1. Costly State Verification

Based on Townsend (1979) and Gale and Hellwig (1985)

1.1. Setup of the model

- Entrepreneur with initial wealth $W_0$ who needs to raise finance for project of outlay $I$.
- Verification of income from project $X_1$ requires costly auditing.

Notation:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_0$</td>
<td>Initial wealth of entrepreneur</td>
</tr>
<tr>
<td>$I$</td>
<td>Investment outlay for project (variable)</td>
</tr>
<tr>
<td>$f(I,s)$</td>
<td>Income from project in state $s$ with investment $I$</td>
</tr>
<tr>
<td>$s$</td>
<td>State variable</td>
</tr>
<tr>
<td>$i$</td>
<td>Opportunity cost of capital</td>
</tr>
<tr>
<td>$c(I,s)$</td>
<td>Verification (auditing) costs in state $s$ with investment $I$</td>
</tr>
<tr>
<td>$E_0$</td>
<td>Equity of entrepreneur in project</td>
</tr>
<tr>
<td>$R(s)$</td>
<td>Return to investors in state $s$</td>
</tr>
<tr>
<td>$W(s)$</td>
<td>Wealth of the entrepreneur in state $s$.</td>
</tr>
<tr>
<td>$d(s)$</td>
<td>Verification dummy</td>
</tr>
<tr>
<td>$B_0$</td>
<td>Finance advanced by investors</td>
</tr>
</tbody>
</table>

Assumptions on technology:

**Axiom 1.** $f_I > 0$, $f_S > 0$, $f_{IS} > 0$, $f_{II} < 0$.

**Axiom 2.** $c \geq 0$, $c_I \geq 0$, $c_{II} \geq 0$, $c_S \geq 0$. 
A contract specifies the following variables:

1. The amount of finance $B_0$ advanced by the investor.
2. The return $R(s)$ the investor receives in state $s$.
3. The amount of equity $E_0$ the entrepreneur invests in the firm.
4. The states where the firm is audited (then the state is verified), represented by a function $d(s)$:
   \[
   d(s) = \begin{cases} 
   1 & \text{if state verified} \\
   0 & \text{otherwise} 
   \end{cases}
   \]
5. The investment outlay $I$.
6. The wealth $W(s)$ of the entrepreneur in state $s$.

Define:

\[
S = \{ s \mid d(s) = 1 \} \\
S^C = \{ s \mid d(s) = 0 \}
\]

Hence, the firm is audited whenever $s \in S$, and not audited whenever $s \in S^C$.

A contract can be represented by a 6-tuple: $C \equiv (W(s), R(s), d(s), E_0, B_0, I)$.
Contracts must be feasible:

**Definition 1.1.** The contract $C \equiv (W(s), R(s), d(s), E_0, B_0, I)$ is said to be feasible if it satisfies the following conditions:

\[
\begin{align*}
0 & \leq I \leq B_0 + E_0 \quad \text{(F1)} \\
0 & \leq E_0 \leq W_0 \quad \text{(F2)} \\
0 & \leq W(s) \quad \text{(F3)} \\
W(s) + R(s) & \leq f(I, s) + (1 + i) (W_0 + B_0 - I) - d(s) c(I, s) \quad \text{(F4)}
\end{align*}
\]

(F1) says that the investment outlay must be financed either by equity or through outside investors.

(F2) is the wealth constraint of the entrepreneur and implies that the entrepreneur cannot borrow against the firm’s future income today.
(F3) Is a non-negative wealth constraint. The model assumes unlimited liability, hence private wealth not invested in the firm $W_0 - E_0$ is included.

(F4) This just says that the total payouts to the entrepreneur and investors cannot exceed the total amount of resources available from the operation, where the amount of initial wealth plus outside finance that is not invested earns income at the rate of interest $i$.

The concept behind this approach is a complete Arrow-Debreu contract subject to additional constraints from asymmetric information.

**Definition 1.2.** The contract $C \equiv (W(s), R(s), d(s), E_0, B_0, I)$ is said to be acceptable to investors if:

$$E(R(s)) \geq (1 + i)B_0$$

(PC(I))

The contract is acceptable for the entrepreneur if:

$$E(W(s)) + (W_0 - E_0)(1 + i) \geq W_0(1 + i)$$

(PC(E))

The objective of the entrepreneur who owns the rights to the project is to maximize expected end of period wealth when choosing the contract $C$:

$$\max_{C \in C} E(W(s)) + (W_0 - E_0)(1 + i)$$

**Remark 1.** Given this objective, the second inequality in (F1) will always be satisfied as an equality since there is no point raising finance for the firm if this is not invested (the firm cannot earn higher interest on capital markets, otherwise the space of available projects would be different). Then: $I = B_0 + E_0$ (hence, substitute out $B_0$) (F4) will always bind, since the objective of the entrepreneur could be increased otherwise, hence:

$$W(s) = f(I, s) + (1 + i)(W_0 - E_0) - d(s)c(I, s) - R(s)$$

and $W(s)$ can be substituted also. We can therefore define the contract as a four-tuple: $C \equiv (R(s), d(s), E_0, I)$ and rewrite the constraints above as:

$$0 \leq I$$

($F1'$)

$$0 \leq E_0 \leq W_0$$

($F2'$)

$$R(s) \leq f(I, s) + (1 + i)(W_0 - E_0) - d(s)c(I, s)$$

($F3'$)
1.2. Direct mechanisms

Define a direct mechanism as the following game:

1. The designer of the mechanism proposes a contract \( C = (R(s), d(s), E_0, I) \).

2. The investor and the entrepreneur decide independently whether to accept or reject the contract.

3. If either rejects the contract, the game ends, the entrepreneur receives \( W_0(1 + i) \) and investors earn interest at a rate \( i \) on their capital; If both accept, the investor advances \( B_0 = I - E_0 \), the entrepreneur advances \( E_0 \) and invests a total amount \( I \) in the project.

4. The entrepreneur observes the state of nature \( s \). She makes an announcement \( m \in S \cup S^C \) to the designer of the mechanism.

5. The entrepreneur makes a payment \( R(m) \) to the investor.

For this mechanism, define the wealth of the entrepreneur as a function of the state and her announcement \( m \):

\[
W(s, m) \equiv f(I, s) + (1 + i)(W_0 - E_0) - d(m)c(I, s) - R(m) - d(m)\{R(s) - R(m)\}
\]

The last term expresses the fact that the transfer to the investor will be \( R(m) \) if \( d(m) = 0 \), and \( R(s) \) if \( d(m) = 1 \). Townsend focuses on “consistent contracts”, Gale and Hellwig on “incentive compatible contracts”:

**Definition 1.3.** The contract \( C = (R(s), d(s), E_0, I) \) is said to be incentive compatible if:

\[
W(s, s) \geq W(s, m) \quad \forall s \quad \forall m
\]

Whenever the contract is incentive compatible, the entrepreneur tells the truth and the mechanism is said to be a direct mechanism with truth-telling.

**Proposition 1.4.** For any contract \( C = (R(s), d(s), E_0, I) \) that leads to an allocation \( \hat{R}(s), \hat{d}(s) \), there exists another contract \( \tilde{C} \) such that the allocation with \( \tilde{C} \) is also \( \tilde{R}(s), \tilde{d}(s) \) and \( \tilde{C} \) can be implemented in a direct mechanism with truth-telling.
Proof:

Set \( \tilde{C} \equiv (\tilde{R}(s), \tilde{d}(s), E_0, I) \) and show that this contract satisfies the requirement.

The general insight behind using incentive compatible or consistent contracts is the revelation principle which holds that there is no loss of generality by restricting attention to incentive compatible contracts.

If a contract is not incentive compatible, then a contract \( C^a \) will lead to an allocation represented by the vector \( (R^a(s), d^a(s), E_0^a, I^a) \neq C^a \). Then define \( C^a \equiv (R^a(s), d^a(s), E_0^a, I^a) \). Hence, if \( C^a \) leads to allocation \( C^a \), one can just as well write a contract \( C^a \) to start with. The proof of the revelation principle then involves only to show that \( C^a \) is incentive compatible. However, it must be, otherwise it couldn’t be an equilibrium outcome of a game.

**Proposition 1.5.** The contract \( C \equiv (R(s), d(s), E_0, I) \) is incentive compatible if and only if (1) \( R(s) \) is constant across all non-verification states:

\[ R(s) = \tilde{R} \quad \forall s \in S^C \]

and (2) the entrepreneur admits bankruptcy:

\[ c(I, s) + R(s) \leq \tilde{R} \]

Before we prove this, note that this is similar to debt: \( \tilde{R} \) is the face value of debt, the firm is audited (declared bankrupt) if the firm doesn’t pay up, otherwise no auditing takes place. Hence, the two conditions say that:

1. The firm cannot discretion about the face value of debt ex post, otherwise it would always choose the lowest one, and

2. The entrepreneur has no incentive to avoid bankruptcy by paying \( \tilde{R} \), otherwise the revelation principle implies that a state where this condition is violated is a non-bankruptcy state.

**Proof:**

Observe that:

\[
W(s, s) - W(s, s') = -c(I, s)(d(s) - d(s')) \quad \text{(1 - d(s'))} (R(s) - R(s'))
\]

and distinguish four cases:
(A) Consider any two states $s \in S^C, s' \in S^C$. Then:

$$W(s, s) \geq W(s, s') \iff R(s) \leq R(s')$$

Hence, the entrepreneur always reports the state to pay a transfer equal to $\min_{s \in S^C} R(s)$. Hence, incentive compatibility requires that $R(s)$ is constant on $S^C$.

(B) Consider any two states $s \in S^C, s' \in S$. Then:

$$W(s, s) \geq W(s, s') \iff c(I, s) \geq 0$$

which is true by assumption.

(C) Consider any two states $s \in S, s' \in S$. Then the entrepreneur’s report is irrelevant.

(D) Consider any two states $s \in S, s' \in S^C$. Then:

$$W(s, s) \geq W(s, s')$$

$$\iff f(I, s) + (1 + i)(W_0 - E_0) - R(s) - c(I, s)$$

$$\geq f(I, s) + (1 + i)(W_0 - E_0) - \bar{R}$$

$$\iff \bar{R} \geq c(I, s) + R(s)$$

which gives the second condition. (A)-(D) shows that conditions (1) and (2) are necessary and sufficient for incentive compatibility. ■

**Definition 1.6.** A contract $C^* \equiv (R(s)^*, d(s)^*, E_0^*, I^*)$ is optimal if and only if it is feasible and acceptable, and there is no other feasible and acceptable contract $C' \equiv (R(s)', d'(s), E_0', I')$ such that $\text{EW}'(s) > \text{EW}(s)$.

**Question:** This argument assumes the point of view of the entrepreneur. Is this restrictive?

**Exercise 1.1.** Show that a contract cannot be optimal unless the constraint $PC(I)$ holds as an equality, hence $ER(s) = (1 + i)B_0$.

Hence, substitute for $ER(s)$. The entrepreneur’s expected wealth is:

$$\text{EW}(s) = E \{f(I, s)\} + (1 + i)(W_0 - E_0) - E \{d(s)c(I, s)\} - ER(s)$$

$$= E \{f(I, s) - d(s)c(I, s)\} + (1 + i)(W_0 - E_0) - (1 + i)B_0$$

$$= E \{f(I, s) - (1 + i)I - d(s)c(I, s)\} - (1 + i)W_0$$
where the last transformation uses $B_0 + E_0 = I$. The first expression is the NPV of the project in time 1 currency, the second expression is a constant and can be ignored.

**Remark 2.** We can now find the optimal contract by maximizing

$$ E \{ f(I, s) - (1 + i) I - d(s) c(I, s) \} $$

subject to the feasibility constraints $(F1)' - (F3')$, the conditions for incentive compatibility (1) and (2) and the participation constraint $PC(E)$.

**Definition 1.7.** A contract is said to have maximum equity participation (MEP) if $E_0 = W_0$.

**Remark 3.** There is no loss of generality in restricting attention to MEP contracts whenever the entrepreneur has unlimited liability.

**Note:** $E_0$ does not show up in the expressions for $W(s)$ above.

**Proof:**
Consider any arbitrary contract $C \equiv (R(s), d(s), E_0, I)$ with $E_0 < W_0$. Consider an alternative contract $C' \equiv (R'(s), d(s), W_0, I)$ where:

\[
\begin{align*}
E_0' &= B_0 - (W_0 - E_0) \\
R'(s) &= R(s) - (1 + i)(W_0 - E_0)
\end{align*}
\]

Then we have:

\[
E \{ R'(s) \} = E \{ R(s) \} - (1 + i)(W_0 - E_0) = B_0(1 + i) - (1 + i)(W_0 - E_0) = E_0'(1 + i)
\]

for the investor and:

$$ EW'(s) = EW(s) $$

for the entrepreneur since $d(s)$ and $I$ are unchanged. Constraints $(F1)' - (F2')$ remain unaffected. $(F3')$ remains satisfied by construction of $C'$. ■

Note, if we had limited liability, then:

$$ R(s) \leq f(I, s) - c(I, s) $$
and equity participation would matter. In this case the gains the entrepreneur
makes on the capital market \( E_0 (1 + i) \) would be protected from investors.

Hence, focus on MEPs. Then \( E_0 = W_0 \) and (F2′) is always satisfied. (F3′)
now reads:

\[
R (s) \leq f (I, s) - d (s) c (I, s)
\]

(F3′) is also unchanged.

**Definition 1.8.** A contract \( C \equiv (R (s), d (s), E_0, I) \) is said to be a standard
debt contract (SDC) if:

\[
R (s) = \begin{cases} 
\bar{R} & \text{if } s \in S^C \\
f (I, s) - c (I, s) & \text{if } s \in S 
\end{cases}
\]

**Proposition 1.9.** Any optimal contract is weakly dominated by a standard
debt contract with maximum equity participation.

**Proof:**
Let \( C^* \) be some optimal contract and \( C^* \equiv (R (s)^*, d (s)^*, W_0, I^*) \) the MEP
contract constructed from this optimal contract as above. Consider a SDC
\( C_{SD} \equiv (R_{SD} (s), d_{SD} (s), W_0, I^*) \) constructed as follows:

\[
d_{SD} (s) = \begin{cases} 
0 & \text{if } f (I^*, s) \geq \bar{R}^* \\
1 & \text{if } f (I^*, s) < \bar{R}^*
\end{cases}
\]

\[
R_{SD} (s) = \begin{cases} 
\bar{R}^* & \text{if } f (I^*, s) \geq \bar{R}^* \Leftrightarrow d_{SD} (s) = 0 \\
f (I^*, s) - c (I^*, s) & \text{if } f (I^*, s) < \bar{R}^* \Leftrightarrow d_{SD} (s) = 1
\end{cases}
\]

Constraints (F1′) and (F2′) are always satisfied by this contract, and (F3″)
is satisfied by construction. For the set of states \( s \) where \( d_{SD} (s) = d^* (s) = 0 \)
\( R_{SD} (s) = R^* (s) \). For the set of states \( s \) where \( d_{SD} (s) = d^* (s) = 1 \)
\( R_{SD} (s) = f (I^*, s) - c (I^*, s) \geq R^* (s) \). For the set of states \( s \) where \( d_{SD} (s) = 0 \), \( d^* (s) = 1 \),

\[
R_{SD} (s) = \bar{R}^* \geq R^* (s) + c (I^*, s)
\]

since the contract is optimal, hence the condition (2) for incentive compatibility
must hold. Finally, observe that there is no state \( \hat{s} \) such that \( d_{SD} (\hat{s}) = 1 \), \( d^* (\hat{s}) = 0 \). Suppose such a state \( \hat{s} \) would exist. Then:

\[
R^* (\hat{s}) = \bar{R}^* \geq f (I^*, s)
\]
However, then $d_{SD}(\hat{s}) = 0$ by construction of the standard debt contract. Hence, we have shown that $R_{SD}(s) \geq R^*(s)$ for all states $s$, hence $E\{R_{SD}(s)\} \geq E\{R^*(s)\}$ and PC(I) is satisfied. Also, we have shown that $d_{SD}(s) \leq d^*(s)$.

Then:

$$E\{W_{SD}(s)\} - E\{W^*(s)\} = E\{f(I^*, s) - (1 + i) I^* - c(I^*, s) d_{SD}(s)\}$$

$$- E\{f(I^*, s) - (1 + i) I^* - c(I^*, s) d^*(s)\}$$

$$= E\{c(I^*, s)[d^*(s) - d_{SD}(s)]\} \geq 0$$

Hence, the standard debt contract weakly dominates.

**Intuition:** Take all returns in bankruptcy states, then auditing is necessary in fewer states, and this increases efficiency (which assumption is critical here?).

**Question:** What else do we need to conclude that debt is optimal.

**Definition 1.10.** A firm is said to be almost solvent with positive probability under contract $C \equiv (R(s), d(s), W_0, I)$ if for any $\hat{\epsilon} > 0$ we have that

$$\Pr(\hat{R} - \hat{\epsilon} \leq f(I, s) \leq \hat{R}) > 0$$

**Proposition 1.11.** Any optimal contract is a standard debt contract almost everywhere if the firm is almost solvent with positive probability and $c(I^*, s) > 0$.

**Proof:**

Consider some arbitrary optimal contract $C^* \equiv (R(s)^*, d(s)^*, W_0, I^*)$ and let $C_{SD} \equiv (R_{SD}(s), d_{SD}(s), W_0, I^*)$ be the standard debt contract as constructed in the proof of the previous proposition, except that $R_{SD}$ is allowed to be lower than $R^*$ now to ensure $E\hat{R}_{SD}(s) = (I - W_0)(1 + i)$, and investors break even, but receive no rents. Note first that:

$$E\{W_{SD}(s)\} - E\{W^*(s)\} = E\{c(I^*, s)[d^*(s) - d_{SD}(s)]\} = 0$$

Hence, since $d_{SD}(s) \leq d^*(s)$ from the proof of the previous proposition, and $c(I^*, s) > 0$ by assumption, $d_{SD}(s) = d^*(s)$ almost everywhere. Then $\hat{R}^* = \hat{R}_{SD}$ otherwise $C^*$ would not be optimal. Suppose otherwise and $\hat{R}^* = \hat{R}_{SD} + \delta$ for some $\delta > 0$. By assumption:

$$\Pr(\hat{R}_{SD} = \hat{R}^* - \delta \leq f(I, s) \leq \hat{R}^*) > 0$$
hence \( d_{SD}(s) \neq d^*(s) \) on a set of positive measure. This is inconsistent with the optimality of \( C^* \), hence it cannot be that \( \bar{R}^* > \bar{R}_{SD} \). \( \bar{R}^* < \bar{R}_{SD} \) is also impossible, otherwise \( ER^*(s) = (I - W_0)(1 + i) < E\bar{R}_{SD}(s) \). Finally, suppose that \( \bar{R}^*(s) < f(I^*, c) - c(I^*, s) \) on a set with positive measure where the state is verified (bankruptcy states):

\[
\hat{S} \equiv \{ s | d^*(s) = 1, \ R^*(s) < f(I^*, c) - c(I^*, s) \}
\]

Then:

\[
\int_S \{ R^*(s) - [f(I^*, c) - c(I^*, s)] \} dH(s) > 0
\]

\[
\Rightarrow \int_S R^*(s) dH(s) < \int_S R_{SD}(s) dH(s)
\]

\[
\Rightarrow \int_{SC} R^*(s) dH(s) > \int_{SC} R_{SD}(s) dH(s)
\]

which immediately implies that \( \bar{R}^* > \bar{R}_{SD} \), which was shown to be inconsistent with \( C^* \) being optimal. Therefore \( \bar{R}^*(s) = \bar{R}_{SD}(s) \) almost everywhere. □

1.3. Stochastic Mechanisms

Consider the following numerical example:

\[
I = 14
\]

\[
i = 0
\]

\[
W_0 = 0
\]

\[
\bar{R} = 22
\]

<table>
<thead>
<tr>
<th>State</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prob</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>( f(I, s) )</td>
<td>10</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>( c(I, s) )</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>( R(s) )</td>
<td>5</td>
<td>15</td>
<td>22</td>
</tr>
<tr>
<td>( W'(s) )</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
</tbody>
</table>

It is clear that investors break even:

\[
ER(s) = (5 + 15 + 22)/3 = 14
\]
The net return to the entrepreneur is:

\[ EW(s) = \frac{8}{3} \]

Evidently, this is incentive compatible, feasible and acceptable from the results above, hence the contract is optimal in the class of deterministic schemes. Now consider the following stochastic scheme. Write the transfer to the investor as a function of \( d(s) \): \( R(s, d(s)) \). Then consider the following stochastic contract, where the state is verified with some probability \( q(s) \):

<table>
<thead>
<tr>
<th>State</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(I, s) )</td>
<td>10</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>( c(I, s) )</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>( R(s, 0) )</td>
<td>10</td>
<td>10</td>
<td>22</td>
</tr>
<tr>
<td>( W(s, 0) )</td>
<td>0</td>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td>( R(s, 1) )</td>
<td>5</td>
<td>15</td>
<td>22</td>
</tr>
<tr>
<td>( W(s, 1) )</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

Subsequent to the entrepreneur’s report \( m \in \{a, b\} \) the state is verified with probability \( q \), but with probability \( 1-q \) no verification takes place. If \( m \in \{a, b\} \) and no verification takes place, then the entrepreneur pays 10 to investors and keeps the rest. There is never any verification if \( m = c \). Then the entrepreneur will tell the truth in state \( c \) if and only if:

\[
W(c, 0) \geq q \times W(c, 1) + (1 - q) \times (f(I, c) - R(a, 0))
\]

\[
\Leftrightarrow 8 \geq q \times 3 + (1 - q) \times 20
\]

\[
\Leftrightarrow q \geq \frac{12}{17}
\]

Evidently, the entrepreneur has no incentive to misreport in states \( a, b \). Investors break even, since \( ER(s, 0) = ER(s, 1) = 14 \). However:

\[
EW(s) = \frac{q}{3} \times (W(a, 1) + W(b, 1)) + \frac{1-q}{3} \times (W(a, 0) + W(b, 0)) + \frac{1}{3} \times W(c, 0)
\]

\[
= \frac{5}{17} \times 10 + \frac{1}{3} \times 8 = \frac{50}{51} + \frac{8}{3}
\]

an improvement of \( 50/51 \) relative to the previous allocation. Hence, it is optimal to keep \( q \) as small as possible.
Exercise 1.2. This analysis was conducted assuming unlimited liability. How would this change if the entrepreneur could limit liability for debt to the assets invested in the firm, so the revenues from investment in the capital market would be protected from the claims of investors?

Exercise 1.3. Generalize the numerical example above and derive the optimal stochastic mechanism for a model with a discrete number of states. Derive the general expression for the mixing probability $q$ in the optimal mechanism.

Exercise 1.4. Suppose the investors could make a decision after the announcement of state by the entrepreneur. Is it sequentially rational for them to verify the state and audit the firm (sometimes? always?).