Ph.D. SEMINAR IN CORPORATE FINANCE

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Moral Hazard refers to scenario where an agent’s unobservable action affects the probability distribution of the outcome, e.g. Probability of accident depends on the care you take while driving. This care is observable. Ideally one would like insure you completely against the risk of an accident. But this affects your incentives to take care. If you are completely insured, why should you take any precautions?

Moral hazard presents a problem as to what the optimal tradeoff is between Need to provide insurance (Risk Sharing) and Need to induce correct actions by providing proper incentives (Incentives).

The term "Moral Hazard" is due to Arrow and is a little of misleading. From an economist’s point of view, if you are completely insured, it is suboptimal for you to take perfect precautions. There is nothing immoral about this.

**PRINCIPAL - AGENT models**

In our example, insurance company is principal and you are the agent. Typically, the scheme of events is as follows:

<table>
<thead>
<tr>
<th>Principal offers contract $S(x)$</th>
<th>Agent knows contract $S(x)$ and takes an unobservable action $a$</th>
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<td></td>
<td>the output $x$ is given by $x = x(a, \theta)$, where $\theta$ is some uncertain state of the world.</td>
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Principal - agent models include:


This list still goes on.

**The H"olmstrom Formulation**

At first, there are some assumptions as below:

1. Action $a \in A \subseteq R$, taken by agents. $X$ is the outcome and $x = x(a, \theta)$; (2). Principal’s utility function is defined as $G(w)$ over wealth and he is risk-averse, but not
necessarily strictly, i.e $G'' < 0$. Agent’s utility function is defined as

$$H(w, a) = U(w) - V(w)$$

and he is strictly risk-averse, i.e $U'' < 0$; (3). Action is costly, i.e $V' > 0$ and it is productive, i.e $X_a \geq 0$; (4). Only $x$ is observable and it should be distributed between the principal and the agent: agent shares $s(x)$ and principal shares $r(x) = x - s(x)$.

Then we need to characterize the Pareto optimal contract.

$$\max_{s(x)} E[G(x - s(x))]$$

s.t.

$$E[H(s(x), a)] \geq \bar{H}$$

$$a \in \arg\max_{a' \in A} E[H(s(x), a')]$$

Constrain (2) reflects a minimum utility constraint. $\bar{H}$ is exogenously given. As we vary $\bar{H}$, we plot the complete Pareto optimal contract curve. Constrain (3) reflects the fact that the agent picks the best action given $s(x)$ in his self-interest.

Note that the contract is only a function of $x$ and not $a$. Only $x$ is observable. Since given $s(x)$, the principal knowing $H(.,.)$ can solve (3), so we could also write:

$$\max_{s(x), a} E[G(x - s(x))]$$

s.t.

$$E[H(s(x), a)] \geq \bar{H}$$

$$a \in \arg\max_{a' \in A} E[H(s(x), a')]$$

Paradoxically, the principal knows exactly what $a$ is. But $a$ is unobservable and cannot be contracted on. Before we study the above problem (the second-best), let us study the case where $a$ is observable (the first-best).

Suppose $a$ is observable. Then the principal’s problem can be simplified as:
\[ \text{Max}_{s(x),a} E[G(x - s(x))] \]

s.t.

\[ E[H(s(x), a)] \geq \bar{H} \]

Contract (3) disappears. For any fixed \( a \), the Lagrangian is:

\[ \text{Max}_{s(x)} E[G(x - s(x)) - \lambda[E[H(s(x), a)] - \bar{H}]] \]

The Kuhn-Tucker conditions are:

\[ -G'(x - s(x)) - \lambda(U'(s(x))) = 0(*) \]

\[ \lambda[E(H(s(x), a)) - \bar{H}] = 0(**) \]

The second constraint reflects the fact that either the constraint is binding or the multiplier \( \lambda \) is 0.

Simplifying the first statement (*):

\[ \frac{G'(x - s(x))}{U'(s(x))} = \lambda \]

Thus the marginal rate of substitution is equated across states of the world is standard for Pareto optimality.

Students often find (*) difficult to understand (why does it hold for each realization of \( x \) ? ). I redo the above with a discrete probability space. Suppose \( \{x_1, ..., x_n\} \) including the whole set of possible outputs; \( P(x_1, a)...P(x_n, a) \) corresponding probabilities; \( P(x_i, a) > 0 \) for any \( a \in A \). (I will explain a little later why we need this) Then the first best problem is

\[ \text{Max}_{s(x_1),...,s(x_n)} \sum_{i=1}^{n} P(x_i)G(x_i - s(x_i)) \]

s.t

\[ \sum_{i=1}^{n} P(x_i)H(s(x_i)) \geq \bar{H} \]
Then the F.O.C w.r.t $s(x_i)$ is
\[ G'(x_i - s(x_i)) = \lambda U'(s(x_i)) \]
for $i=1,...,n$ or
\[ \frac{G'(x_i - s(x_i))}{U'(s(x_i))} = \lambda, \]
for $i=1,...,n$

This is the standard Pareto optimality condition. Equate MRS across states of the world

Having solved for $s(x)$ ( I am returning to the continuum random variable case ), the principal can solve for $a^*$. More carefully we have found $s(x,a)$ and the principal finds the optional $a^*$. The first best contract is
\[
s(x,a) = \begin{cases} 
  s(x,a^*) & \text{if } a = a^*; \\
  -\infty & \text{if } a \neq a^*.
\end{cases}
\]
Since the agent gets $-\infty$ if $a \neq a^*$ and $U(-\infty) = -\infty$, he accepts the contract. This is an example of an "forcing contract".

You can verify those by yourself:

1. The agent will get exactly $\bar{H}$ the first best contract; (2). A first best solution exists.

Unfortunately, in general the agents' action is not observable. Thus one has to solve:
\[
Max_{s(x),a} E[G(x - s(x)]
\]
s.t.
\[
E[H(s(x),a)] \geq \bar{H}
\]
\[
a \in \arg\max_{a' \in A} E[H(s(x),a')]\]

We will also assume that the support of $x$ is the same for all $a$. To see why this is needed, let:
\[
x(a,z) = a + z, \ z \sim \text{unif}(0,1), \ x \sim \text{unif}(a, a+1).
\]

Let $(a^*, s^*(x))$ be the first best solution.

Try $s(x) = \begin{cases} 
  s^*(x) & \text{if } x \geq a^*; \\
  -\infty & \text{if } a \neq a^*.
\end{cases}$

For any action $a < a^*$, $\exists$ a positive probability of $x < a^*$, agent receives utility -$\infty$. He implements $a \geq a^*$. Thus a forcing contract can implement the optimal action $a^*$ and first best contract $s^*(x)$. Hence the support restriction.
Aside: At this stage it is important to prove that risk aversion of agent is important for the agency problem to exist;

Proposition: If the agent is risk neutral, the first best solution can be implemented;

Proof:

At first from intuition: If the agent is risk neutral, no insurance problem exists. The optimal first best contract implies that he must take all the risk and insure the principal completely. Thus he takes all the output risk.

⇒ he takes the optimal action.

Then formal proof:

The first best contract for the optimal action $a^*$ involves a contract $s(x) = x - R(a^*)$, where $R(a^*)$ is constant. Then we must have

$$G'(x - s(x)) = \lambda V'(s(x)) = c\lambda$$

as $v'(s(x)) = c$ (a constant) ⇒

$$G'(x - s(x)) = constant \Rightarrow x - s(x) = R(a^*) \Rightarrow s(x) = x - R(a^*)$$

In fact for any action $a$, the first best contract is of the form:

$$s(x, a) = x - R(a)$$

How is $R(a)$ determined?

Since constraint on minimum utility is binding:

$$\int U(s(x))f(x|a)dx - V(a) = \bar{H}$$

$$\int (d(x - R(a)) + e)f(x|a)dx - v(a) = \bar{H}$$

$$d \int xf(x, a)dx - V(a) - R(a) + de = \bar{H}$$

$$R(a) = d \int xf(x|a)dx - V(a) + de - \bar{H}$$
⇒ Max \(a R(a) \iff \text{Max}_a d \int x f(x|a)dx - V(a) + de\)

But \(\text{Max}_a G(R(a))\) and \(\text{Max}_a R(a)\) have the same maximizer \(a^*\)!

I am done, but if you are not convinced, let \(a^*\) is the first best action which maximizes \(V(R(a))\) over \(a\).

⇒ Max \(a d \int x f(x|a)dx - V(a) + de\)
⇒ Max \(d \int x f(x|a)dx - V(a) + de - R(a^*)\)
⇒ Max \(E[U(s^*(x), a)] - V(a)\)
⇒ it is implemented by agent even if his action is not observable.

Thus risk aversion of agent is important for the agency problem to make sense.

Considering

\[
\int G(x-s(x))f(x|a)dx
\]

s.t

\[
\int [U(s(x)) - V(a)]f(x|a)dx \geq \bar{H}
\]
\[
\int U(s(x))f_a(x|a)dx = V'(a)
\]

We have replaced (3) by its first order condition. The ability to do this is necessary for the first order approach to work.

Then

\[
G'(x-s(x))f(x|a) - \lambda U'(s(x))f(x|a) - \mu V'(s(x))f_a(x|a) = 0(*)
\]

and

\[
\int G(x-s(x))f_a(x|a)dx + \mu \int U(s(x))f_{aa}(x,a)dx - V''(a) = 0
\]

(*) is the really interesting condition:

\[
\frac{G'(x-s(x))}{U'(s(x))} = \lambda + \mu \frac{f_a(x|a)}{f(x|a)}
\]

Perfect risk sharing is not possible — one does not equate the MRS for all \(x\). There is a second term due to the incentive effect. As one expects, this incentive effect is related to \(\frac{f_a(x|a)}{f(x|a)}\), a measure of productivity of action to the importance of that event for risk sharing.
Proposition 1.

If \( v' > 0 \) i.e effort is costly and \( F_a \leq 0 \), i.e higher action implies higher output in terms of first order stochastic dominance.(with \(<\) for a positive measure of \( x \), then \( \mu > 0 \)

Proof:

Consider

\[
\frac{G'(x - s(x))}{U'(s(x))} = \lambda + \mu \frac{f_a(x|a)}{f(x|a)}
\]

Corresponding to this \( \lambda \) in the first best contract situation, \( \exists \) a solution \( s_\lambda(x), r_\lambda(x) \), s.t:

\[
\frac{G'(x - s(x))}{U'(s_\lambda(x))} = \lambda
\]

If \( \mu \leq 0 \) and \( f_a(x|a) \geq 0 \),

\[
\frac{G'(x - s(x))}{U'(s(x))} \leq \frac{G'(x - s_\lambda(x))}{U'(s_\lambda(x))}
\]

Now \( \frac{G'(r(x))}{G'(x-r(x))} = \frac{G'(x-s(x))}{U'(s(x))} \) is decreasing in \( r(x) \).

\[
r(x) \geq r_\lambda(x) \text{ on } x_+ = x|f_a(x,a) \geq 0
\]

; \( r(x) \leq r_\lambda(x) \text{ on } x_- = x|f_a(x,a) < 0 \)

Thus

\[
\int G(r(x))f_a(x|a)dx \geq \int G(r_\lambda(x)f_a(x|a)dx > 0.
\]

The last inequality follows from \( G(r_\lambda(.) \) being an increasing function (this needs \( r_\lambda(x) \) to increase in \( x \), which you can show for yourself).

Exercise: In the first best solution, \( r_\lambda(x), s_\lambda(x) \) increase with \( x \).

and that \( F_a(x,a) \leq 0 \) with \(<\) on a positive measure of \( x \).

But the principal’s first order condition w.r.t \( a \) is

\[
\int G(r(x))f_a(x|a)dx + \mu \int U(s(x)f_{aa}dx - V''(a)) = 0
\]
The first term is +ve, the expression in the brace is -ve, \( \Rightarrow \mu > 0 \), which contradicting our original assumption \( \mu \leq 0 \).
\[ \Rightarrow \mu > 0. \]

(Nowhere here have we talked about the validity of the first order approach. See Roger-son’s Econometrica paper for a statement of the continuous necessity).

**Monotonicity of Contract**

In a real world situation, one would like \( s(x) \) to be increasing in \( x \). The more output one observes, the more likely it is that higher action was taken. Look at:

\[
\frac{G'(x - s(x))}{U'(s(x))} = \lambda + \mu \frac{f_a(x|a)}{f(x|a)}
\]

Suppose \( \frac{f_a(x|a)}{f(x|a)} \) is increasing in \( x \) (this is called the monotone likelihood ratio property). Then \( x' > x \Rightarrow \)

\[
\frac{G'(x - s(x'))}{U'(s(x'))} > \frac{G'(x - s(x))}{U'(s(x))}
\]

Since \( \frac{G'(x-r)}{U'(r)} \) is monotone in \( r \),
\[ \Rightarrow s(x') > s(x). \]

The property that \( \frac{f_a(x|a)}{f(x|a)} \) increasing in \( x \) is extremely useful. We will see it again and again in this course.

**The Value of Additional Information:**

Suppose in addition to \( x \) we can observe \( y \) and contract on it. Should \( y \) be used in the contract? Here are two extreme examples:

1. \( y=a \).
   Of course, one uses \( y \) then. The first best contract can be enforced.

2. \( y=\text{random noise} \)
   Since \( y \) is noise unrelated to the problem at hand, contracting on \( y \) has no incentive effect but increases the risk borne by the two parties. Therefore \( y \) should not be used.
   Once again, there is a tradeoff between: Incentive effects and Risk Sharing effects.
To analyze this, let $F(x,y,a)$ be the joint distribution of $(x,y)$ given $a$ and $f(x,y,a)$ be the density function.

One can show that:

$$\frac{G'(x-s(x,y))}{U'(s(x,y))} = \lambda + \mu \frac{f_a(x,y|a)}{f(x,y|a)}$$

Now instead of $f_a(x|a)f(x|a)$, we have $f_a(x,y|a)f(x,y|a)$

**Definition:**

A signal is said to be valuable if both the principal and agent can be made strictly better off with a contract of the form $s(x,y)$ than they are with a contract of the form $s(x)$.

Consider the ratio $\frac{f_a(x,y|a)}{f(x,y|a)}$. Suppose

$$\frac{f_a(x,y|a)}{f(x,y|a)} = \bar{h}(x,a)(*)$$

for all $(x,y)$ given $a^*$.

Then intuitively the inference measure is independent of $x$ and thus the contract should not depend on $y$.

If $(*)$ holds for all $a$,

$$\ln f(x,y|a) = \int \bar{h}(x,a)da + c = \bar{h}(x,a) + c$$

$$f(x,y|a) = e^{\bar{h}(x,a)}e^c = g(x,y)h(x,a)$$

$$f(x,y,a) = g(x,y)h(x,a)(**)$$

But if one sees $a$ as an unknown parameter, then this is a statement that $x$ is sufficient for $(x,y)$ w.r.t $a$.

For

$$f(a|x,y) = \frac{f(x,y,a)}{\int f(x,y,a)da} = \frac{f(x,y|a)l(a)}{\int f(x,y|a)l(a)da}$$
\[
g(x, y)h(x, a)l(a) \\
\int g(x, y)h(x, a)l(a)da \\
= \frac{h(x, a)l(a)}{\int h(x, a)l(a)da} \\
= m(x, a) \tag{4}
\]

Now this term does not include y! To complete the sufficient statistic proof, one needs to show that the last term is equal to \(f(a|x)\). After all, \(h(x, a) \neq f(x, a)\) necessarily. But in fact \(f(a|x, y) = f(a|x)\) as

\[
E_y[f(a|x, y)] = \int m(x, a)g(y)dy = m(x, a)
\]

(x fixed (so as a))

But \(E_y[f(a|x, y)] = f(a|x)\), so I am done.

Of course, here we don’t have a statitical experiment. But we are using the output to make inferences about the agent’s action. Thus the analogy is compelling.

After all the discussion, the main result is not a surprise.

**Proposition 3:**

Let \(s(x)\) be an optimal sharring rule for which the agent’s choice of action is unique and interior in \(A\). Then \(\exists \) a sharring rule \(s(x,y)\), which strictly Pareto dominates \(s(x)\) iff:

\[
f(x, y|a) = g(x, y)h(x, a) (***)
\]

is false at the Pareto optimal action \(a^*\).

(Hölmstrom restricts attention to cases when (***) holds for all \(a\)). He calls the \(s\) then the signal noninformative.

**Proof:**

A \(\Rightarrow\) B is the same as not B \(\Rightarrow\) not A. ( If not, not B \(\Rightarrow\) not A \(\Rightarrow\) B contradicts).

Suppose y is non informative (** holds), then y is simply adding noise to the contract.

Let’s define \(s(x)\) by

\[
U(s(x)) = \frac{1}{\int g(x, y)dy} \int U(s(x, y)g(x, y)dy
\]
Then
\[\int U(s(x, y)) f(x, y|a) dx dy = \int U(s(x, y)) h(x|a) g(x, y) dx dy = \int U(s(x)) h(x|a) g(x, y) dx dy = \int U(s(x)) f(x, y|a) dx dy \]

(5)

\(s(x)\) results in the same welfare for any action \(a\) to the agent. \(\Rightarrow\) agent still picks \(a^*\).

Now

\[U(s(x)) = \int U(s(x, y)) \frac{g(x, y)}{\int g(x, y) dy} dy \leq U(\int s(x, y) \frac{g(x, y)}{\int g(x, y) dy} dy)\]

\[s(x) \leq \int s(x, y) \frac{g(x, y)}{\int g(x, y) dy} dy\]

\[x - s(x) \geq \int (x - s(x, y)) \frac{g(x, y)}{\int g(x, y) dy} dy\]

\[G(x - s(x)) \geq G(\int (x - s(x, y)) \frac{g(x, y)}{\int g(x, y) dy} dy)\]

\[\geq \int G(x - s(x, y)) \frac{g(x, y)}{\int g(x, y) dy} dy\]

\(\Rightarrow\)

\[\int G(x - s(x)) g(x, y) dy \geq \int G(x - s(x)) g(x, y) dy\]

Integrating over \(x\),

\[\int G(x - s(x)) f(x, y|a) dy \geq \int G(x - s(x, y)) f(x, y|a) dy\]

\(B \Rightarrow A\)

Let \(s(x)\) be the second best solution dependent on \(x\). Suppose \(\delta s(x, y)\) is small variation in the sharring rule. For any \(a, x\) fixed

\[\delta E^A = U'(s(x)) \int \delta s(x, y) f(x, y|a) dy\]
\[ \delta E^p = -G'(x - s(x)) \int \delta s(x,y) f(x,y|a)dy + \mu U'(s(x)) \int \delta s(x,y) f_a(x,y|a)dy \]

Now \( \frac{f_a(x,y|a)}{f(x,y|a)} \) varies with y.

⇒ Let y be a set s.t

\[ \int_y f(x,y|a)dy \equiv f(x,y,a) > 0 \]
\[ \int_{y^c} f(x,y|a)dy \equiv f(x,y^c,a) > 0 \]
\[ \frac{f_a(x,y|a)}{f(x,y|a)} > \frac{f_a(x,y^c|a)}{f(x,y^c|a)} \]

Let \( \delta s(x,y) \) be s.t:

constants

\[ R = \delta s(x,y) > 0 \text{ on } y \]
\[ L = \delta s(x,y) < 0 \text{ on } y^c \]
\[ Rf(x,y,a) + lf(x,y^c,a) = 0 \]

Then considering the variation of agent’s first-order condition at \( a^* \):

\[ \int \int U'(s(x)) \delta s(x,y) f_a(x,y|a)dydx \]
\[ = \int U'(s(x)) \int \delta s(x,y) f_a(x,y|a)dydx \]
\[ = \int U'(s(x)) \int y Rf_a(x,y|a)dy + \int_{y^c} l f_a(x,y|a)dydx \]
\[ (l = -\frac{Rf(x,y,a)}{f(x,y^c,a)}) \]
\[ = \int U'(s(x)) Rf_a(x,y|a) \frac{f_a(x,y|a)}{f(x,y^c,a)} f_a(x,y^c,a)dy \]
\[ = \int U'(s(x)) f(x,y,a) \frac{f_a(x,y^c,a)}{f(x,y^c,a)} f_a(x,y|a)dy \]
\[ > 0 \]  \hspace{1cm} (6)
⇒ The agent at any action has higher benefit to increase action. But by construction the insurance effect is closer to zero.

⇒

\[ \lim_{R \to 0} \frac{\delta \text{Action}^p}{R} \to 0 \]

⇒

\[ \lim_{R \to 0} \frac{\delta E^p}{R} \to 0 \]

Now finishes proof.

The intuition of proof is as follows:

Conditioning on \( y \) has low impact on the risk (the first order effect is zero). However the first order incentive effect is positive. There are benefits to contract on \( y \).