Multi-agent Mechanism Design without Money

Santiago R. Balseiro, Huseyin Gurkan, Peng Sun*

Fuqua School of Business, Duke University
srb43@duke.edu, hg67@duke.edu, psun@duke.edu

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Abstract

We consider a principal repeatedly allocating a single resource in each period to one of multiple agents without relying on monetary payments over an infinite horizon. Agents’ private values are independent and identically distributed. We show that as the discount factor approaches one, the optimal dynamic mechanism without money achieves the first-best efficient allocation (the welfare-maximizing allocation as if values are public). As part of the proof, we provide an incentive compatible dynamic mechanism that achieves asymptotic optimality.

Keywords: dynamic mechanism design, social efficiency, multi-agent games, resource allocation without money

1 Introduction

Mechanism design for resource allocation with asymmetric information have been extensively studied in economics and, more recently, in computer science (see, for example, Nisan et al. 2007) and operations research (see, for example, Vohra 2011, Li et al. 2012, Zhang 2012a,b). Most of these studies allow, and often rely on, monetary transfers as part of the mechanism. In certain problem settings, especially with repeated interactions between agents, monetary transfers may not be practical. For example, monetary transfers may be inconvenient when allocating CPU or memory resources in shared computing environments; using money to manage incentives may sound awkward when an organization is deciding on the allocation of an internal resource, such as scheduling a conference room; in some

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medical resource allocation settings, monetary transfer may be a source of controversy. In the examples above, resource allocation occurs repeatedly, and agents' values for the resource might change over time.

In this paper, we study the problem of a socially maximizing planner repeatedly allocating a single resource without relying on monetary transfers. Specifically, we consider a discrete time infinite horizon setting where agents' private utility valuations for the resource over time are independent and identically distributed. The planner is able to commit to a long term allocation mechanism, but is not able to collect monetary transfers from agents or transfer money between agents. Both the planner and agents share the same time discount factor. The objective of the planner is to maximize allocation efficiency, that is, the expected total discounted utilities from the resource in all periods.

If agents can pay for the resource with money, repeatedly implementing the standard Vickrey–Clarke–Groves (VCG) mechanism achieves the “first-best” allocation (also referred to as the “efficient allocation”). That is, the resource is allocated to the agent with the highest value in every period. Obviously, first-best can also be achieved in settings where valuations are publicly observable. Without monetary payment, however, agents have the natural tendency of claiming that their values for the resource are the highest possible. In this case, if the resource is allocated only once, the planner can do no better than randomly allocating the resource to an agent. Repeated interactions, however, allow the planner to leverage future allocations when eliciting current period values, which may improve efficiency.

In this paper, we design mechanisms without monetary transfers which induce agents to truthfully reveal private information via promises/threats of future favorable/unfavorable allocations. Moreover, we show that our mechanism asymptotically achieves the “first-best” allocation as players become more patient. The fact that the first-best allocation can be approximated may not be surprising, given the Folk Theorem established in Fudenberg et al. (1994). In comparison, however, our paper takes an operational focus. In particular, while Fudenberg et al. (1994) implies the existence of an approximately efficient mechanism as the discount factor is close enough to one, we present a specific, easy to implement, mechanism for given time discount factors. Furthermore, our construction and analysis yields the convergence rate for the approximation. The equilibrium strategy for each agent in the game under our mechanism is also quite simple: each agent truthfully reports the valuation in each period.
1.1 An Overview of Our Approach

Invoking the revelation principle, we focus on direct dynamic mechanisms in which allocations in each period depends on reported private values over time. A direct dynamic mechanism induces a game between agents. Our solution concept for this game is perfect Bayesian equilibrium (PBE). Without loss of generality, we restrict attention to so-called incentive compatible mechanisms, under which all agents reporting truthfully regardless of past history is a PBE.

We consider the set of all achievable utilities, that is, the set of vectors representing all agents’ total discounted expected utilities that can be attained by incentive compatible dynamic mechanisms. Using the set of achievable utilities one could readily optimize any objective involving the total expected utility of each agent, and, in particular, identify the most efficient mechanism. Characterizing the set of achievable utilities by analyzing all dynamic mechanisms directly appears impossible because the dimensionality of the history grows exponentially with time.

Therefore, we provide an alternative characterization of the mechanism and set of achievable utilities using the so-called “promised utility” framework, which allows us to represent long term contracts recursively (Spear and Srivastava [1987], Thomas and Worrall [1990]). In this framework, agents’ total discounted utilities, also referred to as promised utilities, are state variables. In each time period, the planner selects a “stage mechanism” consisting of an allocation function as well as a future promise function, both depending on the current promised utility state and reported values. These functions map the current time period’s reports to an allocation and promised utilities for the next time period, respectively. A stage mechanism is incentive compatible if each agent’s total expected utility from the current period’s allocation and the discounted future promise is maximized by reporting truthfully. Furthermore, the stage mechanism needs to satisfy “promise keeping” constraints, which impose that the total expected utility delivered by the mechanism is equal to the current promised utility. Therefore, implementing an incentive compatible stage mechanism recursively delivers the promised utility for each agent.

Following Abreu et al. [1990], we provide a recursive formulation in the spirit of dynamic programming to characterize the set of achievable utilities. Specifically, we define a Bellman-like operator that maps a target set of future promised utilities to a set of current period promised utilities. The mapping specifies that there exists an incentive compatible stage mechanism that achieves every current
promised utility with future promises lying in the target set. The set of achievable utilities is, therefore, a fixed-point of this Bellman-like operator for sets.

Our main contribution is the construction of an incentive compatible mechanism that can attain, as the discount factor approaches one, the “perfect information” (PI) achievable set, i.e., the set of utilities attainable when values are publicly observable. It is clear that the vector of first-best utilities following efficient allocation is in the PI achievable set. Our approach, therefore, provides a constructive proof that first-best is asymptotically achievable in repeated settings without monetary transfers. Although our mechanism is not necessarily optimal for a fixed discount factor, it is relatively simple to implement, in the sense that one does not need to solve for a fixed-point of the aforementioned Bellman-like operator.

In order to establish our result, we cannot just rely on the previous theoretical characterization of achievable sets. Leveraging convexity of these sets of achievable utilities, we propose an equivalent representation of the sets in terms of their support functions. This representation is amenable to numerical procedures, and allows us to characterize the efficient frontier of a set, which provides a foundation for the proof of our asymptotic results.

More specifically, our mechanism allocates the resource to the agent with the largest weighted value, where weights are dynamically adjusted based on promised utilities. This allocation function is inspired by the PI achievable set: Because every point in the efficient frontier of the PI achievable set can be attained by a weighted allocation, our mechanism seeks to maximize efficiency by setting the weights according to the “closest” point in the efficient frontier of the PI achievable set.

The design of future promise functions resembles the d’Aspremont–Gérard-Varet–Arrow mechanism (d’Aspremont and Gérard-Varet 1979; Arrow 1979) and the risk free transfers of Esö and Futo (1999). Fixing the allocation rule, the interim future promises are uniquely determined from the incentive compatibility constraints. To implement ex-post future promises, we try to minimize the risk of the next time period’s promised utilities lying outside the achievable set. The proof that the proposed mechanism is incentive compatible is geometric in nature and relies on ideas from convex optimization, which is quite involved and resembles the equilibrium construction for the Folk Theorem in Fudenberg et al. (1994). The essential intuition of our mechanism, however, is clear. Reporting a higher value increases an agent’s chance of receiving the resource in the current period, while lowering the agent’s future promised utility. A lower promised utility, in turn, leads to a lower chance of allocating the
resource to this agent in the future.

1.2 Related Literature

There is an extensive literature on dynamic mechanism design problems. Most of the literature focuses on settings in which monetary transfers are allowed. Settings under consideration include dynamically changing populations with fixed information, and fixed populations with dynamically changing information. Under these environments, the efficient outcome can be implemented using natural generalizations of the static VCG mechanism to dynamic setting (see, e.g., Parkes and Singh 2004; Gershkov and Moldovanu 2010; Bergemann and Välimäki 2010). These mechanisms maintain efficient allocation of resources while incentivizing truthful reporting by choosing transfers that are equal to the externality that an agent imposes on others. In our setting, however, incentives cannot be readily aligned with transfers and the efficient outcome is not implementable in general. We refer the reader to the survey by Bergemann and Said (2011) for a more in depth discussion on dynamic mechanism design problems with monetary transfers.

There is a recent stream of studies considering dynamic mechanism design without money. Guo et al. (2015) consider the problem of repeatedly allocating a costly resource to a single agent whose values evolve according to a two-state Markov chain, and they provide the optimal allocation rule. In our setting where the marginal cost of resource is zero, the problem becomes trivial with a single agent. This is because it is optimal to always allocate the resource to the agent in each period. Guo et al. (2009) study the design of dynamic mechanisms with multiple agents and provide a mechanism that achieves at least 75% of the efficient allocation. While their mechanism is guaranteed to attain a fixed proportion of the efficient allocation for all discount factors larger than a threshold, it does not necessarily achieve the first-best allocation as the discount rate converges to one. Johnson (2014) studies a similar problem with multiple agents and discrete private values, and provides some numerical evidence that the optimal mechanism achieves higher social welfare as the discount factor increases. In our paper, we provide a relatively simple mechanism in quasi-closed form and analytically prove that it achieves first-best asymptotically. In addition, agents’ values are continuous in our model, which leads to some technical difficulties in characterizing the optimal mechanism, in exchange for simpler mechanisms without complicated tie-breaking randomization. Gorokh et al. (2016) also study a similar setting with a finite number of periods and discrete values. They provide a mechanism that can be
implemented via artificial currencies and show that the performance of their mechanism approximately achieves the first-best. Different from ours, the mechanism in Gorokh et al. (2016) satisfy incentive compatibility constraints approximately. In particular, it guarantees agents’ truthful reporting only in the limit as the number of periods approaches infinity. In comparison, our mechanism is guaranteed to be incentive compatible.

Our model and analysis relies on the promised utility framework (Spear and Srivastava, 1987; Abreu et al., 1990; Thomas and Worrall, 1990). In settings with momentary transfers, any nonnegative promised utility can be achieved by having the planner transfer money to the agents. Thus, constructing feasible incentive compatible mechanism is relatively straightforward, and the problem of the planner reduces to that of optimizing certain objective. When monetary transfers are not allowed, however, the planner can no longer subsidize agents. Constructing a feasible incentive compatible mechanism is challenging in this case because the planner needs to guarantee that future promises can be delivered exclusively via allocations. Abreu et al. (1990) introduce a recursive approach to study pure strategy sequential equilibria of repeated games with imperfect monitoring. In the paper, they characterize a self-generating set of sequential equilibria payoffs. We extend their recursive approach to characterize a self-generating set of utilities that can be achieved by incentive compatible mechanisms. Although our setting, which is focused on adverse selection (private signal) issues in mechanism design, is different from that of Abreu et al. (1990), some of their proof techniques related to self-generating sets are helpful.

Fudenberg et al. (1994) builds upon the dynamic programming framework of Abreu et al. (1990) to establish Folk Theorems for finite repeated games under imperfect information. In particular, Theorem 8.1 of Fudenberg et al. (1994) implies that in a setting similar to ours (except that the valuation set is finite), agents’ payoff under efficient allocation can be approximated as long as the time discount factor is close enough to one. Fudenberg et al. (1994), however, does not provide an explicit description of such a direct mechanism. For potentially indirect mechanisms (for example, “allocating to the agent with the highest report”), describing agents’ equilibrium reporting strategies to sustain the Folk Theorem approximation appears non-trivial. In comparison, our mechanism is well specified. It turns out that some steps in our construction resemble the steps used in Fudenberg et al. (1994) to prove existence of an asymptotically efficient equilibrium, and we will point them out in our paper. Another advantage of our mechanism is that the equilibrium strategy for agents is straightforward: reporting the true
value in each period. Furthermore, our construction and analysis explicitly characterize the rate of convergence to first-best of the mechanism’s social welfare in terms of the time discount factor.

Another stream of literature that is related to our work is the study of “scrip systems,” which are non-monetary trade economies for the exchange of resources (Friedman et al., 2006; Kash et al., 2007, 2012, 2015; Johnson et al., 2014). In these systems, scrips are used in place of government issued money, and the resource is priced at a fixed amount of scrips whenever trade occurs. The promised utilities in our model can be perceived as scrips. According to our mechanism, the agent who receives the resource in a period sees his promised utility decreases while others’ increase. The exchange of promised utilities according to our mechanism, however, is not fixed. In fact, it depends on the current promised utilities of all agents. From this perspective, our mechanism is more general than the ones considered in the existing studies of scrip systems.

The remaining of the paper is organized as the following. We first introduce the model, its recursive formulation, and various concepts related to self-generating sets in Section 2. In Section 3, we present the main phase of our mechanism. We then focus on the two-agent case and complete the description of the mechanism for the boundary region in Section 4. In Section 5, we describe the intuition behind the truthfulness and asymptotic efficiency of our mechanism. The general case of more than two agents is discussed in Section 6. Finally, Section 7 concludes the paper with comments on potential future directions. Proofs for results in Sections 2 to 6 are presented in Appendices A to E, respectively.

2 Model and Problem Formulation

We consider a discrete time infinite horizon setting where a social planner repeatedly allocates a single resource to one of multiple agents in each period without relying on monetary transfers. We index agents by $i \in \{1, \ldots, n\}$ and denote random vector $\mathbf{v}_t = (v_{i,t})_{i=1}^n$ to represent agents’ (private) values for the resource in period $t \geq 1$. Agents’ values in each period are independent and identically distributed with cumulative distribution function $F(\cdot)$ and density function $f(\cdot)$. Values are supported in the bounded set $[0, \bar{v}]$ and the density is bounded in its domain, i.e., $0 < f \leq f(v) \leq \bar{f} < \infty$ for all $v \in [0, \bar{v}]$. The planner and agents share the same discount factor $\beta \in (0, 1)$. An agent’s overall utility is given by the discounted sum of the valuations generated by the allocations of the resource across the horizon. The objective of the planner is to maximize the expected discounted sum of total valuations
in all periods.

**Notation.** For a sequence of vectors \( \mathbf{a} = ((a_{i,t})_{t=1}^{n})_{i=1}^{\infty} \in \mathbb{R}^{n \times \infty} \), we denote \( a_{i,1:t} = (a_{i,\tau})_{\tau=1}^{t} \in \mathbb{R}^{t} \) to represent the \( i \)th components of the 1-st to the \( t \)-th vectors, and \( \mathbf{a}_{1:t} = ((a_{i,\tau})_{t=1}^{n})_{\tau=1}^{\infty} \in \mathbb{R}^{n \times t} \) the entire 1-st to the \( t \)-th vectors. For a given vector \( \mathbf{x} \), we denote \( \mathbf{x}_{-i} \) to represent the vector obtained by removing \( x_{i} \) from \( \mathbf{x} \), and \( \mathbf{x}^{T} \) its transpose. For any two vectors \( \mathbf{x} \) and \( \mathbf{y} \) in \( \mathbb{R}^{n} \), the inequality \( \mathbf{x} \leq (\geq) \mathbf{y} \) represents that \( x_{i} \leq (\geq) y_{i} \) for each component \( i \). For any function \( g : \mathbb{R}^{n} \rightarrow \mathbb{R} \), we use \( \nabla g \) to represent its gradient. We use \( 1\{\cdot\} \) to represent the indicator function.

### 2.1 Dynamic Mechanisms and Achievable Utilities

We assume that the planner commits to a direct dynamic mechanism. That is, in each period, the agents learn their valuations of the resource, and each reports a value to the planner. The planner, in turn, determines the allocation of the resource for the period based on the entire history of reports and allocations, and publicly announces all agents’ reports and allocations in the end of the period.

**Non-anticipating Mechanisms and Strategies.** Formally, a dynamic mechanism \( \pi \) is a sequence of allocation rules \( \pi = ((\pi_{i,t})_{i=1}^{n})_{t=1}^{\infty} \), where \( \pi_{i,t} \) is the probability that the resource is allocated to agent \( i \) in period \( t \). We denote \( \mathcal{P} \subseteq \mathbb{R}^{n} \) to represent the set of \( n \) dimensional feasible allocations i.e.,

\[
\mathcal{P} \triangleq \{ \mathbf{\pi} \in [0,1]^{n} : \mathbf{\pi} \geq 0, \sum_{i=1}^{n} \pi_{i} \leq 1 \},
\]

and restrict that \( \pi_{i,t} \in \mathcal{P} \) for all \( t \). Any mechanism \( \pi \) induces a dynamic game among agents, in which each agent \( i \) submits a report \( \hat{v}_{i,t} \in [0,\bar{v}] \) in each period \( t \) and receives an allocation \( \pi_{i,t} \). We define the history available at time \( t \), \( \mathbf{h}_{t} = (h_{i,t})_{i=1}^{n} \), as all reports and previous allocations up to time \( t \), where \( h_{i,t} = (\hat{v}_{i,1:t-1}, \pi_{i,1:t-1}) \). We define the set of all possible histories that can be observed in periods \( t \geq 2 \) as \( \mathcal{H}_{t} = \prod_{i=1}^{n} \mathcal{H}_{i,t} \) where \( \mathcal{H}_{i,t} = [0,\bar{v}]^{t-1} \times [0,1]^{t-1} \), and \( \mathcal{H}_{1} = \{\emptyset\} \). We assume that the planner discloses the reports and the allocations after each round, so that the history is publicly observed. A non-anticipating strategy profile \( \sigma = ((\sigma_{i,t})_{t=1}^{\infty})_{i=1}^{n} \) for agents consists of reporting functions for each period, \( \sigma_{i,t} : [0,\bar{v}] \times \mathcal{H}_{t} \rightarrow [0,\bar{v}] \), that depend only on the value \( v_{i,t} \) of agent \( i \) in period \( t \), and the public history \( \mathbf{h}_{t} \) up to that period, i.e., \( \sigma_{i,t}(v_{i,t},\mathbf{h}_{t}) = \hat{v}_{i,t} \). We say

\footnote{Note that if some components of the allocation \( \pi_{t} \) are strictly between 0 and 1, we assume here that \( \pi_{i,t} \), and not the actual resource allocation in the end of the period, is publicly observable. More generally, the planner can decide what information to reveal to the agents in each period, and base the resource allocation rule on previous realized allocations. Given the type of results that we provide in this paper, it is clear that asymptotically the planner can do no better through such manipulations.}
a dynamic mechanism $\pi$ is non-anticipating if $\pi_{i,t}$ depends only on the current period reports $\hat{v}_t$ and the history $h_t$, that is, $\pi_t : [0, \hat{v}]^n \times H_t \to \mathcal{P}$. Since agents’ values in each period are independent, we restrict attention to non-anticipating mechanisms and reporting strategies that depend on past reports and allocations, but not on past values. Because players’ actions and planner’s allocations are only conditioned on previous reports and allocations, and this information is publicly observed, players do not need to form beliefs about the past actions of competitors.

**Direct Mechanisms and Truthful Reporting.** By the Revelation Principle, without loss of generality, we can focus on direct mechanisms in which agents report their values truthfully to the planner. In particular, for the game induced by a mechanism, we consider perfect Bayesian equilibria (PBE) in truthful reporting strategies, with beliefs that assign probability one to the event that other agents report truthfully. Therefore, we enforce *interim incentive compatibility* constraints, which ensure that an agent is better off adopting the truthful reporting strategy than any other reporting strategy when other agents report their values truthfully under mechanism $\pi$.

To elaborate, we introduce some notations. We denote $V_{i,t}$ to represent agent $i$’s utility-to-go in period $t$ when the planner implements mechanism $\pi$, all agents employ strategy profile $\sigma$, agent $i$’s value for the resource is $v_{i,t}$, and the history is $h_t$. That is,

$$V_{i,t}(\pi, \sigma|v_{i,t}, h_t) \triangleq (1 - \beta)E^{\pi,\sigma} \left[ v_{i,t} \pi_{i,t}(\hat{v}_t, h_t) + \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} v_{i,\tau} \pi_{i,\tau}(\hat{v}_\tau, \hat{h}_\tau) | v_{i,t}, h_t \right] ,$$

where $E^{\pi,\sigma}[|v_{i,t}, h_t]$ represents the expectation with respect to histories $(\hat{h}_\tau)_{\tau \geq t}$ induced by the mechanism $\pi$ and the strategy profile $\sigma$, given that the value of agent $i$ at time $t$ is $v_{i,t}$ and an initial history $h_t$. For $\tau \geq t$, we denote $\hat{v}_{i,\tau} = \sigma_{i,\tau}(v_{i,\tau}, \hat{h}_\tau)$ to represent agent $i$’s reported value in period $\tau$, with history $\hat{h}_\tau$ recursively defined as $\hat{h}_\tau = (\hat{h}_{\tau-1}, (\hat{v}_{\tau-1}, \pi_{\tau-1}(\hat{v}_{\tau-1}, \hat{h}_{\tau-1})))$ starting from $\hat{h}_t = h_t$. In order to facilitate comparisons across different discount factor, in the expression for $V_{i,t}$ we multiply by $1 - \beta$ to obtain an “average” discounted utility-to-go.

Using this notation, the interim incentive compatibility constraints on mechanism $\pi$ are given as follows:

$$V_{i,t}(\pi, i|v_{i,t}, h_t) \geq V_{i,t}(\pi, (\sigma_i, i-i)|v_{i,t}, h_t) , \quad \forall v_{i,t}, \sigma_i, h_t, i, t ,$$

where $i$ represents the truthful reporting strategy for agent $i$, i.e., $i_i(v_{i,t}, h_t) = v_{i,t}$. Hereafter, we
say a non-anticipating mechanism $\pi$ is perfect Bayesian incentive compatible (PBIC) if $\pi$ satisfies (1). These constraints enforce that, at every point in time, each agent is better off reporting truthfully when other agents report truthfully, regardless of past reports and allocations.

We next define the set of achievable utilities, $\mathcal{U}_\beta$ as the following.

$$\mathcal{U}_\beta \triangleq \{ u \in \mathbb{R}^n \mid u_i = V_i(\pi,1), \text{ for a PBIC mechanism } \pi \} ,$$

where $V_i(\pi,\sigma) \triangleq \mathbb{E}_{v_i,1}[V_i(\pi,\sigma|v_i,1,\emptyset)]$ is the total expected utility of agent $i$ when the planner implements mechanism $\pi$, and agents employ strategy profile $\sigma$. For any state $u = (u_i)_{i=1}^n$ in $\mathcal{U}_\beta$, there must exists a non-anticipating direct (dynamic) PBIC mechanism $\pi$ which achieves utility $u_i$ for all players $i = 1, \ldots, n$. Specifically, when the planner implements mechanism $\pi$, every player truthfully reporting each period’s value while believing with probability 1 that others report truthfully is a PBE. Moreover, the corresponding expected discounted sum of the valuations generated by $\pi$ for agent $i$ is equal to $u_i$.

Because social welfare is the expected discounted sum of total valuations, the component sum of $u$ corresponds to the social welfare obtained by $\pi$. Therefore, the maximum social welfare that can be obtained by a PBIC mechanism is $J_\beta^* = \max_{u \in \mathcal{U}_\beta} \sum_{i=1}^n u_i$. We denote vector $u^*_\beta \in \arg \max_{u \in \mathcal{U}_\beta} \sum_{i=1}^n u_i$ to represent the utilities of the agents under an optimal mechanism.

The above characterization of $\mathcal{U}_\beta$ allows us to readily optimize any objective involving the total expected utility of each agent. Although in this paper, we only focus on the maximum achievable social welfare, other objectives can be easily accommodated accordingly from the feasible set $\mathcal{U}_\beta$. Unfortunately, characterizing $\mathcal{U}_\beta$ directly by analyzing all non-anticipating mechanisms is not possible in general, because the dimensionality of the history grows exponentially with time. Therefore, in the following section, we provide an equivalent, recursive definition of $\mathcal{U}_\beta$ using the promised utility framework.

### 2.2 Promised Utility Framework

Because the planner has commitment power and values are independent, we can employ the promised utility framework to recursively formulate the set of achievable utilities, $\mathcal{U}_\beta$. We first present the framework and then show its equivalence with the definition (2) in the end of this subsection.
Note that in order to determine if an allocation for the current time period is incentive compatible, the planner needs to understand the impact of today’s actions on the continuation game induced by the mechanism. The promised utility framework builds on the observation that, because agents are expected value maximizers, the next period’s continuation utilities constitute a sufficient statistic for the problem of determining if an allocation for the current time period is incentive compatible. Loosely speaking, we denote state $u_t = (u_{i,t})_{i=1}^n$ to represent the vector of expected discounted total future values starting from period $t$. That is,

$$u_{i,t} = (1 - \beta)\mathbb{E} \left[ \sum_{\tau=t}^{\infty} \beta^{\tau-t} v_{i,\tau} \pi_{i,\tau} \right].$$

In the beginning of period $t$, the planner needs to fulfill the current state $u_t$ as a promise to agents through future allocations. We enforce this recursively by having the planner determine the current period’s allocation of the resource, as well as next period’s promised utilities $u_{t+1}$. For each agent, the expected value of the current’s period allocation plus the next period’s promised utility has to equal the current’s period promised utility.

We refer to the mechanism associated to the allocation of a single resource in a time period as a **stage mechanism**. More formally, for any given state $u \in \mathbb{R}^n$, a stage mechanism is given by an allocation function $p(\cdot|u) : [0,\bar{v}]^n \to \mathcal{P}$ and a **future promise function** $w(\cdot|u) : [0,\bar{v}]^n \to \mathbb{R}^n$, which map the vector of agents’ reports in the current period to an allocation vector $p$ and a next state $w$, respectively. We drop the dependence on $u$ when referring to a fixed state.

A stage mechanism $(p, w)$ should satisfy the following constraints. First, the allocation function should be feasible. That is,

$$\sum_{i=1}^n p_i(v) \leq 1 \quad \text{and} \quad p_i(v) \geq 0, \ \forall v. \quad (FA)$$

Additionally, the mechanism should satisfy the following **promise keeping** constraint,

$$u_i = \mathbb{E}[ (1 - \beta)v_i p_i(v) + \beta w_i(v)], \quad (PK(u))$$

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*We assume that $p$ and $w$ lie in the space $L^\infty \triangleq L^\infty([0,\bar{v}]^n,(\mathbb{R}^n,\|\cdot\|_1))$ of essentially bounded vector functions (see Appendix A.3). This is required to show that the recursive approach delineated in Proposition 2.3 and Corollary 2.1 converges to the set of achievable utilities $U_\beta$. Since the space $L^\infty$ is an equivalence class of functions which agree almost everywhere, all point-wise inequalities are understood to hold almost everywhere (a.e.). To simplify the exposition we drop the “almost everywhere” qualifier in the rest of the paper.*
which guarantees that the promised utility $u$ is fulfilled by the mechanism. Finally, an agent should not have an incentive to misreport the true type. The following interim incentive compatibility constraint imposes that, for each agent, reporting the value truthfully yields an expected utility at least as large as any other strategy, when other players report truthfully. Denote $P_i(v) \triangleq E_{v_i}[p_i(v,v_i)]$ to represent the interim allocation, and $W_i(v) \triangleq E_{v_i}[w_i(v,v_i)]$ the interim future promise of agent $i$. The incentive compatibility constraints are given by:

$$(1 - \beta)vP_i(v) + \beta W_i(v) \geq (1 - \beta)vP_i(v') + \beta W_i(v'), \quad \forall i, v, v'.$$  \hspace{1cm} (IC)$$

Using the constraints defined above, we next define the set operator $B_\beta : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ for a given set $A \subset \mathbb{R}_+^n$ as follows:

$$B_\beta(A) = \{ u \in \mathbb{R}_+^n | \exists (p, w) \text{ satisfying } (IC), (FA), (PK(u)), \text{ and } w(v) \in A \text{ for all } v \}. \hspace{1cm} (3)$$

Essentially, set $B_\beta(A)$ contains all promised utilities $u$ that can be supported by future promise functions $w$ in set $A$, while satisfying feasibility and incentive compatibility constraints. Although the operator $B_\beta$ is analogous to operator $B$ in Abreu et al. (1990, p. 1047), the specific constraints used in our definition are different. Abreu et al. (1990) study sequential equilibria of repeated games with imperfect monitoring, in which each player has a finite action space. In our setting, however, the stage game itself is induced by the stage mechanism selected by the planner. In this game, each agent has an uncountable action space because agents’ private values are continuous.

The operator $B_\beta$ provides a certificate to check whether a given set is a subset of $U_\beta$, according to the following result.

**Proposition 2.1.** If a set $A$ satisfies $A \subseteq B_\beta(A)$, then we have $B_\beta(A) \subseteq U_\beta$.

The proof of Proposition 2.1 is constructive: we show that every point $u \in B_\beta(A)$ lies in $U_\beta$ by constructing a PBIC mechanism that achieves utility $u$. The construction proceeds as follows for a fixed sample path of reports $(\hat{v}_t)_{t \geq 1}$. Because $u \in B_\beta(A)$, we can pick an allocation function $p_1$ and future promise function $w_1$ for the first time period such that $w_1(\hat{v}_1) \in A$. The condition $A \subseteq B_\beta(A)$ allows us to sequentially pick allocations functions $p_t$ and future promise functions $w_t$ for each time period $t > 1$, while guaranteeing that all future states $w_t(\hat{v}_t)$ remain in $A$. The promise keeping constraints imply that the expected utility of the agents under truthful reporting is equal to $u$. The
resulting mechanism is PBIC from (IC), because from the one-shot deviation principle it is sufficient to consider single-period deviations.

Following Abreu et al. (1990), we refer to sets that satisfy $A \subseteq B_\beta(A)$ as self-generating. That is, all the promised utilities in such a set $A$ can be fulfilled with future promises from within the same set. Proposition 2.1 implies that any state in a self-generating set can be achieved by a PBIC mechanism, because this state is in the set $U_\beta$. Furthermore, if there exists a specific mechanism $(p, w)$ that satisfies (IC), (FA), (PK($u$)) for all $u \in A$, and $w(v|u) \in A$ for all $v$ and $u \in A$, then we call the set $A$ to be self-generating with respect to mechanism $(p, w)$.

Remark 2.1. It is worth noting that the “lower triangle” set $L \triangleq \{ u \in \mathbb{R}_+^n \mid \sum_{i=1}^n u_i \leq \mathbb{E}[v]\}$ is self-generating. In fact, for any state $u \in L$, consider the random allocation rule $p^L_i(v|u) \triangleq u_i/\mathbb{E}[v]$ regardless of the value $v$. Such an allocation rule $p^L$, together with the future promise functions $w^L(v|u) \triangleq u$, achieves utilities $u$, i.e., satisfy (PK($u$)). Therefore, the lower triangle set $L$ is self-generating with respect to mechanism $(p^L, w^L)$, and, therefore, is a subset of $U_\beta$ for any discount factor $\beta$.

Proposition 2.1 states that any self-generating set is a subset of the set of achievable utilities, $U_\beta$. The following result further demonstrates that set $U_\beta$ itself is self-generating, and, therefore, a fixed point of the operator $B_\beta$. The result implies that the set of achievable utilities, $U_\beta$, defined according to summations of allocations over an infinite horizon, can be equivalently represented through stage mechanisms. In particular, the stage mechanisms satisfy constraints (PK($u$)) and (IC) in a recursive manner, and with the future promise functions $w$ lying in the same set $U_\beta$.

Proposition 2.2. The set of achievable utilities, $U_\beta$, satisfies $U_\beta \subseteq B_\beta(U_\beta)$. Therefore, $U_\beta = B_\beta(U_\beta)$.

We prove the result by showing that for every point $u \in U_\beta$ there exists a stage mechanism $(p, w)$ satisfying the conditions in (3). Because $u \in U_\beta$, there exists a PBIC mechanism $\pi$ with $u_i = V_i(\pi, 1)$ for all $i$, and we can set the candidate stage mechanism to the allocation and continuation values of $\pi$ for the first time period. That is, for reports for the first period $\hat{v}_1$, set $p(\hat{v}_1) = \pi_1(\hat{v}_1, h_1)$ with $h_1 = \emptyset$ and $w_i(\hat{v}_1) = \mathbb{E}_{v_{i,2}} \left[ V_{i,2}(\pi, 1|v_{i,2}, h_2) \right]$ with $h_2 = (\hat{v}_1, \pi_1(\hat{v}_1, h_1))$. The candidate mechanism trivially satisfies (IC), (FA) and (PK($u$)). We prove $w(\hat{v}_1) \in U_\beta$ by showing that the shifted mechanism obtained by implementing at time $t$ the allocation of mechanism $\pi$ at $t+1$ given fixed first period reports $\hat{v}_1$, is PBIC and achieves utilities $w(\hat{v}_1)$.

Finally, we show that the set $U_\beta$ can be obtained through repeatedly applying operator $B_\beta$. 
Proposition 2.3. Let $U^0 = [0, \mathbb{E}[v]]^n$ and $U^k = B_\beta(U^{k-1})$ for all $k \geq 1$. Then, we have $U^k \subseteq U^{k-1}$ and $\lim_{k \to \infty} U^k = U_\beta$.

The set $U^0$ can be understood as the achievable utilities of a relaxation in which the resource can be simultaneously allocated to all agents in every period. It is not hard to see that the operator $B_\beta$ is monotone, and $B_\beta(U_0) \subseteq U_0$. This implies that the sequence $U^k$ is monotone and converges to a set $U^\infty$. We prove the result by showing that $U_\beta \subseteq U^\infty$ and $U^\infty \subseteq U_\beta$. For the first part, notice that because $U_\beta \subseteq U^0$ and $U_\beta$ is a fixed-point of $B_\beta$, monotonicity of the operator $B_\beta$ implies that $U_\beta \subseteq U^k$ for all $k$, and therefore its limit. For the second part, it suffices to show, following Proposition 2.1, that $U^\infty$ is self-generating. This last step of the proof is technical and relies on the topological properties of the space of stage mechanisms. Specifically, because $U^\infty \subseteq U^k$, for each $u \in U^\infty$ we can construct a sequence of stage mechanisms $(p^k, w^k)$ satisfying (IC), (FA), (PK($u$)), and $w^k(v) \in U^{k-1}$ for all $v$. Because the space of stage mechanisms satisfying these conditions is compact (under an appropriate topology), we can construct a stage mechanism $(p, w)$ satisfying (IC), (FA), (PK($u$)), and $w(v) \in U^\infty$ for all $v$ by taking limits.

2.3 Support Function Representation

The iterative procedure described in Proposition 2.3 is of theoretical interest, but does not directly yield a straightforward numerical procedure due to the difficulty of determining set $B_\beta(A)$ from a set $A$. In this section, we present an equivalent, support function representation of the set $U_\beta$. This support function representation is not only amenable to numerical procedures, but also provides a foundation for the proof of our asymptotic results.

First, we present some basic properties of the set $U_\beta$, which enable us to obtain its support function representation. The next result shows that the set of achievable utilities is convex and compact. The result follows from Proposition 2.3 and the fact that $U^0$ is convex and compact, and that the operator $B_\beta$ preserves convexity and compactness.

Lemma 2.1. The set of achievable utilities, $U_\beta$, is convex and compact, and satisfies $U_\beta = \mathbb{R}^n_+ \cap \text{hyp}(U_\beta)$ where $\text{hyp}(U_\beta) = \{u \in \mathbb{R}^n \mid u \leq \bar{u}, \exists \bar{u} \in U_\beta\}$.

Following [Luenberger (1969) p. 44], any convex set $U_\beta$ can be represented by its support functions.
In particular, for any fixed $\alpha \in \mathbb{R}^n$ such that $\|\alpha\|_1 = 1$, the support function of set $U_\beta$ is given by

$$
\phi_\beta(\alpha) \triangleq \sup_{u \in U_\beta} \alpha^T u.
$$

(4)

Furthermore, we focus on $\alpha \geq 0$, because all nonnegative Pareto dominated points lie in $U_\beta$, following Lemma 2.1. Proposition 2.3 allows us to obtain a recursive representation of the support function $\phi_\beta(\alpha)$. To that end, for a generic function $\psi(\alpha)$, define operator $T_\beta$ as:

$$
[T_\beta \psi](\alpha) = \sup_{(p,w)} \mathbb{E} \left[ \sum_{i=1}^n \alpha_i \left( (1 - \beta)v_i p_i(v) + \beta w_i(v) \right) \right]
$$

s.t. $\text{IC}$, $\text{FA}$,

$$
\tilde{\alpha}^T w(v) \leq \psi(\tilde{\alpha}), \ \forall v \in [0,\bar{v}]^n, \ \tilde{\alpha} \in \mathbb{R}_+^n.
$$

(5)

The definition of operator $T_\beta$ involves an infinite-dimensional linear optimization problem. We can solve it approximately by considering constraints (5) only for $v$ and $\tilde{\alpha}$ on a grid, such that $\tilde{\alpha} \geq 0$ and $\|\tilde{\alpha}\|_1 = 1$. Furthermore, we can define the operator of recursively implementing operator $T_\beta$ as $T_\beta^k \psi \triangleq T_\beta(T_\beta^{k-1} \psi)$. The following result states that $\phi_\beta$ can be obtained by repeatedly applying operator $T_\beta$. The proof, again, follows from Proposition 2.3 and the fact that convergence of convex and compact sets is equivalent to convergence of support functions.

**Corollary 2.1.** The support function $\phi_\beta$ is a fixed point of $T_\beta$, i.e., $\phi_\beta = T_\beta \phi_\beta$. Furthermore, we have $\phi_\beta(\alpha) = \lim_{k \to \infty} [T_\beta^k \phi^0](\alpha)$ for all $\alpha$, starting from $\phi^0(\alpha) = \mathbb{E}[v]$.

Figure 1a demonstrates the set of achievable utilities, $U_\beta$ with two agents. The lines in the graph are the supporting hyperplanes, which are computed from the iterative procedure described in Corollary 2.1. The blank area outlined by the supporting hyperplanes is the set $U_\beta$.

Although the procedure described in Corollary 2.1 generates the achievable set $U_\beta$, it does not directly provide PBIC mechanisms achieving each state in the set. Later in our paper, we provide a mechanism along with a subset of $U_\beta$ such that the subset is self-generating with respect to the mechanism.

In order to establish our main result that efficient allocation is achievable asymptotically as the discount factor approaches one, we first propose a “perfect information achievable set” that provides an upper bound to the set of achievable utilities.
Figure 1: Illustration of achievable sets in a two-agent case. Figure 1a depicts the achievable set \( \mathcal{U}_\beta \) with \( \beta = 0.8 \) and its supporting hyperplanes. Figure 1b depicts efficient frontiers of sets \( \mathcal{U}_\beta \) with \( \beta = 0.6 \) and \( \beta = 0.8 \), and the perfect information achievable set \( \mathcal{U} \) and the lower triangle set \( L \). Here values of \( v \) follow a uniform \([0, 1]\) distribution with \( \mathbb{E}[v] = 0.5 \).

### 2.4 Perfect Information Achievable Set

We define the perfect information (PI) achievable set, \( \mathcal{U} \), as the set of utilities attainable when values are publicly observable by the planner. This set is given by:

\[
\mathcal{U} \triangleq \{ u \in \mathbb{R}_+^n \mid u_i = \mathbb{E}[v_i p_i(v)] \text{ for all } i, \text{ for some } p \text{ satisfying } \{\text{FA}\} \}.
\]  

Clearly, for any \( \beta \in [0, 1) \) we have that \( \mathcal{U}_\beta \subseteq \mathcal{U} \).

Similar to (4), for any \( \alpha \in \mathbb{R}_+^n \) with \( \|\alpha\|_1 = 1 \), define the support function of the convex perfect information achievable set \( \mathcal{U} \) as follows:

\[
\phi(\alpha) \triangleq \sup_{u \in \mathcal{U}} \alpha^T u = \sup_{p \text{ s.t. } \{\text{FA}\}} \sum_{i=1}^n \mathbb{E}_v [\alpha_i v_i p_i(v)] = \mathbb{E}_v \left[ \max_{i=1, \ldots, n} \alpha_i v_i \right],
\]  

where the second equation follows from the definition of the PI achievable set, and the third from optimizing pointwise over values. Numerically, one can start the iterative algorithm of Corollary 2.1 from \( \phi \), instead of \( \phi^0 \), which leads to faster convergence. In fact, Figure 1a is generated by starting from \( \phi \).
The support function $\phi$ satisfies the following properties.

**Proposition 2.4.** The support function $\phi(\alpha)$ given in (7) is convex, differentiable for $\alpha \in \mathbb{R}^n_+$ and twice differentiable for $\alpha \in \mathbb{R}^n_+$ such that $\alpha > 0$. Moreover, the partial derivatives for all $i$ are given by

$$\frac{\partial \phi}{\partial \alpha_i}(\alpha) = \mathbb{E}_v \left[ v_i 1\left\{ \alpha_i v_i \geq \max_{j \neq i} \alpha_j v_j \right\} \right].$$

Differentiability of the support function follows because values are absolutely continuous. The gradient $\nabla \phi$ of the support function for any $\alpha$ corresponds to a point on the efficient frontier of the set $U$. Specifically, for any convex set $A \subset \mathbb{R}^n$, define $E(A)$ to represent its efficient frontier:

$$E(A) \triangleq \{ u \in A \mid \text{for all } u' \in A \text{ with } u' \neq u, \exists i \text{ such that } u'_i < u_i \}.$$  

Because the support function is differentiable and the set $U$ is convex and closed, for every state $u$ on the efficient frontier of $U$ there exists some $\alpha$ such that $\nabla \phi(\alpha) = u$ (see, e.g., Schneider 2013, Corollary 1.7.3 on p. 47).

Furthermore, Proposition 2.4 implies that all points on the efficient frontier of the PI set are achievable by allocations of the form $p_i(v) = 1\{ v_i \geq \max_{j \neq i} v_j \}$. That is, the resource is allocated to the agent with the highest $\alpha$-weighted value. More generally, for any point $u \in U$, not necessarily on the efficient frontier, we can define $\alpha(u)$ as

$$\alpha(u) \in \arg \max_{x : \|x\|_1 = 1, x \geq 0, x^T u - \phi(x)}.$$  

It is easy to verify that for any $u \in E(U)$, we have $\nabla \phi(\alpha(u)) = u$.

The first-best total utility, $J_{FB}$, is achieved by allocating the resource to the agent that values it most in each period, i.e., $J_{FB} = \mathbb{E} \left[ \max_{i=1, \ldots, n} v_i \right]$. Because the first-best utility is attained by the allocation rule $p_i(v) = 1\{ v_i \geq \max_{j \neq i} v_j \}$ and $U_\beta \subseteq U$, we must have

$$J_{FB} = \max_{u \in U} \sum_{i=1}^n u_i \geq \max_{u \in U_\beta} \sum_{i=1}^n u_i = J^*_\beta.$$  

Figure 1b compares the boundaries of sets $U$ and $U_\beta$ for different $\beta$ values. The set $U_\beta$ is monotonically

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3In our setting, the probability of having a tie is zero.
increasing as $\beta$ increases, and, therefore, the planner is able to achieve higher total utility. Furthermore, it appears that as the time discount factor approaches one, the set $\mathcal{U}_\beta$ approaches the full information achievable set $\mathcal{U}$. We now provide a high level summary our approach.

**An Overview of Our Approach.** If the set $\mathcal{U}_\beta$ converges to $\mathcal{U}$ as the discount factor $\beta$ approaches one, inequality (9) implies that the optimal social welfare, $J^*_\beta$, converges to the first-best, $J^{FB}$. However, the set $\mathcal{U}_\beta$ is hard to study directly. Therefore, we design a factor, $F_\beta \in [0, 1]$, to obtain the following “sandwich” condition for set $\mathcal{U}_\beta$:

$$F_\beta \mathcal{U} \subseteq \mathcal{U}_\beta \subseteq \mathcal{U}, \text{ in which } F_\beta \mathcal{U} \equiv \{ F_\beta \mathbf{u} \mid \mathbf{u} \in \mathcal{U} \}.$$ (10)

In particular, the sequence of factors $F_\beta$ is designed to converge to one as $\beta$ converges to one, which implies our result.

Following Proposition 2.1, in order to establish the condition $F_\beta \mathcal{U} \subseteq \mathcal{U}_\beta$, we only need to show that the set $F_\beta \mathcal{U}$ is self-generating, or, $F_\beta \mathcal{U} \subseteq B_\beta(F_\beta \mathcal{U})$. In the rest of the paper, we propose a mechanism $(\hat{p}, \hat{w})$, such that the set $F_\beta$ is self-generating with respect to the mechanism $(\hat{p}, \hat{w})$. Consequently, every state in $F_\beta \mathcal{U}$ is achievable following this incentive compatible mechanism.

## 3 Central Region and Main Phase Mechanism

The **main phase** of the mechanism is defined for a so-called **central region**, $\hat{\mathcal{U}}_\beta$, defined through a pair of scalars $u_\beta \in [0, E[v]]$ and $F_\beta \in [0, 1]$. The specific values of those scalars depend on the number of agents involved and the distribution of values, and are deferred to the following sections. For this section, it is sufficient to keep in mind that $u_\beta$ approaches zero and $F_\beta$ approaches one as $\beta$ approaches one. Define the **central region** $\hat{\mathcal{U}}_\beta$ as

$$\hat{\mathcal{U}}_\beta = F_\beta \mathcal{U} \cap \{ \mathbf{u} \in \mathbb{R}^n \mid u_i \geq u_\beta, \forall i \},$$ (11)

which is a subset of $F_\beta \mathcal{U}$ that includes states that are component-wise higher than the threshold $u_\beta$. Figure 2a illustrates such a central region, where we implement the main phase mechanism, to be described next.

The main phase mechanism consists of allocation functions $\hat{p}(\cdot | \mathbf{u}) : [0, \bar{e}]^n \to \mathcal{P}$ and future promise
functions $\hat{w}(\cdot | u) : [0, \bar{v}] \to F_\beta \mathcal{U}$ for all $u \in \hat{U}_\beta$, which satisfy (IC), (PK($u$)), and $\hat{w}(v | u) \in F_\beta \mathcal{U}$ for all $v$. Next we explain them separately for any state $u$ in the central region.

**Allocation $\hat{p}$.** Note that for any point $u$ on the efficient frontier $E(F_\beta \mathcal{U})$ of the set $F_\beta \mathcal{U}$, the state $u/F_\beta$ is on the efficient frontier $E(\mathcal{U})$. Recall from the discussion in the last section, with perfect information, promised utilities $u/F_\beta$ can be achieved with allocations that weigh values according to $\alpha(u/F_\beta)$ defined in (8). Motivated from this, for any state $u$ in the central region $\hat{U}_\beta$ defined in (11), we define the allocation function to be

$$
\hat{p}_i(v | u) = 1 \left\{ \alpha_i(u/F_\beta) v_i \geq \max_{j \neq i} \alpha_j(u/F_\beta) v_j \right\},
$$

That is, the allocation function $\hat{p}$ corresponds to allocating the resource to the agent with the largest weighted value, where the weights are associated to the efficient frontier of the perfect information achievable set. By Proposition 2.4, the expected utility of this allocation satisfies $E_v [v, \hat{p}_i(v | u)] = \frac{\partial \phi}{\partial \alpha_i} (\alpha(u/F_\beta))$, i.e., the allocation delivers the utility of the point on the efficient frontier of the set $\mathcal{U}$.

Figure 2: The central region and boundary region in a two-agent case. Values of $v$ are uniformly distributed in $[0, 1]$, and $F_\beta = 0.8$ and $u_\beta = 0.1$. The solid curve is the efficient frontier of the perfect information achievable set $\mathcal{U}$. The dashed lines represent the threshold $u_\beta$, and the dotted dashed curve represents the scaled set, $F_\beta \mathcal{U}$. In Figure 2a, the shaded area represents the central region, $\hat{U}_\beta$. In Figure 2b, the shaded area represents the boundary region within the lower triangle set $L$. In this case $E[v] = 0.5$ and $E[v] - u_\beta = 0.4$. 

(a) Central Region

(b) Boundary Region
with normal $\alpha(u/F_\beta)$.

**Remark 3.1.** It is worth illustrating geometrically how the weights $\alpha(u/F_\beta)$ are determined. According to the optimality conditions for $\mathcal{F}$, for any state $u \in F_\beta \mathcal{U}$, we must have $u/F_\beta = \nabla \phi(\alpha(u/F_\beta)) - \mu e$ for some nonnegative scalar $\mu$, where $e$ is the vector of 1’s in $\mathbb{R}^n$. (Here $\mu$ corresponds to the dual variable of the $\|x\|_1 = 1$ constraint.) Furthermore, as explained before, the vector $F_\beta \nabla \phi(\alpha(u/F_\beta))$ is on the efficient frontier of set $F_\beta \mathcal{U}$. Therefore, as illustrated in Figure 3a, the weights $\alpha(u/F_\beta)$ are determined by first projecting the state $u$ along the “45 degree line” up to the point $u + F_\beta \mu e$ on the efficient frontier of $F_\beta \mathcal{U}$. Then we scale this point by $1/F_\beta$ to obtain a point $u/F_\beta + \mu e$ on the efficient frontier of $\mathcal{U}$. Weights $\alpha(u/F_\beta)$ correspond to the normal vector of the hyperplane supporting the point $u/F_\beta + \mu e$ on the efficient frontier of $\mathcal{U}$.

**Future Promise Function $\hat{w}$.** First, it straightforward to determine the interim future promise function $\hat{W}_i(v_i|u) \triangleq \mathbb{E}_{v_i}(\hat{w}(v|u))$ for each agent $i$ from the incentive compatibility constraints. Following the standard argument of Myerson (1981), the (IC) constraints uniquely determine an expression for
\[ \hat{W}_i(v_i|u) = \frac{1}{\beta} \left( u_i + (1 - \beta) \left( \int_0^{v_i} \hat{P}_i(y|u)dy - \hat{P}_i(v_i|u)v_i - \int_0^0 \tilde{F}(y) \hat{P}_i(y|u)dy \right) \right), \forall i. \quad (13) \]

Because the interim allocation is increasing in an agent’s report, equation (13) is necessary and sufficient to guarantee incentive compatibility.

Although the interim future promise functions \( \hat{W} \) in (13) are uniquely determined by \( \hat{p} \), the ex-post future promise functions \( \hat{w} \) may not be uniquely determined given \( \hat{p} \). This is because multiple ex-post future promise functions may correspond to the same interim future promise function. For example, we can set the ex-post future promise function \( \hat{w}_i(v_i, v_{-i}|u) \) to the interim future promise function \( \hat{W}_i(v_i|u) \) for all \( v_{-i} \). This choice, however, does not guarantee that starting from an initial state \( u \in F_{\beta U} \), the future promises remain in \( F_{\beta U} \) for all possible reports. In fact, if the state \( u \) is sufficiently close to the efficient frontier \( E(F_{\beta U}) \), there exist values of \( v \) such that the future promise \( \hat{W} \) falls outside the set \( F_{\beta U} \). If so, state \( u \) cannot be achieved using such a mechanism.

The key consideration, therefore, is that for any value \( v \in [0, \bar{v}] \) and state \( u \in F_{\beta U} \), we must ensure that the future promise function \( \hat{w} \) is feasible, i.e., \( \hat{w}(v|u) \in F_{\beta U} \), so as the set \( F_{\beta U} \) is self-generating with respect to mechanism \((\hat{p}, \hat{w})\). Specifically, the ex-post future promises need to be (i) nonnegative and (ii) within the efficient frontier of \( F_{\beta U} \). In this section, we address the design issue (ii), and leave the issue (i) to Sections 4 and 6. In fact, the threshold \( u_{\beta} \) is designed to resolve issue (i).

In order to guarantee condition (ii), we implement interim future promises so as to minimize the risk of the next time period’s promise utilities lying outside the achievable set \( F_{\beta U} \). The ex-post future promise function for any state \( u \) in the central region \( \hat{U}_{\beta} \) and value \( v \in [0, 1] \) is given by:

\[ \hat{w}_i(v|u) = \hat{W}_i(v_i|u) - \frac{1}{n - 1} \sum_{j \neq i} \frac{\alpha_j(u/F_{\beta})}{\alpha_i(u/F_{\beta})} \left[ \hat{W}_j(v_j|u) - \mathbb{E}_{\beta_j} [\hat{W}_j(\bar{v}_j|u)] \right]. \quad (14) \]

The design of the future promise functions resembles the d’Aspremont–Gérard-Varet–Arrow mechanism (d’Aspremont and Gérard-Varet, 1979; Arrow, 1979) and the risk free transfers of Esö and Futo (1999). In the context of a risk-averse seller in a static setting, Esö and Futo (1999) propose a similar transfer rule that gives rise to a constant ex-post revenue. Analogously, our choice of future promise
functions guarantees that the weighted sum of the ex-post future promises is constant. The following result shows that the future promises lie in a plane with normal $\alpha(u/F_\beta)$.

**Proposition 3.1.** For any given state $u \in \hat{U}_\beta$, the future promise vector $\hat{w}(v|u)$ lies within a plane in $\mathbb{R}^n$ for all $v$. Specifically, the plane is described by the following equation,

$$\alpha(u/F_\beta)^T(u - \hat{w}(v|u)) = \frac{1 - \beta}{\beta} \alpha(u/F_\beta)^T(\nabla\phi(\alpha(u/F_\beta)) - u) \geq 0.$$ 

The fact that our future promises lie on a plane resembles the concept of “enforceability with respect to hyperplanes” in Fudenberg et al. (1994). Furthermore, our plane being parallel to a support function of the (PI) achievable set echoes the construction of future promises given in the proof of Lemma 6.1 in Fudenberg et al. (1994).

The properties stated in Proposition 3.1 are useful towards establishing that future promises $\hat{w}(v|u)$ fall in the set $F_\beta\mathcal{U}$. Recall that $\alpha(u/F_\beta)$ gives the normal of the supporting hyperplane of a point $u$ on the efficient frontier of $F_\beta\mathcal{U}$. Interpreting $\alpha(u/F_\beta)^T(u - x)$ as the “signed directional distance” of a point $x \in \mathbb{R}^n$ to the hyperplane supporting $u$, we obtain that $\alpha(u/F_\beta)^T(u - \hat{w}(v|u))$ measures how close the future promises are to the hyperplane supporting $u$. Among all ex-post future promise functions which satisfy the interim future promise functions $\hat{W}_i(v_i|u)$, the one according to expression (14) is “pushed” the farthest from the efficient frontier $E(F_\beta\mathcal{U})$. This is formalized in the next result.

**Proposition 3.2.** Fix a state $u \in E(\hat{U}_\beta)$, and let $\hat{W}_i(v_i|u)$ be the interim future promise given in (13). Then, the ex-post future promise $\hat{w}(v|u)$ defined in (14) is an optimal solution of the following optimization problem:

$$\max_{\hat{w}(:)} \min_v \alpha(u/F_\beta)^T(u - \hat{w}(v))$$

s.t. $E_{v\sim i}[\hat{w}_i(v_i, v_{-i})] = \hat{W}_i(v_i|u)$ $\forall v_i, i$

where $\alpha(\cdot)$ is defined in (8).

Proposition 3.1 shows that the signed directional distance between $\hat{w}(v|u)$ and the hyperplane supporting the projection of $u$ onto $E(F_\beta\mathcal{U})$ is independent of $v$ and strictly positive. This provides some indication that $\hat{w}$ lies within the efficient frontier of $F_\beta\mathcal{U}$. As in Fudenberg et al. (1994), we show that this is the case by carefully balancing the “step size” $\hat{w}(v|u) - u$ against the curvature of the efficient frontier at the projection of $u$ onto $E(F_\beta\mathcal{U})$. We formally establish this for the two-agent case.
in the next section.

Figure 3b demonstrates, in a two-agent case, the future promised function $\hat{w}(v|u)$ starting from a state $u \in \hat{U}_\beta$. The figure shows that the realized future promise $\hat{w}(v|u)$ ranges on a line when we vary values of $v$. Moreover, we also observe that for some values of $v$, the ex-post future promise $\hat{w}(v|u)$ may fall out of the central region $\hat{U}_\beta$, but remains in the set $F_\beta U$.

### 4 Boundary Region and Mechanism for the Two Agent Case

In the previous section we have described the mechanism $(\hat{p}, \hat{w})$ for the central region. In order to complete the description of the mechanism, we need to specify the mechanism for the “boundary region,” or, when any of the initial state $u_i$ is below the threshold $u_\beta$. As it turns out, the two-agent case is much simpler to describe compared with the general $n > 2$ case. In this section, we focus on the two-agent case. We generalize the analysis to larger $n$ in Section 6.

We start by specifying the scalars $F_\beta$ and $u_\beta$ as follows:

$$F_\beta = 1 - \frac{u_\beta}{E[v]}$$

and

$$u_\beta = \xi (1 - \beta),$$

where $\xi$ is a constant scalar independent of $\beta$, and is given in equation (33) in the appendix. In fact, the threshold $u_\beta$ is determined so that the efficient frontier $E(F_\beta U)$ intersects with each axis at the point $E[v] - u_\beta$. Therefore, for any $u$ in the boundary region of $F_\beta U$, that is, when either $u_1$ or $u_2$ is below the threshold $u_\beta$, the state $u$ must be within the lower triangle set $L$. See Figure 2b for an illustration.

We have already described the mechanism $(\hat{p}, \hat{w})$ for the central region in the previous section. For a state $u$ in the boundary region, because it is also in the lower triangle region, we define the mechanism to be, simply, random allocation. That is,

$$\hat{p}(v|u) = p^L(v|u) = u/E[v], \quad \text{and} \quad \hat{w}(v|u) = w^L(v|u) = u,$$

if $\exists u_i < u_\beta, \ i \in \{1, 2\}$.

Following Remark 2.1, the mechanism $(\hat{p}, \hat{w})$ satisfies $[IC], [FA], [PK(u)]$ for $u$ in the boundary region. Furthermore, it is obvious that future promises remain in the set $F_\beta U$ in this region. Therefore, the boundary region is self-generating with respect to mechanism $(\hat{p}, \hat{w})$. 

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Now we have completed the description of mechanism \((\hat{p}, \hat{w})\) for all initial state \(u \in F_\beta U\), we are ready to proceed with the claim that the set \(F_\beta U\) is self-generating with respect to the mechanism \((\hat{p}, \hat{w})\). What remains to be shown is that following the definitions of \(F_\beta\) and \(u_\beta\) in [16], the future promises indeed remain in set \(F_\beta U\) according to the main phase mechanism. This is formally established in the following result.

**Proposition 4.1.** Let \(\beta \in (0, 1)\) be such that \(F_\beta \geq 0.5\). For any \(\beta \geq \beta\) and initial state \(u \in F_\beta U\), the ex-post future promise function \(\hat{w}(v|u) \in F_\beta U\).

The above key result of our paper relies crucially on the design of the constant \(\xi\) in [16], which determines the scaling factor \(F_\beta\) and the lower bound \(u_\beta\). First of all, the lower bound \(u_\beta\) has to be high enough so that the future promises \(\hat{w}(v|u)\), defined in [14], remain positive. To achieve this we first provide an upper bound on \(\|\hat{w}(v|u) - u\|_2\), which measures the distance between current and future promised utilities, in terms of \(\beta\) and other model parameters. Using this bound we show that future promises remain positive for all \(u\) with \(u_i \geq u_\beta\), when \(\xi\) is suitably chosen.

Second, and perhaps more importantly, we need to make sure that the boundary of the convex central region \(F_\beta U\) (its efficient frontier) is “flat” enough so that starting from a point \(u\) very close to the boundary, the next point \(\hat{w}(v|u)\) does not fall outside of it. Bounding the curvature of the efficient frontier \(E(F_\beta U)\) is not easy. We instead work with the signed distance function between a point \(u\) and the convex set \(U\), which is defined as

\[
I_U(u) = \max_{x: \|x\|_1 = 1, x \geq 0} x^T u - \phi(x).
\] (17)

The signed distance \(I_U(u)\) measures the distance of a point to the efficient frontier of the PI set \(U\). It is positive if the point lies outside the set, zero if on the efficient frontier of the set, and negative otherwise. Following [5], we obtain that \(I_U(u) = \alpha(u)^T u - \phi(\alpha(u)) = \alpha(u)^T (u - \nabla \phi(\alpha(u)))\), because \(\phi(\alpha) = \nabla \phi(\alpha)^T \alpha\). Recall that \(\nabla \phi(\alpha)\) corresponds to a point on the efficient frontier of the PI set with normal \(\alpha\). Therefore, the signed distance function measures the “directional distance” of a point \(u\) to the “closest” point on the efficient frontier.

The signed distance between any point \(u\) and the scaled set \(F_\beta U\) is given by \(I_U(u/F_\beta)\). We design the constant \(F_\beta\) so that \(I_U(\hat{w}(v|u)/F_\beta) \leq 0\), which guarantees that future promises \(\hat{w}(v|u)\) lie in the scaled set \(F_\beta U\). Here are some intuitions using the shorthand notation \(\hat{w} = \hat{w}(v|u)\). Consider the
following quadratic approximation of the signed distance function at the current state $u/F_\beta$,

$$I_{\mathcal{U}}\left(\frac{\hat{w}}{F_\beta}\right) \approx I_{\mathcal{U}}\left(\frac{u}{F_\beta}\right) + \nabla I_{\mathcal{U}}\left(\frac{u}{F_\beta}\right)^T \left(\frac{\hat{w} - u}{F_\beta}\right) + \frac{1}{2} \left(\frac{\hat{w} - u}{F_\beta}\right)^T \text{Hess} I_{\mathcal{U}}\left(\frac{u}{F_\beta}\right) \left(\frac{\hat{w} - u}{F_\beta}\right),$$

where $\text{Hess} I_{\mathcal{U}}(u) \in \mathbb{R}^{n \times n}$ represents the Hessian of function $I_{\mathcal{U}}$ evaluated at $u$. The zeroth-order term is nonnegative because the current state $u$ lies in $F_\beta \mathcal{U}$. The Envelope Theorem implies that $\nabla I_{\mathcal{U}}(u) = \alpha(u)$. Thus, the first-order term is negative and independent of $\hat{w}$, according to Proposition 3.1. Because the signed distance function $I_{\mathcal{U}}$ is convex, the second-order term is nonnegative. We control the contribution of the second-order term by bounding the maximum eigenvalue of the Hessian matrix in terms of model parameters and using the aforementioned bound on $\|\hat{w} - u\|_2$. As it turns out, our design of the constant $\xi$ allows us to ensure that the signed distance of $\hat{w}$ is at most 0. Therefore, future promises $\hat{w}$ fall below the efficient frontier $\mathcal{E}(F_\beta \mathcal{U})$ starting from any point $u \in F_\beta \mathcal{U}$.

Proposition 4.1 implies that starting from any state $u \in F_\beta \mathcal{U}$, we can construct a sequence of allocation and future promises which delivers that initial state. That is, every state in $F_\beta \mathcal{U}$ is achievable following this mechanism. In particular, consider the state that maximizes $u_1 + u_2$ in $F_\beta \mathcal{U}$. The mechanism $(\hat{p}, \hat{w})$ is able to achieve the social welfare given by

$$J_\beta \triangleq \max_{u \in F_\beta \mathcal{U}} (u_1 + u_2) = F_\beta \max_{u \in \mathcal{U}} (u_1 + u_2) = F_\beta J_{FB},$$

because $u \in F_\beta \mathcal{U}$ if and only if $u/F_\beta \in \mathcal{U}$. Because the set $F_\beta \mathcal{U}$ is self-generating, it is a subset of $\mathcal{U}_\beta$, following Proposition 2.1. Therefore, the total social welfare satisfies $J_\beta \leq J_\beta^* = \max_{u \in \mathcal{U}_\beta} (u_1 + u_2)$. Overall, we have the following main theorem.

**Theorem 4.1.** Let $\beta \in (0, 1)$ be such that $F_\beta \geq 0.5$. For any $\beta \geq \beta$, we have

$$F_\beta J_{FB} = J_\beta \leq J_\beta^* \leq J_{FB}.$$  

It is clear that $F_\beta$ as defined in (16) approaches one as $\beta$ approaches one. Therefore, the achievable set $F_\beta \mathcal{U}$ approaches the perfect information set $\mathcal{U}$, and, correspondingly, the optimally achievable social welfare $J_\beta$ approaches the first-best social welfare $J_{FB}$. In particular, the convergence rate of our mechanism $(\hat{p}, \hat{w})$ to efficiency is $1 - F_\beta = O(1 - \beta)$. Additionally, because $F_\beta \mathcal{U}$ converges to the PI achievable set $\mathcal{U}$ as $\beta$ approaches one, every point in the PI achievable set is asymptotically achievable.

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according to our mechanism.

Now that we have completed the description of our mechanism, in the next section we provide some economic insights into our mechanism, before we proceed with the mechanism for the general $n$ agent case.

5 Economic Insights

In this section, we shed light on the economic insights derived from our mechanism. In particular, we explain, intuitively, why our main phase mechanism is dynamically incentive compatible (inducing agents to report truthfully) and approximately efficient (approaching first-best social welfare as the discount factor converges to one).

Incentive Compatibility

The main phase mechanism achieves incentive compatibility by introducing intertemporal substitution of consumptions. That is, according to our mechanism, reporting a high value today reduces the chance of receiving the resource in the future. Consequently, if today’s value is not as high, agents may forego today’s allocation in view of more valuable future opportunities. This effect stems from the following results.

First, according to (12), in each period the allocation is determined by the comparison of weighted values among agents. The weights are determined by the promised utilities $u$. Following Remark 3.1, it is clear that for any $u$ in the central region $F_\beta\mathcal{U}$, if agent $i$’s promised utility is larger than agent $j$’s, i.e., $u_i > u_j$, then $\alpha_i > \alpha_j$. This implies that the higher an agent’s promised utility, the higher the chance that the agent receives the resource. Furthermore, $(IC)$ implies that higher valuation $v_i$ also increases agent $i$’s chance of receiving the resource. This is summarized in the following result.

Proposition 5.1. Agent $i$’s allocation $\hat{p}_i(v|u)$ is non-decreasing in the agent’s report $v_i$ (for fixed $v_{-i}$ and $u$), and non-decreasing in the agent’s promised utility $u_i$ (for fixed $v$ and $u_{-i}$).

Incentive compatibility implies that the interim future promise function of each agent is non-increasing in his own report. That is, an agent’s future promised utility tends to be lower if the current period report is higher. The following result characterizes the ex-post future promise function.
Proposition 5.2. The future promise function $\hat{w}_i(v|u)$ is non-increasing in $v_i$ and non-decreasing in $v_j$ for $j \neq i$ (for fixed $u$).

As a result, reporting a higher value entails a higher chance of receiving the resource in the current period, at the expense of a lower future promised utility. This, in turn, translates to a lower chance of receiving the resource in the future. This provides an intuitive explanation on how the mechanism ensures that agents do not report higher than their true values.

Furthermore, it is interesting to see how other agents’ reports affect a focal agent’s future promise. Figure 3b illustrates such a point. Compare, for example, points $A$ and $C$, or $B$ and $D$. As agent 2’s value increases from 0 to 1, agent 1’s future promised utility $w_1$ increases. Intuitively, if another agent other than $i$ reports a higher value, it decreases agent $i$’s chance of receiving the resource in the current period. Agent $i$ is then compensated with a higher future utility, to be fulfilled by future allocations.

Efficiency

The main phase mechanism starts at the state $F_\beta u^*$, where $u^*$ is the vector of agents’ utilities under the efficient allocation. That is, $u_i^* = E_v [v_i \mathbf{1} \{v_i \geq \max_{j \neq i} v_j\}]$. Therefore, the mechanism starts at the state with the largest component sum in the scaled set $F_\beta U$. At this state, the allocation is efficient. In fact, if a state $u$ has equal components, i.e., $u_i = u_j$ for all $i, j$, then the allocation $\hat{p}(v|u)$ is always efficient, because the weights $\alpha_i(u/F_\beta)$ are the same for all $i$.

In Figure 4, we plot sample trajectories of promised utilities starting at state $F_\beta u^*$ following our mechanism. As we can see, the sample trajectories concentrate around the 45 degree line. Correspondingly, the weights $\alpha_i$ of all agents tend to be close to each other along these trajectories. As a result, the resource is allocated almost efficiently, until it reaches the boundary region. In Figure 4a, all trajectories reach the boundary region within 450 time periods.

As agents become more patient and the discount factor $\beta$ increases, the step size between current promised utility $u$ and future promises $\hat{w}$ decreases (see Lemma C.2 in the Appendix). In Figure 4b, the first 250 steps of each trajectory are marked black while later steps are colored grey. As we can see, after 250 time periods the promised utilities are still concentrated around the initial state, and it takes longer to reach the boundary region in this case.

Figure 5 illustrates the expected social welfare per round generated by our mechanism, which is
Figure 4: Each figure demonstrates 100 sample trajectories.

given by $\mathbb{E}_{u_t,v_t} \left[ \sum_{i=1}^{n} v_{i,t} \hat{p}_i(v_t|u_t) \right]$, as a function of time. During the first time periods, the mechanism is in the main phase and the expected social welfare per round is very close to that of the efficient allocation. Over time, trajectories drift to the boundary region where the expected social welfare per round drops significantly because there the allocation is highly inefficient. The boundary mechanism, albeit inefficient, is necessary to ensure that the incentive compatibility and promise keeping constraints are sustained. Furthermore, as the discount factor increases, the mechanism remains in the main phase for more time periods. As we can see from Figure 5a, it takes between 400 and 500 periods for all trajectories to reach the boundary region. In comparison, for a higher discount factor, Figure 5b shows that it takes between 1000 and 1500 time periods to reach the boundary region. As the time discount factor approaches one, the boundary mechanism is pushed further into the future and the mechanism allocates approximately efficiently for most of the horizon.

6 Generalization to $n$ Agents

In this section, we generalize the analysis to settings with $n$ agents, where $n > 2$. In this case, we need to carefully design the scaling factor and lower bound, and the boundary mechanism depending on the
number \( n \) of agents involved. We first describe the difficulties that arise in the general case, and then provide our mechanism and analysis.

First of all, the boundary mechanism for the two-agent setting no longer works for the general setting. When there are only two agents, in the boundary region we use the randomization mechanism \((\hat{p}^L, \hat{w}^L)\). If \( n > 2 \), however, the boundary region is not always contained within the lower triangle set \( L \) anymore, as long as the lower bound for the \( n \) agent case, \( u^{(n)}_\beta \), approaches zero with \( \beta \) approaching one. As a result, the randomized allocation is no longer feasible for the boundary region.

Specifically, consider, for example, a three-agent setting as illustrated in Figure 6. Here we focus on a situation where one agent’s promised utility is below the threshold \( u^{(3)}_\beta \). In Figure 6a, we plot the efficient frontier of the PI set \( E(\mathcal{U}) \), its scaled version \( E(F_\beta \mathcal{U}) \), and the plane corresponding to \( u_1 = 0.01 < u^{(3)}_\beta = 0.0167 \). Figure 6b demonstrates the intersections between the plane with efficient frontiers of sets \( \mathcal{U}, F_\beta \mathcal{U}, \) and \( L \), respectively. All points on the intersection of the plane and \( F_\beta \mathcal{U} \) lie in the boundary region. The state \( u \) represented by the circle, however, is outside the efficient frontier of the lower triangle set. At this state, \( u_1 + u_2 + u_3 > E[v] \), which implies that the randomized allocation \( \hat{p}^L \) is no longer feasible.

While the two-agent mechanism allocates the resource randomly when some agent’s promise utility lies in the boundary region, the three-agent mechanism shall allocate randomly only to the player...
with low promised utility and implement the two-agent mechanism for the other players. Because the two-agent mechanism has been shown to attain every point in the two-agent PI achievable set as the discount factor increases, this construction allows us to attain points otherwise not achievable with random allocations.

![Efficient frontiers](image1.png)  
![Level set](image2.png)

**Figure 6:** In Figure 6a, the outer surface with a solid mesh is $E(U)$. The inner surface with a dotted mesh is the scaled efficient frontier $E(F_{\beta}U)$. The horizontal plane represents the slice corresponding to $u_1 = 0.01$, which is lower than the threshold $u_1^{(3)} = 0.0167$. Figure 6b plots the intersection of the surfaces in Figure 6a with level set $u_1 = 0.01$ in the $(u_2, u_3)$ space. In both figures, the value distributions are assumed to be uniform $[0, 1]$, $F_{\beta} = 0.8$, and $u_1^{(3)} = 0.0167$.

More generally, suppose the main phase mechanism defined in Section 3 carries the promised utilities into a state in the boundary region, which means that at least some promised utility, say $u_i$, is below the threshold $u_{i}^{(n)}$. By allocating the resource from this period on to agent $i$ with probability $u_i/E[v]$, we can guarantee that agent $i$’s future promise $\hat{w}_i(u|v)$ remains at $u_i$. As such, we convert the problem into one at a lower dimensional space. Therefore, when there are more than two players, the boundary region mechanism can be defined recursively, depending on how many players are still involved.

**Mechanism.** We refer to the agents with promise utilities above the threshold $u_{i}^{(n)}$ as the active agents, in which $u_{i}^{(n)}$ is defined as

$$u_{i}^{(n)} = \xi^{(n)}(1 - \beta)\frac{1}{n+1}.$$
Here, $\xi^{(n)}$ is a constant scalar independent of $\beta$, provided in Definition E.2 in the appendix.

At any point in time, the allocation and future promise for an inactive agent $i$ are, simply, $\hat{p}_i(v|u) = u_i/E[v]$ and $\hat{w}_i(v|u) = u_i$, respectively. The mechanism for the active agents resembles the main phase mechanism described in Section 3.

Consider a case with $k$ active agents. That is, $k$ out of the $n$ components in the state vector $u \in \mathbb{R}^n$ are above the threshold $u^{(n)}_\beta$. Denote the subvector $u^{(k)} \in \mathbb{R}^k$ to represent the promised utilities for the active agents. Further define the total probability that the resource is allocated to an active agent to be

$$s(u) = 1 - \sum_{i=1}^{n} u_i \mathbf{1}\{u_i < u^{(n)}_\beta\}/E[v].$$

Consider the $k$ dimensional PI set $U^{(k)}$, the corresponding support function $\phi^{(k)}$, and weights $\alpha^{(k)}$, as defined in (6), (7), and (8), respectively. (In these definitions the state vector $u$ corresponds to a $k$ dimensional vector.) Similar to Section 3, we scale the PI set $U^{(k)}$ with a factor $s(u) F^{(k,n)}_\beta$, in which $F^{(k,n)}_\beta$ is defined as the following

$$F^{(k,n)}_\beta \triangleq 1 - n(k-1)u^{(n)}_\beta/E[v].$$

Note that the scaling of the PI set needs to involve the factor $s(u)$, because with probability $1 - s(u)$ the resource is allocated to the inactive agents.

For the $n$ agent case, the constants $u_\beta$ and $F_\beta$ in Section 3 correspond to $u^{(n)}_\beta$ and $F^{(n,n)}_\beta$, respectively. In the boundary region with $k$ active agents, the allocation of our mechanism for an active agent $i$ is defined similarly to the main phase mechanism (12) for the $k$ agent case. That is,

$$p_i^{(k)}(v|u) = s(u) \mathbf{1}\left\{ \alpha_i^{(k)} \left( \frac{u^{(k)}}{s(u) F^{(k,n)}_\beta} \right) v_i \geq \max_{j \neq i} \alpha_j^{(k)} \left( \frac{u^{(k)}}{s(u) F^{(k,n)}_\beta} \right) v_j \right\}. \quad (18)$$

The corresponding future utility is defined similar to (14), as

$$w_i^{(k)}(v|u) = W_i^{(k)}(v_i|u) - \frac{1}{k-1} \sum_{j \neq i} \alpha_j^{(k)} \left( \frac{u^{(k)}}{s(u) F^{(k,n)}_\beta} \right) \left\{ W_j^{(k)}(v_j|u) - E_{\tilde{v}_j} \left[ W_j^{(k)}(\tilde{v}_j|u) \right] \right\}. \quad (19)$$
Here, and the interim future promise function $W_i^{(k)}$ is as defined in (13) of Section 3, with the interim allocation function defined accordingly.

To summarize, at the beginning of each time period, the number of active agents $k$ is updated to reflect the remaining number of active agents. Then our mechanism $(\hat{p}, \hat{w})$ is defined as the following. The allocation is given by

$$\hat{p}_i(v|u) = \begin{cases} p_i^{(k)}(v|u), & \text{if } u_i \geq u_{n}^{(n)}, \\ u_i/E[v], & \text{otherwise}. \end{cases}$$

(20)

The future promise function is given by

$$\hat{w}_i(v|u) = \begin{cases} w_i^{(k)}(v|u), & \text{if } u_i \geq u_{n}^{(n)}, \\ u_i, & \text{otherwise}. \end{cases}$$

(21)

**Self-generating Set.** Note that when the number of agents $n$ is larger than 2, the set $F_{\beta}^{(n,n)}U^{(n)}$ is not self-generating with respect to our mechanism. This is because as the number of active agents decreases to $k < n$, the PI achievable set $U^{(k)}$ is not a scaled version of the intersection between the $n$-dimensional PI set and a subspace $\mathbb{R}^k$. In Figure 6b, for example, the solid curve marks the boundary of the intersection between the 3-dimensional PI achievable set $U^{(3)}$ and the subspace $u_1 = 0.01$. However, this set is different from the efficient frontier of the 2-dimensional PI achievable set $U^{(2)}$, even with scaling. Consequently, when the number of active agents drops to $k < n$, there is a priori no guarantee that the promise utility lies in the set $s(u)F_{\beta}^{(k,n)}U^{(k)}$.

Therefore, we define the following set $\Omega_{\beta}$, which we show to be self-generating with respect to our mechanism $(\hat{p}, \hat{w})$. Some additional notations are in order. For a vector $u \in \mathbb{R}^n$ and set $\kappa \subset \{1, \ldots, N\}$, we denote $u^{(\kappa)}$ to represent the subvector corresponding to components of $u$ in the index set $\kappa$. We denote the complement of $\kappa$ by $\bar{\kappa}$. An inequality between a vector and a scalar means that each component of the vector satisfies this inequality. The set $\Omega_{\beta}$ is given by

$$\Omega_{\beta} = \bigcup_{\kappa \subseteq \{1, \ldots, N\}} \left\{ u \in \mathbb{R}^n \mid u^{(\kappa)} \geq u_{\beta}^{(n)}, u^{(\bar{\kappa})} < u_{\beta}^{(n)}, \text{ and } u^{(\kappa)} \in s(u)F_{\beta}^{(k,n)}U^{(k)} \right\} .$$

That is, for any state $u$ in $\Omega_{\beta}$ with $k$ active agents, the scaled PI achievable set $s(u)F_{\beta}^{(k,n)}U^{(k)}$ is
contained in the $\Omega_\beta$ set. The following proposition confirms that our construction of the lower bounds ensures that $\Omega_\beta$ is indeed self-generating.

**Proposition 6.1.** The set $\Omega_\beta$ is self-generating with respect to the mechanism $(\hat{p}, \hat{w})$ defined in (20)-(21).

In order to show that $\Omega_\beta$ set is self-generating with respect to our mechanism $(\hat{p}, \hat{w})$, we need to argue that for all $u$ in $\Omega_\beta$, the mechanism given in (20)-(21) satisfies the conditions given in (3). The (IC), (FA), and (PK(u)) constraints follow by construction. The main step of the proof involves showing that future promises satisfy $\hat{w}(v|u) \in \Omega_\beta$ for every report $v$.

For any state $u$ in $\Omega_\beta$ with $k$ active agents, it is clear that future promised utilities of inactive agents remain in $\Omega_\beta$. In the proof of Proposition 6.1 we first show that the subvector of future promises satisfies

$$w^{(k)}(v|u) \in s(u)F_\beta^{(k,n)}U^{(k)}.$$  \hspace{1cm} (22)

This step extends the geometric approach of Proposition 4.1 to higher dimensions by using the fact that the mechanism for active agents also satisfies the properties in Section 3. (In particular, in Appendix G we prove extensions of Propositions 3.1 and 3.2 that account for the scaling factor.)

Condition (22) alone is not sufficient to establish our result because $\hat{w}(v|u)$ could involve fewer active agents than $u$. That is, in the next step the number of active agents may decrease to $k' < k$. We need to show the stronger result that the subvector $w^{(k')}(v|u)$, consisting of the components of $\hat{w}(v|u)$ above the threshold $u^{(n)}_\beta$, lies in $s(\hat{w}(v|u))F_\beta^{(k',n)}U^{(k')}$. Therefore, we design the constant $\xi^{(n)}$ accordingly such that the intersection between $s(u)F_\beta^{(k,n)}U^{(k)}$ and the $k'$ dimensional space of active agents is contained in $s(\hat{w}(v|u))F_\beta^{(k',n)}U^{(k')}$. With this argument, (22) is sufficient to guarantee that all future promises lie in $\Omega_\beta$.

Now we are ready to state the next theorem, which is the main result of this section.

**Theorem 6.1.** There exists $\beta \in (0,1)$ such that for any $\beta \geq \beta$ the mechanism $(\hat{p}, \hat{w})$ for $n$ agents satisfies

$$F_\beta^{(n,n)}J^{FB}_\beta \leq J_\beta \leq J^*_\beta \leq J^{FB}.$$  

Because the scaling factor $F_\beta^{(n,n)}$ converges to one as $\beta$ approaches one, Theorem 6.1 implies that the maximum expected discounted social welfare achieved by the mechanism $(\hat{p}, \hat{w})$ approaches to
first-best. In particular the convergence rate is \( 1 - F^{(n,n)}_{\beta} = O \left( (1 - \beta)^{\frac{1}{n+4}} \right) \). Similar to the two agent case, because \( \omega^{(n)}_{\beta} \) converges to zero as \( \beta \) converges to one, set \( \Omega_{\beta} \) converges to the PI achievable set \( \mathcal{U} \) as \( \beta \) approaches one. Thus, every point in the PI achievable set is asymptotically achievable following our mechanism.

Note that the convergence rate to first-best given in Theorem 6.1 for \( n = 2 \) is slower than the one in Theorem 4.1. The proof for the two-agent case given in Section 4 leverages some special structure of the problem, which is not present in the general case. One may be able to provide better convergence rates by tightening the analysis. We leave this to future research.

7 Conclusion and Extensions

In this paper, we study resource allocation with asymmetric information and no monetary transfer in a dynamic setting. The marginal cost for the resource is zero in each period. Therefore, the mechanism designer focuses on allocation efficiency. We propose a mechanism that achieves asymptotic efficiency as the time discount factor approaches one.

Our analytical framework is focused on self-generating sets of agents’ promised utilities. The essence of our approach is to establish that a self-generating set with respect to our mechanism expands as the discount factor increases, and eventually approaches the set of utilities achievable when values are publicly observable.

There are a number of potential extensions to our work. For example, in our current mechanism, future promises are determined only through the interim, and not ex-post, allocation. If we perceive promised utilities as money, our mechanism requires the planner to introduce lotteries, which may not be appealing in practice. It is, therefore, interesting to explore whether it is possible to asymptotically achieve efficiency with mechanisms in which future promises depend on the ex-post allocations. Such a mechanism, if exists, establishes an indirect implementation without lotteries. That is, the agent who receives the resource in a period pays with future promises, while other agents are potentially compensated by higher future promises.

Along the line of thinking about ex-post versus interim promised utilities, we can also consider varying the incentive compatibility constraint. Currently we enforce incentive compatibility at the “interim” level. That is, truthful reporting is each agent’s best strategy, taking expectations with
respect to other agents’ values and assuming competitors report truthfully. Alternatively, one can enforce certain “ex-post” incentive compatibility. That is, truthful reporting could be weakly dominant for every player in each period regardless of other players’ reports (and assuming all agents report truthfully in the future). The optimal social welfare under ex-post incentive compatibility is less than or equal to that of our setting for any fixed discount factor, because there are more constraints in the definition of the self-generating set. It remains to see if one can still establish asymptotic efficiency in this case.

Another extension is to consider a positive marginal production cost for the resource in each period. If the cost is positive, the mechanism designer needs to trade-off efficiency with production cost. In this case the planner may have to withhold the resource if agents’ valuations are low. In such a setting, merely studying the achievable set of promised utilities is no longer sufficient. In a companion paper, we establish that asymptotic optimality is still achievable in that environment.
References


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A Appendix for Section 2

A.1 Proof of Proposition 2.1

Proof. We need to show that if a set \( A \) satisfies \( A \subseteq B_\beta(A) \), then \( B_\beta(A) \subseteq U_\beta \). We prove this by constructing, for each point \( u \in B_\beta(A) \), a mechanism \( \pi \) satisfying (1) such that \( u_i = V_i(\pi, 1) \).

Step 1 (constructing mechanism \( \pi \)). Let \((\hat{v}_t)_{t \geq 1}\) be a sequence of reports. We construct a mechanism recursively that depends exclusively on past reports (and not on the allocations) as follows. First, let \( u_1 = u \). Because \( u_1 \in B_\beta(A) \), there exists a pair of functions \((p^u_1, w^u_1)\) satisfying (IC), (PK(\(u_1\)), and (FA) and \( w^u_1(v) \in A \) for all \( v \). We proceed for \( t > 1 \) by setting \( u_t = w^u_{t-1}(\hat{v}_{t-1}) \). Because \( u_t \in A \subseteq B_\beta(A) \), there exists a pair of functions \((p^u_t, w^u_t)\) satisfying (IC), (PK(\(u_t\)), and (FA) and \( w^u_t(v) \in A \) for all \( v \). The mechanism is thus given by \( \pi_t(v, h_t) = p^u_t(v) \) with the history \( h_t = v_{1:t-1} \) given by the past reports (which recursively determines the promise utility \( u_t \)).

Step 2 (mechanism \( \pi \) satisfies \( u_i = V_i(\pi, 1) \)). This result follows from the fact that \((p^u_t, w^u_t)\) satisfy (PK(\(u_t\))) for all \( t \geq 1 \). In particular, we replace the continuation values with each \( w^u_t(v) \) by using the promise keeping constraints. More formally, when agents bid truthfully and \( \hat{v}_t = v_t \), we obtain by iterating up to a time \( \tau < \infty \)

\[
u_i = (1 - \beta)E[v_{i,1}p^u_{1}(v_1)] + \beta E[w^u_{1}(v_1)] = (1 - \beta)E[v_{i,1}p^u_{1}(v_1)] + \beta E[u_{2,1}]
\]

\[
= (1 - \beta) \sum_{t=1}^{\tau} \beta^{t-1}E[v_{i,t}p^u_{t}(v_t)] + \beta^{\tau} E[u_{i,\tau+1}]
\]

\[
= (1 - \beta) \sum_{t=1}^{\tau} \beta^{t-1}E[v_{i,t}p^u_{t}(v_t)] + \beta^{\tau} E[u_{i,\tau+1}],
\]

where the first, second and third equalities follow from the fact that \((p^u_t, w^u_t)\) satisfy (PK(\(u_t\))) for all \( t \geq 1 \) and using that \( u_t = w^u_{t-1}(v_{t-1}) \) together with the fact that values are identically distributed; and the fourth equality follows from our definition of the mechanism \( \pi \). Because values have finite means and \( \beta \in (0, 1) \), we obtain that the series in absolutely convergent. Taking \( \tau \) to infinity, we conclude that \( u_i = V_i(\pi, 1) \).

Step 3 (mechanism \( \pi \) is PBIC). Fix a time period \( t \geq 1 \). We need to show that

\( V_{i,t}(\pi, 1|v_{i,t}, h_t) \geq V_{i,t}(\pi, (\sigma, 1:i-1)|v_{i,t}, h_t), \) (23)

for every history \( h_t \) and strategy \( \sigma_i \). By the one-shot deviation principle, it suffices to show that agent \( i \) has no incentive to deviate at time \( t \). Therefore, we restrict attention to strategies \( \sigma_i \) that reports \( \sigma_i,v_i,h_t = \hat{v}_{i,t} \) at time \( t \) and truthfully at time \( \tau > t \). Additionally, because the mechanism \( \pi \) only depends on the past reports via the promise utility \( u_t \), it suffices to consider \( u_t \) as the state.

Because under the strategies \( (\sigma, 1:i-1) \) the other agents report truthfully at time \( t \) and all agents report truthfully at times \( \tau > t \), we obtain using a similar argument to that of step 2 that

\[
V_{i,t}(\pi, (\sigma, 1:i-1)|v_{i,t}, h_t) = (1 - \beta)v_{i,t}E_{v_{i,t}}[p^u_{i,t}(\hat{v}_{i,t}, v_{i,t})] + \beta E_{v_{i,t}}[w^u_{i,t}(\hat{v}_{i,t}, v_{i,t})],
\]

where \( \hat{v}_{i,t} = \sigma_{i,t}(v_{i,t}, h_t) \) is the report of agent \( i \) at time \( t \). Therefore, we can write (23) as

\[
(1 - \beta)v_{i,t}E_{v_{i,t}}[p^u_{i,t}(v_t)] + \beta E_{v_{i,t}}[w^u_{i,t}(v_t)] \geq (1 - \beta)v_{i,t}E_{v_{i,t}}[p^u_{i,t}(\hat{v}_{i,t}, v_{i,t})] + \beta E_{v_{i,t}}[w^u_{i,t}(\hat{v}_{i,t}, v_{i,t})],
\]

39
which follows because the function \( p^w_i \) satisfies (IC) and agents’ values are identically distributed.

\[ \Box \]

### A.2 Proof of Proposition 2.2

**Proof.** Recall that the set of achievable utilities is given by

\[ \mathcal{U}_\beta \triangleq \{ u \in \mathbb{R}^n | u_i = V_i(\pi,1) \} , \text{ for a PBIC mechanism } \pi \],

while the operator \( B_\beta(\cdot) \) is defined in (3). We prove that \( \mathcal{U}_\beta = B_\beta(\mathcal{U}_\beta) \) by showing that \( \mathcal{U}_\beta \subseteq B_\beta(\mathcal{U}_\beta) \), since by Proposition 2.1 the latter implies that \( B_\beta(\mathcal{U}_\beta) \subseteq \mathcal{U}_\beta \). For a given point \( u \in \mathcal{U}_\beta \), we show that \( u \in B_\beta(\mathcal{U}_\beta) \) by constructing an allocation function \( p(\cdot) \) and a future promise function \( w(\cdot) \), such that \((p,w)\) satisfies (IC), (FA), (PK(u)) and \( w(v) \in \mathcal{U}_\beta \) for all \( v \).

**Step 1 (constructing stage mechanism \((p,w)\)).** Because \( u \in \mathcal{U}_\beta \), there exists a PBIC mechanism \( \pi \) satisfying (P) and \( \pi_t \in \mathcal{P} \) for all \( t \), and \( u_i = V_i(\pi,1) \). We use the first period allocation and the continuation values induced by mechanism \( \pi \) to construct the stage mechanism \((p,w)\). For all \( v \), we set \( p(v) = \pi_1(v,\emptyset) \) and

\[ w_i(v) = (1 - \beta) \sum_{\tau = 2}^{\infty} \beta^{\tau - 2} E_{\tau}^i [v_i,\pi_i,\pi(v,\emptyset)] = (v_i,\pi_1(v,\emptyset))] . \tag{24} \]

Note that the function \( p \) trivially satisfies (FA) since \( p(v) = \pi_1(v,\emptyset) \) and \( \pi_1(v,\emptyset) \in \mathcal{P} \) for all \( v \). The promise keeping constraint holds because

\[ u_i = V_i(\pi,1) = (1 - \beta) E_{v_i} [p_i(v)] + \beta E_{v_i} [w_i(v)] , \]

where the last equation follows from our definition of the allocation function \( p \) and the future promise function \( w \). Thus the stage mechanism \((p,w)\) satisfies (PK(u)).

**Step 2 (stage mechanism \((p,w)\) satisfies (IC)).** Since the mechanism \( \pi \) is PBIC, we obtain that at time \( t = 1 \) agent \( i \) has no incentive to misreport his value:

\[ V_{i,1}(\pi,(\sigma_i,1-i))|v_{i,1},\emptyset) \leq V_{i,1}(\pi,1|v_{i,1},\emptyset) , \forall v_{i,1} . \]

Consider the strategy \( \sigma_i \) that reports \( \sigma_i(v_{i,1},\emptyset) = \hat{v}_{i,1} \) at time \( t \) and truthfully at time \( t > 1 \). We obtain using our definition of the stage mechanism \((p,w)\) that

\[ V_{i,1}(\pi,(\sigma_i,1-i)|v_{i,1},\emptyset) = (1 - \beta) v_{i,1} E_{v_{i-1}} [p_i(\hat{v}_{i,1},v_{i-1})] + \beta E_{v_{i-1}} [w_i(\hat{v}_{i,1},v_{i-1})] , \]

and

\[ V_{i,1}(\pi,1|v_{i,1},\emptyset) = (1 - \beta) v_{i,1} E_{v_{i-1}} [p_i(v_{i,1},v_{i-1})] + \beta E_{v_{i-1}} [w_i(v_{i,1},v_{i-1})] . \]

This implies that the stage mechanism \((p,w)\) satisfies (IC).

**Step 3 (for all \( v \), \( w(v) \in \mathcal{U}_\beta \)).** Fix a report \( \hat{v}_1 \in [0,\bar{v}]^n \) for period 1. We prove that \( w(\hat{v}_1) \in \mathcal{U}_\beta \), by constructing a PBIC mechanism \( \tilde{\pi} \) such that \( w_i(\hat{v}_1) = V_i(\tilde{\pi},1) \). The proposed mechanism \( \tilde{\pi} \) involves implementing at time \( t \) the allocation of mechanism \( \pi \) at \( t + 1 \) given that the reports of the first period \( \hat{v}_1 \), that is, we shift all allocations one time period. For any history \( h_t \in \mathcal{H}_t \) for the shifted mechanism \( \tilde{\pi} \) consider the history \( h_{t+1} \in \mathcal{H}_{t+1} \) for the original mechanism \( \pi \) given by
identically distributed, and the third equation follows from our definition of the shifted mechanism and using that values are identically distributed. The second term is given by

\[ w_i(\hat{v}_1) = (1 - \beta) \sum_{\tau=2}^{\infty} \beta^{\tau-2} E_{\pi,1,\tau} [v_{i,\tau} \pi_{i,\tau}(v_{\tau}, h_{\tau})] h_2 = (\hat{v}_1, \pi_1(\hat{v}_1, \emptyset)) \]

\[ = (1 - \beta) \sum_{\tau=1}^{\infty} \beta^{\tau-1} E_{\pi,1,\tau} [v_{i,\tau} \pi_{i,\tau+1}(v_{\tau}, h_{\tau+1})] h_2 = (\hat{v}_1, \pi_1(\hat{v}_1, \emptyset)) \]

\[ = (1 - \beta) \sum_{\tau=1}^{\infty} \beta^{\tau-1} E_{\pi,1,\tau} [v_{i,\tau} \pi_{i,\tau}(v_{\tau}, \hat{h}_\tau)] |\emptyset| = V_i(\bar{\pi}, 1), \]

where the second equation follows from shifting the index in the summation and using that values are identically distributed, and the third equation follows from our definition of the shifted mechanism and its history.

Second, we show that the shifted mechanism \( \bar{\pi} \) is PBIC. For any strategy \( \bar{\sigma}_i \) for the shifted mechanism \( \bar{\pi} \), consider the strategy \( \sigma_i \) for the original mechanism \( \pi \), given by \( \sigma_{i,t+1}(\cdot, h_{t+1}) = \bar{\sigma}_{i,t}(\cdot, h_t) \). If \( \sigma_{i,t} \) shifts reports by one time period (since we consider the original mechanism at times \( t > 1 \), the reports of the strategy \( \sigma_{i,t} \) at time \( t = 1 \) are irrelevant). Because players only condition their actions on previous reports and allocations, and this information is publicly observed, we claim that for all \( v_i, \bar{\sigma}_i, h_t, i \) and \( t \geq 1 \)

\[ V_{i,t} (\bar{\pi}, (\bar{\sigma}_i, 1_{-i})|v_i, h_t) = V_{i,t+1} (\pi, (\sigma_i, 1_{-i})|v_i, h_{t+1}) \cdot \]

This readily implies that the shifted mechanism \( \bar{\pi} \) is PBIC, since the original mechanism \( \pi \) is PBIC.

We conclude by proving the claim. We have that \( V_{i,t+1} (\pi, (\sigma_i, 1_{-i})|v_i, h_{t+1}) = (I) + (II) \) where the first term is given by

\[ (I) = (1 - \beta) v_i E_{\pi,1_{-i}} [\bar{\pi}_{i,t+1}((\sigma_{i,t+1}(v_i, h_{t+1}), v_{-i,t+1}), h_{t+1})|v_i, h_{t+1}] \]

\[ = (1 - \beta) v_i E_{\pi,1_{-i}} [\bar{\pi}_{i,t}((\bar{\sigma}_{i,t}(v_i, h_t), v_{-i,t}), h_t)|v_i, h_t] \] ,

where the second equality follows from our definition of the shifted strategy, mechanism and histories and using that values are identically distributed. The second term is given by

\[ (II) = (1 - \beta) \sum_{\tau=t+2}^{\infty} \beta^{\tau-t-1} E_{\pi,1_{-i}} [v_{i,\tau} \pi_{i,\tau}((\sigma_{i,\tau}(v_{i,\tau}, h_{\tau}), v_{-i,\tau}), h_{\tau})|v_i, h_{t+1}] \]

\[ = (1 - \beta) \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E_{\pi,1_{-i}} [v_{i,\tau} \pi_{i,\tau+1}((\sigma_{i,\tau+1}(v_{i,\tau}, h_{\tau+1}), v_{-i,\tau}), h_{\tau+1})|v_i, h_{t+1}] \]

\[ = (1 - \beta) \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E_{\pi,1_{-i}} [v_{i,\tau} \pi_{i,\tau}((\sigma_{i,\tau}(v_{i,\tau}, \hat{h}_{\tau}), v_{-i,\tau}), \hat{h}_{\tau})|v_i, h_t] \]

where the second equation follows from shifting the index in the summation and using that values are identically distributed, and the third equation follows from our definition of the shifted mechanism, the shifted strategies and the shifted history. We thus conclude that \( (I) + (II) = V_{i,t} (\bar{\pi}, (\bar{\sigma}_i, 1_{-i})|v_i, h_t) \) and the claim follows.
A.3 Proof of Proposition 2.3

We provide some preliminary results before proving Proposition 2.3. First, we endow the space of stage mechanism with a topology and prove that the set of mechanism satisfying our constraints is compact. Second, we show that the operator $B$ is monotone, and preserves convexity and compactness.

We assume that the functions of the stage mechanism $p$ and $w$ lie in the space $L^\infty \triangleq L^\infty([0,\bar{v}]^n, (\mathbb{R}^n, || \cdot ||_1))$ of essentially bounded vector functions from $[0,\bar{v}]^n$ to $(\mathbb{R}^n, | \cdot |_1)$, i.e., a measurable vector function $\rho : [0,\bar{v}]^n \rightarrow \mathbb{R}^n$ lies in $L^\infty$ if $\text{ess sup}_v ||\rho(v)||_1 < \infty$ where $||\rho(v)||_1 = \sum_{i=1}^n |\rho_i(v)|$. We denote $L^1 \triangleq L^1([0,\bar{v}]^n, (\mathbb{R}^n, || \cdot ||_\infty))$ to represent the pre-dual of $L^\infty$, i.e., a measurable vector function $\psi : [0,\bar{v}]^n \rightarrow \mathbb{R}^n$ lies in $L^1$ if $\int_{[0,\bar{v}]^n} ||\psi(v)||_\infty dv < \infty$ where $||\psi(v)||_\infty = \max_{i=1}^n |\psi_i(v)|$. Since the spaces $L^\infty$ and $L^1$ are equivalence classes of functions which agree almost everywhere, all point-wise inequalities are understood to hold almost everywhere (a.e.). To simplify the exposition we drop the “almost everywhere” qualifier.

We endow $L^\infty$ with the weak-* topology, the coarsest topology under which every element $\psi \in L^1$ corresponds to a continuous (linear) map. A sequence $\rho^m \in L^\infty$ converges to $\rho$ in the weak-* topology if

$$\int \rho^m(v) \cdot \psi(v) dv \rightarrow \int \rho(v) \cdot \psi(v) dv \quad \forall \psi \in L^1,$$

where $\rho(v) \cdot \psi(v) = \sum_{i=1}^n \rho_i(v)\psi_i(v)$ denotes the standard inner product in $\mathbb{R}^n$. In this case, one writes $\rho^m \rightharpoonup \rho$ as $m \rightarrow \infty$.

Lemma A.1. The stage mechanisms satisfy the following properties:

1. Let $A \subset \mathbb{R}^n$ be a closed and convex set. The set $\{w \in L^\infty : w(v) \in A \text{ for almost all } v \in [0,\bar{v}]^n\}$ is weak-* closed.

2. Let $A \subset \mathbb{R}^n$ be a compact and convex set. The set $\mathcal{A} = \{(p, w) \in L^\infty \times L^\infty : \sum_{i=1}^n p_i(v) \leq 1, p(v) \geq 0, \text{ and } w(v) \in A \text{ for almost all } v \in [0,\bar{v}]^n\}$ is weak-* compact.

3. The set $\{(p, w) \in L^\infty \times L^\infty : (p, w) \text{ satisfy (IC)}\}$ is weak-* closed.

4. The function $E : L^\infty \times L^\infty \rightarrow \mathbb{R}^n$ given by $E(p, w) = (\mathbb{E}_v[(1-\beta)v_i p_i(v) + \beta w_i(v)])_{i=1}^n$ is weak-* continuous.

We next provide some properties on the operator $B_\beta$ defined in [3].

Lemma A.2. The operator $B_\beta : 2^{\mathbb{R}^n} \rightarrow 2^{\mathbb{R}^n}$ satisfies the following properties:

1. ($B_\beta$ is monotone) If $A \subseteq Y \subseteq \mathbb{R}^n$ then $B_\beta(A) \subseteq B_\beta(Y)$.

2. ($B_\beta$ preserves convexity) If $A \subset \mathbb{R}^n$ is convex, then $B_\beta(A)$ is convex.

3. ($B_\beta$ preserves compactness) If $A \subset \mathbb{R}^n$ is compact and convex, then $B_\beta(A)$ is compact.

We defer the proofs of these results to the end of the subsection. We are now in position to prove the main result.

Proof of Proposition 2.3. Let $U^\infty = \lim_{m \rightarrow \infty} U^m$ with $U^0 = [0,\mathbb{E}[v]]^n$ and $U^m = B_\beta(U^{m-1})$ for all $m \geq 1$. We prove this result in two steps. We first prove that the limit $U^\infty$ exists and that $U_{\beta} \subseteq U^\infty$. We then show that $U^\infty \subseteq U_{\beta}$.
Step 1. We first claim that (i) $B_β(\mathcal{U}_0) \subseteq \mathcal{U}_0$ and (ii) $\mathcal{U}_β \subseteq \mathcal{U}_\infty$. Claim (i), together with Lemma A.2 item 1, implies that $\mathcal{U}_m \subseteq \mathcal{U}_{m-1}$, and thus $\mathcal{U}_\infty = \lim_{m \to \infty} \mathcal{U}_m$ exists. Moreover, because $\mathcal{U}_β$ is self-generating we obtain by (ii) that $\mathcal{U}_β \subseteq \mathcal{U}_m$ for all $m$. This implies that $\mathcal{U}_β \subseteq \mathcal{U}_\infty$. We next prove the claims.

We show that $B_β(\mathcal{U}_0) \subseteq \mathcal{U}_0$ by proving that for all $u \in B_β(\mathcal{U}_0)$ we have $u \in \mathcal{U}_0$. Because $u \in B_β(\mathcal{U}_0)$, there exist functions $(p, w)$ satisfying (IC), (PK(u)) and (FA), such that $w(v) \in \mathcal{U}_0$ for all $v$. Since $p$ satisfies (FA), it follows that $u^1 := (\mathbb{E}[v_i p_i(v)])_{i=1}^n$ satisfies $u^1 \in \mathcal{U}_0$. Note that $\mathcal{U}_0$ is a convex set. Thus, the fact that $w(v) \in \mathcal{U}_0$ implies that $u^2 := (\mathbb{E}[v_i(v)])_{i=1}^n$ satisfies $u^2 \in \mathcal{U}_0$. Finally, (PK(u)) implies that $u = (1 - \beta)u^1 + \beta u^2$ and hence $u \in \mathcal{U}_0$ because $u$ is a convex combination of two points in $\mathcal{U}_0$.

We argue that $\mathcal{U}_β \subseteq \mathcal{U}_\infty$. Fix a point $u \in \mathcal{U}_β$. Recognizing the definition of $\mathcal{U}_β$ in (2) it follows that $u_i = V_i(\pi, 1)$ for some PBIC mechanism $\pi$. Because mechanism $\pi$ satisfies $\pi_t \in \mathcal{P}$, we conclude that $u_i \in [0, \mathbb{E}[v]]$ for all $i$. The claim follows.

Step 2. In this step, we prove the other direction, i.e., $\mathcal{U}_\infty \subseteq \mathcal{U}_β$. By Proposition 2.1, it is sufficient to prove that $\mathcal{U}_\infty \subseteq B_β(\mathcal{U}_\infty)$. Fix a point $u \in \mathcal{U}_\infty$. In order to prove that $u \in B_β(\mathcal{U}_\infty)$, we need to show that there exists a pair of functions $(p, w)$ satisfying (IC), (FA), (PK(u)) and $w(v) \in \mathcal{U}_\infty$ for all $v$.

As shown in the first step, $\mathcal{U}_\infty = \bigcap_{m \geq 1} \mathcal{U}_m$. Therefore, $u \in \mathcal{U}_m$ for all $m$, and there exists a sequence of pairs of functions $(p^m, w^m)$ which satisfy (IC), (FA), (PK(u)) and $w^m(v) \in \mathcal{U}_{m-1} \subseteq \mathcal{U}_0$. Because $\mathcal{U}_0$ is convex and compact, we obtain by Lemma A.1 item 2 that $(p^m, w^m)$ lie in the weak-* compact set $\mathcal{M}_{\mathcal{U}_0}$ with the set $\mathcal{M}_{\mathcal{U}_0}$ defined in Lemma A.1 item 2. By passing to a subsequence if necessary we obtain that $p^m$ and $w^m$ weak-* converge to some $(p, w) \in \mathcal{M}_{\mathcal{U}_0}$.

By Lemma A.1 item 3 the set of incentive compatible stage mechanisms is weak-* closed, and thus $(p, w)$ is incentive compatible. For a fixed $m$, we have $w^j(v) \in \mathcal{U}_m$ for all $j > m$ because the sequence $\mathcal{U}_m$ is non-increasing. Because $\mathcal{U}_0$ is convex and compact, we obtain that $\mathcal{U}_m$ is convex and compact because the operator $B_β$ preserves convexity and compactness from Lemma A.2 items 2 and 3. Because the set $\mathcal{U}_m$ is closed and convex, we obtain by Lemma A.1 item 1 that weak limit verifies $w(v) \in \mathcal{U}_m$. Therefore, $w(v) \in \bigcap_{m \geq 1} \mathcal{U}_m = \mathcal{U}_\infty$.

We conclude by showing that $(p, w)$ satisfies the promise keeping constraint with $u$. For each $m$ we have that the promised keeping constraint is equivalently given by $u = E(p^m, w^m)$ with the function $E$ defined in Lemma A.1 item 1. Because the function $E$ is weak-* continuous, we obtain by taking limits that $u = E(p, w)$. Thus, $u \in B_β(\mathcal{U}_\infty)$ since the functions $(p, w)$ satisfy (IC), (FA), (PK(u)) and $w(v) \in \mathcal{U}_\infty$ for all $v$.

A.3.1 Proof of Lemma A.1

Proof. We prove each item at a time.

Item 1. Let $A \subseteq \mathbb{R}^n$ be a closed and convex set. We need to show that the set $\mathcal{C} = \{w \in \mathcal{L}^\infty : w(v) \in A \text{ for almost all } v \in [0, \bar{v}]^n\}$ is weak-* closed. Convexity of $A$ implies that $\mathcal{C}$ is convex. Because $A$ is closed, we obtain that the set $\mathcal{C}$ is (strongly) closed. Because $\mathcal{C}$ is convex and (strongly) closed, we obtain that $\mathcal{C}$ is weak-* closed by Hahn-Banach separation theorem (see Aliprantis and Border 2006, Theorem 5.98 in p. 214).

Item 2. Let $A \subseteq \mathbb{R}^n$ be a compact and convex set. We need to show that $\mathcal{M}_A = \{(p, w) \in \mathcal{L}^\infty \times \mathcal{L}^\infty : \sum_{i=1}^n p_i(v) \leq 1, p(v) \geq 0, \text{ and } w(v) \in A \text{ for almost all } v \in [0, \bar{v}]^n\}$ is weak-* compact. We write
$\mathcal{M}_A = \mathcal{M}_p \times \mathcal{M}_w$, where $\mathcal{M}_p = \{ \rho \in L^\infty : \sum_{i=1}^n p_i(v) \leq 1, p(v) \geq 0 \text{ for all } v \in [0, \bar{v}]^n \}$ and $\mathcal{M}_w = \{ w \in L^\infty : w(v) \in A \text{ for almost all } v \in [0, \bar{v}]^n \}$.

Because the pre-dual $L^1$ is normed, following Alaoglu’s compactness theorem (see Aliprantis and Border 2006 Theorem 5.105 in p. 218), we obtain that the closed unit ball in $L^\infty$ given by $\{ \rho \in L^\infty : \text{ess sup}_v \| \rho(v) \|_1 \leq 1 \}$ is weak-* compact. Because $A \subset \mathbb{R}^n$ is a compact and convex set, item 1 implies that $\mathcal{M}_w$ is weak-* compact because the intersection of a weak-* compact set and a weak-* closed set is weak-* compact. A similar argument shows that $\mathcal{M}_p$ is weak-* compact. The result follows by Tychonoff product theorem (see Aliprantis and Border 2006 Theorem 2.61 in p. 52) because the product space $\mathcal{M}_A = \mathcal{M}_p \times \mathcal{M}_w$ is also weak-* compact.

**Item 3** Let us assume that the interim functions of the stage mechanism for a single agent $W_i(v_i)$ and $P_i(v_i)$ lie in the space $L^\infty,1 \triangleq L^\infty([0, \bar{v}],[\mathbb{R},|\cdot|])$ of essentially bounded vector functions from $[0, \bar{v}]$ to $(\mathbb{R},|\cdot|)$, i.e., a measurable function $\rho : [0, \bar{v}] \rightarrow \mathbb{R}$ lies in $L^\infty,1$ if $\text{ess sup}_v |\rho(v)| < \infty$. Let $T : L^\infty \rightarrow (L^\infty,1)^n$ be the interim operator that projects an element $\rho \in L^\infty$ to $(T\rho)_i(v_i)$ for each $i = 1, \ldots, n$ by taking expectations over the other agents’ values. This operator is given by

$$
(T\rho)_i(v_i) = \int_{[0,\bar{v}]^{n-1}} \rho_i(v_i, v_{-i}) \prod_{j \neq i} f(v_j) dv_{-i}.
$$

We claim that $T$ is weak-* continuous. Define the space of interim incentive compatible mechanisms for a single agent as follows:

$$
C^1 = \{(\rho, \omega) \in L^\infty,1 \times L^\infty,1 : (1 - \beta)v \rho(v) + \beta \omega(v) \geq (1 - \beta)v \rho(v') + \beta \omega(v') \text{ for almost all } v, v' \in [0, \bar{v}] \}.
$$

Note that $C^1$ is convex and (strongly) closed. This follows because the constraints in $C^1$ correspond to closed halfspaces, so the set of points satisfying these constraints is closed and convex. This implies that $C^1$ is weak-* closed (see the proof of Lemma A.1 Item 1).

Let $C = \{(p, w) \in L^\infty \times L^\infty : (p, w) \text{ satisfy } (IC)\}$. Then, we have that $C = (T \times T)^{-1}((C^1)^n)$. Because $T$ is weak-* continuous, then the cartesian product of functions $T \times T$ is weak-* continuous in the product topology. Because $T \times T$ is weak-* continuous and the cartesian product $(C^1)^n$ is weak-* closed, $C$ is weak-* closed (see Aliprantis and Border 2006 Theorem 2.27 in p. 36). The result follows.

We now prove the claim that the interim operator $T : L^\infty \rightarrow (L^\infty,1)^n$ is weak-* continuous. Following Proposition 4 in Anderson and Nash (1987 p. 37), it is sufficient to show that the adjoint $T^*$ of the operator $T$ maps the pre-dual of $(L^\infty,1)^n$ into the pre-dual of $L^\infty$. Let $L^{1,1} \triangleq L^{1,1}([0, \bar{v}],[\mathbb{R},|\cdot|])$ denote the pre-dual of $L^\infty,1$, i.e., a measurable function $\psi : [0, \bar{v}] \rightarrow \mathbb{R}$ lies in $L^{1,1}$ if $\int_{[0,\bar{v}]} |\psi(v)| dv < \infty$. The pre-dual of $(L^\infty,1)^n$ is given by the space $(L^{1,1})^n$ with the norm of a typical element $(\psi_i(v_i))_{i=1}^n \in (L^{1,1})^n$ given by $\max_{i=1,\ldots,n} \int_{[0,\bar{v}]} |\psi_i(v_i)| dv_i$. Specifically, we need to show that $T^*(\psi_i)_{i=1}^n \in L^1$ where $(\psi_i)_{i=1}^n \in (L^{1,1})^n$.

We start with determining the adjoint $T^* : (L^{1,1})^n \rightarrow L^1$. By definition, the adjoint satisfies the following property for all $\rho \in L^\infty$ and $(\psi_i)_{i=1}^n \in (L^{1,1})^n$:

$$
\sum_{i=1}^n \int_{[0,\bar{v}]} (T\rho)_i(v_i) \psi_i(v_i) dv_i = \int_{[0,\bar{v}]^n} T^*((\psi_i)_{i=1}^n)(\nu) \cdot \rho(\nu) d\nu.
$$

(26)
We need to find $T^*$ satisfying the above condition. We claim that the adjoint is given by

$$T^*((\psi_i)_{i=1}^n)(v) = \left(\psi_i(v_i) \prod_{j \neq i} f(v_j)\right)_{i=1}^n.$$  \hspace{1cm} (27)

This follows from equations (26) and (25) because

$$\sum_{i=1}^n \int_{[0,1]} (T \rho)_{i}(v_i) \psi_i(v_i) dv_i = \sum_{i=1}^n \int_{[0,1]} \int_{[0,1]^{n-1}} \rho_i(v_i, v_{-i}) \prod_{j \neq i} f(v_j) dv_{-i} \psi_i(v_i) dv_i$$

$$= \int_{[0,1]^n} \sum_{i=1}^n \left(\psi_i(v_i) \prod_{j \neq i} f(v_j)\right) \rho_i(v) dv,$$

where we used Fubini’s theorem to change the order of integrals, because these functions are integrable; and exchanged summation with integration, because the sum is finite, using Tonelli’s Theorem because the functions are nonnegative and using that the second equation from exchanging summation with integration because the sum is finite, using Tonelli’s Theorem because the functions are nonnegative and using that the third inequality because $|x|_1 \leq n |x|_\infty$ for $x \in \mathbb{R}^n$. Finally, since $(\psi_i)_{i=1}^n \in (L^{1,1})^n$, the statement follows.

**Item 4** We need to show that the function $E(p, w) = (\mathbb{E}_v[(1 - \beta) v_i p_i(v) + \beta w_i(v)])_{i=1}^n$ is weak-* continuous. Consider $p^m \to p$ and $w^m \to w$ as $m \to \infty$. For a fixed $i$, consider the function $\psi^p \in L^1$ given by $\psi^p_i(v) = v_i \prod_{i=1}^n f(v_j)$ and $\psi^p_j(v) = 0$ for all $j \neq i$, and the function $\psi^w \in L^1$ given by $\psi^w_i(v) = \prod_{i=1}^n f(v_j)$ and $\psi^w_j(v) = 0$ for all $j \neq i$. By definition, we know that

$$\mathbb{E}_v[v_i p^m_i(v)] \to \mathbb{E}_v[v_i p_i(v)]$$

and $\mathbb{E}[w_i^m(v)] \to \mathbb{E}[w_i(v)].$

This implies that $E(p^m, w^m) \to E(p, w)$ as $m \to \infty$.  

**A.3.2 Proof of Lemma A.2**

**Proof.** We prove each item at a time.

**Item 1** (B) is monotone. Fix a point $u \in B(\mathcal{X})$ and let $(p, w)$ be the functions corresponding to $u$ satisfying [IC], [PK(u)], [FA], and $w(v) \in \mathcal{X}$ for all $v$. Since $\mathcal{X} \subseteq \mathcal{Y}$, it follows that $w(v) \in \mathcal{Y}$.
for all \( v \). Thus, the existence of the functions \((p, w)\) satisfying the previous requirements implies that \( u \in B_\beta(Y) \).

**Item 2 \((B_\beta \text{ preserves convexity})\).** Fix points \( u^1, u^2 \in B_\beta(A) \). We need to show that \( u = \lambda^1 u^1 + \lambda^2 u^2 \in B_\beta(A) \) for \( \lambda^1, \lambda^2 \geq 0 \) with \( \lambda^1 + \lambda^2 = 1 \). For each \( u^j \) with \( j = \{1, 2\} \) there exists some functions \((p^j, w^j)\) satisfying \((\text{IC}), (\text{PK}(u^j)), (\text{FA})\), and \( w^j(v) \in A \) for all \( v \). Because constraints are linear, we have that functions \((p, w)\) given by \( p(v) = \lambda^1 p^1(v) + \lambda^2 p^2(v) \) and \( w(v) = \lambda^1 w^1(v) + \lambda^2 w^2(v) \) satisfy \((\text{IC}), (\text{PK}(u)), (\text{FA})\). We conclude that \( u \in B_\beta(A) \) because \( w(v) \in A \) since \( A \) is convex.

**Item 3 \((B_\beta \text{ preserves compactness})\).** Recall that, by Heine-Borel theorem, the set \( A \subset \mathbb{R}^n \) is compact if and only if \( A \) is closed and bounded. Suppose that \( A \) is closed and bounded. We need show that \( B_\beta(A) \) is closed and bounded. Boundedness follows trivially from the promise keeping constraint because the allocation is bounded and values have finite means. We next prove closedness.

Consider a sequence \( u^m \in B_\beta(A) \) converging to some \( u \in \mathbb{R}^n \). We need to show that \( u \in B_\beta(A) \). For each \( m \) there exists functions \((p^m, w^m)\) which satisfy \((\text{IC}), (\text{FA}), (\text{PK}(u^m))\) and \( w^m(v) \in A \). Because \( A \) is convex and compact, we obtain by Lemma \[A.1\] item 2 that \((p^m, w^m)\) lie in the weak-* compact set \( \mathcal{M}_A \). By passing to a subsequence if necessary we obtain that \( p^m \) and \( w^m \) weak-* converge to some \((p, w)\) \( \in \mathcal{M}_A \). By Lemma \[A.1\] item 3 the set of incentive compatible stage mechanisms is weak-* closed, and thus \((p, w)\) is incentive compatible. Because \( A \) is closed and convex, we obtain by Lemma \[A.1\] item 1 that \( w(v) \in A \) for almost all \( v \in [0, \bar{v}]^n \). We conclude by showing that \((p, w)\) satisfies the promise keeping constraint with \( u \). For each \( m \) we have that the promised keeping constraint is equivalently given by \( u^m = E(p^m, w^m) \) with the function \( E \) defined in Lemma \[A.2\] item 4. Because the function \( E \) is weak-* continuous, we obtain by taking limits that \( u = E(p, w) \). Thus, \( u \in B_\beta(A) \) since the functions \((p, w)\) satisfy \((\text{IC}), (\text{FA}), (\text{PK}(u))\) and \( w(v) \in A \).

\[\square\]

### A.4 Proof of Lemma \[2.1\]

**Proof.** We first prove that \( U_\beta \) is convex and compact, and then show that \( U_\beta = \mathbb{R}_+^n \cap \text{hyp}(U_\beta) \).

**Step 1 \((U_\beta \text{ is convex and compact})\).** From Proposition \[2.3\] we have \( U^\infty = \bigcap_{m \geq 1} U^m \) with \( U^0 = [0, E[v]^n] \) and \( U^m = B_\beta(U^{m-1}) \) for all \( m \geq 1 \). Because \( U^0 \) is convex and compact, we obtain that \( U^m \) is convex and compact because the operator \( B_\beta \) preserves convexity and compactness from Lemma \[A.2\] items 2 and 3. Therefore, \( U^\infty \) is convex and compact, since the intersection of arbitrary convex and compact sets in \( \mathbb{R}^n \) is convex and compact. Convexity and compactness of \( U_\beta \) follows because \( U^\infty = U_\beta \) from Proposition \[2.3\].

**Step 2 \((U_\beta = \mathbb{R}_+^n \cap \text{hyp}(U_\beta))\).** Note that \( U_\beta \) is a subset of \( \mathbb{R}_+^n \) by definition. Moreover, any set is in its hypo-graph. Thus, we get \( U_\beta \subseteq \mathbb{R}_+^n \cap \text{hyp}(U_\beta) \). To prove the converse, fix a point \( u \in \mathbb{R}_+^n \cap \text{hyp}(U_\beta) \). We need to show that \( u \in U_\beta \). Because \( u \in \mathbb{R}_+^n \cap \text{hyp}(U_\beta) \), this implies that \( u \geq 0 \) and there exists \( \bar{u} \in U_\beta \) such that \( \bar{u} \geq u \). Let \( \bar{u} \) be the mechanism corresponding to \( u \).

Consider the \( 2^n \) states that can be obtained by replacing each subset of components of \( \bar{u} \) with zero. For example, when \( n = 2 \), we get \( \bar{u}^0 = (0, 0) \), \( \bar{u}^1 = (0, \bar{u}_2) \), \( \bar{u}^2 = (\bar{u}_1, 0) \), \( \bar{u}^4 = \bar{u} \). All these \( 2^n \) states are in \( U_\beta \). In particular, for a state \( \bar{u}^m \), we can construct a PBIC mechanism \( \bar{v}^m \) satisfying \( \bar{u}^m_i = V_i(\bar{v}^m, 1) \) for all \( i \) by setting \( \bar{v}^m_i = 0 \) if \( \bar{u}^m_i = 0 \) and \( \bar{v}^m_i = \bar{v}_i \) otherwise. Therefore, \( \bar{u}^m \in U_\beta \) for all \( m \). The polytope whose extreme points are the states \( \bar{u}^m \) given by \( \{\bar{u} \in \mathbb{R}_+^n : 0 \leq \bar{u}_i \leq \bar{u}_i, \forall i\} \) is a subset of \( U_\beta \) because all extreme points are in \( U_\beta \) and \( U_\beta \) is convex. The result follows because \( u \) lies in this polytope. \[\square\]
A.5 Proof of Corollary 2.1

Proof. Because the set $U_β$ is convex by Lemma 2.1 we get that $U_β$ can be characterized in terms of its support function $φ_β$. Proposition 2.2 implies that the support function $φ_β$ is a fixed point of $T$. In step 2 of the proof of Proposition 2.3 we showed that the sequence of sets $U^m = B_β(U^0)$ is convex and compact, and converges to the set $U_β$, which is convex and compact by Lemma 2.1. Therefore, we obtain from Salinetti and Wets (1979, Corollary 4.A in p.31) that $φ_β(α) = \lim_{m→∞} [T^m φ^0](α)$ for all $α$.

\[\]

A.6 Proof of Proposition 2.4

Proof. As noted by Boyd and Vandenberghe (2009), the convexity of support function follows because it is the point-wise supremum of a family of linear functions. We next show that $φ$ is twice differentiable in the following steps.

\[\]

Step 1. In this step, we show that $φ(α)$ is differentiable and provide its derivative with respect to $α$. Shapiro et al. (2014, Theorem 7.46, p. 371) show that a finite valued, well defined function, $f(α) = E_v[F(α, v)]$, is differentiable at a point $α_0$ if and only if the random function, $F(α, v)$, is convex and differentiable at $α_0$ with probability 1. Here, the convexity of the random function means that $F(·, v)$ is convex for almost every $v$. In our case, the random function is $\max_i(α_i v_i)$, and the corresponding expected value function is $φ(α)$. First, note that $φ(α)$ is well defined and finite valued because $0 ≤ φ(α) ≤ \max_j α_j E[\max_i v_i]$ for all $α ∈ \mathbb{R}^n$. Second, the random function $\max_i α_i v_i$ is a convex function of $α$ for all $v$ because it is maximum of linear functions of $α$ (see Boyd and Vandenberghe 2009). Moreover, it is differentiable at a fixed $α_0$ with probability 1, because for a given $v$, the partial derivative of $\max_i α_i v_i$ with respect to $α_i$ at $α_0$ does not exist only when $α_{0,i} v_i = α_{0,j} v_j$ for some $j ≠ i$; and the probability of this event is 0 for a continuous distribution of $v$.

Therefore, the derivative of $φ(α)$ exists and is given by

\[h_i(α) = \frac{∂φ}{∂α_i}(α) = E_v\left[v_i 1\{α_i v_i ≥ \max_j α_j v_j\}\right].\]

Step 2. We next consider the second derivative. Taking expectation of $v_i 1\{α_i v_i ≥ \max_j α_j v_j\}$ with respect to $v_{−i}$ in $h_i(α)$, we obtain $h_i(α) = E_{v_i}\left[v_i \prod_{j ≠ i} F\left(\frac{α_i v_i}{α_j}\right)\right]$. The derivatives of the integrand with respect to $α_i$ and $α_j$ exist for almost all $v_i$, and are given by

\[\]

\[\]

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Moreover, these derivatives are locally integrable, that is, for all compact intervals \([α, \overline{α}]\) contained in \(\mathbb{R}_{>0}^n\) the following integrals are finite.

\[
\int_{α}^{\overline{α}} \int_{0}^{β} \sum_{j\neq i} \frac{v_j^2}{α_j} f \left( \frac{α_i v_i}{α_j} \right) 1\{α_i v_i \leq α_j \bar{v} \} \prod_{k \neq i, j} F \left( \frac{α_i v_i}{α_k} \right) f(v_i) dv_i dα ≤ \bar{v}^3 f^2(n-1) \prod_{j=1}^{n} \overline{α}_j < ∞
\]

\[
\int_{α}^{\overline{α}} \int_{0}^{β} \frac{α_i v_i^2}{α_j} f \left( \frac{α_i v_i}{α_j} \right) 1\{α_i v_i \leq α_j \bar{v} \} \prod_{k \neq i, j} F \left( \frac{α_i v_i}{α_k} \right) f(v_i) dv_i dα ≤ \frac{\max}{\min} \bar{α}_j \bar{v}^3 f^2 \prod_{j=1}^{n} \overline{α}_j < ∞
\]

In these inequalities, we use the fact that the density function \(f(\cdot)\) is bounded by \(\bar{f}\) and the support of the values are bounded by \(\bar{v}\).

Therefore Leibniz rule implies that \(h_i(α)\) is differentiable and hence \(φ(α)\) is twice differentiable. Finally, we provide the components of the Hessian matrix of \(φ(α)\).

\[
\frac{∂h_i}{∂α_i}(α) = \frac{1}{(α_i)^2} \sum_{j \neq i} \frac{1}{α_j} \int_{0}^{\min(α_i, α_j)} x^2 f \left( \frac{x}{α_i} \right) f \left( \frac{x}{α_j} \right) \prod_{t \neq i, j} F \left( \frac{x}{α_t} \right) dx,
\]

\[
\frac{∂h_i}{∂α_j}(α) = -\frac{α_i}{(α_j)^2(α_i)^2} \int_{0}^{\min(α_i, α_j)} x^2 f \left( \frac{x}{α_i} \right) f \left( \frac{x}{α_j} \right) \prod_{t \neq i, j} F \left( \frac{x}{α_t} \right) dx.
\]
Appendix for Section 3

B.1 Proof of Proposition 3.1

Proof. We show that there exist \( \alpha \) and \( \beta \) such that \( \alpha^T \hat{w}(v|u) + \beta = 0 \). Suppose that \( \alpha = \alpha(u/F_\beta) \) and \( \beta = \sum_{i=1}^n [(1 - \beta)(\nabla \phi(a))_i - u_i] / \beta \).

Using the definition of \( \hat{w}(v|u) \), we obtain \( \alpha^T \hat{w}(v|u) = \sum_{i=1}^n a_i \mathbb{E}_{\hat{v}_i}[\hat{W}_i(\hat{v}_i|u)] \). Moreover, \( \mathbb{P}(u) \) implies that \( \beta \mathbb{E}_{\hat{v}_i}[\hat{W}_i(\hat{v}_i|u)] = \left[u_i - (1 - \beta)\mathbb{E}[v_i1\{a_i \geq \max_{j \neq i} a_j v_j\}] \right] \). Finally, recall that the support function satisfies \( \nabla \phi(a) \) satisfies \( \phi(a) \). Combining these three observations, it follows that \( \alpha^T \hat{w}(v|u) + \beta = 0 \).

The second part of the proposition follows from the following properties of the support function \( \phi(a) \). Note that \( x^T \nabla \phi(x) = \phi(x) \) for all \( x \) by the definition of \( \phi(a) \). Furthermore, for a vector \( x \), the support function satisfies \( \phi(x) \geq x^T z \) for all \( z \in U_\beta \) (see Schneider, 2013). Because \( u \in U_\beta \), we obtain the following inequality, which concludes the proof.

\[
(\nabla \phi(a) - \beta)^T a = \nabla \phi(a)^T a - u^T a = \phi(a) - u^T a \geq 0.
\]

B.2 Proof of Proposition 3.2

Proof. For simplicity, we use \( \alpha \) to represent \( \alpha(u/F_\beta) \) in this proof. First, we reformulate problem \([15]\) as the following linear programming problem:

\[
\begin{align*}
\max_{\tilde{w}(\cdot)} & \quad z \\
\text{st.} & \quad \mathbb{E}[\tilde{w}_i(v_i, v_{-i})] = \hat{W}_i(v_i|u) \quad \forall v_i, i \\
& \quad z \leq \alpha^T (u - \tilde{w}(v)) \quad \forall v \quad (28) \\
& \quad (29)
\end{align*}
\]

Next, we find an upper bound for this problem by relaxing the constraints over \( z \). Because \( u \in \mathcal{E}(U_\beta) \), we have \( u/F_\beta \in \mathcal{E}(U) \), which implies \( u/F_\beta = \nabla \phi(a) \), i.e., \( u_i = F_\beta \mathbb{E}[v_i1\{a_i \geq \max_{j \neq i} a_j v_j\}] \). Moreover, \( u_i = (1 - \beta)\mathbb{E}[v_i1\{a_i \geq \max_{j \neq i} a_j v_j\}] + \beta \mathbb{E}[\hat{W}_i(v_i|u)] \), following \( \mathbb{P}(u) \). These two observations imply that \( u_i = \tau \mathbb{E}[\hat{W}_i(v_i|u)] \), where \( \tau = F_\beta - (1 - \beta) \). Therefore, constraint \((29)\) becomes

\[
z \leq \sum_{i=1}^n a_i (\tau \mathbb{E}[\hat{W}_i(v_i|u)] - \tilde{w}_i(v)) \quad \forall v.
\]

Because \( z \) has to satisfy this constraint for all \( v \), we can relax it by replacing these constraints with a single constraint where the right hand side is the expectation over \( v \). This operation corresponds to taking expectation of \( \tilde{w}_i(v) \). Following constraint \((28)\), we obtain \( z \leq \sum_{i=1}^n a_i(\tau - 1)\mathbb{E}[\hat{W}_i(v_i|u)] \). Therefore, it follows that \( \sum_{i=1}^n a_i(\tau - 1)\mathbb{E}[\hat{W}_i(v_i|u)] \) is an upper bound for the optimal value.

To show \( \hat{w}(v|u) \) is an optimal solution, we first show that \( \hat{w}(v|u) \) is feasible, and the objective function evaluated at \( \hat{w}(v|u) \) is equal to this upper bound. By its definition, the expectation of \( \tilde{w}_i(v_i|u) \) with respect to \( v_{-i} \) is \( \hat{W}_i(v_i|u) \), which implies feasibility. By Proposition 3.1, we also know that \( a^T (u - \hat{w}(v|u)) = \sum_{i=1}^n a_i(\tau - 1)\mathbb{E}[\hat{W}_i(v_i|u)] \), which, in turn, implies that \( \hat{w}(v|u) \) is an optimal solution for \([15]\).
C Appendix for Section 4

In Section 4, we introduce the signed distance function for the perfect information achievable set $\mathcal{U}$, in (17). In fact, this function can be defined for any convex set $\mathcal{C}$. In Appendix C.1 we provide the definition and a key property of the function, which is used to show that a point belongs to a set. We then focus on the signed distance function for (scaled) perfect information achievable sets, and prove the main results of Section 4.

C.1 Signed Distance Function for Convex Sets

For a convex set $\mathcal{C} \in \mathbb{R}^n$, Schneider (2013) defines its support function as

$$\psi(\lambda) \triangleq \sup\{\lambda^T x : x \in \mathcal{C}\}.$$ 

We further define the following function

$$I_C(x) \triangleq \sup_{\lambda \geq 0, \|\lambda\|=1} \{x^T \lambda - \psi(\lambda)\}.$$ 

The following lemma demonstrates that function $I_C$ is indeed a signed distance function.

**Lemma C.1.** For any convex set $\mathcal{C} \subset \mathbb{R}^n$, we have

$$I_C(x) = \begin{cases} \leq 0 & \text{if } x \in \text{hyp}(\mathcal{C}), \\ > 0 & \text{if } x \not\in \text{hyp}(\mathcal{C}), \end{cases}$$

where $\text{hyp}(\mathcal{C}) = \{x \in \mathbb{R}^n : x \leq \bar{x}, \exists \bar{x} \in \mathcal{C}\}$.

**Proof.** We prove this result by considering two separate cases which determines the sign of the indicator function.

**Step 1.** In this step, we assume that $x \in \text{hyp}(\mathcal{C})$. The definition of the $\text{hyp}(\mathcal{C})$ implies the existence of $\bar{x} \in \mathcal{C}$ satisfying $x \leq \bar{x}$. Fixing an arbitrary nonnegative vector $\lambda$ satisfying $\|\lambda\| = 1$, we have

$$\lambda^T x \leq \lambda^T \bar{x} \leq \psi(\lambda).$$

Therefore, we have $I_C(x) \leq 0$.

**Step 2.** Now assume that $x \not\in \text{hyp}(\mathcal{C})$. The separating hyperplane theorem implies that there exists a nonzero vector $\lambda$ satisfying $\lambda^T x > \lambda^T \bar{x}$ for all $\bar{x} \in \text{hyp}(\mathcal{C})$. We argue that $\lambda \geq 0$, following the fact that $\text{hyp}(\mathcal{C})$ is unbounded from below. Otherwise, for the component $\lambda_i < 0$, we can always find a low enough negative $\bar{x}_i$ to violate $\lambda^T x > \lambda^T \bar{x}$.

Let $\bar{\lambda} \triangleq \lambda/\|\lambda\|$, such that $\|\bar{\lambda}\| = 1$. Moreover, for all $\bar{x} \in \text{hyp}(\mathcal{C})$, we have

$$\bar{\lambda}^T x > \bar{\lambda}^T \bar{x}.$$ 

These observations together imply that $\bar{\lambda}^T x > \psi(\bar{\lambda})$, thus $I_C(x) > 0$. □

C.2 The Signed Distance Function for the Scaled Perfect Information Achievable Set

In this section, we consider the perfect information achievable set $\mathcal{U}$ defined in (6). We provide the signed distance function for this set and the sets obtained by scaling it down. First, we start by showing
that the perfect information achievable set and the sets obtained by scaling it down are convex.

**Proposition C.1.** The perfect information achievable set $\mathcal{U}$ defined in (6) is a convex set. Moreover, for any $s \in [0, 1]$, the set $s\mathcal{U}$ is also convex, and $s\mathcal{U} = \text{hyp}(s\mathcal{U}) \cap \mathbb{R}_+^n$.

**Proof.** For any two points $u', u'' \in \mathcal{U}$ and a scalar $x \in (0, 1)$, let $p'$ and $p''$ be the allocation functions corresponding to the points $u'$ and $u''$, respectively. Then, the point $xu' + (1-x)u''$ is in $\mathcal{U}$ because $xp' + (1-x)p''$ is an allocation satisfying (FA), and $xu' + (1-x)u'' = E[v_i(xp'_i(v) + (1-x)p''_i(v))]$. Therefore, $\mathcal{U}$ is convex.

Any set is a subset of its hypograph. And because $U$ is convex, $s\mathcal{U}$ is also convex. Thus, it is sufficient to show that $\text{hyp}(s\mathcal{U}) \cap \mathbb{R}_+^n \subseteq s\mathcal{U}$. Let $u \in \text{hyp}(s\mathcal{U}) \cap \mathbb{R}_+^n$. This implies that $u \geq 0$ and there exists $\tilde{u} \in s\mathcal{U}$ such that $u \geq \tilde{u}$. Let $\tilde{p}$ be the allocation corresponding to $\tilde{u}$.

Consider the $2^n$ points that can be obtained by replacing each subset of components of $\tilde{u}$ with zero. For example, when $n = 2$, we get $\tilde{u}^0 = (0, 0)$, $\tilde{u}^1 = (0, \bar{u}_2)$, $\tilde{u}^2 = (\bar{u}_1, 0)$, $\tilde{u}^4 = \tilde{u}$. All these $2^n$ states are in $s\mathcal{U}$. In particular, for a state $\tilde{u}^m$, we can construct a feasible allocation $\tilde{p}^m$ satisfying $\tilde{u}^m = E[v_i\tilde{p}^m_i(v)]$ for all $i$ by setting $\tilde{p}^m_i = 0$ if $\bar{u}_i = 0$ and $\tilde{p}^m_i = \tilde{p}_i$ otherwise. Therefore, $\tilde{u}^m \in s\mathcal{U}$ for all $m$. The polytope whose extreme points are the states $\tilde{u}^m$ given by $\{\tilde{u} \in \mathbb{R}_+^n : 0 \leq \bar{u}_i \leq \tilde{u}_i \ \forall i\}$ is a subset of $s\mathcal{U}$ because all extreme points are in $s\mathcal{U}$ and $s\mathcal{U}$ is convex. The result follows because $u$ lies in this polytope.

We now provide the signed distance function corresponding to $F\mathcal{U}$ for a given scalar $F$ as follows:

$$J(u, F) \triangleq I_{\mathcal{U}}\left(\frac{u}{F}\right) = \sup_{\|x\|_1 = 1, x \geq 0} \left\{ \frac{u^T x}{F} - \phi(x) \right\}, \tag{30}$$

where $\phi(\cdot)$ is the support function of $\mathcal{U}$ given in (7).

By Lemma C.1 in Appendix C.1 for any nonnegative $u$, it follows that

$$J(u, F) = \begin{cases} 
\leq 0 & u \in \text{hyp}(F\mathcal{U}), \\
> 0 & u \notin \text{hyp}(F\mathcal{U}). 
\end{cases}$$

**Corollary C.1.** For a point $u \in \mathbb{R}_+^n$, $u \in F\mathcal{U}$ if and only if $J(u, F) \leq 0$.

**Proof.** Proposition C.1 implies that $F\mathcal{U} = \text{hyp}(F\mathcal{U}) \cap \mathbb{R}_+^n$, thus for any $u \geq 0$, $u \in F\mathcal{U}$ if and only if $u \in \text{hyp}(F\mathcal{U})$. Moreover, Lemma C.1 implies that $J(u, F) \leq 0$ if and only if $u \in \text{hyp}(F\mathcal{U})$.

Finally, we provide two properties of $J(u, F)$ by the following propositions. Before stating these results, we introduce the following matrix that would be used in later results:

$$\Pi(a) \triangleq \text{Hess}(\phi(a)) + ee^T, \tag{31}$$

where $a > 0$ and $e$ is the $n$ dimensional vector of ones.

For any scalar $F \in (0, 1)$, define set $\mathcal{S}(F) \subseteq \mathbb{R}^n$ as

$$\mathcal{S}(F) \triangleq \{z + ae : \forall a \geq 0, z \in F\mathcal{U}, \text{ and } z > 0\}.$$ 

Set $\mathcal{S}(F)$ is useful in the following result as well as later in the Appendices.

**Proposition C.2.** Let $u \in \mathcal{S}(F)$. We have $\alpha(u/F) > 0$, in which $\alpha(u/F)$ is defined in (8).
Proof. Consider a point \( u = z + a e \) for some fixed \( z \in \mathcal{F} \), \( z > 0 \) and \( a \geq 0 \). The optimization problem in (8) becomes
\[
\sup_{\|x\|_1=1, x \geq 0} \left\{ \frac{(z + ae)^T x}{F} - \phi(x) \right\} = \frac{a}{F} + \sup_{\|x\|_1=1, x \geq 0} \left\{ z^T x - \phi(x) \right\}.
\]

Therefore, \( \alpha((z + ae)/F) = \alpha(z/F) \). Now focus on \( z > 0 \) such that \( z \in \mathcal{F} \). The Lagrangian is given by
\[
\mathcal{L}(x, \mu, \zeta) = x^T z - F\phi(x) - \mu(1 - e^T x) + \zeta^T x,
\]
where \( \mu \) and \( \zeta \) are the Lagrange multipliers corresponding to constraints \( \|x\|_1 = 1 \) and \( x \geq 0 \), respectively. Optimality implies that
\[
\nabla_x \mathcal{L}(\alpha(z/F), \mu^*, \zeta^*) = z - F\nabla\phi(\alpha(z/F)) + \mu^* e + \zeta^* = 0.
\]
As discussed in Remark [3.1], the optimal value \( \mu^* \) is nonnegative for \( z \in \mathcal{F} \). Suppose now there exists a component of \( \alpha(z/F) \) such that \( \alpha_i(z/F) = 0 \). In this case, the optimality condition for \( i \) implies \( z_i + \mu^*_i + \xi^*_i = 0 \), because \( (\nabla\phi(\alpha(u/F)))_i = 0 \). This is a contradiction because \( z_i > 0 \), \( \mu^*_i \geq 0 \) and \( \xi^*_i \geq 0 \). Therefore, we have the result. \( \square \)

**Proposition C.3.** Let \( F \) be a positive scalar in \([0, 1]\) and \( u \in \mathcal{S}(F) \). Suppose also that \( \Pi(\alpha) \) is positive definite for all \( \alpha > 0 \). Then, the optimal solution \( \alpha(u/F) \) is unique. Moreover, the function \( \mathcal{J}(u, F) \) is twice differentiable, and the first and second derivatives are given as follows:
\[
\nabla \mathcal{J}(u, F) = \frac{\alpha(u/F)}{F} \quad \text{and} \quad \text{Hess}(\mathcal{J}(u, F)) = -\frac{1}{F^2} \Pi(\alpha(u/F))^{-1} \left( I - \frac{ee^T \Pi(\alpha(u/F))^{-1}}{e^T \Pi(\alpha(u/F))^{-1} e} \right),
\]
where \( I \) is the \( n \times n \) identity matrix.

Proof. We prove this result in three steps.

**Step 1.** In this step, we show that the optimal solution \( \alpha(u/F) \) of the maximization problem in \( \mathcal{J}(u, F) \) is unique. In Proposition [C.2], we show that an optimal solution is \( \alpha(u/F) > 0 \) for \( u \in \mathcal{S}(F) \), thus we can ignore the non-negativity constraints. Because \( x > 0 \), Proposition [2.4] implies that the objective is twice-differentiable with hessian \(-\text{Hess}(\phi(x))\). Note that the objective function is not strictly concave over \( \mathbb{R}_{>0}^n \) because \( \text{Hess}(\phi(\alpha)) \alpha = 0 \) for any \( \alpha \neq 0 \). However, we can show that the restriction of the objective function to the feasible set \( \|x\|_1 = 1 \) is strictly concave. Let \( y \in \mathbb{R}^n \) be a feasible direction satisfying \( e^T y = 0 \). We obtain using (31) that
\[
y^T \text{Hess}(\phi(x)) y = y^T \Pi(x) y - (e^T y)^2 = y^T \Pi(x) y > 0,
\]
where the last inequality follows because \( \Pi(x) \) is positive definite for all \( x > 0 \). This implies that the optimal solution is unique.

**Step 2.** In this step, we show that \( \mathcal{J}(u, F) \) is differentiable with respect to \( u \) and provide \( \nabla \mathcal{J}(u, F) \). As noted by Milgrom and Segal (2002) in Corollary 3, the gradient of \( \mathcal{J}(u, F) \) is well defined because (i) \( \{ x : \|x\|_1 = 1, x \geq 0 \} \) is a convex set, (ii) \( x^T u/F - \phi(x) \) is a concave function of \( x \) and \( u \), and (iii) there is an optimal solution, \( \alpha(u/F) \), such that \( \nabla \mathcal{J}(u, F) \) exists. Then, the gradient at \( u \) is given by
\[
\nabla \mathcal{J}(u, F) = \frac{\alpha(u/F)}{F}.
\]

**Step 3.** In this step, we show the existence of the second derivative and provide the Hessian matrix of \( \mathcal{J}(u, F) \). In the previous step, the gradient of \( \mathcal{J}(u, F) \) is given by \( \alpha(u/F)/F \). Therefore, the Hessian matrix consists of the derivatives of the optimal solution \( \alpha(u/F) \) with respect to \( u \). We next characterize the optimal solution using the KKT conditions of the optimization problem.
Because \( x > 0 \), we can write the Lagrangian of (30) as \( \mathcal{L}(x, \mu) = x^T u - F(x) - \mu(1 - e^T x) \). Optimality implies \( \nabla_x \mathcal{L}(\alpha(u/F), \mu^*) = u - F \nabla \phi(\alpha(u/F)) + \mu^* e = 0 \) and \( \nabla_{\mu} \mathcal{L}(\alpha(u/F), \mu^*) = e^T \alpha(u/F) - 1 = 0 \). Introduce an auxiliary function \( Q : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) as follows:

\[
Q_i(x, \mu) \triangleq F(h_i(x) - \mu) = z_i, \quad \forall i = 1, \ldots, n, \\
Q_{n+1}(x, \mu) \triangleq -F \sum_{i=1}^{n} x_i = z_{n+1}.
\]

For a given \( u \), it follows that \( Q(\alpha(u/F), \mu^*) = [u^T, -F]^T \). Inverse function theorem implies that if the Jacobian determinant of \( Q \) is non-zero, then \( Q \) is invertible. Specifically, if the Jacobian determinant of \( Q \) is non-zero, \( Q \) is invertible and thus the unique optimal solution is obtained by \( Q^{-1}(u, -F) = [\alpha(u/F)^T, \mu^*]^T \). We next argue that the Jacobian determinant of \( Q \), denoted by \( \det(J_Q(\alpha(u/F), \mu^*)) \), is non-zero. The Jacobian matrix of \( Q \), \( J_Q(\alpha(u/F)) \), is given as follows:

\[
J_Q(\alpha(u/F), \mu^*) = F \begin{bmatrix} \text{Hess}(\phi(\alpha(u/F))) & -e \\ -e^T & 0 \end{bmatrix}.
\]

Note that \( J_Q(\alpha(u/F), \mu^*) \) is a block matrix thus its determinant is expressed in terms of the determinants of \( \text{Hess}(\phi(\alpha(u/F))) \) and \( \Pi(\alpha(u/F)) \). We have that \( \det(\Pi(a)) \neq 0 \) for all \( a > 0 \) because the matrix is positive definite. Moreover, it follows that \( \det(\text{Hess}(\phi(\alpha(u/F)))) = 0 \) because \( \text{Hess}(\phi(\alpha))a = 0 \) for any \( a \neq 0 \). Therefore, we obtain

\[
\det(J_Q(\alpha(u/F), \mu^*)) = \det(\text{Hess}(\phi(\alpha(u/F)))) - \det(\Pi(\alpha(u/F))) = -\det(\Pi(\alpha(u/F))) \neq 0.
\]

The inverse function theorem further implies that \( Q^{-1} \) is continuously differentiable.

Moreover, the theorem implies \( J_Q^{-1}(u, -F) = [J_Q(\alpha(u/F), \mu^*)]^{-1} \), where \( J_Q^{-1} \) is the Jacobian matrix of inverse function \( Q^{-1} \). This system of equations implies the existence of \( \text{Hess}(\Lambda(u, F)) \) because \( \text{Hess}(\Lambda(u, F)) \) is the upper left sub-matrix of \( J_Q^{-1}(u, -F) \). Specifically, denoting \( c_i = \frac{\partial \mu^*}{\partial z_{n+1}} \) and \( b_i = \frac{\partial \alpha_i(u/F)}{\partial z_{n+1}} \), we can equivalently express \( J_Q^{-1}(u, -F) = [J_Q(\alpha(u/F), \mu^*)]^{-1} \) as follows:

\[
F^{-1} \text{Hess}(\Lambda(u, F)) = \Pi(\alpha(u/F))^{-1} \left( I - e^T \Pi(\alpha(u/F))^{-1} e \right) \bigg( e^T \Pi(\alpha(u/F))^{-1} e \bigg),
\]

Following Lemma 3 in [Pringle and Rayner (1970)], we obtain the following closed form expression for \( \text{Hess}(\Lambda(u, F)) \):

\[
F^{-2} \text{Hess}(\Lambda(u, F)) = \Pi(\alpha(u/F))^{-1} \left( I - e^T \Pi(\alpha(u/F))^{-1} e \right) .
\]

**Proposition C.4.** Let \( F \) be a positive scalar in \([0, 1]\), \( u' \in S(F) \) and \( u'' \in H \) such that \( u'' > 0 \). Suppose \( \Pi(a) \) is positive definite for \( a > 0 \). Then \( \Lambda(u', F) \) is expressed via its quadratic expansion around \( u'' \) as follows:

\[
\Lambda(u', F) = \Lambda(u'', F) + \nabla \Lambda(u'', F)^T (u' - u'') + \frac{1}{2} (u' - u'')^T \text{Hess}(\Lambda(\Theta, F))(u' - u''),
\]

where \( \Theta \) is a point between \( u' \) and \( u'' \).

**Proof.** When \( \Pi(a) \) is positive definite for all \( a > 0 \), Proposition C.3 provides the gradient and the
such that

\[ \Theta \in \mathcal{S}(F). \]

We can express \( \Theta \) as a convex combination of \( u' \) and \( u'' \), i.e., \( \Theta = \omega u' + (1 - \omega) u'' \) for some \( \omega \in (0, 1) \) because \( \Theta \) is a point between \( u' \) and \( u'' \). We know that there exist \((x'', a'')\) such that \( u'' = x'' + a'' e \) where \( x'' \in \mathcal{U} \) and \( a'' \geq 0 \), and \( a'' \geq 0 \). Thus it follows that \( \Theta = \omega u' + (1 - \omega)x'' + a'' e \). Because \( \mathcal{U} \) is convex (see Proposition \[C.1\]), we have that \( \omega u' + (1 - \omega)x'' \in \mathcal{U} \) and \( \Theta \in \mathcal{S}(F) \). Therefore, following Taylor’s theorem, we can express \( \mathcal{J}(u', F) \) by the quadratic expansion around \( u'' \) as follows:

\[ \mathcal{J}(u', F) = \mathcal{J}(u'', F) + \nabla \mathcal{J}(u'', F)^T (u' - u'') + \frac{1}{2}(u' - u'')^T \text{Hess}(\mathcal{J}(\Theta, F))(u' - u''). \]

\[ \square \]

### C.3 Proof of Proposition 4.1

Before the proof, we provide the value of the constant used in (16). Following this definition, we provide the proof of the proposition.

**Definition C.1.** Define constants \( F_\beta \) and \( u_\beta \) such that \( F_\beta = 1 - u_\beta / \mathbb{E}[v] \) and \( u_\beta = \xi (1 - \beta) \), in which scalar \( \xi \) is given by

\[ \xi \triangleq \frac{12 \nu}{\beta^2 v^3 f^2}, \]  

where

\[ \nu \triangleq \max \left( 2 \left[ \frac{\tilde{f} \nu^2}{2} - \frac{\tilde{f}}{2} \mathbb{E}[v^2] + \mathbb{E}[v] \right]^2, \left( \mathbb{E}[v] + \frac{\tilde{f}}{2} \mathbb{E}[v^2] \right)^2 + (\mathbb{E}[v] + \tilde{v})^2, \frac{f^2 v^6}{36} \right). \]

We determine the value of \( \beta \) such that \( F_\beta \geq 1/2 \) for all \( \beta \geq \beta \).

**Proof of Proposition 4.1.** We prove this result in two steps. We defer the proofs of lemmas used in these steps to the end of this section.

**Step 1.** In this step, we prove that \( \hat{w}(v|u) \geq 0 \) for all \( u \in \mathcal{F}_\beta \mathcal{U} \). For the fixed \( u \), for simplicity, we use \( \hat{w} \) to represent \( \hat{w}(v|u) \) given in (14). Suppose that for an \( i \), \( u_i \leq u_\beta \), then \( \hat{w} = w^i(v|u) = u \) and \( \hat{w} \) is clearly nonnegative because \( u \in \mathcal{F}_\beta \mathcal{U} \). Suppose that \( u_i \geq u_\beta \) for all \( i \), i.e., \( u \) lies in the central region \( \mathcal{U}_\beta \). We show that the largest jump from the initial state \( u \) to \( \hat{w} \) is less than \( u_\beta \), thus \( \hat{w} \) is always nonnegative. The following lemma provides a bound on the largest jump.

**Lemma C.2.** For any \( u \in \mathcal{U}_\beta \) and \( \beta \geq \beta \), we have \( \|\hat{w} - u\|_2 \leq \left( \frac{1 - \beta}{\beta} \right)^\nu \sqrt{v} \).

This lemma enables us to show that \( u_\beta \geq \|\hat{w} - u\|_2 \) for all \( u \in \mathcal{U}_\beta \) because it provides a uniform bound on this norm which is independent of \( u \) and \( v \). Using the fact that \( \nu \geq \frac{\tilde{f} v^6}{36} \) and \( \xi = \frac{12 \nu}{\beta v^3 f^2} \), we get

\[ \|\hat{w} - u\|_2 \leq \frac{\sqrt{v}}{\beta} (1 - \beta) = \frac{\nu}{\beta \sqrt{v}} (1 - \beta) \leq \frac{6 \nu}{\beta v^3 f^2} (1 - \beta) \leq \frac{\xi}{2} (1 - \beta) = \frac{u_\beta}{2}. \]

Therefore, it follows that \( \hat{w}_i \geq u_i - \|\hat{w} - u\|_2 \geq u_\beta - \|\hat{w} - u\|_2 > 0 \) for all \( i \).

**Step 2.** Now we show that \( \hat{w} \in \mathcal{F}_\beta \mathcal{U} \). For the case that \( u \) is in the boundary region of \( \mathcal{F}_\beta \mathcal{U} \), \( \hat{w} = w^i(v|u) = u \), thus clearly \( \hat{w} \in \mathcal{F}_\beta \mathcal{U} \). Next, assume that \( u \) is in the central region \( \mathcal{U}_\beta \). Consider the signed distance function \( \mathcal{J} \) defined in (30). For simplicity, we use \( \mathcal{J}(x) \) to represent \( \mathcal{J}(x, F_\beta) \). Following Corollary \[C.1\], we have \( \mathcal{J}(\hat{w}) \leq 0 \) if and only if \( \hat{w} \in \mathcal{F}_\beta \mathcal{U} \) because \( \hat{w} \geq 0 \). In order to show
\[ \mathcal{J}(\hat{w}) \leq 0, \] we consider the quadratic expansion around \( u \) as in Proposition \text{C.4} (with scalar \( F = F_\beta \) in the proposition). We first prove that the necessary conditions for Proposition \text{C.4} to hold.

**Lemma C.3.** The future promise \( \hat{w} \) lies in \( \mathcal{S}(F_\beta) \) for any \( u \) in \( \mathcal{U}_\beta \).

This lemma implies that the set \( \mathcal{S}(F_\beta) \) contains all \( \Theta \) between \( \hat{w} \) and \( u \) because it is convex. Following Proposition \text{C.2} we have that \( \alpha(\Theta/F_\beta) > 0 \).

**Lemma C.4.** The matrix \( \Pi(a) \) is positive definite for all \( a > 0 \).

Because both \( u \) and \( \Theta \) are in \( \mathcal{S}(F_\beta) \), Lemma \text{C.4} enables us to invoke Proposition \text{C.4}. Therefore, we next analyze each item in the quadratic expansion at a time. First, \( u \in F_\beta \mathcal{U} \) implies that \( \mathcal{J}(u) \leq 0 \).

Thus, to show \( \mathcal{J}(\hat{w}) \leq 0 \), it is sufficient to show that

\[ \nabla \mathcal{J}(u)^T(\hat{w} - u) + \frac{1}{2}(\hat{w} - u)^T \text{Hess}(\mathcal{J}(\Theta))(\hat{w} - u) \leq 0. \tag{36} \]

In order to show this inequality holds, we find a lower bound for \( -\nabla \mathcal{J}(u)^T(\hat{w} - u) \), and an upper bound for \( (\hat{w} - u)^T \text{Hess}(\mathcal{J}(\Theta))(\hat{w} - u) \). Then, we compare these bounds. We start by considering the first-order term. The following lemma provides the lower bound.

**Lemma C.5.** For any \( \beta \geq \beta \), the gradient of \( \mathcal{J}(u) \) satisfies

\[ \frac{(1 - \beta)^2 \xi}{4 F_\beta^2} \leq -\nabla \mathcal{J}(u)^T(\hat{w} - u). \]

This lemma mainly stems from the fact that \( \nabla \mathcal{J}(u) = \alpha(u/F_\beta)/F_\beta \) (see Proposition \text{C.3}) and the future promises lie in a plane with normal \( \alpha(u/F_\beta) \) (see Proposition \text{3.1}). Following this lemma, it is sufficient to show that \( \frac{1}{2}(\hat{w} - u)^T \text{Hess}(\mathcal{J}(\Theta))(\hat{w} - u) \leq \frac{(1 - \beta)^2 \xi}{4 F_\beta^2} \). Therefore, we next find an upper bound for the term with the Hessian in the following lemma.

**Lemma C.6.** For any \( \beta \geq \beta \), the Hessian of \( \mathcal{J}(u) \) satisfies

\[ \frac{1}{2}(\hat{w} - u)^T \text{Hess}(\mathcal{J}(\Theta))(\hat{w} - u) \leq \left( \frac{1 - \beta}{\beta} \right)^2 \frac{3 \nu}{F_\beta^2 v^2} \]

where \( \Theta \) is an arbitrary point between \( u \) and \( \hat{w} \).

In the proof of this lemma, following the min-max theorem, we first bound \( (\hat{w} - u)^T \text{Hess}(\mathcal{J}(\Theta))(\hat{w} - u) \) by the multiplication of the maximum eigenvalue of \( \text{Hess}(\mathcal{J}(\Theta)) \) and \( \|\hat{w} - u\|_2^2 \). Then, we replace the norm square, \( \|\hat{w} - u\|_2^2 \), with its bound provided in Lemma \text{C.2}.

Combining the bounds provided by Lemmas \text{C.5} and \text{C.6} we obtain that \tag{36} holds because \( \frac{12 \nu}{\beta F_\beta^2 v^2} \leq \xi \). Therefore, we prove \( \mathcal{J}(\hat{w}) \leq 0 \) implying that \( \hat{w} \in F_\beta \mathcal{U} \) for \( u \) such that \( u_i \geq \bar{u}_\beta \) for all \( i \).

\[ \square \]

**C.4 Proof of Theorem \text{[4.1]}**

**Proof.** This result follows from Proposition \text{[4.1]} and Proposition \text{2.1}. The set \( F_\beta \mathcal{U} \) characterized by the constant in \tag{33} is a subset of \( \mathcal{U}_\beta \) by Proposition \text{[2.1]}. Thus, the maximum achievable social welfare \( J_\beta = F_\beta J_{FB} \), is less than \( J^* \). \[ \square \]
C.5 Proofs of Lemmas C.2, C.3, C.4, C.5 and C.6

Proof of Lemma C.2. Proposition 3.1 states that future promises lie on a line for different \( v \)'s when \( n = 2 \). The extreme points of this line correspond to \((0, \bar{v})\) and \((\bar{v}, 0)\) because \( \hat{w}_i(v | u) \) is non-increasing in \( v_i \) and non-decreasing in \( v_j \), according to Proposition 5.2. Therefore, it follows that

\[
\| \hat{w} - u \|_2^2 \leq \max \left( \| \hat{w}((\bar{v}, 0) | u) - u \|_2^2, \| \hat{w}((0, \bar{v}) | u) - u \|_2^2 \right).
\]

Let \( \gamma = \alpha (u / F_\beta) \) and suppose that \( \gamma_1 \leq \gamma_2 \) without loss of generality.

Step 1 (Bounding \( \| \hat{w}(((\bar{v}, 0) | u) - u \|_2^2 \)). In this step, using (13) and (14), we bound \( (\hat{w}_1((\bar{v}, 0) | u) - u_1)^2 \) and \( (\hat{w}_2((\bar{v}, 0) | u) - u_2)^2 \) separately. First, we obtain more compact expressions for these terms and using these expressions, we get the bounds.

\[
\hat{w}_1((\bar{v}, 0) | u) = \frac{u_1}{\beta} - \frac{1 - \beta}{\beta} \left[ \int_0^\nu \frac{\gamma_1 y}{\gamma_2} F(y) F \left( \frac{\gamma_1 y}{\gamma_2} \right) dy - \int_0^{\nu_2} F(y) \left( \frac{\gamma_2 y}{\gamma_1} \right) dy + \frac{\gamma_2}{\gamma_1} \left[ \nu_2 \left( \frac{\gamma_2 y}{\gamma_1} \right) F \left( \frac{\gamma_2 y}{\gamma_1} \right) \right] \right].
\]

Changing the order of integrations, we rewrite \( \hat{w}_1((\bar{v}, 0) | u) \) as follows:

\[
\hat{w}_1((\bar{v}, 0) | u) = \frac{u_1}{\beta} - \frac{1 - \beta}{\beta} \left[ \int_0^\nu \frac{\gamma_1 y}{\gamma_2} F(y) dy + \mathbb{E}[v] \right].
\]

We are now ready to bound \( (\hat{w}_1((\bar{v}, 0) | u) - u_1)^2 \).

\[
\left( \frac{\beta}{1 - \beta} \right)^2 (\hat{w}_1((\bar{v}, 0) | u) - u_1)^2 = \left[ u_1 - \frac{\gamma_1 \nu_1}{\gamma_2} \int_0^\nu f \left( \frac{\gamma_1 y}{\gamma_2} \right) F(y) dy - \mathbb{E}[v] \right]^2 \leq \left( \frac{\nu_1}{2} \right)^2 \left( \mathbb{E}[v] + \frac{\nu_2}{2} \mathbb{E}[v^2] + \mathbb{E}[v] \right)^2.
\]

This inequality follows from the fact that (i) \( \gamma_1 / \gamma_2 \) is less than 1, (ii) the p.d.f. is less than \( \bar{f} \) in its support, and (iii) we can remove \( u_1 \) because \( \mathbb{E}[v] \geq u_1 \) and they have different signs.

Next, we derive \( \hat{w}_2((\bar{v}, 0) | u) \) following similar steps and bound \( (\hat{w}_2((\bar{v}, 0) | u) - u_2)^2 \).

\[
(\hat{w}_2((\bar{v}, 0) | u) - u_2)^2 = \left( \frac{1 - \beta}{\beta} \right)^2 \left[ u_2 + \frac{\gamma_1 \nu_1}{\gamma_2} \int_0^{\nu_1} f \left( \frac{\gamma_1 y}{\gamma_2} \right) y F(y) dy - \int_0^{\nu_2} \bar{F}(y) \left( \frac{\gamma_2 y}{\gamma_1} \right) dy \right]^2 \leq \left( \frac{1 - \beta}{\beta} \right)^2 \left[ \max \left( u_2 + \frac{\gamma_1 \nu_1}{\gamma_2} \int_0^{\nu_1} f \left( \frac{\gamma_1 y}{\gamma_2} \right) y F(y) dy, \int_0^{\nu_2} \bar{F}(y) \left( \frac{\gamma_2 y}{\gamma_1} \right) dy \right) \right]^2 \leq \left( \frac{1 - \beta}{\beta} \right)^2 \left[ \max \left( \mathbb{E}[v] + \frac{\nu_1}{2} \mathbb{E}[v^2], \mathbb{E}[v] \right) \right]^2 = \left( \frac{1 - \beta}{\beta} \right)^2 \left( \mathbb{E}[v] + \frac{\nu_2}{2} \mathbb{E}[v^2] \right)^2.
\]

Here, we first group the terms with same sign inside the square brackets and bound them separately. We use the fact that \( \gamma_1 / \gamma_2 \leq 1 \) and the p.d.f. is less than \( \bar{f} \) in its support to obtain the second inequality.

By using the bounds on \( \hat{w}_i((\bar{v}, 0) | u) \) for \( i = 1, 2 \), we obtain

\[
\| \hat{w}((\bar{v}, 0) | u) - u \|_2^2 \leq \left( \frac{1 - \beta}{\beta} \right)^2 (\nu_1 + \nu_2).
\]
Step 2 (Bounding $\|\hat{w}((0,\bar{v})|u) - u\|^2_2$). In this step, using (13) and (14), we bound $(\hat{w}_1((0,\bar{v})|u) - u_1)^2$ and $(\hat{w}_2((0,\bar{v})|u) - u_2)^2$ separately, following similar steps as above.

$$\hat{w}_1((0,\bar{v})|u) = \frac{u_1}{\beta} - \frac{1 - \beta}{\beta} \left\{ \int_0^{v_2} \hat{F}(y) F\left(\frac{\gamma_1 y}{\gamma_2}\right) dy + \frac{\gamma_2}{\gamma_1} \left[ \int_0^{v_2} F\left(\frac{\gamma_2 y}{\gamma_1}\right) dy - F\left(\frac{\gamma_2 \bar{v}}{\gamma_1}\right) \bar{v} - \mathbb{E}_{v_2} \left[ \int_0^{v_2} \hat{F}(y) F\left(\frac{\gamma_1 y}{\gamma_2}\right) dy \right] \right\}$$

Because $u_1 \leq \mathbb{E}[v], \gamma_1/\gamma_2 \leq 1$ and the p.d.f. is bounded by $\bar{f}$ in its support, we obtain the following inequality:

$$(\hat{w}_1((0,\bar{v})|u) - u_1)^2 = \left( \frac{1 - \beta}{\beta} \right)^2 \left( u_1 + \frac{\gamma_1}{\gamma_2} \int_0^{v_2} \hat{F}(y) F\left(\frac{\gamma_1 y}{\gamma_2}\right) dy \right)^2 \leq \left( \frac{1 - \beta}{\beta} \right)^2 \left[ \mathbb{E}[v] + \frac{\bar{f}}{2} \mathbb{E}[v^2] \right]^2.$$ 

Next, we derive a more compact expression for $\hat{w}_2((0,\bar{v})|u)$.

$$\hat{w}_2((0,\bar{v})|u) = \frac{u_2}{\beta} + \frac{1 - \beta}{\beta} \left[ \int_0^{v_2} F\left(\frac{\gamma_2 y}{\gamma_1}\right) dy - F\left(\frac{\gamma_2 \bar{v}}{\gamma_1}\right) \bar{v} - \int_0^{v_2} \hat{F}(y) F\left(\frac{\gamma_2 y}{\gamma_1}\right) dy \right]$$

$$- \frac{1 - \beta}{\beta} \frac{\gamma_1}{\gamma_2} \int_0^{v_2} \hat{F}(y) f\left(\frac{\gamma_1 y}{\gamma_2}\right) y dy$$

$$= \frac{u_2}{\beta} + \frac{1 - \beta}{\beta} \left[ \int_0^{v_2} F(y) F\left(\frac{\gamma_2 y}{\gamma_1}\right) dy - F\left(\frac{\gamma_2 \bar{v}}{\gamma_1}\right) \bar{v} - \int_0^{v_2} \hat{F}(y) f\left(\frac{\gamma_1 y}{\gamma_2}\right) y dy \right].$$

Using a similar approach to previous steps, we now bound $(\hat{w}_2((0,\bar{v})|u) - u_2)^2$.

$$(\hat{w}_2((0,\bar{v})|u) - u_2)^2 \leq \left( \frac{1 - \beta}{\beta} \right)^2 \left[ \max \left( u_2 + \int_0^{v_2} F(y) F\left(\frac{\gamma_2 y}{\gamma_1}\right) dy, \int_0^{v_2} y f(y) F\left(\frac{\gamma_2 y}{\gamma_1}\right) dy + F\left(\frac{\gamma_2 \bar{v}}{\gamma_1}\right) \bar{v} \right) \right]^2$$

$$\leq \left( \frac{1 - \beta}{\beta} \right)^2 \left( \mathbb{E}[v] + \bar{v} \right)^2.$$ 

It follows that $\|\hat{w}((0,\bar{v})|u) - u\|^2_2 \leq \left( \frac{1 - \beta}{\beta} \right)^2 (\nu_3 + \nu_4)$. 

Combining the bounds on $\|\hat{w}((0,\bar{v})|u) - u\|^2_2$ and $\|\hat{w}((0,\bar{v})|u) - u\|^2$, we obtain the following bound on the largest jump, i.e., $\|\hat{w}(v)|u\) - u\|^2 \leq \left( \frac{1 - \beta}{\beta} \right)^2 \max (\nu_1 + \nu_2, \nu_3 + \nu_4) \leq \left( \frac{1 - \beta}{\beta} \right)^2 \nu$. 

Proof of Lemma C.3. First, when $n = 2$, it is obvious that

$$\mathcal{S}(F_{\beta}) = \{ z : |z_1 - z_2| < F_{\beta} \mathbb{E}[v], z > 0 \}. $$

Without loss of generality, assume $\hat{w}_1 \geq \hat{w}_2$. Because $u \in \hat{U}_{\beta}$, we know that $u_1 < F_{\beta} \mathbb{E}[v]$ and $u_2 \geq y_{\beta}$,
which further implies
\[ \hat{w}_1 - F_{\beta}E[v] \leq u_1 - F_{\beta}E[v] + \| \hat{w} - u \|_2 < \| \hat{w} - u \|_2, \] and \( \hat{w}_2 - u_{\beta} \geq u_2 - u_{\beta} - \| \hat{w} - u \|_2 \geq -\| \hat{w} - u \|_2. \]

Combining these two inequalities, we obtain \( \hat{w}_1 - \hat{w}_2 - F_{\beta}E[v] + u_{\beta} < 2\| \hat{w} - u \|_2. \) Equation (35) further implies that \( u_{\beta} \geq 2\| \hat{w} - u \|_2, \) and, therefore, \( \hat{w}_1 - \hat{w}_2 < F_{\beta}E[v]. \)

**Proof of Lemma C.4.** The Hessian matrix of \( \phi(a) \) is provided in Proposition 2.4 for \( a > 0. \) Denote
\[ z \triangleq \int_0^{\tilde{v} \min(a_1,a_2)} x^2 f \left( \frac{x}{a_1} \right) f \left( \frac{x}{a_2} \right) dx. \] Then, \( \Pi(a) \) is given as follows:
\[ \Pi(a) = \left[ \begin{array}{cc} \frac{x^2}{a_1^2a_2} + 1 & \frac{x^2}{a_1^2a_2} + 1 \\ \frac{x^2}{a_1a_2^2} + 1 & \frac{x^2}{a_1a_2^2} + 1 \end{array} \right]. \]

Following Sylvester’s criterion, a matrix is positive definite if its leading principal minors are all positive. Because \( \frac{z}{a_1^3a_2} + 1 > 0, \) it is sufficient to show \( \det(\Pi(a)) > 0. \) Thus, we next derive this determinant and show that it is positive. Because the p.d.f. \( f(\cdot) \) is lower bounded in the support \( z \geq \frac{1}{f(\hat{v} \min(a_1,a_2))} / 3. \) Thus, we have that
\[ \det(\Pi(a)) = \frac{z}{a_1^3a_2} \geq \frac{\left( \frac{1}{f(\hat{v} \min(a_1,a_2))} \right)^3}{3a_1^3a_2} = \frac{\frac{1}{f(\hat{v} \min(a_1,a_2))}}{3 \max(a_1^3,a_2^3)} > 0. \]

**Proof of Lemma C.5.** Lemma C.4 implies we can invoke Proposition C.3 to show that the gradient of \( J \) is well defined and \( \nabla J(u) = \alpha(u/F_{\beta})/F_{\beta} \) as given in (32). Moreover, using Proposition 3.1, we can alternatively express \( \alpha(u/F_{\beta})^T(u - \hat{w}) \) as follows:
\[ -\nabla J(u)^T (w - u) = \frac{\alpha(u/F_{\beta})^T (u - \hat{w})}{F_{\beta}} = \frac{\alpha(u/F_{\beta})^T \nabla \phi(\alpha(u/F_{\beta})) - u}{F_{\beta}} \left( 1 - \frac{1}{\beta} \right). \]

The first-order conditions for the optimization problem \( \sup_{\| x \|_1 = 1, x \geq 0} \left\{ \frac{x^T u}{F_{\beta}} - \phi(x) \right\} \) enable us to express \( u \) as \( u = F_{\beta} \nabla \phi(\alpha(u/F_{\beta})) - \mu e, \) where \( \mu \) is the dual variable corresponding to constraint \( \| x \|_1 = 1 \) and \( e \) is the 2 dimensional vector of ones. Therefore, we get the following equation.
\[ -\nabla J(u)^T (w - u) = \frac{\alpha(u/F_{\beta})^T ((1 - F_{\beta}) \nabla \phi(\alpha(u/F_{\beta})) + \mu e)}{F_{\beta}} \left( 1 - \frac{1}{\beta} \right). \]

Note that \( x^T \nabla \phi(x) = \phi(x) \), and the dual variable \( \mu \) is nonnegative because \( u \in U_{\beta} \) and \( F_{\beta} \nabla \phi(\alpha(u/F_{\beta})) \) corresponds to a state in the efficient frontier of \( U_{\beta}. \) Moreover \( F_{\beta} \geq 1/2 \) and \( \phi(x) \geq E[v]/2 \) for all \( x \in \mathbb{R}^d_+ \) by definition. Thus, using these observations we obtain
\[ -\nabla J(u)^T (w - u) \geq \frac{(1 - F_{\beta}) (1 - \beta)}{\beta} \phi \left( \frac{u}{F_{\beta}} \right) \geq \frac{(1 - \beta) (1 - F_{\beta}) E[v]}{4 \beta F_{\beta}^2} = \frac{(1 - \beta)^2 \xi}{4 \beta F_{\beta}^2}. \]
Proof of Lemma C.6 \[ \text{Lemma C.4 enables us to use Proposition C.3 to obtain that } 
\]
\[ \text{Hess}(\mathcal{J}(\Theta)) = \frac{1}{F^2_\beta} \Pi(\alpha(\Theta/F_\beta))^{-1} \left( I - \frac{ee^T \Pi(\alpha(\Theta/F_\beta))^{-1}}{e^T \Pi(\alpha(\Theta/F_\beta))^{-1} e} \right). \]

Denote \( \lambda = \alpha(\Theta/F_\beta) \) and \( z \triangleq \int \min(\lambda_1, \lambda_2) x^2 f \left( \frac{x}{\lambda_1} \right) f \left( \frac{x}{\lambda_2} \right) \text{d}x \), then \( \Pi(\lambda) = \text{Hess}(\phi(\lambda)) + ee^T \) by definition. Therefore, we can evaluate the matrix multiplication operations using the definitions and we obtain
\[ \text{Hess}(\mathcal{J}(\Theta)) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{1}{F^2_\beta} \frac{\lambda_1 \lambda_2^3}{z}. \]

The largest eigenvalue of matrix \( \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \) is 2. Because the p.d.f. \( f(\cdot) \) is lower bounded in the support, we have that \( z \geq \frac{1}{6} \left( \frac{v}{\min(\lambda_1, \lambda_2)} \right)^3 \). These observations and the fact that \( \max(\lambda_1, \lambda_2) \leq 1 \) imply that the largest eigenvalue of \( \text{Hess}(\mathcal{J}(\Theta)) \) is bounded by \( \frac{6}{F^2_\beta v^4} \). Finally, using Lemma C.2, we can also bound \( \|\hat{w} - u\|_2^2 \) and obtain the following inequality:
\[ \frac{1}{2}(\hat{w} - u)^T \text{Hess}(\mathcal{J}(\Theta))(\hat{w} - u) \leq \frac{3}{F^2_\beta v^2} \|\hat{w} - u\|_2^2 \leq \left( \frac{1 - \beta}{\beta} \right)^2 \frac{3\nu}{F^2_\beta v^4}. \]
D Appendix for Section 5

D.1 Proof of Proposition 5.1

Proof. Agent $i$′s allocation is given in (12) as $\hat{p}_i(v|u) = 1\{\alpha_i(u/F_\beta)v_i \geq \max_{j \neq i} \alpha_j(u/F_\beta)v_j\}$ where $\alpha(u/F_\beta) = \arg\max_{\|x\|_1=1,x \geq 0} \left\{ \frac{u^Tx}{F_\beta} - \phi(x) \right\}$.

Step 1 ($\hat{p}_i(v|u)$ is non-decreasing in $v_i$). Let $v \geq v'$, and fix $v_{-i} \in [0,\bar{v}]^{n-1}$ and $u$. If $\alpha_i(u/F_\beta)v' \geq \max_{j \neq i} \alpha_j(u/F_\beta)v_j$ then $\alpha_i(u/F_\beta)v \geq \max_{j \neq i} \alpha_j(u/F_\beta)v_j$ because $\alpha(u/F_\beta) \geq 0$. Therefore, whenever $1\{\alpha_i(u/F_\beta)v' \geq \max_{j \neq i} \alpha_j(u/F_\beta)v_j\} = 1$, it follows that $1\{\alpha_i(u/F_\beta)v \geq \max_{j \neq i} \alpha_j(u/F_\beta)v_j\} = 1$. Hence, $\hat{p}_i(v, v_{-i}|u) \geq \hat{p}_i(v', v_{-i}|u)$.

Step 2 ($\hat{p}_i(v|u)$ is non-decreasing in $u_i$). Consider the following optimization problem,

$$\sup_{\|x\|_1=1,x \geq 0} \left\{ \frac{u^Tx}{F_\beta} - \phi(x) \right\}.$$ 

Change variables with $a = x/x_i$, such that $a_i = 1$, and, for any $x$ satisfying $\|x\|_1 = 1$, $x_i = 1/\left(1 + \sum_{j \neq i} x_j\right)$. Consequently, the optimization problem above is equivalent to $\sup_{a_{-i} \geq 0} \Xi(u_i, a_{-i})$, in which

$$\Xi(u_i, a_{-i}) \triangleq \frac{1}{1 + \sum_{j \neq i} a_j} \left( \frac{u_i + u^T_{-i}a_{-i}}{F_\beta} - \phi\left((a_{-i}, 1)\right) \right),$$

where we use that $\phi(\cdot)$ is homogeneous of degree one. We have

$$\frac{\partial^2 \Xi}{\partial u_i \partial a_j} = -\frac{1}{F_\beta} < 0.$$ 

Therefore, function $\Xi$ is submodular in $u_i$ and $a_j$, which implies that as $u_i$ increases, the optimal $a_j^*$ decreases. As a result, $\hat{p}_i(v|u) = 1\{v_i \geq \max_{j \neq i} a_j^*v_j\}$ increases with $u_i$. □

D.2 Proof of Proposition 5.2

Proof. In this proof, we first show that agent $i$′s interim allocation function is a non-decreasing function of his value, $v_i$. Using this observation, we show that agent $i$′s interim future promise function is a non-increasing function of his value $v_i$. Finally, using the definition of $\hat{w}_i(v|u)$, we conclude the proof.

The interim allocation $\tilde{P}_i(v_i|u) = \prod_{j \neq i} P\left(\frac{\alpha_i(u/F_\beta)v_i}{\alpha_j(u/F_\beta)}\right)$ is a non-decreasing function of $v_i$ because it is a product of non-decreasing functions of $v_i$. The interim future promise function $\tilde{W}_i(v_i|u)$ is given in (13) as

$$\tilde{W}_i(v_i|u) = \frac{1}{\beta} \left( u_i + (1 - \beta) \left( \int_0^{v_i} \tilde{P}_i(y|u)dy - \tilde{P}_i(v_i|u)v_i - \int_0^{v_i} F(y)\tilde{P}_i(y|u)dy \right) \right), \forall i.$$ 

We next show that $\tilde{P}_i(v_i|u)$ being non-decreasing implies that $\tilde{W}_i(v_i|u)$ is a non-increasing function of $v_i$. Let $v \geq v'$. Then, it follows that $\tilde{P}_i(v|u) \geq \tilde{P}_i(y|u)$ for all $v \geq y$. Using this fact, we obtain the
following inequality.

\[ v\hat{P}_i(v|u) - v'\hat{P}_i(v'|u) \geq v\hat{P}_i(v|u) - v'\hat{P}_i(v|u) = \int_{v'}^{v} \hat{P}_i(v|u)dy \geq \int_{v'}^{v} \hat{P}_i(y|u)dy \]

Rearranging the terms in the inequality \( v\hat{P}_i(v|u) - v'\hat{P}_i(v'|u) \geq \int_{v'}^{v} \hat{P}_i(y|u)dy \), we obtain

\[ \int_{0}^{v'} \hat{P}_i(y|u)dy - v'\hat{P}_i(v'|u) \geq \int_{0}^{v} \hat{P}_i(y|u)dy - v\hat{P}_i(v|u). \]

This observation implies that \( \int_{0}^{v_i} \tilde{P}_i(y|u)dy - v_i\tilde{P}_i(v_i|u) \), and, hence, \( \tilde{W}_i(v_i|u) \) is a non-increasing function of \( v_i \).

Because \( \tilde{W}_i(v_i|u) \) is non-increasing in \( v_i \), the future promise function \( \tilde{w}_i(v|u) \) given in (14) as

\[ \tilde{w}_i(v|u) = \tilde{W}_i(v_i|u) - \frac{1}{n-1} \sum_{j \neq i} \alpha_j(u/F) \left[ \tilde{W}_j(v_j|u) - E_{\tilde{v}_j} [\tilde{W}(\tilde{v}_j|u)] \right], \]

is non-increasing in \( v_i \), and non-decreasing in \( v_j \), for all \( j \neq i \). \( \square \)
E Appendix for Section 6

E.1 Proof of Proposition 6.1

Before proving the proposition, we provide the values of constants used to define our mechanism. Following this definitions, we provide the proof of the proposition.

Definition E.1. The functions $K_1(k)$ and $K_2(k)$ are defined as follows:

$$
K_1(k) \triangleq \left(3c_s(k)c_n(k)\right)^{\frac{1}{2}}
$$

$$
K_2(k) \triangleq \frac{k3^{k+1}(c_s(k))^2(c_n(k))^{k+3}}{\mathbb{E}[v]c_n(k)},
$$

where

$$
c_n(k) \triangleq \sqrt{k(\mathbb{E}[v] + \bar{v})},
$$

$$
c_s(k) \triangleq kf\mathbb{E}[v^2],
$$

$$
c_n(k) \triangleq \frac{(f\bar{v})^{k+2}(k-1)}{k(k-1)f^2\bar{v} + f^{k+2}\bar{v}k^2 + (k-1)\bar{v}^2f^2 k + 1}.
$$

Definition E.2. We define the sequence $\left\{ F^{(k,n)}_{\beta} \right\}_{k=1}^{n}$ and the constant $u^{(n)}_{\beta}$ as follows:

$$
F^{(k,n)}_{\beta} \triangleq 1 - \frac{n(k-1)u^{(n)}_{\beta}}{\mathbb{E}[v]} \quad \text{and} \quad u^{(n)}_{\beta} \triangleq \xi^{(n)}(1-\beta)^{\frac{1}{n+1}},
$$

where $\xi^{(n)} \triangleq \max\left[ \frac{K_1(n)}{\beta^{3/2}}, \max_{k=2...n} \left( \frac{\mathbb{E}[v]K_2(k)}{n(k-1)\beta^{2}} \right)^{\frac{1}{k+4}} \right]$, and $K_1(k)$ and $K_2(k)$ are defined in Definition E.1. Furthermore, we determine $\beta$ such that for all $\beta \geq \beta$, $F^{(k,n)}_{\beta} \in [0.5, 1]$ for all $k = 2, \ldots, n$, and $nu^{(n)}_{\beta} \leq \mathbb{E}[v]$.

Proof of Proposition 6.1. We prove this result in three steps. We defer the proofs of lemmas used in these steps to the end of this appendix.

Following Proposition 2.1, it is sufficient to show that $\Omega_{\beta} \in B_{\beta}(\Omega_{\beta})$. Specifically, we need to show that for an arbitrary $u \in \Omega_{\beta}$, there exists $(\hat{p}, \hat{w})$ satisfying (IC), (FA), (PK($u$)) and $w(v) \in \Omega_{\beta}$ for all $v \in [0, \bar{v}]^n$. Fix $u \in \Omega_{\beta}$ and consider our mechanism $(\hat{p}, \hat{w})$. By construction of $(\hat{p}, \hat{w})$, (IC), (FA), (PK($u$)) are satisfied. We will analyze the last condition.

Step 1. First, we show that $\hat{w}(v|u) \geq 0$. Note that when $k \leq 1$ we have $\hat{w}(v|u) = u$, and the result holds trivially. Therefore, the proof focuses on situations with $k > 1$. For simplicity, we use $w^{(k)}$ to represent the future promises $w^{(k)}(v|u)$. For an inactive agent $i$ (i.e., $u_i < u^{(n)}_{\beta}$), $\hat{w}_i(v|u) = u_i \geq 0$, thus we focus on $w^{(k)}$, and the bound on the jump from $u^{(k)}$ to $w^{(k)}$ which is provided by the following lemma.

Lemma E.1. We have that $\|w^{(k)} - u^{(k)}\|_2 \leq \left( \frac{1-\beta}{\beta} \right) \frac{(K_1(k))^2}{3u^{(n)}_{\beta}} s(u)$.
Following this lemma, we can argue that \( \|w^{(k)} - u^{(k)}\|_2 < \frac{u^{(n)}}{3} \) which implies \( w^{(k)} > 0 \) and \( \hat{w}(v|u) \geq 0 \). In particular, using the fact that \((1 - \beta) < (1 - \beta)\frac{2}{n+1} \) for \( n \geq 1 \), \( (K_1(k))^2 / \beta \leq (\frac{\xi(n)}{2})^2 \), and \( s(u) \leq 1 \) we have that

\[
\|w^{(k)} - u^{(k)}\|_2 \leq \left( \frac{1 - \beta}{\beta} \right) \frac{(K_1(k))^2}{3u^{(n)}} \leq \left( \frac{1 - \beta}{\beta} \right) \frac{2}{3} \frac{(\xi(n))^2}{3u^{(n)}} = \frac{(u^{(n)})^2}{3u^{(n)}} \leq \frac{u^{(n)}}{3}.
\]

\( (37) \)

**Step 2.** In this step, we show that \( w^{(k)} \in s(u)F^{(k,n)}_{\beta}U^{(k)} \). We consider the indicator function \( \mathcal{J} \) defined in \( \text{(30)} \) with scalar \( s(u)F^{(k,n)}_{\beta} \). For simplicity, we define \( \mathcal{J}(x) = \mathcal{J}(x, s(u)F^{(k,n)}_{\beta}) \) in this proof. Corollary \( \text{C.1} \) states that \( \mathcal{J}(w^{(k)}) \leq 0 \) if and only if \( w^{(k)} \in s(u)F^{(k,n)}_{\beta}U^{(k)} \), because \( w^{(k)} \geq 0 \) as shown earlier. In order to show \( \mathcal{J}(w^{(k)}) \leq 0 \), we consider the quadratic expansion of \( \mathcal{J}(w^{(k)}) \) around \( u^{(k)} \) provided in Proposition \( \text{C.4} \). We next prove that the necessary conditions for Proposition \( \text{C.4} \) to hold.

**Lemma E.2.** The future promise \( w^{(k)} \) is in \( S\left(s(u)F^{(k,n)}_{\beta}\right) \).

Following this lemma, we obtain that the set \( S\left(s(u)F^{(k,n)}_{\beta}\right) \) contains all \( \Theta^{(k)} \) between \( w^{(k)} \) and \( u^{(k)} \) because it is convex. Therefore, Proposition \( \text{C.2} \) implies that \( \alpha(\Theta^{(k)}/s(u)F^{(k,n)}_{\beta}) > 0 \).

**Lemma E.3.** The matrix \( \Pi(a) \) is positive definite for all \( a > 0 \), and its minimum eigenvalue is \( c_{\min}(a) (\min_i \alpha_i)^{k+1} \).

This lemma enables us to invoke Proposition \( \text{C.4} \). Thus, we can analyze the terms in the quadratic expansion. We have that \( \mathcal{J}(u^{(k)}) \leq 0 \) because \( u^{(k)} \in s(u)F^{(k,n)}_{\beta}U^{(k)} \). In order to show, \( \mathcal{J}(w^{(k)}) \leq 0 \), it is sufficient to show the following inequality holds.

\[
\nabla \mathcal{J} (u^{(k)})^T (w^{(k)} - u^{(k)}) + \frac{1}{2} (w^{(k)} - u^{(k)})^T \text{Hess} \left( \mathcal{J} (\Theta^{(k)}) \right) (w^{(k)} - u^{(k)}) \leq 0.
\]

We first consider the first-order term. The following lemma provides a lower bound for \( -\nabla \mathcal{J} (u^{(k)})^T (w^{(k)} - u^{(k)}) \).

**Lemma E.4.** The gradient of \( \mathcal{J}(x) \) evaluated at \( u^{(k)} \) satisfies

\[
\frac{(1 - \beta) (1 - F^{(k,n)}_{\beta}) \mathbb{E}[u]}{2k (F^{(k,n)}_{\beta})^2} \leq -\nabla \mathcal{J} (u^{(k)})^T (w^{(k)} - u^{(k)}) .
\]

We next consider the second-order term.

**Lemma E.5.** We have that

\[
\frac{1}{2} (w^{(k)} - u^{(k)})^T \text{Hess} \left( \mathcal{J} (\Theta^{(k)}) \right) (w^{(k)} - u^{(k)}) \leq \frac{1}{2} \frac{1}{F^{(k,n)}_{\beta}} \frac{1}{(u^{(n)})^{k+3}} \left( \frac{1 - \beta}{\beta} \right)^2 \mathbb{E}[v]K_2(k) .
\]

For \( n \geq k \), it follows that \( (1 - \beta) \frac{k+4}{n+1} \geq (1 - \beta) \), and \( \xi(n) (n(k-1)\beta)^2 \frac{1}{n+1} \geq (\mathbb{E}[v]K_2(k))^{\frac{1}{n+1}} \), thus we have that \( (1 - F^{(k,n)}_{\beta}) (u^{(n)})^{k+3} \beta^2 \geq (1 - \beta)K_2(k) \). Finally, we use this observation to replace
Recall that for an inactive agent $w^{(k)}$ in the upper bound provided in Lemma E.5 and show that this upper bound is smaller than the lower bound on the term with gradient (negative term) given in Lemma E.1. Consequently, it follows that $w^{(k)} \in s(u)F_{\beta}^{(k,n)}u^{(k)}$

Note that $w^{(k)} \in s(u)F_{\beta}^{(k,n)}u^{(k)}$ does not necessarily guarantee that $\hat{w}(v|u)$ lies in the set $\Omega_{\beta}$ because $\hat{w}(v|u)$ could involve fewer active agents than $u$. That is, in the next round the number of active agents may decrease to $k' < k$. Therefore, in the following step, we prove that $w^{(k')}(v|u) \in s(\hat{w}(v|u))F_{\beta}^{(k',n)}u^{(k')}$

**Step 3.** For simplicity, we use vector $w^{(k')}$ to represent $w^{(k')}(v|u)$. The support function of the set $s(u)F_{\beta}^{(k,n)}u^{(k)}$ is $s(u)F_{\beta}^{(k,n)}\phi^{(k)}(\cdot)$. According to Schneider (2013), p. 44, the support function satisfies that $z \in s(u)F_{\beta}^{(k,n)}u^{(k)}$ if and only if $z^T x^{(k)} \leq s(u)F_{\beta}^{(k,n)}\phi^{(k)}(x^{(k)})$ for all $x^{(k)} \in \mathbb{R}^k$. Therefore, the fact that $w^{(k)} \in s(u)F_{\beta}^{(k,n)}u^{(k)}$ implies that $(w^{(k)})^T x^{(k)} \leq s(u)F_{\beta}^{(k,n)}\phi^{(k)}(x^{(k)})$ for all $x^{(k)} \in \mathbb{R}^k$.

To prove that $w^{(k')} \in s(\hat{w}(v|u))F_{\beta}^{(k',n)}u^{(k')}$, we show that $(w^{(k')})^T x^{(k')} \leq s(\hat{w}(v|u))F_{\beta}^{(k',n)}\phi^{(k')}(x^{(k')})$ for an arbitrary $x^{(k')} \in \mathbb{R}^{k'}$. Fix $x^{(k')} \in \mathbb{R}^{k'}$. Let $x \in \mathbb{R}^k$ be such that $x_i = x_i^{(k')}$ for all $i \leq k'$ and $x_i = 0$ otherwise. Therefore, we obtain $x^T w^{(k')} = (x^{(k')})^T w^{(k')}$, and $\phi^{(k')}(x) = \phi^{(k')}(x^{(k')})$. Now, using the property of the support functions, we obtain $(w^{(k')})^T x^{(k')} \leq s(u)F_{\beta}^{(k',n)}\phi^{(k')}(x^{(k')})$.

We next prove that $s(\hat{w}(v|u))F_{\beta}^{(k',n)} \geq s(u)F_{\beta}^{(k,n)}$. Using the definitions, we get the following inequality for $F_{\beta}^{(k,n)}$ and $F_{\beta}^{(k',n)}$:

$$F_{\beta}^{(k,n)} = F_{\beta}^{(k,n)} - F_{\beta}^{(k,n)} + F_{\beta}^{(k',n)} = F_{\beta}^{(k',n)} - \frac{n(k-k')u_{\alpha}^{(n)}}{E[v]} \leq F_{\beta}^{(k',n)} \left(1 - \frac{n(k-k')u_{\alpha}^{(n)}}{E[v]}\right).$$

Recall that for an inactive agent $i$ (i.e., $u_i < u_{\beta}^{(n)}$) our future promise function $\hat{w}(v|u)$ is such that $\hat{w}(v|u) = u_i$. If an agent is inactive for a given initial state $u$, then he remains to be inactive for a given future promise $\hat{w}(v|u)$. Therefore, we can bound $s(\hat{w}(v|u))$ in terms of $s(u)$ as follows:

$$s(\hat{w}(v|u)) = s(u) - \frac{\sum_{i=k'+1}^{k} \hat{w}_i(v|u)}{E[v]} \geq s(u) \left(1 - \frac{(k-k')u_{\alpha}^{(n)}}{s(u)E[v]}\right) \geq s(u) \left(1 - \frac{n(k-k')u_{\alpha}^{(n)}}{E[v]}\right).$$

where (a) follows from the fact that $u_{\beta}^{(n)} \geq \hat{w}_i(v|u)$ for the inactive agents and (b) follows because $s(u) \geq 1/n$ (since $E[v] \geq u_{\beta}^{(n)}$ and $k > 1$). Therefore, it follows that $s(\hat{w}(v|u))F_{\beta}^{(k',n)} \geq s(u)F_{\beta}^{(k,n)}$.

Finally, this observation implies that $(w^{(k')})^T x^{(k')} \leq s(\hat{w}(v|u))F_{\beta}^{(k',n)}\phi^{(k')}(x^{(k')})$, and $w^{(k')}(v|u) \in s(\hat{w}(v|u))F_{\beta}^{(k',n)}u^{(k')}$. Hence, we obtain that

$$\hat{w}(v|u) \in \left\{ u \in \mathbb{R}^n : u^{(k')} \geq u_{\beta}^{(n)}, u^{(k')} < u_{\beta}^{(n)}, u^{(k')} \in s(u)F_{\beta}^{(k',n)}u^{(k')} \right\} \subseteq \Omega_{\beta}. \quad \Box$$

**E.2 Proof of Theorem 6.1**

**Proof.** The set $\Omega_{\beta}$ is a subset of $U_{\beta}^{(n)}$ by Proposition 6.1. Thus, the maximum achievable social welfare $J_{\beta} = F_{\beta}^{(n,n)}J_{FB}$, is less than $J_{\beta}$. \[\square\]
E.3 Proofs of Lemmas E.1, E.2, E.3, E.4 and E.5

In the following proofs, we consider \( u \in \Omega_{\beta} \) with \( k \) active agents and the \( k \) dimensional subvector \( u^{(k)} \) represent the promised utilities for the active agents. We use \( \Theta^{(k)} \) to represent an arbitrary \( k \) dimensional point between \( u^{(k)} \) and \( w^{(k)}(v|u) \). Here the set \( \Omega_{\beta} \) is characterized by constants \( u^{(n)}_\beta \) and \( \left\{ F^{(k,n)}_\beta \right\}_{k=1}^n \) following Definition E.2. Further, define vectors

\[
\gamma = \alpha^{(k)} \left( \frac{u^{(k)}}{s(u)F^{(k,n)}_{\beta}} \right) \quad \text{and} \quad \lambda = \alpha^{(k)} \left( \frac{\Theta^{(k)}}{s(u)F^{(k,n)}_{\beta}} \right).
\]  

(38)

**Proof of Lemma E.1.** For simplicity, we use the short notation \( w^{(k)} \) to represent the future promises \( w^{(k)}(v|u) \). Because \( \|w^{(k)} - u^{(k)}\|_2 \leq \sqrt{k}\|w^{(k)} - u^{(k)}\|_\infty \), we focus on an upper bound for \( \|w^{(k)} - u^{(k)}\|_\infty = \max_i |w_i^{(k)} - u_i^{(k)}| \). We start with the following expansion.

\[
w_i^{(k)} - u_i^{(k)} = \left( \frac{1 - \beta}{\beta} \right) \left[ u_i^{(k)} + \int_0^{v_i} P_i^{(k)}(y|u)dy - P_i^{(k)}(v_i|u) + \int_0^\bar{y} \bar{F}(y)P_i^{(k)}(y|u)dy \right]
- \frac{1}{k-1} \sum_{j \neq i} \frac{\gamma_j}{\gamma_i} \left[ W_j^{(k)}(v_j|u) - \mathbb{E}_{\hat{v}_j} \left[ W_j^{(k)}(\hat{v}_j|u) \right] \right],
\]

where \( P_i^{(k)}(y|u) = s(u) \prod_{j \neq i} F \left( \frac{\gamma_i y}{\gamma_j} \right) \) and \( W_j^{(k)}(v_j|u) - \mathbb{E}_{\hat{v}_j} \left[ W_j^{(k)}(\hat{v}_j|u) \right] \) is given by

\[
\left( \frac{1 - \beta}{\beta} \right) \left\{ \int_0^{v_j} P_j^{(k)}(y|u)dy - P_j^{(k)}(v_j|u)v_j - \mathbb{E}_{\hat{v}_j} \left[ \int_0^{\bar{y}_j} P_j^{(k)}(y|u)dy \right] + \mathbb{E}_{\hat{v}_j} \left[ P_j^{(k)}(\hat{v}_j|u)\hat{v}_j \right] \right\}.
\]

We prove this result in three steps. In the first two steps, we consider two cases: \( |w_i^{(k)} - u_i^{(k)}| = w_i^{(k)} - u_i^{(k)} \) and \( |w_i^{(k)} - u_i^{(k)}| = u_i^{(k)} - w_i^{(k)} \), and get a bound on the norm which depends on \( 1/\min_i \gamma_i \). In the last step, we get an upper bound for \( 1/\min_i \gamma_i \) that depends on \( u^{(n)}_\beta \).

**Step 1 (Bounding \( w_i^{(k)} - u_i^{(k)} \)).** Here, we remove negative terms to obtain the following bound.

\[
w_i^{(k)} - u_i^{(k)} \leq \left( \frac{1 - \beta}{\beta} \right) \left[ u_i^{(k)} + \int_0^{v_i} P_i^{(k)}(y|u)dy + \frac{1}{k-1} \sum_{j \neq i} \frac{\gamma_j}{\gamma_i} \left( P_i^{(k)}(v_i|u)v_i + \mathbb{E}_{\hat{v}_j} \left[ \int_0^{v_j} P_j^{(k)}(y|u)dy \right] \right) \right].
\]

Note that \( P_i^{(k)}(v|u) \leq s(u) \) for all \( v \) by definition, and \( u_i^{(k)} \leq s(u)\mathbb{E}[v] \). Moreover, using the fact that the sum of \( \gamma_i \)'s adds up to 1, and \( v_i \) and \( v_j \) are less than \( \hat{v} \), we get the following bound.

\[
w_i^{(k)} - u_i^{(k)} \leq \left( \frac{1 - \beta}{\beta} \right) s(u)(\mathbb{E}[v] + \bar{v}) \left[ 1 + \frac{1 - \gamma_i}{(k-1)\gamma_i} \right] \leq s(u) \left( \frac{1 - \beta}{\beta} \right) \left( \frac{\mathbb{E}[v] + \bar{v}}{\min_i \gamma_i} \right).
\]
Step 2 (Bounding $u_i^{(k)} - w_i^{(k)}$). Similar to the previous step, we obtain

$$u_i^{(k)} - w_i^{(k)} \leq \left( \frac{1 - \beta}{\beta} \right) \left[ P_i^{(k)}(v_i u_i) + \int_0^v F(y) P_i^{(k)}(y u_i) dy \right] + \frac{1}{k - 1} \sum_{j \neq i} \gamma_j \gamma_i \left( \int_0^{\gamma_j} P_j^{(k)}(y u_i) dy + \mathbb{E}[\hat{P}_j^{(k)}(\hat{v}_j u_i)] \right),$$

$$\leq s(u) \left( \frac{1 - \beta}{\beta} \right) \left( \frac{\|v\| + \bar{v}}{\min_{i} \gamma_i} \right).$$

Summarizing the previous two steps, we obtain $\|w^{(k)} - u^{(k)}\|_{\infty} \leq s(u) \left( \frac{1 - \beta}{\beta} \right) \left( \frac{\|v\| + \bar{v}}{\min_{i} \gamma_i} \right)$. Recognizing $\|w^{(k)} - u^{(k)}\|_2 \leq \sqrt{k} \|w^{(k)} - u^{(k)}\|_{\infty}$, we obtain

$$\|w^{(k)} - u^{(k)}\|_2 \leq s(u) \left( \frac{1 - \beta}{\beta} \right) \left( \frac{\sqrt{k}(\|v\| + \bar{v})}{\min_{i} \gamma_i} \right) \leq s(u) \left( \frac{1 - \beta}{\beta} \right) \left( \frac{c_N(k)}{\min_{i} \gamma_i} \right).$$

Step 3 (Bounding $1/\min_i \gamma_i$). In this step, we find an upper bound for $1/\min_i \gamma_i$ as follows:

$$u_i^{(k)} \overset{(a)}{=} s(u)f_{\beta,n}(\gamma_i v_i) \overset{(b)}{=} \gamma_i k \mathbb{E}[v^2] = c_S(k) \gamma_i.$$

As discussed in Remark 3.1, $u_i^{(k)} = s(u)f_{\beta,n}(\nabla \phi^{(k)}(\gamma)) - \mu$ for some nonnegative scalar $\mu$. Using the expression derived in Proposition 2.4 for $(\nabla \phi^{(k)}(\gamma))$, we get (a). Because the p.d.f. $f(\cdot)$ is bounded in the support by $f$ and the c.d.f. $F(\cdot)$ is always less than one, $\mathbb{E}[v_i \{ \gamma_i v_i \geq \max_{j \neq i} \gamma_j v_j \}]$ is bounded by $\gamma_i k \mathbb{E}[v^2] / \max_{j \neq i} \gamma_j$. Together with this observation, using the fact that $\max_{j \neq i} \gamma_j \geq 1/k$, and $s(u) \leq 1$ and $F_{\beta,n}(k) \leq 1$, we obtain (b). Taking the minimum over $i$'s in both sides, we get $c_S(k) \min_i \gamma_i \geq \min_i u_i^{(k)} \geq u_{\beta}^{(n)}$. Consequently, we have that

$$\frac{\|w^{(k)} - u^{(k)}\|_2}{s(u)} \leq \left( \frac{1 - \beta}{\beta} \right) \left( \frac{c_S(k) c_{n}(k)}{\min_i \gamma_i} \right) \leq \left( \frac{1 - \beta}{\beta} \right) \left( \frac{K_1(k)}{3u_{\beta}^{(n)}} \right).$$

Proof of Lemma 3.3: Equation (37) implies that $\|w^{(k)} - u^{(k)}\|_2 \leq u_{\beta}^{(n)}/3$. Because $u^{(k)} \geq u_{\beta}^{(n)}$, we must have $w^{(k)} > 0$. If the point $w^{(k)} \in s(u)F_{\beta,n}(k)U^{(k)}$, the result holds immediately. Now suppose $w^{(k)} \not\in s(u)F_{\beta,n}(k)U^{(k)}$, and, therefore, there must exist a component $j$ such that $w_j^{(k)} > u_j^{(k)}$. Select $\bar{a}$ such that $\bar{a} = \inf\{a : w^{(k)} - a e \leq u^{(k)}\}$, therefore $w^{(k)} - \bar{a} e \in S(s(u)F_{\beta,n}(k))$ if $w^{(k)} - \bar{a} e > 0$. The definition of $\bar{a}$ implies that there must exist a component $i$ such that $w_i^{(k)} - \bar{a} = u_i^{(k)} > 0$. We next focus on components $j \neq i$. Using the fact that $\bar{a} = w_i^{(k)} - u_i^{(k)}$, we have $w_j^{(k)} - \bar{a} = w_j^{(k)} - w_i^{(k)} + u_i^{(k)}$. Thus, we only need to show that $w_j^{(k)} - w_i^{(k)} + u_i^{(k)} > 0$. Because $u_i^{(k)} + \|w^{(k)} - u^{(k)}\|_2 \geq w_i^{(k)}$ and
Following functions:

Step 1. In this step, we provide a lower bound for the minimum eigenvalue of definite. We show this result in three steps.

Denote \(\Pi(\alpha) = M + \ee^T\) is also well defined. We identify a lower bound on the minimum eigenvalue of the matrix \(\Pi(\alpha)\) which is positive and thus implying that \(\Pi(\alpha)\) is positive definite. We show this result in three steps.

**Step 1.** In this step, we provide a lower bound for the minimum eigenvalue of \(M + \ee^T\). Define the following functions:

\[
\omega(y) \triangleq \frac{\prod F \left( \frac{y}{a_i} \right)}{y^2} \quad \text{and} \quad \Phi \triangleq \int_0^v \omega(y)dy ,
\]

\[
\eta(y) \triangleq y^2 \frac{F(y)}{F(v)} 1\{y \leq v\} \quad \text{and} \quad g_i(y) \triangleq \frac{1}{a_i} \eta \left( \frac{y}{a_i} \right) .
\]

Let \(G\) be a matrix whose entries are given by \(G_{ij} = \langle a_i g_i, a_j g_j \rangle_\omega \triangleq \int_0^v a_i g_i(y) a_j g_j(y) \omega(y)dy\).

Using the weighted norm notation in Definition \(E.3\) we can equivalently express \(M\) as follows:

\[
M = \text{diag} \left( \sum_j a_j^2 \langle g_i, g_j \rangle_\omega \right) - G .
\]

Here, the first term defines a diagonal matrix whose \(i^{\text{th}}\) diagonal component is \(a_i g_i \langle g_i, g_j \rangle_\omega\). Define \(M(y) = \text{diag} \left( \sum_j a_j^2 g_i(y) g_j(y) \right) - G(y)\) to alternatively represent \(M\), i.e., \(M = \int_0^v M(y) \omega(y)dy\).

Next, consider the following optimization problem, whose objective value is the minimum eigenvalue of the matrix \((M + \ee^T)\),

\[
\Psi \triangleq \min_{\|x\|_2 \geq 1} x^T(M + \ee^T)x = \min_{\|x\|_2 \geq 1} \int_0^v (x^T M(y)x) \omega(y)dy + (x^T e)^2 .
\]

A straightforward lower bound for \(\Psi\) is obtained by taking the minimization to inside the integral. To do that, we also need to take the second term, \((x^T e)^2\), to inside the integral by properly adjusting the weight as follows:

\[
\hat{e} \triangleq \frac{e}{\sqrt{\Phi}} .
\]
Therefore, we have
\[
\Psi \geq \int_0^\infty \left[ \min_{\|x\|_2 \geq 1} (x^T M(y)x) + (x^T \hat{e})^2 \right] \omega(y)dy. \tag{40}
\]

**Step 2.** Denote \( \triangle(y) \) to represent \( \min_{\|x\|_2 \geq 1} (x^T M(y)x) + (x^T \hat{e})^2 \). Then, the inequality (40) is represented as \( \Psi \geq \int_0^\infty \triangle(y) \omega(y)dy \). We continue by analyzing \( \triangle(y) \).

\[
\triangle(y) = \min_{\|x\|_2 \geq 1} (x^T M(y)x) + (x^T \hat{e})^2
= \min_{\|x\|_2 \geq 1} \left( \sum_j a_j^2 g_j(y) \right) \left( \sum_j x_j^2 g_j(y) \right) - \left( \sum_j x_j g_j(y) a_j \right)^2 + \left( \sum_j x_j \frac{1}{\sqrt{\Phi}} \right)^2
= \min_{\|x\|_2 \geq 1} \|a\|^2 \|x\|^2 - \langle x, a \rangle^2 + \left( \sum_j x_j \frac{1}{g_j \sqrt{\Phi}} \right)^2.
\]

Denote \( e_g \triangleq \left( \frac{1}{\sqrt{\Phi g_i}} \right) \). Then, \( \triangle(y) \) is given as follows:
\[
\triangle(y) = \min_{\|x\|_2 \geq 1} \|a\|^2 \|x\|^2 - \langle x, a \rangle^2 + \langle x, e_g \rangle^2. \tag{41}
\]

We next provide a lower bound for \( \triangle(y) \) by using the expression (41). We first consider a relaxation obtained by replacing the constraint \( \|x\|_2 \geq 1 \) with \( \|x\|_2^2 \geq \min_i g_i \) as follows:

\[
\triangle(y) \geq \min_{\|x\|_2^2 \geq \min_i g_i} \|a\|^2 \|x\|^2 - \langle x, a \rangle^2 + \langle x, e_g \rangle^2 = \min_i g_i \|a\|^2 + \min_{\|x\|_2^2 \geq \min_i g_i} \langle x, e_g \rangle^2 - \langle x, a \rangle^2,
\]

\[
= \min_i g_i \left[ \|a\|^2 + \min_{\|x\|_2^2 \geq 1} \langle x, e_g \rangle^2 - \langle x, a \rangle^2 \right].
\]

In order to express this lower bound on \( \triangle(y) \), there is a need for a closed-form solution for the optimization problem over the differences of weighted dot products. The following lemma provides a closed-form solution for it.

**Lemma E.6.** The optimal value of the optimization problem \( \min_{\|x\|_2^2 \geq 1} \langle x, e_g \rangle^2 - \langle x, a \rangle^2 \) is given by
\[
\frac{1}{2} \left( \|e_g\|^2 + \|a\|^2 - \sqrt{\left( \|e_g\|^2 + \|a\|^2 \right)^2 - \left( 2 \langle e_g, a \rangle \right)^2} \right).
\]

We defer the proof of this lemma to the supplementary appendix. Using the closed-form solution given in the lemma, the lower bound on \( \triangle(y) \) is updated as follows:

\[
\triangle(y) \geq \frac{\min_i g_i}{2} \left[ \|e_g\|^2 + \|a\|^2 - \sqrt{\left( \|e_g\|^2 + \|a\|^2 \right)^2 - \left( 2 \langle e_g, a \rangle \right)^2} \right]
\geq \frac{\min_i g_i}{2} \left( \|e_g\|^2 + \|a\|^2 \right) \frac{(2 \langle e_g, a \rangle)^2}{2 \left( \|e_g\|^2 + \|a\|^2 \right)^2} = \frac{\min_i g_i \langle e_g, a \rangle^2}{\|e_g\|^2 + \|a\|^2}.
\]
We obtain the second inequality by taking the common multiplier \( \|e_g\|^2_g + \|a\|^2_g \) outside the square brackets because \( 1 - \sqrt{1-y^2} \geq y^2/2 \). We now consider each term separately to simplify the lower bound.

\[
(e_g, a)_g = \sum_j a_j \frac{1}{g_j \sqrt{\Phi}} g_j = \frac{1}{\sqrt{\Phi}}, \quad \|e_g\|^2_g = \sum_j \frac{1}{g_j^2} g_j = \frac{1}{\Phi} \sum_j \frac{1}{g_j}, \quad \text{and} \quad \|a\|^2_g = \sum_j a_j^2 g_j.
\]

Using these terms, we define the following lower bound.

\[
\triangle(y) \geq \frac{\min_i g_i(y)}{\sum_j \frac{1}{g_j(y)} + \Phi \sum_j a_j^2 g_j(y)} = \frac{\min_i \frac{1}{a_i} \left( \frac{y}{a_i} \right)^2 \frac{f(\frac{y}{a_i})}{f(\frac{y}{a_i})} \mathbb{1}\{y \leq a_i \bar{v}\}}{\sum_j \frac{1}{g_j(y)} + \Phi \sum_j a_j^2 g_j(y)} \geq \frac{y \min_i \frac{1}{a_i} f \mathbb{1}\{y \leq a_i \bar{v}\}}{\frac{k}{g(y)} + \Phi \bar{g}(y)},
\]

where \( g \) and \( \bar{g} \) are lower and upper bounds for any \( g_i \), respectively, and given by

\[
g(y) \triangleq \frac{y f \mathbb{1}\{y \leq \min_i a_i \bar{v}\}}{f} \quad \text{and} \quad \bar{g}(y) \triangleq \frac{\bar{v} f \mathbb{1}\{y \leq \bar{v}\}}{y f}.
\]

Replacing \( g(y) \) and \( \bar{g}(y) \) in the right-hand side, we obtain the following lower bound for \( \triangle(y) \):

\[
\triangle(y) \geq \frac{y^2 f^2 \mathbb{1}\{y \leq \min_i a_i \}}{\frac{k}{f^2} + \Phi \bar{v}^2 f^2 \mathbb{1}\{y \leq \bar{v}\} \mathbb{1}\{y \leq \min_i a_i \}}.
\]

**Step 3.** In this step, we find an upper bound for \( \Phi \) to obtain a lower bound of \( \triangle(y) \) which we denote by \( \Gamma(y) \). Then, we integrate \( \Gamma(y)\omega(y) \) to bound \( \Psi \). In particular, we use the upper bound \( \bar{f} \) on the p.d.f. \( f(\cdot) \) in its support to get the first inequality.

\[
\Phi \leq \int_0^{\min_i a_i} \frac{\prod_i \bar{f} \left( \frac{y}{a_i} \right)}{y^2} dy + \int_{\min_i a_i}^{\bar{v}} \frac{1}{y^2} dy \leq \left( \frac{\bar{f}}{\min_i a_i} \right)^k \frac{(\bar{v} \min_i a_i)^{k-1}}{(k-1)} + \frac{1}{\bar{v} \min_i a_i} = \left( \frac{\bar{v}}{\min_i a_i} \right)^k (k-1).
\]

When we replace the upper bound on \( \Phi \) in the right-hand side of \((42)\), we get another lower bound for \( \triangle(y) \). Define the piece-wise function

\[
\Gamma(y) \triangleq \frac{y^2 f^2 \mathbb{1}\{y \leq \min_i a_i \}}{\frac{k}{f^2} + \frac{(\bar{f})^k + (k-1)}{\bar{v} (\min_i a_i) (k-1)} \bar{v}^2 f^2 \mathbb{1}\{y \leq \bar{v}\} \mathbb{1}\{y \leq \min_i a_i \}} = \begin{cases} 
\frac{y^2 (k+1) (k) (\min_i a_i)}{f^k} & \text{if } y \leq \min_i a_i \bar{v} \\
0 & \text{otherwise}.
\end{cases}
\]

Summarizing the previous steps, we obtain \( \triangle(y) \geq \Gamma(y) \).

Recall that in \((40)\), we have the following inequality \( \Psi \geq \int_0^\bar{v} \triangle(y) \omega(y) dy \). Because \( \Gamma(y) \) is less
than \( \Delta(y) \), we can lower bound \( \Psi \) as follows.

\[
\Psi \geq \int_0^\theta \Gamma(y) \omega(y) dy = \frac{(k+1)c_u(k)(\min_i a_i)}{\int_{k+1}^{k+1} s} \prod_{i} F \left( \frac{y}{a_i} \right) dy
\]

Then, this lower bound on \( \Psi \) is a lower bound on the minimum eigenvalue of the matrix \( (M + e e^T) \).

Proof of Lemma \( \text{E.4} \) Proposition \( \text{C.3} \) provides \( \nabla \bar{J}(u(k)) \) because Lemma \( \text{E.3} \) implies that \( \Pi(a) \) is positive definite for all \( a > 0 \) and \( u(k) \in S(s(u)F_{(k,n)}^\beta) \). Following Proposition \( \text{G.1} \) which extends Proposition \( \text{3.1} \) to account for the scaling factor, we obtain the following equations.

\[
- \nabla \bar{J}(u(k))^T (w(k) - u(k)) = \gamma^T (u(k) - w(k)) = \gamma^T (s(u)\nabla \phi(k)(\gamma) - u(k)) = \frac{1 - \beta}{\beta}.
\]

Using the fact that \( u(k) + \mu e = s(u)F_{(k,n)}^\beta \nabla \phi(k)(\gamma) \) with \( \mu \geq 0 \) (see Remark \( \text{3.1} \)), it follows that \( u(k) \leq s(u)F_{(k,n)}^\beta \nabla \phi(k)(\gamma) \) and \( \nabla \nabla \phi(k)(x) = \phi(k)(x) \). Moreover \( F_{(k,n)}^\beta \geq 1/2, 1 \geq \beta \) and \( \phi(k)(x) \geq \mathbb{E}[v]/k \) for all \( x \in \mathbb{R}_+^k \) by definition. Thus, we obtain

\[
- \nabla \bar{J}(u(k))^T (w(k) - u(k)) \geq \frac{(1 - \beta)}{2k} \mathbb{E}[v].
\]

Proof of Lemma \( \text{E.5} \) We can invoke Proposition \( \text{C.3} \) to get \( \text{Hess}(\bar{J}(\Theta(k))) \) because \( \Theta(k) \in S(s(u)F_{(k,n)}^\beta) \) and Lemma \( \text{E.3} \) shows that \( \Pi(a) \) is positive definite for all \( a > 0 \).

\[
\text{Hess}(\bar{J}(\Theta(k))) = \frac{1}{(s(u)F_{(k,n)}^\beta)^2} \Pi(\lambda)^{-1} \left( I - e e^T \lambda^{-1} \right) e e^T \lambda^{-1} e.
\]

We prove this result in two steps. For simplicity, we use \( A \) to represent \( (s(u)F_{(k,n)}^\beta)^2 \text{Hess}(\bar{J}(\Theta(k))) \) and \( H \) to represent \( \Pi(\lambda)^{-1} \).

Step 1. In this step, we find an upper bound for \( \sup_{||x||_2=1} x^T Ax \) which depends on \( \min_i \lambda_i \). Using \( (43) \), we can rewrite \( x^T Ax \) as follows:

\[
x^T Ax = \frac{(e^T H e)x^T H x - x^T H e e^T H x}{e^T H e}.
\]

Note that \( H \) is symmetric, thus eliminating the negative term \( -x^T H e e^T H x \) yields the following bound,

\[
x^T Ax \leq \frac{(e^T H e)x^T H x}{e^T H e} = x^T H x = x_H.
\]

Here, \( x_H \) is the largest eigenvalue of \( H \), and in the last equality we use that \( ||x||_2^2 = 1 \). In Lemma \( \text{E.3} \) we further show that the minimum eigenvalue of \( \lambda(\lambda) \) is \( c_u(k) (\min_i \lambda_i)^{k+1} \). Therefore the largest
implies that Θ

is nonnegative and eigenvalue of

the right-hand side of (44), and have that

Therefore, we have that

Step 2. In this step, we find an upper bound for 1/ min

. Optimality of λ implies that

if we knew Θ

is in s(u)F

, then it would follow

µ is nonnegative and Θ

could lie outside of s(u)F

. In that case, µ becomes negative. Thus, it could be the case that Θ

for all i. Despite this fact, we know −µ is less than ∥u

− w

∥2 because Θ

is between u

and w

. Therefore, we have that

Because the p.d.f. f(·) is bounded by 

and the c.d.f. is always less than one, 

can be bounded by λi fE[v

/ max

1 ≤ k, and 

≤ 1. Thus, it follows that 

we get Θ

≥ u

− w

∥2 for all i. Because u

≥ u

, it follows Θ

≥ u

− w

∥2. Using the fact that u

≥ 3∥w

− u

∥2 we get Θ

≥ u

− u

/ 3 = 2u

/ 3 for all i, and so

≥ 2u

/ 3. Combining these bounds we obtain that min

≥ u

/ (3c

).

Finally, using Lemma E.1.3 to bound ∥w

− u

∥2 and min

, we can bound the right-hand side of (44), and have that

E[v]K2(k)k. □
Supplementary Appendix

F Proof of Lemma E.6

Proof. The optimization problem in the statement of the lemma can be reformulated by the following change of variables and parameters. Let \( e_{gi} \sqrt{g_i} = v_i \) and \( a_i \sqrt{g_i} = w_i \), and \( x_i \sqrt{g_i} = z_i \) for all \( i \). Then it follows that

\[
\min_{\|x\|_g^2 \geq 1} \langle x, e_{gi} \rangle^2 - \langle x, a_i \rangle^2 = \min_{\|z\|_g^2 \geq 1} \langle z, v \rangle^2 - \langle z, w \rangle^2 = \min_{\|z\|_g^2 \geq 1} z^T (vv^T - ww^T) z.
\]

Following the min-max theorem, the optimal value of this problem is given by the minimum eigenvalue of the matrix \( vv^T - ww^T \). To find the eigenvalues, we express \( v \) and \( w \) with two orthonormal unit vectors \( v = c_1 u_1 \) and \( w = c_2 u_1 + c_3 u_2 \), respectively. This implies that \( c_1 = \|v\|_2 \) and \( u_1 = v / \|v\|_2 \), which further implies that \( c_2 = u_1^T w \) and \( c_3 = u_2^T w \). Finally, we have \( u_2 = (w - c_2 u_1) / \|w - c_2 u_1\|_2 \). In an orthonormal basis starting with \( u_1 \) and \( u_2 \), the matrix \( vv^T - ww^T \) has first two rows and columns

\[
\begin{bmatrix}
    c_2^2 - c_3^2 & -c_2 c_3 \\
    -c_2 c_3 & -c_3^2
\end{bmatrix},
\]

and everything else is 0. The minimum eigenvalue emerges as follows:

\[
\frac{1}{2} \left[ c_1^2 - c_2^2 - c_3^2 - \sqrt{(c_1^2 + c_2^2 + c_3^2)^2 - 4(c_1^2 c_2^2)} \right] = \frac{1}{2} \left[ \|v\|_2^2 - c_2^2 - c_3^2 - \sqrt{(\|v\|_2^2 + c_2^2 + c_3^2)^2 - 4(\langle v, w \rangle)^2} \right].
\]

Therefore, we have \( \|v\|_g^2 = \|e_{gi}\|_g^2 \) and \( v^T w = \langle a, e_{gi} \rangle_g \). To complete the proof, we show that \( c_2^2 + c_3^2 = \|w\|_g^2 = \|a\|_g^2 \). In fact,

\[
c_2^2 + c_3^2 = \frac{(v^T w)^2}{\|v\|_2^2} + \frac{\|v\|_2^2 \|w\|_2^4 + (w^T v)^4 - 2\|v\|_2^2 \|w\|_2^2 (w^T v)^2}{\|v\|_2^2 - (v^T w) v^T v} = \frac{\langle e_{gi}, a \rangle_g^2}{\|e_{gi}\|_g^2} + \frac{\|e_{gi}\|_g^2 \|a\|_g^2 + \langle e_{gi}, a \rangle_g^4 - 2\|e_{gi}\|_g^2 \|a\|_g^2 \langle e_{gi}, a \rangle_g^2}{\|e_{gi}\|_g^2 \|a\|_g^2 + \|e_{gi}\|_g^2 \|a\|_g^2 - \langle e_{gi}, a \rangle_g^2} = \frac{\|e_{gi}\|_g^2 \|a\|_g^2 - \langle e_{gi}, a \rangle_g^2}{\|e_{gi}\|_g^2 \|a\|_g^2 - \langle e_{gi}, a \rangle_g^2} = \|a\|_g^2.
\]

□

G Generalizations of Proposition 3.1 and Proposition 3.2

In this section, we provide generalizations of Proposition 3.1 and Proposition 3.2 to scaled settings. Specifically, we consider these propositions for the scaled main phase mechanism \((p(k), w(k))\) which is introduced in Section 6. Note that, Proposition 3.1 and Proposition 3.2 are special cases with \( k = n \).

Proposition G.1. Suppose we are given a constant \( \beta \beta^{(n)} \in [0, E[v]] \) and a sequence \( \{\omega_{\beta, n}^{(k)} \in (0, 1)\} \). Let \( \Omega_{\beta, n} \) is determined by those constants. For any given state \( u \in \Omega_{\beta, n} \) with \( k \) active agents, the future promise vector \( w(k)(v|u) \) lies within a plane in \( \mathbb{R}^k \). Specifically, the plane is described by the following equation.

\[
(u(k) - w(k)(v|u))^\top a = \frac{1 - \beta}{\beta} \left(s(u)\nabla\phi(k)(a) - u(k)\right)^\top a \geq 0
\]

App. 1
where $a = \alpha^{(k)}(u^{(k)}/(s(u)F_{\beta}^{(k,n)})$.

**Proof.** We first show that $b$ such that $a^T w^{(k)}(v|u) + b = 0$. Let $b = \sum_{i=1}^{k} [(1 - \beta)s(u)(\nabla \phi^{(k)}(a))_i - w_i^{(k)}] / \beta$. Using the definition of $w^{(k)}(v|u)$, we obtain the following equation.

$$a^T w^{(k)}(v|u) = \sum_{i=1}^{k} a_i E_{\tilde{v}_i}[W_i^{(k)}(\tilde{v}_i|u)].$$

Moreover, $(\text{PK}(u))$ implies that

$$\beta E_{\tilde{v}_i}[W_i^{(k)}(\tilde{v}_i|u)] = \left[ v_i^{(k)} - (1 - \beta)s(u)E[v_i \mathbf{1}\{a_i v_i \geq \max_{j \neq i} a_j v_j\}] \right].$$

Recall that $(\nabla \phi^{(k)}(a))_i = E[v_i \mathbf{1}\{a_i v_i \geq \max_{j \neq i} a_j v_j\}]$. Combining these three observations, it follows that $a^T w^{(k)}(v|u) + b = 0$.

The second part of the proposition follows from the following properties of the support function $\phi^{(k)}(\cdot)$. Note that $a^T \nabla \phi^{(k)}(x) = \phi^{(k)}(x)$ for all $x$ by the definition of $\phi^{(k)}(\cdot)$. Furthermore, for a vector $x$ the support function satisfies that $\phi^{(k)}(x) \geq x^T z$ for all $z \in \mathcal{U}^{(k)}$ (see Schneider, 2013). Because $u^{(k)} \in s(u)F_{\beta}^{(k,n)} \mathcal{U}^{(k)}$, we obtain the following inequality, which concludes the proof.

$$(s(u)\nabla \phi^{(k)}(a) - u^{(k)})^T a = s(u)\nabla \phi^{(k)}(a)^T a - (u^{(k)})^T a = s(u)\phi^{(k)}(a) - (u^{(k)})^T a \geq 0.$$ \hfill \Box

**Proposition G.2.** Suppose we are given a constant $u_{i_{\beta}}^{(n)} \in [0, E[v]]$ and a sequence $\{F_{\beta}^{(k,n)} \in (0, 1)\}_{k=1}^{n}$. Let $\Omega_{\beta}$ is determined by those constants. Fix a state $u \in \Omega_{\beta}$ with $k$ active agents. Then, the ex-post future promise $w^{(k)}(v|u)$ is an optimal solution of the following optimization problem:

$$\max_{w^{(k)}} \frac{\alpha^{(k)}(u^{(k)}/(s(u)F_{\beta}^{(k,n)}))}{(u^{(k)} - \tilde{w}(v))}$$

$$s.t. E_{v_i}[\tilde{w}_i(v_i, v_{-i})] = W_i^{(k)}(v_i|u) \quad \forall v_i, i = 1, \ldots, k \quad (45)$$

**Proof.** For simplicity, we use $a$ to represent $\alpha^{(k)}(u^{(k)}/(s(u)F_{\beta}^{(k,n)}))$ in this proof. First, we reformulate problem (45) as the following linear programming problem:

$$\max \quad z$$

$$s.t. \quad E[\tilde{w}_i(v_i, v_{-i})] = W_i^{(k)}(v_i|u) \quad \forall v_i, i = 1, \ldots, k \quad (46)$$

$$z \leq a^T (u^{(k)} - \tilde{w}(v)) \quad \forall v \quad (47)$$

Next, we find an upper bound for this problem by relaxing the constraints over $z$.

The support function of set $s(u)F_{\beta}^{(k,n)} \mathcal{U}^{(k)}$ is $s(u)F_{\beta}^{(k,n)} \phi^{(k)}(\cdot)$, where $\phi^{(k)}(x) = E[\max_{i \leq k} x_i v_i]$. Because $u^{(k)} \in \mathcal{E}(s(u)F_{\beta}^{(k,n)} \mathcal{U}^{(k)})$, we have $u^{(k)} / s(u)F_{\beta}^{(k,n)} \in \mathcal{E}(\mathcal{U}^{(k)})$, which implies $u^{(k)} / s(u)F_{\beta}^{(k,n)} = \nabla \phi^{(k)}(a)$, i.e., $u_i^{(k)} = s(u)F_{\beta}^{(k,n)}E[v_i \mathbf{1}\{a_i v_i \geq \max_{j \neq i} a_j v_j\}]$. Moreover, $u_i^{(k)} = (1 - \beta)s(u)E[v_i \mathbf{1}\{a_i v_i \geq \max_{j \neq i} a_j v_j\}] + \beta E[W_i^{(k)}(v_i|u)]$, following (PK(u^{(k)})). These two observations imply that $u_i^{(k)} = \cdots$. 

*App. 2*
\( \tau \mathbb{E}[W_i^{(k)}(v_i|u)] \), where \( \tau = \frac{\beta s(u) F_{\beta}^{(k,n)}}{s(u) F_{\beta}^{(k,n)} - (1 - \beta)} \).

Therefore, the constraint (47) becomes

\[
z \leq \sum_{i=1}^{k} a_i (\tau \mathbb{E}[W_i^{(k)}(v_i|u)] - \tilde{w}_i(v)) \text{ for all } v.
\]

Because \( z \) has to satisfy this constraint for all \( v \), we can relax it by replacing these constraints with a single constraint where the right hand side is the expectation over \( v \). This operation corresponds to taking expectation of \( \tilde{w}_i(v) \). Following constraint (46), we obtain

\[
z \leq \sum_{i=1}^{k} a_i (\tau - 1) \mathbb{E}[W_i^{(k)}(v_i|u)]
\]

Therefore, \( \sum_{i=1}^{k} a_i (\tau - 1) \mathbb{E}[W_i^{(k)}(v_i|u)] \) is an upper bound for the optimization problem (45). To show \( w_i^{(k)}(v|u) \) is an optimal solution, we first show that \( w_i^{(k)}(v|u) \) is feasible, and the objective function evaluated at \( w_i^{(k)}(v|u) \) is equal to this upper bound.

By its definition, the expectation of \( w_i^{(k)}(v|u) \) with respect to \( v_{-i} \) is \( W_i^{(k)}(v_i|u) \), which implies feasibility. By Proposition G.1, we also know that \( a^T(u^{(k)} - w^{(k)}(v|u)) = \sum_{i=1}^{k} a_i (\tau - 1) \mathbb{E}[W_i^{(k)}(v_i|u)] \) which, in turn, implies that \( w^{(k)}(v|u) \) is an optimal solution for (45). \( \square \)