, does not hold for the standard search problem without congestion. Indeed, the agent never identifies a customer as type $\bar{\tau}$ if her current subjective probability is greater than $p_0$, or as type $\bar{\tau}$ if the subjective probability falls below $p_0$ in the absence of congestion. In particular, the agent should never identify the customer as type $\bar{\tau}$ after a positive test. This is because the expected profit of identifying a type $\tau$ customer after $k$ positive tests,
Figure 2: Optimal Policies for a Diagnostic System with Congestion; $p_0 = 0.2, \beta_k = 0.6$ for $k \geq 0, \rho = 0.1, \bar{v} = \nu = 0, \bar{c} = 50$, and $c_w(x) = x$. 
\( r(p_k) \), is decreasing in \( k \), since \( p_k \) increases with \( k \). Thus, the profit would have been higher had the agent identified type \( \tau \) at the outset rather than after positive tests.

Hence, if we ignore congestion and manage the system as a standard search problem, one may design a control policy that ignores the possibility of diagnosing a customer as type \( \tau \) after a positive test, as is typically done in practice (to the best of our knowledge). It turns out that the best policy within this class also has nonincreasing thresholds, and is therefore optimal if and only if \( p_0 \geq \theta \) (Corollary 1). When \( p_0 < \theta \), the structure of this policy is quite intuitive and might yield a near optimal level of performance. Numerical studies, however, show that policies of this type can significantly impair performance with an average relative gap to the optimal policy exceeding 15% (see Appendix E for more details).

3.2 The Two Sided Test Case

In specification (2S), tests are symmetrical and identical with \( \alpha_k = \beta_k = \beta \). In this case, the difference between the number of positive and negative results, also referred to as the “intensity of preference”, is a sufficient statistic. The system state is then given by \((x, d)\) where \( d \) represents the intensity of preference. As \( d \) increases (resp. decreases) the customer is more likely to be of type \( \bar{\tau} \) (resp. \( \bar{\tau} \)). We denote \( p_d \) to represent the corresponding subjective probability of a customer being type \( \bar{\tau} \), where \( p_0 \) is the base rate of type \( \bar{\tau} \) before any test, or, when \( d = 0 \). Probability \( p_d \) is nondecreasing in \( d \), and can be obtained recursively following the Bayes’ updating rule (1) and (2),

\[
p_{d+1} = \frac{\beta p_d}{\beta p_d + (1 - \beta)(1 - p_d)} \quad \text{or, equivalently,} \quad p_{d-1} = \frac{(1 - \beta)p_d}{(1 - \beta)p_d + \beta(1 - p_d)},
\]

(7)

While tests correctly identify a customer’s true type with the same probability \( \beta \), we do not assume the cost structure to be symmetric. Therefore \( \bar{c} \) and \( \bar{\bar{c}} \) do not necessarily equal to \( \bar{c} \) and \( c \), respectively. Similar to the one sided case, we can define \( d_\theta \) as the smallest \( d \) such that it is better to diagnose the customer as type \( \bar{\tau} \), or, \( d_\theta \equiv \min\{d; p_d \geq \theta\} \).

The Bellman’s equation is,

\[
g + J(x, d) = \max \left\{ -c_w(x) + \lambda J(x+1, d) + \mu (p_d \beta + (1 - p_d)(1 - \beta)) J(x, d+1) + \mu (p_d(1 - \beta) + (1 - p_d)\beta) J(x, d-1), \right. \\
\left. \quad g + \bar{\tau}(p_d) + J(x-1, 0), \quad g + \bar{\tau}(p_d) + J(x-1, 0) \right\} \quad \text{for} \ x \geq 1 \ \text{and any} \ d, \quad (8)
\]

\[
g + J(0, 0) = \lambda J(1, 0) + \mu J(0, 0) \quad \text{for} \ x = d = 0 .
\]

The three alternatives in the above Bellman equation are similar to those in the
Bellman equation (6). It is, however, worth mentioning a distinction between the two systems. In the one sided case, a negative test result automatically implies customer identification ($τ$), and therefore the corresponding reward $v$ is considered as a possible consequence of the “continuation” alternative. Such a treatment allows $k$ to be a sufficient statistic, and the state space expressed in $x$ and $k$. In the two sided case, however, when the next event is a negative test result (with probability $\mu(p_d(1 - \beta) + (1 - p_d)\beta)$, the system evolves to state $x, d - 1$. All possible rewards are obtained through $\bar{r}(p)$ and $\bar{r}(p)$ from actively stopping the search and identifying the customer to be one type or the other.

The characterization of the optimal policy follows steps similar to the one sided case (see Appendices B and C). This leads to the following result,

**Theorem 2.** (2S) For diagnostic systems with congestion, the optimal rule can be characterized by two queue length dependent thresholds $\bar{d}(x)$ and $\bar{d}(x)$, such that it is optimal to continue testing when $d(x) < d < \bar{d}(x)$. Otherwise it is optimal to stop testing and identify the customer as type $\tau$ when $d \leq \bar{d}(x)$, or as type $\bar{\tau}$ when $d \geq \bar{d}(x)$. Furthermore, $\bar{d}(x) < d_\theta \leq \bar{d}(x)$ for all $x$, and $\bar{d}(x)$ is nondecreasing in $x$ while $\bar{d}(x)$ is nonincreasing in $x$.

Hence, the optimal policy corresponds to a queue length dependent interval, similar to the one sided case, albeit in the intensity of preference $d$ instead of the number of tests $k$. The length of this interval decreases as the number of customers in the system increases. Similar to the one sided case, the optimal policy can also be stated in terms of limits on the congestion level. The agent continues the search as long as the queue length is below a given threshold. As soon as the number of customers in the system exceeds this threshold, the agent identifies the customer as $\tau$ if the intensity of preference is lower than $d_\theta$, and as $\bar{\tau}$ otherwise. Furthermore, the maximum number of customers allowed in the system is unimodal and achieves its maximum at either $d_\theta$ or $d_\theta - 1$.

Figure 3 depicts an example of the optimal policy. Under the optimal policy, the agent may identify the customer as type $\bar{\tau}$ even though all tests thus far are negative. For instance, in Figure 3, the agent stops the diagnosis process and identifies the customer as type $\bar{\tau}$ if the first test returns a negative result such that $d = -1$, while the queue length reaches $x = 7$. For the same system without congestion, agents would never stop the process and diagnose the customer as type $\bar{\tau}$ for negative values of $d$, that is, when $p_d < p_0$. This is because the cost of performing an additional test remains constant in the standard search problem but increases with the queue length in congested diagnosis services.

### 3.3 Congestion-Dependent Intervals of Probabilities

We conclude this section by stating our results in terms of subjective probabilities. This framework provides a unified description of Theorems 1 and 2, and helps derive a conjecture
Figure 3: Optimal Policy for a Two Sided Diagnostic System with Congestion; \( p_0 = 0.3 \), \( \beta = 0.6 \), \( \rho = 0.3 \), \( \bar{v} = 1000 \), \( v = 200 \), \( \bar{c} = c = 0 \) and \( c_w(x) = x \).

for the optimal policy of the general case with asymmetric double sided tests.

Since \( p_k \) and \( p_d \) are monotone in \( k \) and \( d \), we can represent the state space using the subjective probability \( p \) instead of \( k \) and \( d \), for cases (1S) and (2S), respectively. This leads to a unified representation of the system in states \((x, p)\), the number of customers in the system and the subjective probability for the customer in service to be of type \( \bar{\tau} \).

Due to the one to one correspondence between \( p_k \) (resp. \( p_d \)) and \( k \) (resp. \( d \)), Theorems 1 and 2 imply that the optimal policies for both cases can be represented as a congestion dependent intervals of subjective probabilities. That is, for each queue length \( x \), there are two thresholds, \( \overline{p}(x) \) and \( \underline{p}(x) \), such that it is optimal to continue testing if the current subjective probability \( p \) falls in the interval \((\underline{p}(x), \overline{p}(x))\). Otherwise, it is optimal to stop and identify the customer as type \( \bar{\tau} \) when \( p \geq \overline{p}(x) \), or \( \tau \) when \( p \leq \underline{p}(x) \). Furthermore, \( \underline{p}(x) \) is nondecreasing and \( \overline{p}(x) \) is nonincreasing in \( x \). The length of the interval for continuing testing, \( \overline{p}(x) - \underline{p}(x) \), is therefore nonincreasing in \( x \).

Figure 4 depicts the example as illustrated in Figure 1, in the \((x, p)\) space. Note that the agent is indifferent between type \( \bar{\tau} \) and \( \tau \) when \( p = \theta \). This is also where the value of additional information is the highest. Hence, when the subjective probability \( p \) is close to \( \theta \), the agent is willing to bear higher congestion costs for additional tests. On the other hand, when \( p \) is away from \( \theta \), the value of additional tests is low and the agent aborts the search for low levels of congestion. Thus, the optimal policy takes the form of an interval around \( \theta \), which shrinks as congestion intensifies.
Finally, our findings suggest a structure for the optimal policy of the case where $\alpha_k \neq \beta_k$. Indeed, a generalization of the previous result consists in considering two monotone thresholds in $p$, $\bar{p}(x, k)$ and $p(x, k)$ which depend on both queue length and number of performed tests. For a given number of tests $k$, the length of continuation interval, $\bar{p}(x, k) - p(x, k)$, should then also decrease in $x$.

4. Numerical Study

When making a diagnosis, current congestion levels are typically ignored in practice. Instead, management often resorts to more efficient and faster tests. In the following, we show numerically that not limiting congestion in diagnostic services can significantly impair performance. Further, improving test validity can sometimes decrease accuracy, a phenomenon which never occurs in the standard search problem without congestion. We also show that improving service speed (test completion rate) may not necessarily reduce congestion under the optimal policy.

4.1 The Cost of Not Limiting Congestion

The optimal rule for systems with congestion dynamically limits the maximum number of customers present in the system, which requires tracking the current level of congestion. An alternative, simpler approach commonly seen in practice consists in fixing the maximum number of performed tests or particular test results regardless of how many people are waiting. This means, for instance, using the same check lists or protocols independently of
the congestion level. We refer to heuristics of this type as fixed threshold policies.

In the one sided case, a fixed threshold policy stops testing a customer only when a negative test result comes back, or the number of completed tests \( k \) has reached a fixed threshold. In the two sided case, a fixed threshold policy is characterized by two thresholds on \( d \), which are non-positive and non-negative, respectively. A customer is released only when \( d \) reaches either threshold. In other words, a fixed threshold policy directly applies the optimal structure for a standard search problem without congestion, independently of the number of waiting customers. This means, in particular, that a fixed threshold policy never diagnoses a customer as type \( \bar{\tau} \) (resp. \( \tau \)) after a negative (resp. positive) test. This also corresponds to the type of static policies studied by Wang et al. (2008) in the context of continuous diagnosis processes.

In the following, we consider the optimal rule within this class of policies, i.e., we evaluate the best threshold values, accounting for waiting costs. We also analyze the relative gap \( \Delta \equiv |(g^* - g^{FT})/g^*| \), where \( g^* \) and \( g^{FT} \) denote the average profits of the optimal policy and the best fixed threshold policy, respectively. We obtain the optimal policy and \( g^* \) numerically using the standard relative value iteration algorithm (Bertsekas 2007a, Chapter 7.4). The average profit of the best fixed threshold policy can be evaluated directly using M/G/1 queueing formulas without value iteration. We present the details in Appendix F.

In the following, we set \( \beta_k = \beta \) and \( \mu_k = \mu \) for all \( k \). Furthermore, we set waiting costs to be linear with \( c_w(x) = x \). This corresponds to normalizing the different rewards and mismatch costs. Our model thus has seven degrees of freedom \( (p_0, \beta, \bar{v}, \bar{c}, v, c, and \rho) \).

We choose model parameters \( \beta, p_0, \rho, \bar{c} + \bar{v} \) and \( \theta \) according to Table 1. The values of \( \theta \) and \( \bar{c} + \bar{v} \) determine the value of \( \bar{c} + \bar{v} \). Recall that the optimal policy does not change with \( \bar{c}, \bar{v}, \bar{c}, \bar{v} \) when \( \bar{c} + \bar{v} \) and \( \bar{c} + \bar{v} \) remain constant. The corresponding optimal profit, however, varies. We therefore consider four extreme cases: \( \bar{c} = \bar{c} = 0, \bar{v} = \bar{v} = 0, \bar{c} = \bar{v} = 0 \) and \( \bar{c} = \bar{v} = 0 \). This leads to a total of 6400 cases \( (= 4^3 \times 5^2 \times 4) \). Among these cases, a total of 5236 have a non-degenerate optimal policy for the one sided system. For the two sided system there are 3544 such cases out of the 6400 total cases.

Table 2 reports the summary statistics of the relative gaps of the best fixed-threshold policy for the 5236 one sided cases and 3544 two sided cases. The table reveals that although

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_0 )</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
</tr>
<tr>
<td>( \bar{c} + \bar{v} )</td>
<td>10</td>
<td>20</td>
<td>50</td>
<td>100</td>
</tr>
<tr>
<td>( \theta )</td>
<td>0.1</td>
<td>0.3</td>
<td>0.5</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Table 1: Models parameters for the numerical study.
Table 2: Distribution of the fixed threshold policy relative sub-optimality gap.

<table>
<thead>
<tr>
<th></th>
<th>10th</th>
<th>25th</th>
<th>50th</th>
<th>75th</th>
<th>90th</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>one sided</td>
<td>0.32%</td>
<td>1.87%</td>
<td>7.77%</td>
<td>25.65%</td>
<td>80.58%</td>
<td>21.43%</td>
</tr>
<tr>
<td>two sided</td>
<td>0.70%</td>
<td>2.32%</td>
<td>6.67%</td>
<td>18.84%</td>
<td>100%</td>
<td>20.41%</td>
</tr>
</tbody>
</table>

in 25% of the cases the fixed threshold policy performs very well (with sub-optimality below 2.5%), generally speaking, the fixed threshold policy does not perform well, with an average sub-optimality above 20% among our test cases. In the worst 10% cases, failing to control the system dynamically can be quite costly.

4.2 Improving Service Capability

The firm has several levers it can use to improve the service capability of diagnostic systems. For example, an increase in the elicitation rate or test validity should make it possible to identify customer types more quickly. In this section, we explore analytically and numerically the impact of such changes on accuracy and congestion.

4.2.1 Test Validity

In practice, firms can train their employees or introduce new pieces of equipment to increase test validity. The intended outcome is often to provide greater accuracy. With congestion, however, the optimal accuracy measure may not always increase with test validity. Figure 5(a) depicts the impact of test validity $\beta$ (with $\beta_k = \beta$ for all $k$) on type $\tau$ accuracy measure $\delta^*$ in a single sided case. In particular, as $\beta$ increases from 0.7 to 0.75, for example, the probability of correctly identifying a type $\tau$ customer under the optimal policy drops from 87% to 75%, which is due to the sudden drop of the accuracy measure when $\beta$ crosses value 0.713. Note that type $\bar{\tau}$ accuracy is not affected at this point, as shown in Figure 5(b). The average queue length, on the other hand, decreases from 0.19 to 0.11, offsetting the worsening of type $\tau$ accuracy (Figure 5(c)). These changes in accuracy and congestion measures can further be explained by changes in the underlying optimal policies. For instance, as $\beta$ increases right above 0.713, the corresponding policy decreases the maximum number of performed tests when a single customer is present in the system (similar shifts occur when $\beta$ is around 0.565 and 0.806 as detailed in Appendix D).

We observe decreases in accuracy for higher test validity $\beta$ in the two sided case as well. Figure 6(a), for example, depicts the effect of test validity on type $\tau$ accuracy for the two sided case, with $\rho = 0.4$, $\bar{c} + \bar{v} = 500$, $c + v = 200$, $c_w(x) = x$ and $p_0 = 0.45$. Here, again, better tests can impair an accuracy measure. When $\beta$ is low (close to 0.5), it is optimal to diagnose all customers as type $\bar{\tau}$ without performing any test. Therefore, 100% of the type $\bar{\tau}$ customers are identified. Consequently, the other accuracy measure is null (Figure 6(b)).
Figure 5: Impact of Test Validity $\beta$ on Accuracy and Congestion Measures (1S). $\rho = 0.1$, $\bar{c} = c = 0$, $\bar{v} = 5$, $v = 50$, $c_w(x) = x$ and $p_0 = 0.9$. 
and no congestion is present (Figure 6(c)). As $\beta$ increases up to 0.675, however, accuracy measure $\delta^*$ demonstrates a general decreasing trend with $\beta$. Figures 6 shows that in this case, the decrease in $\tau$ accuracy is offset by the increase in $\bar{\tau}$ accuracy measure but not a reduction in queue length. The increase in congestion when $\beta$ increases from 0.5 to 0.75 in Figure 6(c), also reveals that congestion under the optimal policy can, in fact, intensify with higher test validity.

### 4.2.2 Test Completion Rate

The classical approach to reduce waiting times in congested systems is to improve service rate through hiring more experienced employees, training them, improving technology, etc. However, under the optimal policy, improving the service rate can sometimes intensify congestion.

Figure 7 depicts this phenomenon for the one sided case, with $\beta = 0.6$, $\bar{c} + \bar{v} = 500$, $\bar{c} + \bar{v} = 400$, $c_w(x) = x$ and $p_0 = 0.4$. When $\rho$ increases from 0.42 to 1, the average queue length in the system decreases from above 4 to about 2.65. For the two sided case, Figure 8 shows that the average queue length decreases from 3.2 when $\rho = 0.31$ to 1.93 when $\rho = 0.99$, with $\beta = 0.7$, $\bar{c} + \bar{v} = 1000$, $\bar{c} + \bar{v} = 400$, $c_w(x) = x$ and $p_0 = 0.5$. The intuition behind this effect is that an increase in $\mu$ allows one to elicit more cues. This can improve accuracy but also may intensify congestion. Similar to Figure 5(a), the drops of congestion measures in Figures 7 and 8 are due to changes in the corresponding policies.

Hopp et al. (2007) also demonstrated that increasing service capacity can increase congestion in their framework. The general reason driving their result is that, given a particular tradeoff between different performance measures in the objective, it may be optimal to improve one measure at the expense of others when changing the system parameters. The same logic holds in our context for the effect of improving test validity and completion rate.

### 5. Conclusion

This paper is the first to study how to dynamically perform diagnoses under time pressure in the form of congestion. We formulate this problem as a Markov Decision Process and demonstrate that the agent should continue performing tests as long as her subjective probability belongs to a given interval, the size of which decreases as congestion intensifies. As a result, the agent should sometimes stop the diagnosis process and identify the customer as a given type even when all tests performed thus far indicate otherwise. Decisions like these never occur in diagnostic systems with no congestion. For the one sided case, our results further imply that the agent should let the congestion level first increase with the number of performed tests. Only when enough tests have been run should the number of customers in the system decrease.

Natural extensions of our system should help explore many other questions related to
Figure 6: Impact of Test Validity $\beta$ on Accuracy and Congestion Measures (2S). $\rho = 0.4$, $\bar{c} = c = 0$, $\bar{v} = 500$, $v = 200$, $c_w(x) = x$ and $p_0 = 0.45$. 
Figure 7: Impact of Service Rate on Congestion (1S). $\beta = 0.6$, $\bar{c} = \bar{c} = 0$, $\bar{v} = 400$, $\underline{v} = 500$, $c_w(x) = x$ and $p_0 = 0.4$.

Figure 8: Impact of Service Rate on Congestion (2S). $\beta = 0.7$, $\bar{c} = \bar{c} = 0$, $\bar{v} = 1000$, $\underline{v} = 400$, $c_w(x) = x$ and $p_0 = 0.5$. 

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forming judgements under congestion. For instance, the diagnostic process could include non-homogeneous costs of performing tests. Multi-server systems are also relevant as they may shed light on staffing rules for diagnostic services. In addition, agents may be in charge of both making the diagnosis and taking follow-up actions, which would raise the design problem of striking the right balance between providing the diagnostic and subsequent services.

On the other hand, these extensions do not always yield tractable models. For instance, our preliminary results suggest that the optimal policy does not have a simple structure when accounting for unequal false negative and positive probabilities, or when considering more than two customer types. In such settings, implementing an optimal policy appears very challenging in practice. Some of our general findings, however, should hold. In particular, the region of the state space where the agent continues testing should shrink as congestion intensifies. Also, because of congestion, the system may still diagnose the customer to be of a given type, even though all tests performed thus far indicate another type. Another fruitful direction consists in exploring simple heuristics that perform well. This would allow one to analyze more general systems and answer other important managerial questions for diagnostic systems, such as capacity planning.

Finally, our approach constitutes a very promising framework for understanding how individuals make actual decisions when tasks can accumulate. In fact, extensions of our diagnostic process have actually been proposed to represent how individuals decide (Busemeyer and Rapoport 1988). Psychologists have long recognized the importance of time pressure in human decision making. However, the cognitive environments psychologists consider are not typical of service organizations. In particular, situations where time pressure takes the form of accumulation of tasks have systematically been ignored. Our model naturally lends itself to experimental studies. Our results also offer a normative benchmark against which performances can be compared.
References


Appendix

Appendices A to C contain the proofs to Theorems 1 and 2, while Appendices D to F provide supporting materials to the different numerical examples and studies of the paper.

For the proofs, we first provide the derivation to obtain the optimality equation (6) in Appendix A. We then focus on the corresponding infinite horizon total discounted models in Appendix B, and show various properties of the value function which are required for the structural results. Then, in Appendix C, we extend the results to the long run average case, which complete the proof.

A. Derivation of the Optimality Equations

We construct Equation (6) which is the optimality equation for the one sided model. The derivation of Equation (8) for the two sided model is very similar. Consider the congested system with one sided tests. At state \((x, k)\), the set of actions is \(A = \{0, 1, \ldots, x\}\), which represents the number of customers simultaneously removed from the system. Action \(a = 0\) also states to keep the customer in service. Let \(J(x, k)\) represent the value function.

Following uniformization, the optimality equation for maximizing the long run average profit becomes

\[
g + J(x, k) = \max_{a \in A} Q(x, k, a)
\]

in which

\[
Q(x, k, 0) = \mu (1 - p_k) \beta_k v - c_w(x) + \lambda J(x + 1, k) + \mu (1 - p_k) \beta_k J(x - 1, 0) + \\
\mu (1 - \beta_k + p_k \beta_k) J(x, k + 1),
\]

\[
Q(x, k, a) = \max \{\bar{r}(p_k), r(p_k)\} + (a - 1) \max \{\bar{r}(p_0), r(p_0)\} + \mu (1 - p_0) \beta_0 v - c_w(x - a) + \\
\lambda J(x - a + 1, 0) + \mu (1 - p_0) \beta_0 J(x - a - 1, 0) + \\
\mu (1 - \beta_0 + p_0 \beta_0) J(x - a, 1), \text{ for } 1 \leq a < x,
\]

\[
Q(x, k, x) = \max \{\bar{r}(p_k), r(p_k)\} + (x - 1) \max \{\bar{r}(p_0), r(p_0)\} + \lambda J(1, 0) + \mu J(0, 0).
\]

Note that \(Q(x, k, a) = \max \{\bar{r}(p_k), r(p_k)\} + Q(x - 1, 0, a - 1)\), which further implies that for any \(k \geq 0\),

\[
g + J(x - 1, 0) + \max \{\bar{r}(p_k), r(p_k)\} = \max \{\bar{r}(p_k), r(p_k)\} + \max_{a:0 \leq a \leq x - 1} Q(x - 1, 0, a)
\]

\[
= \max_{a:0 \leq a \leq x - 1} Q(x, k, a + 1)
\]

\[
= \max_{a:1 \leq a \leq x} Q(x, k, a),
\]

which implies Equation (6).
B. Analysis of the Total Discounted Profit Model

In this appendix, we consider a different version of the models in Section 3, in which the objective is to maximize the expected total discounted profit of the infinite horizon MDP. The proofs focus on establishing value function properties that enable the structures of the optimal policy. We let $\gamma$ represent the discount rate, and assume, without loss of generality, that $\lambda + \mu + \gamma = 1$ in both one sided and two sided models. Later, in Appendix C, we show that all the results from this section can be extended to the corresponding long-run average profit models by letting $\gamma$ go to zero.

In particular, we first show, in Appendix B.1, that although the state space is countably infinite, under the optimal policy, the recurrent states must be finite, for both (1S) and (2S). The boundedness of the recurrent state space is important for the long run average proof in Appendix C. Then, Appendices B.2 and B.3 present the value function properties for (1S) and (2S), respectively.

First, similar to Appendix A, we can derive the optimality equation for the one sided case as follows,

$$
\Gamma J(x, k) = \max \left\{ \mu(1 - p_k)\beta_k v - c_w(x) + \lambda J(x + 1, k) + \mu(1 - p_k)\beta_k J(x - 1, 0) + \mu(1 - \beta_k + p_k\beta_k)J(x, k + 1), \bar{r}(p_k) + J(x - 1, 0), r(p_k) + J(x - 1, 0) \right\}, \text{ for } x \geq 1 \text{ and } k \geq 0 ; (9)
$$

$$
\Gamma J(0, 0) = \lambda J(1, 0) + \mu J(0, 0).
$$

The optimality equation for the two sided case can be represented as,

$$
\Gamma J(x, d) = \max \left\{ -c_w(x) + \lambda J(x + 1, d) + \mu(\beta p_d + (1 - \beta)(1 - p_d))J(x, d + 1) + \mu((1 - \beta)p_d + \beta(1 - p_d))J(x, d - 1), \bar{r}(p_d) + J(x - 1, 0), r(p_d) + J(x - 1, 0) \right\}, \text{ for } x \geq 1 \text{ and } d ; (10)
$$

$$
\Gamma J(0, 0) = \lambda J(1, 0) + \mu J(0, 0).
$$

The two linear functions $\bar{r}(p)$ and $r(p)$ represent the expected profit of announcing the customer as type $\bar{r}$ and $r$, respectively, when the subjective probability is $p$. $\bar{r}(p)$ is increasing in $p$, while $r(p)$ decreases with $p$. Figure 9 illustrates these two functions for $\bar{v} = 5$, $\bar{c} = 4$, $\bar{v} = 7$, and $\bar{c} = 2$. The two functions, shown by the dashed lines, intersect at $p = \theta$, and their upper envelope is marked by the solid line.

B.1 Boundedness of the State Space

Define $\hat{c} = \inf_{x \geq 0} \{c_w(x + 1) - c_w(x)\}$ and $v_0 = (1 - p_0)\bar{v} + p_0\bar{v}$. Next, we prove that when the total discounted profit objective is considered with small enough discount rate, any
optimal policy traverses through only a finite number of states which can be represented in a bounded box.

Lemma 1. (1S) For the one sided model, let \( \hat{k} \) be the smallest value of \( k \) such that \( (1-p_k) < \frac{\hat{c}}{\hat{v}+\hat{c}} \). Any policy \( u \) that tests a customer at state \((x, k)\) with \( k \geq \hat{k} \) can not be optimal.

Proof: Consider state \((x, k)\) with \( k \geq \hat{k} \) such that according to policy \( u \), the agent continues testing for a period of length \( T_c \). Note that \( T_c \) is a random variable, and \( T_c \geq T_1 \), where \( T_1 \) is the random variable representing the time until the next event (arrival of a new customer or termination of a test). Moreover, \( T_1 \) is exponentially distributed with rate \( \lambda + \mu \).

Now consider another policy \( u' \) which is the same as \( u \) other than at state \((x, k)\) where, instead of conducting more tests, it does the followings: (i) releases the current customer and announces her as type \( \tilde{\tau} \); (ii) does not do anything during the next \( T_c \) period of time while simply keeping the remaining customers waiting; (iii) follows policy \( u \) afterward.

We want to show, using a sample path argument, that the expected profit under policy \( u' \) is always higher than policy \( u \) and hence, \( u \) can not be optimal. Since \( u \) and \( u' \) are exactly the same after \( T_c \), it suffices to compare their profit only during \( T_c \). First, note that the size of the queue is always the same under both policies. More precisely, the queue length at each point in time during \( T_c \) consists of \( x - 1 \) customers already in the queue, plus the new arrivals to the system up to that point. Thus, the queue-related waiting costs incurred to the system is equal for both policies. What is different for \( u \) and \( u' \) is the waiting cost.
for the customer under service and the expected profit that the system obtains from this customer.

Under policy $u$, the presence of the customer under service during $T_c$ incurs an additional holding cost which is at least:

$$\int_0^{T_c} \hat{c} e^{-\gamma t} \, dt = \frac{\hat{c}}{\gamma} \left(1 - e^{-\gamma T_c}\right)$$

Furthermore, we have:

$$T_c \geq T_1 \Rightarrow E\left[e^{-\gamma T_c}\right] \leq E\left[e^{-\gamma T_1}\right] \leq \lambda + \mu \Rightarrow E\left[\frac{\hat{c}}{\gamma} \left(1 - e^{-\gamma T_c}\right)\right] \geq \frac{\hat{c} \left(1 - \lambda - \mu\right)}{\gamma} = \hat{c}$$

The expected profit obtained from the current customer is at most $E\left[e^{-\gamma T_c} \left((1 - p_k)\bar{v} + p_k\tilde{v}\right)\right]$ under policy $u$, which is further bounded from above by:

$$E\left[e^{-\gamma T_c} \left((1 - p_k)\bar{v} + p_k\tilde{v}\right)\right] \leq (\lambda + \mu) \left((1 - p_k)\bar{v} + p_k\tilde{v}\right)$$

On the other hand, under policy $u'$, no additional holding cost is incurred as the current customer is released at the beginning of the period. Further, the expected profit obtained from this customer is $\bar{r}(p_k) = p_k\tilde{v} - (1 - p_k)\bar{c}$.

When $k \geq \hat{k}$, we have:

$$(1 - p_k)(\bar{v} + \bar{c}) < \hat{c}$$

Adding the above inequality with $(\lambda + \mu)p_k\tilde{v} < p_k\tilde{v}$ and then subtracting $\gamma(1 - p_k)\bar{v}$ from the left hand side gives us:

$$(\lambda + \mu) \left((1 - p_k)\bar{v} + p_k\tilde{v}\right) - \hat{c} < p_k\tilde{v} - (1 - p_k)\bar{c} = \bar{r}(p_k)$$

This shows that $u'$ is preferred over $u$, and hence $u$ cannot be optimal.

Lemma 2. (2S) For the two sided model, let $\bar{d}$ be the largest value of $d$ such that $p_d < \hat{c}/(\bar{v} + \bar{c})$. Similarly, let $\bar{d}$ be the smallest value of $d$ such that $(1 - p_d) < \hat{c}/(\bar{v} + \bar{c})$. Any policy $u$ that tests a customer at state $(x, d)$ with $d \leq \bar{d}$ or $d \geq \bar{d}$ can not be optimal.

Proof: Proof is exactly similar to that of Lemma 1. For $d \leq \bar{d}$, policy $u$ is outperformed by a policy $u'$ which announces the customer as type $\bar{v}$ instead of conducting more tests.

For $d \geq \bar{d}$, policy $u$ is outperformed by a policy $u'$ which announces the customer as type $\bar{v}$ instead of conducting more tests.
Lemmas 1 and 2 bound the number of performed tests in the one sided case and the intensity of preference in the two sided case. As a result, we can introduce \( \bar{w} \) as a finite upper bound on the expected service time when the system operates under the optimal policy. The next lemma bounds the number of customers in the system for both one sided and two sided models with \( \gamma \) small enough.

**Lemma 3.** (1S and 2S) If \( r(p_0) \geq 0 \), let \( \bar{\gamma} = 1 \), and \( \bar{x} = \lceil \frac{\lambda v_0 \bar{w} + (\bar{v} + \bar{c}) + (1 - \lambda) r(p_0)}{\bar{c}} \rceil \); if \( r(p_0) < 0 \), let \( \bar{\gamma} = -\bar{c} / (2r(p_0)) \), and \( \bar{x} = \lceil \frac{2(\lambda v_0 \bar{w} + (\bar{v} + \bar{c}) - \lambda r(p_0)) - 1}{\bar{c}} \rceil \).

Suppose \( \gamma < \bar{\gamma} \). Any policy \( u \) for the one sided model that tests a customer at state \((x, k)\) with \( x \geq \bar{x} \) can not be optimal. Similarly, any policy \( u \) for the two sided model that tests a customer at state \((x, d)\) with \( x \geq \bar{x} \) can not be optimal.

**Proof:** We present the proof for the one sided case. The proof for the two sided case is exactly similar. Consider state \((x, k)\) with \( x \geq \bar{x} \) such that according to policy \( u \), the agent continues testing for a period of length \( T_c \). Note that \( T_c \) is a random variable with \( E[T_c] \leq \bar{w} \) and \( T_c \geq T_1 \), where \( T_1 \) is the random variable representing the time until the next event (arrival of a new customer or termination of a test). Moreover, \( T_1 \) is exponentially distributed with rate \( \lambda + \mu \).

Let \( x_{T_c} \) denote the number of arrivals to the system during \( T_c \). We have \( E[x_{T_c}] = \lambda E[T_c] \). By ignoring the holding cost for these new customers, we have the following lower bound on the holding cost incurred to the system during \( T_c \) under policy \( u \):

\[
\int_0^{T_c} c_w(x)e^{-\gamma t}dt = \frac{c_w(x)}{\gamma} \left( 1 - e^{-\gamma T_c} \right)
\]

Moreover, the discounted profit obtained from the current customer is bounded from above by \( e^{-\gamma T_c} \left( (1 - p_k) \bar{w} + p_k \bar{v} \right) \) (as shown in the proof of Lemma 1). The state of the system is \( (x + x_{T_c} - 1, 0) \) after \( T_c \).

On the other hand, consider a policy \( u' \) which releases the current customer now and announces her as type \( \tilde{z} \), gets the profit of \( r(p_k) \) and reaches state \( (x - 1, 0) \). It then follows policy \( u \) at state \( (x - 1, 0) \).

We still can not compare the performance of \( u \) and \( u' \) because the system is not in the
same state under these two policies. If policy \( u \), after releasing the current customer, could identify the first \( x_{Tc} \) customers in line in zero time and get the maximum profit out of them (\( v_0 \) per customer), then the system would obtain a profit of \( e^{-\gamma T_{c} \left(x_{T_{c}}v_{0}\right)} \), and would transit to state \((x - 1, 0)\). Although this can never happen under any policy, but it can be used to give an upper bound on the performance of \( u \) for our purpose.

Putting all together, this means that under policy \( u \) and after \( T_{c} \) period of time, the system obtains a profit which is bounded from above by:

\[
e^{-\gamma T_{c} \left(x_{T_{c}}v_{0} + (1 - p_k)v + p_k\bar{v}\right)} - \frac{c_w(x)}{\gamma} \left(1 - e^{-\gamma T_{c}}\right)
\]

and reaches state \((x - 1, 0)\). Whereas, policy \( u' \) obtains the profit of \( r(p_k) \) and enters state \((x - 1, 0)\) right away. In other words:

\[
J^u(x, k) \leq E \left[e^{-\gamma T_{c} \left(x_{T_{c}}v_{0} + (1 - p_k)v + p_k\bar{v}\right)} + e^{-\gamma T_{c}}J^u(x - 1, 0) - \frac{c_w(x)}{\gamma} \left(1 - e^{-\gamma T_{c}}\right)\right]
\]

\[
J^{u'}(x, k) = r(p_k) + J^u(x - 1, 0)
\]

where \( J^u(x, k) \) is the value function under policy \( u \) when the system is at state \((x, k)\).

First we note that:

\[
E \left[e^{-\gamma T_{c} \left(x_{T_{c}}\right)}\right] = E_{T_{c}} \left[E_{x_{T_{c}}} \left[e^{-\gamma T_{c} \left(x_{T_{c}}\right)}|T_{c}\right]\right] = E \left[\lambda T_{c} e^{-\gamma T_{c}}\right]
\]

Furthermore:

\[
E \left[\lambda T_{c} e^{-\gamma T_{c}}\right] \leq \lambda E[T_{c}] \leq \lambda \bar{w}
\]

\[
\Rightarrow J^u(x, k) \leq \lambda v_0 \bar{w} + E \left[e^{-\gamma T_{c} \left((1 - p_k)v + p_k\bar{v}\right)} + e^{-\gamma T_{c}}J^u(x - 1, 0) - \frac{c_w(x)}{\gamma} \left(1 - e^{-\gamma T_{c}}\right)\right]
\]

\[
\leq \lambda v_0 \bar{w} + (\lambda + \mu) \left((1 - p_k)v + p_k\bar{v}\right) + E \left[e^{-\gamma T_{c}}J^u(x - 1, 0) - \frac{c_w(x)}{\gamma} \left(1 - e^{-\gamma T_{c}}\right)\right]
\]

To show that \( u' \) is preferred over \( u \), it is enough to show that \( J^{u'}(x, k) \) is greater than the upper bound for \( J^u(x, k) \).

If \( r(p_0) \geq 0 \), the definition of \( \bar{x} \) implies:

\[
x \geq \bar{x} = \left[\frac{\lambda v_0 \bar{w} + (\bar{v} + \bar{c}) + (1 - \lambda)r(p_0)}{\bar{c}}\right] \Rightarrow
\]

\[
\bar{c}x \geq \lambda v_0 \bar{w} + (\bar{v} + \bar{c}) + (1 - \lambda)r(p_0)
\]

\[
\geq \lambda v_0 \bar{w} + (\bar{v} + \bar{c}) + (\gamma - \lambda)r(p_0) \Rightarrow
\]

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Adding $\gamma \tau(p_0)x$ to the left hand side:

$$(\hat{c} + \gamma \tau(p_0))x \geq \lambda v_0 \bar{w} + (\bar{v} + \bar{c}) + (\gamma - \lambda)\tau(p_0)$$

And if $\tau(p_0) < 0$, the definition of $\bar{x}$ implies:

$$x \geq \bar{x} = \left\lfloor \frac{2(\lambda v_0 \bar{w} + (\bar{v} + \bar{c}) - \lambda \tau(p_0))}{\hat{c}} \right\rfloor \Rightarrow \hat{c}x \geq 2(\lambda v_0 \bar{w} + (\bar{v} + \bar{c}) - \lambda \tau(p_0))$$

Adding $2\gamma \tau(p_0)$ to the right hand side:

$$\geq 2(\lambda v_0 \bar{w} + (\bar{v} + \bar{c}) - \lambda \tau(p_0) + \gamma \tau(p_0)) \Rightarrow (\text{Since } \gamma \leq \bar{\gamma})$$

$$(\hat{c} + \gamma \tau(p_0))x \geq \lambda v_0 \bar{w} + (\bar{v} + \bar{c}) + (\gamma - \lambda)\tau(p_0)$$

Therefore, in both cases, we have the following inequality:

$$((\hat{c} + \gamma \tau(p_0))x \geq \lambda v_0 \bar{w} + (\bar{v} + \bar{c}) + (\gamma - \lambda)\tau(p_0) \Rightarrow \hat{c}x \geq 2(\lambda v_0 \bar{w} + (\bar{v} + \bar{c}) - \lambda \tau(p_0)) + \gamma \tau(p_0)) \Rightarrow (\text{Since } \hat{c}x \leq c_w(x))$$

$$c_w(x) + \lambda \tau(p_0) + \gamma(x - 1)\tau(p_0) \geq \lambda v_0 \bar{w} + p_k(\bar{v} + \bar{c}) - \gamma((1 - p_k)\bar{w} + p_k \bar{v})$$

By applying the inequality:

$$E[e^{-\gamma T_c}] \leq E[e^{-\gamma T_1}] = \lambda + \mu \Rightarrow E[1 - e^{-\gamma T_c}] \geq 1 - (\lambda + \mu) = \gamma$$

It follows that:

$$\left(\frac{c_w(x)}{\gamma} + \frac{\lambda}{\gamma} \tau(p_0) + (x - 1)\tau(p_0)\right) E[1 - e^{-\gamma T_c}] \geq \lambda v_0 \bar{w} + p_k(\bar{v} + \bar{c}) - \gamma((1 - p_k)\bar{w} + p_k \bar{v})$$

Using the following lower bound for $J^u(x - 1, 0)$:

$$J^u(x - 1, 0) \geq (x - 1)\tau(p_0) + \frac{\lambda}{\gamma} \tau(p_0)$$

which is obtained by declaring all customers, including those who would arrive later as type
\( \tau \), gives us:

\[
\left( J^u(x-1,0) + \frac{c_w(x)}{\gamma} \right) E \left[ 1 - e^{-\gamma T_c} \right] \geq \lambda v_0 \bar{w} + p_k (\bar{v} + \bar{c}) - \gamma \left( (1-p_k)\bar{v} + p_k \bar{v} \right) \Rightarrow \\
\tau(p_k) + \left( J^u(x-1,0) + \frac{c_w(x)}{\gamma} \right) E \left[ 1 - e^{-\gamma T_c} \right] \geq \lambda v_0 \bar{w} + (\lambda + \mu) \left( (1-p_k)\bar{v} + p_k \bar{v} \right) \Rightarrow \\
\tau(p_k) + J^u(x-1,0) \geq \lambda v_0 \bar{w} + (\lambda + \mu) \left( (1-p_k)\bar{v} + p_k \bar{v} \right) + E \left[ e^{-\gamma T_c} J^u(x-1,0) - \frac{c_w(x)}{\gamma} \left( 1 - e^{-\gamma T_c} \right) \right] \Rightarrow \\
J^u'(x,k) \geq J^u(x,k)
\]

Therefore, for \( \gamma \leq \bar{\gamma} \) and \( x \geq \bar{x} \), the above inequality is valid and this violates the optimality of \( u \). This implies that the policy \( u \) is not optimal and can be improved.

The above lemmas imply that for the discount rate small enough (that is, when \( \gamma \leq \bar{\gamma} \)), we can consider the set \( S_1 = \{(x,k) : 1 \leq x \leq \bar{x}; 0 \leq k \leq \hat{k}\} \cup \{(0,0)\} \) as the state space of the problem in the one sided case, and \( S_2 = \{(x,d) : 1 \leq x \leq \bar{x}; \bar{d} \leq d \leq \hat{d}\} \cup \{(0,0)\} \) as the state space of the problem in the two sided case, and hence conclude that the state space for the problem in both cases is finite.

B.2 Properties of the Optimal Value Function: One Sided Case (1S)

When the objective is to maximize the total discounted profit with small enough discount rate, and tests are one sided, the optimal value function has some monotonicity properties in \( x \) and \( k \). In this appendix we prove the following properties of the optimal value function \( J^* \).

\textbf{C1: } \( J^*(x,k) - J^*(x-1,0) \searrow x \)

\textbf{C2: } \( J^*(x,k) - \bar{\tau}(p_k) \searrow k \)

\textbf{C3: } \( J^*(x,k) \leq J^*(x-1,0) + p_k \bar{v} + (1-p_k)\bar{v} \)

\textbf{C4: } \( J^*(x,k) - \tau(p_k) \nearrow k \)

In order to prove the above properties, we first prove the convergence of the value iteration algorithm and existence of the optimal value function (Lemma 4). Then, after showing Condition \textbf{C1}, we need to recast the Bellman equation using \( J^* \) on the right hand side, and again show the convergence of the value iteration algorithm (Lemma 7). The remaining Lemmas show Conditions \textbf{C2-C4} based on the recast Bellman equation. (Throughout this appendix, we always assume that \( \gamma \leq \bar{\gamma} \).)
Lemma 4. The optimal value function, $J^*$, satisfies the Bellman’s Equation (9) such that $\Gamma J^* = J^*$, and can be obtained by the value iteration algorithm starting from any arbitrary function $J_0$, i.e.:

$$\lim_{k \to \infty} \Gamma^{(k)} J_0 = J^*$$

Proof: Since we have a maximization problem and the instantaneous costs $\mu(1-p_k)\beta_k\underline{v} - c_w(x), \underline{r}(p_k)$ and $\bar{r}(p_k)$ are bounded from above, the Negativity Assumption holds and the result follows. (Bertsekas 2007b, Volume II, Proposition 3.1.6)

Lemma 5. For any strictly-increasing function $c_w(x)$ and $\gamma \leq \bar{\gamma}$, we have $c_w(x) + \gamma J^*(x - 1, 0) \uparrow x$.

Proof: From the Bellman’s Equation (9) we have,

$$J^*(x, 0) \geq J^*(x - 1, 0) + \underline{r}(p_0) \Rightarrow \gamma J^*(x, 0) \geq \gamma J^*(x - 1, 0) + \gamma \underline{r}(p_0)$$

Since $\gamma \leq \bar{\gamma}$, we have $\hat{c} + \gamma \underline{r}(p_0) > 0$ (see Lemma 3). It follows that:

$$\gamma J^*(x, 0) + \hat{c} > \gamma J^*(x - 1, 0) \Rightarrow \gamma J^*(x, 0) + c_w(x + 1) - c_w(x) > \gamma J^*(x - 1, 0)$$

$$\Rightarrow c_w(x + 1) + \gamma J^*(x, 0) > c_w(x) + \gamma J^*(x - 1, 0)$$

Lemma 6. The optimal value function, $J^*$, satisfies Condition C1.

Proof: Let $T$ be a random variable representing the time needed to go from state $(x + 1, k)$ to state $(x, 0)$ when the system operates under the optimal policy.

Now suppose the system is at state $(x, k)$ and consider a policy $u$ which, instead of following the optimal policy, does the following: whenever the system enters state $(x, k)$, policy $u$ adds a dummy customer to the system (so the state of the system becomes $(x + 1, k)$) and then follows the optimal policy. Policy $u$ does this until the system reaches state $(x, 0)$ (x includes the dummy customer). At this point, policy $u$ disregards the dummy customer (so the state of the system changes to $(x - 1, 0)$) and starts following the optimal policy. We have,

$$J^u(x, k) \geq \left( J^*(x + 1, k) - E \left[ e^{-\gamma T} J^*(x, 0) \right] \right) + E \left[ e^{-\gamma T} J^*(x - 1, 0) \right] + E \left[ \int_0^T \hat{c} e^{-\gamma t} dt \right]$$

The right hand side of the above inequality is the value function under policy $u$ except that we have used a lower bound for the waiting cost of the dummy customer during $T$. 
Since $J^u(x, k)$ provides a lower bound for $J^*(x, k)$, we have

$$J^*(x, k) \geq \left( J^*(x + 1, k) - E \left[ e^{-\gamma T} J^*(x, 0) \right] \right) + E \left[ e^{-\gamma T} J^*(x - 1, 0) \right] + E \left[ \int_0^T \hat{c} e^{-\gamma t} dt \right]$$

$$\Rightarrow J^*(x, k) - E \left[ e^{-\gamma T} J^*(x, 0) \right] \geq \left( J^*(x + 1, k) - E \left[ e^{-\gamma T} J^*(x, 0) \right] \right) + E \left[ \int_0^T \hat{c} e^{-\gamma t} dt \right]$$

Furthermore, from Lemma 5 we know that $\gamma J^*(x - 1, 0) + \hat{c} x \nearrow x$, which implies

$$J^*(x - 1, 0) + \frac{\hat{c} x}{\gamma} \leq J^*(x, 0) + \frac{\hat{c}(x + 1)}{\gamma}$$

$$\Rightarrow -E \left[ 1 - e^{-\gamma T} \right] \left( J^*(x - 1, 0) + \frac{\hat{c} x}{\gamma} \right) \geq -E \left[ 1 - e^{-\gamma T} \right] \left( J^*(x, 0) + \frac{\hat{c}(x + 1)}{\gamma} \right)$$

Adding this inequality to the one obtained above gives us:

$$J^*(x, k) - J^*(x - 1, 0) \geq J^*(x + 1, k) - J^*(x, 0)$$

Before deriving the monotonicity properties of $J^*$ in $k$, we need to make the following assumption which is without loss of generality:

**Assumption 1.** Parameters $\bar{v}, v, \bar{c}$ and $c$ are such that: $r_0 := \max \{ \bar{r}(p_0), r(p_0) \} \leq 0$.

Recall that changing parameters $\bar{v}, v, \bar{c}$ and $c$ does not affect the optimal policy as long as the total rewards $\bar{v} + \bar{c}$ and $v + c$ remain unchanged. Assumption 1 is without loss of generality, because for any given $\bar{v}, v, \bar{c}$ and $c$, we can define, for example, $\bar{v}' = v' = 0$ and $\bar{c}' = \bar{c} + \bar{v}$ and $c' = c + v$ in order to satisfy this assumption.

Next, recast the Bellman’s Equation (9) in the following form:

$$\Delta J(x, k) = \max \left\{ TJ(x, k), \bar{r}(p_k) + J^*(x - 1, 0), r(p_k) + J^*(x - 1, 0) \right\} , \text{ where } \ (11)$$

$$TJ(x, k) = \mu(1 - p_k)\beta_k(v + J^*(x - 1, 0)) - c_w(x) + \lambda J(x + 1, k) + \mu(1 - \beta_k + p_k\beta_k)J(x, k + 1)$$

**Lemma 7.** The optimal value function, $J^*$, satisfies the Bellman’s Equation (11) such that $\Delta J^* = J^*$. Moreover, $J^*$ can be obtained by performing value iteration on the above Bellman’s equation starting from any arbitrary $J_0$, i.e.:

$$\lim_{k \to \infty} \Delta^{(k)} J_0 = J^*$$

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**Proof:** Similar to the proof of Lemma 4, we need to show that the instantaneous costs, $\mu(1 - p_k)\beta_k(\gamma + J^*(x - 1, 0)) - c_w(x)$, $\bar{r}(p_k) + J^*(x - 1, 0)$ and $\underline{r}(p_k) + J^*(x - 1, 0)$ are bounded from above. For this, it suffices to show that $J^*(x - 1, 0)$ is less than a constant. First assume that $x - 1 \leq \bar{x}$, where $\bar{x}$ is defined in Lemma 3. We have

$$J^*(x - 1, 0) \leq \bar{x}v_0 + J^*(0, 0) \leq \bar{x}v_0 + \frac{\lambda}{\gamma}v_0 < \infty,$$

where $v_0 = p_0\bar{v} + (1 - p_0)\gamma$. This upper bound is obtained by correctly identifying all the existing and future customers in zero time.

Next, let $x - 1 > \bar{x}$. From Lemma 3, it is optimal to stop at state $(x - 1, 0)$, or,

$$J^*(x - 1, 0) = (x - \bar{x} - 1)r_0 + J^*(\bar{x}, 0).$$

From Assumption 1,

$$J^*(x - 1, 0) \leq J^*(\bar{x}, 0) \leq \bar{x}v_0 + J^*(0, 0) \leq \bar{x}v_0 + \frac{\lambda}{\gamma}v_0 < \infty.$$

\[\Box\]

**Lemma 8.** For $x \geq 1$ we have $\lambda(J^*(x, 0) - J^*(x - 1, 0)) \leq c_w(x) + \gamma J^*(x - 1, 0)$.

**Proof:** The inequality holds for $x = 1$ because $J^*(1, 0) - J^*(0, 0) = (\gamma/\lambda)J^*(0, 0)$ by the boundary condition of the Bellman’s equation. The left hand side is nonincreasing from Lemma 6 and the right hand side is nondecreasing from Lemma 5.

\[\Box\]

**Lemma 9.** For $k \geq 0$ we have

$$(\beta_k(1 - p_k) - \beta_{k+1}(1 - p_{k+1}))p_{k+1}(\bar{v} - v) \leq \bar{r}(p_{k+1}) - \bar{r}(p_k) + (1 - \beta_{k+1} + p_{k+1}\beta_{k+1})(\bar{r}(p_{k+1}) - \bar{r}(p_{k+2}))$$

**Proof:** The proof can be established through the following sequence of inequalities:

$$p_k = (1 - \beta_k + p_k\beta_k)p_{k+1} \Rightarrow$$

$$\beta_k(1 - p_k)p_{k+1} = p_{k+1} - p_k \Rightarrow$$

$$(\beta_k(1 - p_k) - \beta_{k+1}(1 - p_{k+1}))p_{k+1} = (1 - \beta_{k+1} + p_{k+1}\beta_{k+1})p_{k+1} - p_k \Rightarrow$$

$$(\beta_k(1 - p_k) - \beta_{k+1}(1 - p_{k+1}))p_{k+1} = (1 - \beta_{k+1} + p_{k+1}\beta_{k+1})p_{k+1} - p_{k+1} + p_{k+1} - p_k \Rightarrow$$

$$(\beta_k(1 - p_k) - \beta_{k+1}(1 - p_{k+1}))p_{k+1} = (1 - \beta_{k+1} + p_{k+1}\beta_{k+1})(p_{k+1} - p_{k+2}) + p_{k+1} - p_k \Rightarrow$$

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\[
(\beta_k(1-p_k)-\beta_{k+1}(1-p_{k+1}))p_{k+1}\bar{v} = (1-\beta_{k+1}+p_{k+1}\beta_{k+1})(p_{k+1}-p_{k+2})\bar{v}+(p_{k+1}-p_k)\bar{v} \Rightarrow
\]
\[
(\beta_k(1-p_k)-\beta_{k+1}(1-p_{k+1}))p_{k+1}\bar{v} \leq (1-\beta_{k+1}+p_{k+1}\beta_{k+1})(p_{k+1}-p_{k+2})(\bar{v}+\zeta)+(p_{k+1}-p_k)(\bar{v}+\zeta) \Rightarrow
\]
\[
(\beta_k(1-p_k)-\beta_{k+1}(1-p_{k+1}))p_{k+1}\bar{v} \leq (1-\beta_{k+1}+p_{k+1}\beta_{k+1})(\bar{r}(p_{k+1})-\bar{r}(p_{k+2}))+\bar{r}(p_{k+1})-\bar{r}(p_k) \Rightarrow
\]
\[
(\beta_k(1-p_k)-\beta_{k+1}(1-p_{k+1}))p_{k+1}(\bar{v}-\bar{v}) \leq (1-\beta_{k+1}+p_{k+1}\beta_{k+1})(\bar{r}(p_{k+1})-\bar{r}(p_{k+2}))+\bar{r}(p_{k+1})-\bar{r}(p_k)
\]

\[\blacksquare\]

**Lemma 10.** If function J satisfies Conditions C2-C3, so does \(\Delta J\).

**Proof:** First we prove the lemma for C3:

Because \(\bar{r}(p_k) < p_k\bar{v}\) and \(\underline{r}(p_k) < (1-p_k)\bar{v}\), if \(\Delta J(x, k) = \bar{r}(p_k) + J^*(x-1, 0)\) or \(\Delta J(x, k) = \underline{r}(p_k) + J^*(x-1, 0)\), the result follows immediately. Therefore we only need to show that \(TJ(x, k) \leq p_k\bar{v} + (1-p_k)\bar{v} + J^*(x-1, 0)\). From C3 we have

\[
TJ(x, k) = \mu(1-p_k)\beta_k(\bar{v}+J^*(x-1, 0)) - c_w(x) + \lambda J(x+1, k) + \mu(1-\beta_k+p_k\beta_k)J(x, k+1)
\]

\[
\leq \mu(1-p_k)\beta_k(\bar{v}+J^*(x-1, 0)) - c_w(x) + \lambda J(x+1, k) + \mu(1-\beta_k+p_k\beta_k)(p_{k+1}\bar{v}+(1-p_{k+1})\bar{v}+J^*(x-1, 0))
\]

\[
= \mu(p_k\bar{v} + (1-p_k)\bar{v}) + \mu J^*(x-1, 0) - c_w(x) + \lambda J(x+1, k)
\]

\[
\leq (\lambda + \mu)(p_k\bar{v} + (1-p_k)\bar{v}) + \mu J^*(x-1, 0) - c_w(x) + \lambda J^*(x, 0) \quad \text{(with Lemma 8)}
\]

\[
\leq (\lambda + \mu)(p_k\bar{v} + (1-p_k)\bar{v}) + \mu J^*(x-1, 0) + (\lambda + \gamma)J^*(x-1, 0)
\]

\[
\leq p_k\bar{v} + (1-p_k)\bar{v} + J^*(x-1, 0)
\]

This completes the proof for C3. Next we prove the lemma for C2:

With C3 we have:

\[
J(x, k+1) \leq p_{k+1}\bar{v} + (1-p_{k+1})\bar{v} + J^*(x-1, 0) \Rightarrow
\]

\[
(\beta_k(1-p_k)-\beta_{k+1}(1-p_{k+1}))J(x, k+1) \leq (\beta_k(1-p_k)-\beta_{k+1}(1-p_{k+1}))(p_{k+1}\bar{v}+(1-p_{k+1})\bar{v}+J^*(x-1, 0)) \Rightarrow
\]

\[
\beta_{k+1}(1-p_{k+1})(p_{k+1}\bar{v}+(1-p_{k+1})\bar{v}+J^*(x-1, 0)) - \beta_{k+1}(1-p_{k+1})J(x, k+1) \leq
\]

\[
\beta_k(1-p_k)(p_{k+1}\bar{v}+(1-p_{k+1})\bar{v}+J^*(x-1, 0)) - \beta_k(1-p_k)J(x, k+1) \Rightarrow
\]

\[
\beta_{k+1}(1-p_{k+1})(p_{k+1}\bar{v}+(1-p_{k+1})\bar{v}+J^*(x-1, 0)) + (1-\beta_{k+1}+p_{k+1}\beta_{k+1})J(x, k+1) \leq
\]

\[
\beta_k(1-p_k)(p_{k+1}\bar{v}+(1-p_{k+1})\bar{v}+J^*(x-1, 0)) + (1-\beta_k+p_k\beta_k)J(x, k+1) \Rightarrow
\]

\[
\beta_{k+1}(1-p_{k+1})p_{k+1}(\bar{v}-\bar{v}) + \beta_{k+1}(1-p_{k+1})((\bar{v}+J^*(x-1, 0)) + (1-\beta_{k+1}+p_{k+1}\beta_{k+1})J(x, k+1) \leq
\]

\[
\beta_k(1-p_k)p_{k+1}(\bar{v}-\bar{v}) + \beta_k(1-p_k)((\bar{v}+J^*(x-1, 0)) + (1-\beta_k+p_k\beta_k)J(x, k+1) \quad \text{(with C2)} \Rightarrow
\]
Lemma 12. If function $J$ satisfies Condition $C_4$, and $J(x,k) \geq r(p_k) + J^*(x-1,0)$, so does $\Delta J$.

Proof: First, $\Delta J(x,k) \geq r(p_k) + J^*(x-1,0)$ follows directly from the definition of $\Delta$ for any function $J$. 

\[ \Delta J(x,k+1) - \tilde{r}(p_{k+1}) \leq \Delta J(x,k) - \tilde{r}(p_k) \]

Lemma 11. For $k \geq 0$ we have

\[ (\beta_{k-1}(1-p_{k-1}) - \beta_k(1-p_k))p_k(\tilde{v} + \tilde{e}) = r(p_{k-1}) - r(p_k) + (1 - \beta_k + p_k \beta_k) (r(p_{k+1}) - r(p_k)) \]

Proof: Through simplifications, the equality above is equivalent to

\[ p_k = \frac{p_{k-1}}{1 - \beta_{k-1} + p_{k-1} \beta_{k-1}}. \]
Next,

\[ J(x, k) \geq r(p_k) + J^*(x - 1, 0) \Rightarrow J(x, k) \geq \nu - p_k(\bar{\nu} + \bar{c}) + J^*(x - 1, 0) \Rightarrow \]

\[ J(x, k) - \nu - J^*(x - 1, 0) + p_k(\bar{\nu} + \bar{c}) \geq 0 \Rightarrow \]

\[ (\beta_{k-1}(1 - p_{k-1}) - \beta_k(1 - p_k))(J(x, k) - \nu - J^*(x - 1, 0) + p_k(\bar{\nu} + \bar{c})) \geq 0 \Rightarrow \]

\[ \beta_{k-1}(1 - p_{k-1})(\nu + J^*(x - 1, 0) - J(x, k)) \leq \]

\[ \beta_k(1 - p_k)(\nu + J^*(x - 1, 0) - J(x, k)) + (\beta_{k-1}(1 - p_{k-1}) - \beta_k(1 - p_k))p_k(\bar{\nu} + \bar{c}) \Rightarrow \]

\[ \beta_{k-1}(1 - p_{k-1})(\nu + J^*(x - 1, 0)) + (1 - \beta_{k-1} + p_{k-1}\beta_{k-1})J(x, k) \leq \]

\[ \beta_k(1 - p_k)(\nu + J^*(x - 1, 0)) + (1 - \beta_k + p_k\beta_k)J(x, k) + \]

\[ (\beta_{k-1}(1 - p_{k-1}) - \beta_k(1 - p_k))p_k(\bar{\nu} + \bar{c}) \Rightarrow \text{(with C4)} \]

\[ \beta_{k-1}(1 - p_{k-1})(\nu + J^*(x - 1, 0)) + (1 - \beta_{k-1} + p_{k-1}\beta_{k-1})J(x, k) \leq \]

\[ \beta_k(1 - p_k)(\nu + J^*(x - 1, 0)) + (1 - \beta_k + p_k\beta_k)J(x, k + 1) - r(p_k) \Rightarrow \]

\[ \mu\beta_{k-1}(1 - p_{k-1})(\nu + J^*(x - 1, 0)) + \mu(1 - \beta_{k-1} + p_{k-1}\beta_{k-1})J(x, k) - \mu r(p_{k-1}) \leq \]

\[ \mu\beta_k(1 - p_k)(\nu + J^*(x - 1, 0)) + \mu(1 - \beta_k + p_k\beta_k)J(x, k + 1) - \mu r(p_k) \Rightarrow \text{(with C4)} \]

\[ \mu\beta_{k-1}(1 - p_{k-1})(\nu + J^*(x - 1, 0)) + \lambda J(x + 1, k - 1) + \mu(1 - \beta_{k-1} + p_{k-1}\beta_{k-1})J(x, k) - (\lambda + \mu r(p_{k-1}) \leq \]

\[ \mu\beta_k(1 - p_k)(\nu + J^*(x - 1, 0)) + \lambda J(x + 1, k) + \mu(1 - \beta_k + p_k\beta_k)J(x, k + 1) - (\lambda + \mu r(p_k) \Rightarrow \]

\[ TJ(x, k - 1) - (\lambda + \mu r(p_{k-1}) \leq TJ(x, k) - (\lambda + \mu r(p_k) \Rightarrow \]

\[ TJ(x, k - 1) - r(p_{k-1}) \leq TJ(x, k) - r(p_k) \]

Because \( r(p_{k-1}) - r(p_k) \leq \bar{r}(p_k) - r(p_k) \):

\[ \Delta J(x, k - 1) - r(p_{k-1}) \leq \Delta J(x, k) - r(p_k) . \]

Lemmas 10 and 12 imply that for any discount value \( \gamma \leq \bar{\gamma} \), the optimal value function has properties C2-C4.
B.3 Properties of the Optimal Value Function: Two Sided Case (2S)

Similar to Appendix B.2, in this section we derive the following properties of the optimal value function for the two sided case when the discount rate is below $\gamma$:

\[C5: \quad J^*(x, d) - J^*(x - 1, 0) \searrow x\]

\[C6: \quad J^*(x, d) - \bar{r}(p_d) \nearrow d\]

\[C7: \quad J^*(x, d) - \tilde{r}(p_d) \searrow d\]

We can obtain results parallel to Lemmas 4 and 5 in Appendix B.2 following the same logic. That is, the optimal value function, $J^*$, satisfying the Bellman’s Equation (10) exists and can be obtained by the value iteration. Further, $J^*$ has the following property:

\[c_w(x) + \gamma J^*(x - 1, 0) \nearrow x\]

**Lemma 13.** The optimal value function $J^*$ satisfies Condition C5.

**Proof:** The proof for this lemma follows the same logic as Lemmas 4, 5 and 6 in Appendix B.2.

**Lemma 14.** If function $J$ satisfies Condition C6, so does $\Gamma J$.

**Proof:** For any $0 \leq p \leq 1$, let $p^+ = \beta p + (1 - \beta)(1 - p)$ and $p^- = (1 - \beta)p + \beta(1 - p)$. We start from the following equality (simple algebra shows that this equality holds):

\[\bar{r}(p_d) - p^+_d \bar{r}(p_{d+1}) - p^-_d \bar{r}(p_{d-1}) = \bar{r}(p_{d-1}) - p^+_d \bar{r}(p_d) - p^-_{d-1} \bar{r}(p_{d-2})\]

Replacing $p^-$ with $1 - p^+$ and some straightforward algebra gives us:

\[(p^+_d - p^-_{d-1})(\bar{r}(p_{d-1}) - \bar{r}(p_d)) = p^+_d (\bar{r}(p_{d+1}) - \bar{r}(p_d)) - \bar{r}(p_d) + \bar{r}(p_{d-1}) + (1 - p^+_d)(\bar{r}(p_{d-1}) - \bar{r}(p_{d-2}))\]

With C6:

\[(p^+_d - p^-_{d-1})(J(x, d - 1) - J(x, d)) \leq p^+_d (\bar{r}(p_{d+1}) - \bar{r}(p_d)) - \bar{r}(p_d) + \bar{r}(p_{d-1}) + (1 - p^+_d)(\bar{r}(p_{d-1}) - \bar{r}(p_{d-2}))\]

Adding $J(x, d - 1)$ to both sides implies:

\[p^+_d J(x, d) + (1 - p^-_{d-1})(J(x, d - 1) - \bar{r}(p_{d-1}) + \bar{r}(p_{d-2})) - \bar{r}(p_{d-1}) \leq \]

\[p^+_d (J(x, d) + \bar{r}(p_{d+1}) - \bar{r}(p_d)) + (1 - p^+_d)J(x, d - 1) - \bar{r}(p_d)\]
With C6:

\[ p_{d-1}^+ J(x, d) + (1 - p_{d-1}^+) J(x, d - 2) - r(p_{d-1}) \leq p_d^+ J(x, d + 1) + (1 - p_d^+) J(x, d - 1) - r(p_d) \]

After multiplying both sides by \( \mu \) and another use of C6, it follows that:

\[
\lambda J(x + 1, d - 1) + \mu p_{d-1}^+ J(x, d) + \mu p_{d-1}^- J(x, d - 2) - (\lambda + \mu) r(p_{d-1}) \leq \\
\lambda J(x + 1, d) + \mu p_d^+ J(x, d + 1) + \mu p_d^- J(x, d - 1) - (\lambda + \mu) r(p_d)
\]

Finally, adding the inequality \(-\gamma r(p_{d-1}) \leq -\gamma r(p_d)\) to the above inequality leads to:

\[
\Gamma J(x, d - 1) - r(p_{d-1}) \leq \Gamma J(x, d) - r(p_d)
\]

And this completes the proof.

Lemma 15. If function \( J \) satisfies Condition C7, so does \( \Gamma J \).

Proof: Starting from the following equality, proof is very similar to that of Lemma 14:

\[
\bar{r}(p_d) - p_d^+ \bar{r}(p_{d+1}) - p_d^- \bar{r}(p_{d-1}) = \bar{r}(p_{d-1}) - p_{d-1}^+ \bar{r}(p_d) - p_{d-1}^- \bar{r}(p_{d-2})
\]

Lemmas 14 and 15 imply that for any discount value \( \gamma \leq \bar{\gamma} \), the optimal value function has properties C6-C7.

C. Extension to the Long-Run Average Profit Model

In this appendix, we show that all the results derived for the total discounted profit model in Appendix B still hold when we change the objective to maximizing the long-run average profit. Then, we use this fact to provide the proofs for the main results stated in the paper.

Proposition 1. There exists a unique optimal value function \( J^* \) and a unique optimal long-run average profit \( g^* \) such that \( J^* \) and \( g^* \) satisfy the Bellman’s Equation (6). Furthermore, \( J^* \) has the following properties:

(i) \( J^*(x, k) - J^*(x - 1, 0) \downarrow x \)

(ii) \( J^*(x, k) - r(p_k) \uparrow k \)

(iii) \( J^*(x, k) - \bar{r}(p_k) \downarrow k \)

The optimal policy that achieves the optimal long-run average profit \( g^* \) is stationary.
Proof: In Appendix B.1 we show that for $\gamma \leq \tilde{\gamma}$, the recurrent states under the optimal policy form a finite set, and in Appendix B.2 we prove that the optimal value function for the discounted model with the discount rate below $\tilde{\gamma}$ has the above properties. It follows that a Blackwell optimal policy exists, and is optimal to the discounted profit problem for all values of $\gamma$ small enough, and has the properties (i)-(iii). Moreover, this policy is also optimal to the long-run average profit problem among all policies (Bertsekas 2007b, Chapter 4, Proposition 4.1.7).

Proposition 2. There exists a unique optimal value function $J^*$ and a unique optimal long-run average profit $g^*$ such that $J^*$ and $g^*$ satisfy the Bellman’s Equation (8). Furthermore, $J^*$ has the following properties:

1. $J^*(x, d) - J^*(x - 1, 0) \leq x$
2. $J^*(x, d) - \bar{r}(p_d) \geq d$
3. $J^*(x, d) - \bar{r}(p_d) \leq d$

The optimal policy that achieves the optimal long-run average profit $g^*$ is stationary.

Proof: The proof is similar to that of Proposition 1.

Proof of Theorem 1:

The structure of the optimal policy can be derived from properties of the optimal value function stated in Proposition 1. First, suppose that it is optimal to stop and announce $\tau$ at state $(x, k)$. Then we should have $J^*(x, k) - J^*(x - 1, 0) - \tau(p_k) = 0$. Following Property (ii), $J^*(x, k - 1) \leq J^*(x - 1, 0) + \tau(p_{k-1})$. Together with Equation (6), this inequality holds as equality which implies that it is also optimal to stop and announce type $\tau$ at state $(x, k - 1)$. Similarly, Property (iii) implies that if it is optimal to stop and announce $\bar{\tau}$ at state $(x, k)$, then it is also optimal to stop and announce $\bar{\tau}$ at state $(x, k + 1)$. Therefore, for any given queue length $x$, there exist thresholds $k(x)$ and $\bar{k}(x)$ such that the search stops at state $(x, k)$ when $k \leq k(x)$ (announcing $\tau$) or $k \geq \bar{k}(x)$ (announcing $\bar{\tau}$). Thresholds $k(x)$ can not exceed $k_\theta - 1$ since announcing $\tau$ is suboptimal at state $(x, k)$ with $k \geq k_\theta$.

Using Property (i) and a similar argument, we conclude that if it is optimal to stop and announce type $\bar{\tau}$ (resp. $\tau$) at state $(x, k)$, it is also optimal to stop and announce type $\bar{\tau}$ (resp. $\tau$) at state $(x + 1, k)$. Thus, the monotonicity properties of the thresholds $k(x)$ and $\bar{k}(x)$ in $x$ immediately follows.
Proof of Corollary 1:
The condition \( p_0 \geq \theta \) is equivalent to \( k_\theta = 0 \). The necessity directly follows from the structure of the optimal policy described in Theorem 1. To prove the sufficiency, consider state \((\bar{x}, 0)\). According to Lemma 3, the optimal policy should release the customer at this state without performing any test. Since the optimal policy is fully characterized by thresholds \( \tilde{k}(x) \), we should have \( \tilde{k}(\bar{x}) = 0 \). That is, the optimal decision at this state is to announce the customer as type \( \bar{\tau} \). This implies that announcing \( \bar{\tau} \) dominates announcing \( \tau \) at this state which further implies that \( \bar{r}(p_0) \geq r(p_0) \), and this is equivalent to \( p_0 \geq \theta \).

Proof of Theorem 2:
Using the properties of the optimal value function stated in Proposition 2, the logic of the proof follows that of Theorem 1, and therefore it omitted.

D. Change of Optimal Policy with Test Validity
In this appendix, we provide detailed explanations for Figure 5 of Section 4.2.1. Figure 10 depicts the optimal continuation regions with different \( \beta \) for this one sided example studied in Section 4.2.1, with parameters \( \rho = 0.1, \bar{c} = c = 0, \bar{v} = 5, v = 50, c_w(x) = x \) and \( p_0 = 0.9 \). The continuation states are marked by bold dots on the grid, with the triangles facing up (resp. down) indicating the states for announcing type \( \bar{\tau} \) (resp. \( \tau \)). In fact, when \( \beta \) takes any value in the interval \([0.501, 0.565]\), the optimal policy remains the same, and is depicted in Figure 10(a). In this interval, the accuracy and congestion measures change smoothly with \( \beta \). When \( \beta \) increases beyond 0.565, on the other hand, the optimal policy changes to the one depicted in Figure 10(b). As we can see, the optimal continuation region is expanded from the one depicted in Figure 10(a). Similarly, when \( \beta \) takes values from interval \([0.566, 0.713], [0.714, 0.806], \) or \([0.807, 0.999]\), the optimal policy remains the same, and is depicted by Figures 10(b), (c) and (d), respectively. We may focus on the change of \( \beta \) around 0.713. Comparing policies depicted in Figures 10(b) and (c), it is clear that the continuation region is getting smaller from (b) to (c). Therefore the congestion measure improves, which is reflected in Figure 5(c). Similarly, the (smaller) increases of congestion as \( \beta \) increases over 0.565 and 0.806 may be explained by comparing between Figures 10(a) and (b), and Figures 10(c) and (d), respectively.

E. Simple Dynamic Threshold Policy
In this appendix, we further the discussion about a simple heuristic for the one sided system in Section 3.1, following intuition from the no congestion case. Formally, we define a Simple
Figure 10: Optimal Policy Varies with Test Validity $\beta$ (1S). $\rho = 0.1$, $\bar{c} = c = 0$, $\bar{v} = v = 5$, $\bar{v} = v = 50$, $c_w(x) = x$ and $p_0 = 0.9$.

Dynamic Threshold (SDT) policy as a policy that never stops testing and announces type $\tau$. That is, under a SDT policy, a customer is either revealed to be type $\tau$ by a negative test result, or is announced as type $\bar{\tau}$ after a sequence of positive test results.

The best simple dynamic threshold policy has a nonincreasing threshold structure. This is because restricting to a SDT policy is equivalent to not allowing a type $\bar{\tau}$ customer to be announced as $\tau$, which is further equivalent to setting $\bar{c}$ to be $\infty$. The corresponding $\theta$ is therefore set to 0, which implies that $k_\theta = 0$. Therefore the threshold is monotone.

We have conducted a computational test using the same parameter combinations according to Table 1 in Section 4.1. We also consider four extreme cases: $\bar{c} = c = 0$, $\bar{v} = v = 0$, $\bar{v} = v = 0$ and $\bar{c} = c = 0$, which leads to a total of 6400 cases ($= 4^3 \times 5^2 \times 4$). Exactly half of these cases are such that $p_0 \geq \theta$. For these cases, our numerical results show that the simple dynamic policy is optimal, confirming Corollary 1. For the other half of the cases, the set of parameters for which the optimal policy is not degenerate reduce further to 2044 cases.

Table 3 reports the summary statistics of the relative gaps of the simple dynamic threshold policy for the remaining 2044 cases. The relative gap for SDT is above 50% in more than 200 cases (over 10% of 2044). In other words, adjusting the threshold of the search problem without congestion to regulate congestion can dramatically impair performance. On aver-
Table 3: Distribution of simple dynamic threshold policy relative sub-optimality gap.

<table>
<thead>
<tr>
<th></th>
<th>$10^{th}$</th>
<th>$25^{th}$</th>
<th>$50^{th}$</th>
<th>$75^{th}$</th>
<th>$90^{th}$</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sub-Optimality</td>
<td>$0%$</td>
<td>$0.2%$</td>
<td>$3.3%$</td>
<td>$13.37%$</td>
<td>$53.1%$</td>
<td>$15.09%$</td>
</tr>
</tbody>
</table>

The performance of SDT also sheds light on other practical heuristics. For instance, the agent may observe the queue length only after each service completion, and not during the diagnostic process. The agent needs, then, to commit to the maximum number of tests she intends to run on the customer before starting the diagnosis. This class of heuristics appears appropriate for many practical settings. The best heuristic within this class have a structure similar to that of the optimal SDT: the agent never identifies a customer as type $\tau$ after positive tests and the maximum number of tests are nonincreasing in the queue length. Note, however, that the performance of SDT dominates the performance of these heuristics. In particular, choosing the maximum number of tests before starting the search process can significantly hurt the system’s performance. For these cases, enabling the agent to observe the congestion level can result in substantial benefits.

F. Fixed Threshold Policies

Under the fixed threshold policy, both the one sided and two sided systems are equivalent to an M/G/1 queue (see Gross and Harris 1998). For given thresholds $\bar{k}$ for the one sided system, or $(\bar{d}, \bar{d})$ for the two sided system, we can evaluate the performance of the heuristic policies directly, without running the value iteration algorithm.

F.1 one sided System

Denote random variable $\tilde{k}$ to represent the number of elicited cues that are needed to reach a decision for a given customer. We have

$$p(k) \equiv P(\tilde{k} = k) = \left( \prod_{i=0}^{k-2} (1 - \beta_i + p_i \beta_i) \right) \beta_{k-1} (1 - p_{k-1}) \quad \text{for } k < \bar{k}$$

$$p(\bar{k}) \equiv P(\tilde{k} = \bar{k}) = \prod_{i=0}^{\bar{k}-2} (1 - \beta_i + p_i \beta_i) \quad \text{otherwise.}$$
(Here $\prod_{i=a}^{b}(1 - \beta_i + p_i \beta_i) = 1$ if $a > b$.) We deduce,

$$L_k = \tilde{\rho} + \frac{\bar{\rho}^2 + \lambda^2 \sigma_s^2}{2(1 - \tilde{\rho})}, \quad \delta_k = 1 - \prod_{i=0}^{k-2} (1 - \beta_i), \quad \tilde{\delta}_k = 1,$$

with $\tilde{\rho} = \rho E[\tilde{k}]$ and, using the variance decomposition formula,

$$\sigma_s^2 = \frac{1}{\mu^2} \left( E[\tilde{k}] + \sum_{k=1}^{\bar{k}} p(k)k^2 - E[\tilde{k}]^2 \right)$$

For linear waiting cost $c_w(x) = c_w x$, the long run average profit is a linear combination of $L_k$, $\delta_k$ and $\tilde{\delta}_k$ following Equation (4). Then we search through $\tilde{k}$ to find the best threshold policy.

**F.2 two sided System**

Similar to before, we need to derive the first two moments of the service time for each customer in order to use the M/G/1 formula.

Denote $\bar{s}(d)$ to be the expected time until completing service when a type $\bar{\tau}$ customer is at state $d$. Similarly, let $s(d)$ represent the corresponding expected time for type $\tau$. For any $d \in (d, \bar{d})$, we have

$$\bar{s}(d) = 1/\mu + \beta \bar{s}(d + 1) + (1 - \beta) \bar{s}(d - 1),$$

$$s(d) = 1/\mu + (1 - \beta) s(d + 1) + \beta s(d - 1).$$

Combined with $\bar{s}(d) = \bar{s}(\bar{d}) = 0$ and $s(d) = s(\bar{d}) = 0$, we have a linear system of equations to solve for $\bar{s}(d)$ and $s(d)$.

Denote $\bar{v}(d)$ (resp. $v(d)$) to represent the variance of the uncertain time until completion of service when a type $\bar{\tau}$ (resp. $\tau$) customer is at state $d$. Using the variance decomposition formula, we have

$$\bar{v}(d) = 1/\mu^2 + \beta \bar{v}(d + 1) + (1 - \beta) \bar{v}(d - 1) + \beta(1 - \beta) (\bar{s}(d + 1) - \bar{s}(d - 1))^2,$$

$$v(d) = 1/\mu^2 + (1 - \beta) v(d + 1) + \beta v(d - 1) + \beta(1 - \beta) (s(d + 1) - s(d - 1))^2.$$

Together with $\bar{v}(d) = \bar{v}(\bar{d}) = 0$ and $v(d) = v(\bar{d}) = 0$, we again have a linear system of equations for the vectors $\bar{v}(d)$ and $v(d)$.

Next, define:

$$m_s = p_0 \bar{s}(0) + (1 - p_0) s(0)$$
\[ \sigma_s^2 = p_0 \bar{v}(0) + (1 - p_0) q(0) + p_0(1 - p_0)(\bar{s}(0) - s(0))^2 \]

Denoting \( \bar{\rho} = \lambda m_s \) and \( q = (1 - \beta)/\beta \), we have

\[ L_{(d, \bar{d})} = \bar{\rho} + \frac{\bar{\rho}^2 + \lambda^2 \sigma_s^2}{2(1 - \bar{\rho})}, \quad \bar{\delta}_{(d, \bar{d})} = \frac{q^d - 1}{q^d - q^{\bar{d}}}, \quad \bar{\delta}_{(d, d)} = \frac{q^{-\bar{d}} - 1}{q^{-\bar{d}} - q^{-d}}. \]

Searching through various combinations of \((d, \bar{d})\), we can obtain the best fixed threshold policy.