Constrained-Efficient Capital Reallocation*

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Abstract

We analyze the constrained-efficient allocation in an equilibrium model of investment and capital reallocation with heterogeneous firms facing collateral constraints. The model features two types of pecuniary externalities: collateral externalities, because the resale price of capital affects firms' ability to borrow, and distributive externalities, because buyers of old capital are more financially constrained than sellers, consistent with empirical evidence. We show analytically and quantitatively that the equilibrium price of old capital is inefficiently high in general, because the distributive pecuniary externality exceeds the collateral externality, by a factor of two in the calibrated model. New investment generates a positive aggregate externality by reducing the future price of old capital, fostering reallocation toward more constrained firms. The constrained-efficient allocation induces a consumption-equivalent welfare gain of 5% compared to the competitive equilibrium, and can be implemented with subsidies on new capital and taxes on old capital.

Keywords: Capital reallocation; Pecuniary externalities; Collateral; Constrained efficiency; Investment subsidies

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1 Introduction

Financial frictions, such as collateral constraints, distort the level of aggregate investment and the allocation of capital across firms. What is the allocation of capital that maximizes welfare given financial frictions? To address this question, we develop an equilibrium model of investment and capital reallocation with collateral constraints. We then characterize the constrained-efficient allocation, that is, the allocation that would arise if a benevolent planner made investment decisions on behalf of firms, using the same markets and subject to the same frictions firms face in the competitive equilibrium. By characterizing this benchmark and comparing it with the decentralized equilibrium, we find that the constrained-efficient allocation features a higher level of investment, a larger volume of capital reallocation, and, crucially, a lower resale price of capital.

In our framework, heterogeneous firms face collateral constraints on borrowing as well as costs of issuing equity. Over time, they accumulate net worth and respond to productivity shocks by investing in new capital, or by acquiring old capital from other firms. Old capital is reallocated in a competitive secondary market. Importantly, the model is consistent with the key facts about capital reallocation: On average, older assets flow to more financially constrained and more productive firms. These firms have a high marginal value of current net worth. Thus, they take advantage of the fact that old capital is cheaper and has hence a lower financing need than new capital, because it has a lower future residual value. On the other hand, larger, less financially constrained firms tend to acquire newer investment goods, as they effectively discount the future resale value of capital at a lower rate. These firms account for most of the formation of new capital in the economy, and typically resell their capital on the secondary market as it ages.

Because of financial frictions, the competitive-equilibrium price of old capital does not coincide with its social value: Financial frictions manifest themselves as pecuniary externalities. Specifically, our economy encompasses both collateral externalities, because the resale value of capital affects firms' ability to borrow, and distributive externalities, because buyers and sellers of old capital have different valuations of internal funds. We show that the price of old capital, which serves as collateral, affects the aggregate value of these externalities with opposite sign. On the one hand, a *higher* resale price of capital relaxes collateral constraints. On the other hand, because buyers of old capital tend to be more financially constrained than sellers, a *lower* price of old capital redistributes resources toward firms with a higher marginal product of capital.

Our main result is that this distributive externality is larger than the collateral externality in stationary equilibrium. As a consequence, the equilibrium price of old capital is higher than the constrained-efficient price. An additional unit of new investment today increases the supply of old capital in the future, thereby reducing its price and creating a positive externality on future constrained firms, who are net buyers of old capital. In the decentralized equilibrium, investing firms do not take this effect into account. A subsidy on new investment may thus lead to a more efficient allocation.

Importantly, a low price of old capital is optimal, despite its negative effect on the value of collateral. The economic intuition is that the buyers of old capital are the most constrained firms, whereas the firms that purchase new capital and borrow against its collateral value are less constrained or unconstrained. Thus, the marginal value of net worth of firms that benefit from the distributive externality of a lower price of old capital is higher than the marginal value of net worth of the firms that are negatively affected by the collateral externality of a lower price of old capital.

To formalize this result, we consider a planner who faces the same constraints and has access to the same markets as private firms, but, crucially, internalizes all pecuniary externalities. The planner needs to respect all individual budget constraints and cannot redistribute net worth across firms, that is, cannot "remove" financial frictions. We solve for the constrained-efficient allocation and compare it with the stationary competitive equilibrium. We show, both analytically and quantitatively, that the price of old capital is inefficiently high in competitive equilibrium. The constrained-efficient allocation induces a lower price of old capital, allowing financially constrained firms to produce at a higher scale and to grow their net worth faster.

Our analysis is organized in two parts. First, we consider a stylized infinite-horizon model of capital reallocation and pecuniary externalities with over-lapping generations of firms and capital that lasts for two periods. In this model, we characterize the stationary equilibrium analytically and obtain a formal result on the sign of the inefficiency in equilibrium: The distributive externality is larger than the collateral externality. Importantly, this result holds independently of specific assumptions about the distribution of net worth. We then provide a closed-form solution for the constrained-efficient allocation, as well as a Ramsey implementation of this allocation with proportional subsidies on purchases of new capital, and taxes on purchases of old capital. We also show that our key analytical results obtain under several relevant generalizations of our assumptions, namely entrepreneurial risk aversion, heterogeneity in productivity, and when both firms and capital goods are long-lived.

Second, we consider a richer quantitative model with persistent idiosyncratic productivity shocks and long-lived firms and capital, which nests our stylized model. We calibrate the model to match empirical moments related to US firm dynamics and financing costs, and use it to perform a quantitative efficiency analysis, with a main focus on the stationary equilibrium. We find that the distributive externality is over twice as large as the collateral externality in competitive equilibrium. Moreover, output and consumption are respectively 10% and 7% lower than in the first-best allocation. The constrained-efficient allocation recovers approximately 70% of these losses (7 percentage points of output and 5 percentage points of consumption), by substantially decreasing the price of old capital. This outcome can be implemented in competitive equilibrium, with a mix of subsidies on new investment and taxes on purchases of old capital. We also consider the case of a single policy instrument, namely a new-investment subsidy, and find that it should be positive, again to induce a lower price of old capital. However, this restriction on the policy instrument set may have a significant welfare cost, as it only allows the planner to recover a smaller fraction of the welfare losses induced by financial frictions.

The paper proceeds as follows. Section 2 discusses the related literature. Section 3 presents our main theoretical results in a stylized model of capital reallocation. Section 4 introduces the model with idiosyncratic productivity shocks and characterizes the constrained-efficient allocation. Section 5 presents our quantitative results. Section 6 discusses additional analyses, including transition dynamics. Section 7 concludes.

2 Related Literature

This paper contributes to several strands of literature, specifically on capital reallocation and the role of secondary markets, on pecuniary externalities with collateral constraints, on constrained efficiency in dynamic heterogeneous-agent economies, and on the effect of financial frictions on capital misallocation.[1]

Capital reallocation and secondary markets. Several papers study the reallocation of durable assets across heterogeneous producers, starting with Eisfeldt and Rampini (2006). A robust empirical finding of this literature is that financially constrained agents tend to buy assets in the secondary market. In particular, Eisfeldt and Rampini (2007) analyze investment in new and used capital in the presence of financial frictions, and present empirical evidence that more financially constrained firms tend to acquire older investment goods, using both the Annual Capital Expenditure Survey and micro data on commercial trucks. More recently, Ma, Murfin, and Pratt (2020) leverage a large dataset on equipment transactions to document a negative correlation between firm age and capital age. We relate our quantitative results to their estimates. Gavazza, Lizzeri, and Roketskiy (2014) provide

[1]To focus on the effects of collateral constraints on the efficiency of investment and capital reallocation, we abstract from adverse selection (as in the seminal paper of Akerlof, 1970), illiquidity due to search frictions (as in, for example, Gavazza, 2011, 2016), and heterogeneity not due to differences in net worth or productivity (as in, for example, Bond, 1983).

a quantitative analysis of the welfare gains due to secondary markets for durable goods in the presence of consumer heterogeneity. Gavazza and Lanteri (forthcoming) emphasize the role of secondary markets in reallocating used consumer durable goods from wealthier to poorer households and argue that this mechanism contributes to the transmission of credit shocks. Lanteri (2018) analyzes the market for used investment goods in a quantitative business-cycle model with heterogeneous firms subject to idiosyncratic productivity shocks. Rampini (2019) analyzes the effects of asset durability on the financing of investment with collateral constraints.[2] We build on his model and develop a quantitative framework with idiosyncratic productivity shocks and a general depreciation schedule for capital. Different from the existing literature on capital reallocation, our focus is on efficiency.[3]

Pecuniary externalities and constrained efficiency. Several papers study pecuniary externalities related to asset prices in economies with collateral constraints—as introduced by Kiyotaki and Moore (1997)—or other financial frictions. In a seminal contribution, Lorenzoni (2008) develops a finite-horizon model with production heterogeneity between borrowers and lenders and aggregate shocks, and emphasizes how financial frictions may induce an inefficient level of borrowing and investment. Dávila and Korinek (2018) show that, in general, financial frictions may give rise to both distributive externalities, that is, externalities between sellers and buyers of assets, and collateral externalities, that is, externalities deriving from the dependence of financial constraints on asset prices, and that prices could be too high or too low.[4] In quantitative analyses of models with pecuniary externalities stemming from asset prices, the literature typically focuses on collateral externalities, abstracting from distributive externalities by assuming a representative producer: Bianchi and Mendoza (2018) and Jeanne and Korinek (2019) analyze infinite-horizon small open economy models with a representative firm and an asset in fixed supply. In these models, the price of collateral is too low in states of the world in which collateral constraints bind, and optimal policy can improve efficiency by increasing collateral values.[5]

[2]Rampini and Viswanathan (2010, 2013) study a dynamic model of firm financing with tangible assets serving as collateral, deriving the collateral constraints from limited enforcement without exclusion.

[3]Cooper and Schott (2020) analyze capital reallocation and aggregate fluctuations by formulating a planning problem, but abstract from financial frictions and the related inefficiency. Cui (2017) studies the effects of financing constraints and partial irreversibility on the cyclicality of capital liquidation. Ai, Li, and Yang (2020) study the link between financial intermediation and capital reallocation. See Eisfeldt and Shi (2018) for a survey of the literature on capital reallocation.

[4]See Diamond (1967) for an early contribution on efficiency in the presence of market incompleteness. He and Kondor (2016) study the role of pecuniary externalities in liquidity management for the efficiency of investment over the cycle. Kurlat (2020) considers the role of asymmetric information about capital quality for pecuniary externalities and the efficiency of investment.

[5]Michelacci and Pozzi (2017) characterize the efficient price of land in a small-open-economy model with collateral constraints and measure the collateral externality using data on land prices and economic activity in Italy. A related literature in international macroeconomics analyzes endowment economies in which the relative price of non-tradable goods affects the value of collateral, which is assumed to be income, instead

We contribute to this literature by analyzing an infinite-horizon model of investment with heterogeneous firms, consistent with the key facts about capital reallocation. We build on the analysis of externalities of Dávila and Korinek (2018) and show that, in the stationary equilibrium of our economy, the distributive externality is larger than the collateral externality. The price of collateral is too high from the perspective of a planner, because the most financially constrained firms are net buyers of old capital, that is, collateral. A related literature analyzes constrained efficiency in dynamic general-equilibrium models with incomplete markets, with a focus on distributive externalities through wages and interest rates: Dávila, Hong, Krusell, and Ríos-Rull (2012) analyze constrained efficiency in the Aiyagari (1994) model; Park (2018) extends their framework to characterize the efficient allocation of human capital; Itskhoki and Moll (2019) analyze optimal development policies that redistribute between workers and entrepreneurs in an economy with financial constraints.[6] Relative to this literature, we apply the same notion of constrained efficiency, but the focus of our paper is on efficiency in investment and capital reallocation. To our knowledge, we provide the first analysis of optimal investment subsidies in the presence of financial frictions.[7]

Financial frictions and capital misallocation. A large literature studies the role of financial frictions for the allocation of capital across heterogeneous firms. See, for instance, Buera, Kaboski, and Shin (2011), Midrigan and Xu (2014), and Moll (2014).[8] These papers provide theoretical and quantitative insights on the efficiency gains that could be achieved by *removing* financial frictions. We focus on what gains could be achieved if a benevolent planner were to face the same set of financial constraints as private agents.[9] In so doing, we build a bridge between the quantitative literature on capital misallocation and the

of the value of capital. See, for instance, Bianchi (2011), Benigno, Chen, Otrok, Rebucci, and Young (2013), and Ottonello, Perez, and Varraso (2019). Bianchi and Mendoza (2020) survey both strands of this literature, with capital or income as collateral, and connect them in a model with endogenous investment, in which the price of capital is tied to the price of non-tradable goods.

[6]Nuño and Moll (2018) develop tools to study constrained efficiency in economies with heterogeneous agents in continuous time.

[7]While we focus on a Ramsey implementation of the constrained-efficient allocation, Kilenthong and Townsend (forthcoming) propose a market-based approach to implementing efficient allocations in the presence of pecuniary externalities. Related to our analysis of investment taxes and subsidies, Dávila and Hébert (2020) study the optimal design of corporate taxation in the presence of financial frictions. Parodi (2020) provides a quantitative analysis of optimal subsidies on consumer durable goods in presence of partial irreversibility. Samaniego and Sun (2019) analyze the long-run effects of vintage-specific investment subsidies in a vintage capital model.

[8]Restuccia and Rogerson (2008) and Hsieh and Klenow (2009) provide early contributions on the aggregate effects of capital misallocation across heterogeneous producers. David and Venkateswaran (2019) quantify the roles of different types of frictions, including financial ones, for capital misallocation.

[9]Ai, Bhandari, Chen, and Ying (2019) develop an optimal contracting model subject to agency frictions. The optimal allocation features dispersion in marginal products of capital across firms and can be implemented with state-contingent securities and collateral constraints.

theoretical literature on efficiency in presence of pecuniary externalities. Thus, our results provide guidance for the design of second-best policies, such as investment subsidies.[10]

3 Capital Reallocation and Pecuniary Externalities

In this section, we describe a stylized model of capital reallocation with new and old capital. We analytically characterize the constrained efficiency of the allocation of capital in the presence of financial frictions that induce distributive and collateral externalities. We show that the distributive externality dominates the collateral externality; the price of old capital in stationary competitive equilibrium is too high from the perspective of efficiency. The economic intuition is as follows. The most financially constrained firms buy old capital due to its lower financing need; firms that buy new capital are less constrained or unconstrained, and while some of these firms benefit from a higher price of old capital, since they borrow against the resale value of their investment in terms of old capital, the severely constrained firms benefit from a lower price of old capital considerably more.

3.1 Environment

Time is discrete and the horizon infinite, that is, $t = 0, 1, 2, \ldots$. There is an infinitely-lived, risk-neutral representative household with preferences

$$\sum_{t=0}^{\infty} \beta^t C_t, \tag{1}$$

where $\beta \in (0, 1)$ is the discount factor and C_t is consumption.

There are over-lapping generations of firms and the representative household owns all firms. At each date, a continuum of firms with measure one is born. Firms live at two dates, make an investment decision when young and produce when old. Each firm has access to a production function f with $f(0) = 0$, $f_k > 0$, and $f_{kk} < 0$; investing capital $k_t > 0$ at date t yields output $f(k_t)$ at date $t + 1$. Output can be used to make new capital goods and it takes one unit of output to make a unit of new capital goods. Capital goods are productive for two periods and then fully depreciate. We refer to capital goods with two periods of useful life as "new" capital (denoted k_t^N) and to capital goods with a single residual period of productive life as "old" (denoted k_t^O). New and old capital goods are perfect substitutes in production and we define the total capital of a firm as $k_t \equiv k_t^N + k_t^O$.

[10] Relatedly, Gourio and Miao (2010) and Jo and Senga (2019) use quantitative models with heterogeneous firms to study the effects of dividend taxes and credit subsidies respectively.

3.2 Frictionless Economy and First Best

We start by considering a frictionless economy in which the representative household can choose investment in each firm without facing any financial frictions. We index firms of each generation by $w \in \mathcal{W} = [w_{min}, w_{max}]$ with distribution $\pi(w)$.[11] The aggregate resource constraint for the frictionless economy is

$$\int f\left(k_{t-1}^N(w) + k_{t-1}^O(w)\right) d\pi(w) = C_t + \int k_t^N(w) d\pi(w); \tag{2}$$

aggregate output equals consumption of the representative household plus aggregate investment in new capital goods. Aggregate investment in new capital at date $t-1$ determines the aggregate stock of old capital at date t

$$\int k_{t-1}^N(w) d\pi(w) = \int k_t^O(w) d\pi(w). \tag{3}$$

The first best (FB) allocation maximizes the utility of the representative household (1) by choosing aggregate consumption C_t and an allocation of new and old capital $k_t^N(w)$ and $k_t^O(w)$, $\forall w \in \mathcal{W}$, subject to the resource constraints (2) and (3), and taking as given $k_{-1}^N(w)$ and $k_{-1}^O(w)$, $\forall w \in \mathcal{W}$. The first-order conditions with respect to new and old capital satisfy

$$1 = \beta\left[f_k(k_t^{FB}) + q_{t+1}^{FB}\right] \tag{4}$$

$$q_t^{FB} = \beta f_k(k_t^{FB}), \tag{5}$$

where we use q_t^{FB} to denote the shadow value of old capital k_t^O in terms of date t consumption. Thus, q_t^{FB} can be interpreted as the first-best valuation (or price) of old capital. The economy is in steady state from date 1 onwards. Notice that the allocation of total capital is the same for all firms. By combining equations (4) and (5), we get that in a steady state $q^{FB} = \frac{1}{1+\beta}$, and the optimal scale of production for all firms is $k^{FB} = f_k^{-1}\left(\frac{1}{\beta(1+\beta)}\right)$. In the spirit of Jorgenson (1963), we can define the frictionless user cost of new and old capital as $u_N^{FB} \equiv 1 - \beta q^{FB}$ and $u_O^{FB} \equiv q^{FB}$, and note that $u_N^{FB} = u_O^{FB} \equiv u^{FB}$. The user cost would be the rental rate in a frictionless rental market and we define it as of the beginning of the period. The allocation of new and old capital across firms is indeterminate, but must satisfy $\int k_{t+1}^O(w) d\pi(w) = \int k_t^N(w) d\pi(w) = \frac{k^{FB}}{2}$.

[11] We will later interpret w as the initial net worth of each firm.

3.3 Financial Frictions and Competitive Equilibrium

We now consider a competitive equilibrium with financial frictions. Firms are subject to the following financial frictions. Firms are born with exogenous net worth w distributed over the interval $[w_{min}, w_{max}]$ according to an exogenous non-degenerate distribution $\pi(w)$, with $0 < w_{min} < q^{FB}k^{FB}$ and $k^{FB} < w_{max}$, with positive mass in a neighborhood of w_{min} and w_{max}. We index firms by their net worth, but suppress the dependence on net worth wherever appropriate.

Firms can raise additional internal funds from the representative household, that is, issue equity by paying negative dividends $d < 0$, at a cost $\phi(-d)$ incurred by the household, such that $\phi(-d) = 0$ if $d \geq 0$, $\phi(-d) > 0$ if $d < 0$. We denote the marginal cost of equity issuance by $\phi_d(-d) \equiv \frac{\partial \phi(-d)}{\partial(-d)}$ and assume it is positive, increasing, and convex. Specifically, $\phi_d \geq 0$, $\phi_d(0) = 0$ and $\phi_{dd} \equiv \frac{\partial^2 \phi(-d)}{\partial(-d)^2} \geq 0$ (see, for example, Gomes, 2001).

Firms can also borrow from the representative household at rate $R \equiv \beta^{-1}$, but borrowing is subject to a collateral constraint. The collateral constraint requires that debt repayments cannot exceed a fraction $\theta \in [0, 1)$ of the future resale value of new capital purchases. That is, the collateral value of new capital goods is the future price at which these can be sold as old capital next period. Old capital purchases have no future resale value, as old capital fully depreciates at the end of the period.[12] Rampini and Viswanathan (2010, 2013) show how to derive such collateral constraints in an economy with limited enforcement without exclusion, in which firms can default on their promises and retain their output, a fraction $1 - \theta$ of their capital, as well as access to the markets for capital goods and financing.

Given their initial net worth w and the price of old capital q_t, firms maximize the present discounted value of their dividends net of equity issuance costs, that is, their value to the household, by choosing dividends d_{0t} and $d_{1,t+1}$, new and old capital k_t^N and k_t^O,and borrowing b_t, to solve

$$\max_{\{d_{0t}, d_{1,t+1}, b_t, k_t^N, k_t^O\} \in \mathbb{R}^3 \times \mathbb{R}_+^2} d_{0t} - \phi(-d_{0t}) + \beta d_{1,t+1} \tag{6}$$

subject to the budget constraints for current and next period,

$$w_{0t} + b_t = d_{0t} + k_t^N + q_t k_t^O \tag{7}$$

$$f(k_t^N + k_t^O) + q_{t+1} k_t^N = d_{1,t+1} + \beta^{-1} b_t, \tag{8}$$

[12] In Section 3.8 we consider a model with standard geometric depreciation, in which both new and old capital serve as collateral.

and the collateral constraint

$$\theta q_{t+1} k_t^N \geq \beta^{-1} b_t. \tag{9}$$

Denote the multipliers on the budget constraints by μ_{0t} and $\beta\mu_{1,t+1}$, on the collateral constraint by $\beta\lambda_t$, and on non-negativity constraint for new and old capital by $\underline{\nu}_t^N$ and $\underline{\nu}_t^O$, respectively. The optimal demand for new capital, old capital, and borrowing, as functions of initial net worth w, satisfy the following first-order conditions

$$\begin{aligned} 1+\phi_{d,t} &= \beta\left[f_k(k_t)+q_{t+1}\right]+\beta\theta\lambda_t q_{t+1}+\underline{\nu}_t^N && (10) \\ q_t(1+\phi_{d,t}) &= \beta f_k(k_t)+\underline{\nu}_t^O && (11) \\ 1+\phi_{d,t} &= 1+\lambda_t, && (12) \end{aligned}$$

where $k_t = k_t^N + k_t^O$. Moreover, the firm's marginal value of net worth at date t is $\mu_{0,t} = 1+\phi_{d,t} \geq 1$, that is, equals one plus the marginal cost of raising additional equity. In contrast, the firm's marginal value of net worth at date $t+1$ is $\mu_{1,t+1} = 1$, as the firm pays out all its remaining net worth as dividends to the representative household when it exits. Finally, the premium on internal funds $\phi_{d,t} = \lambda_t$, that is, equals the multiplier on the collateral constraint.

A *stationary competitive equilibrium* is a set of policy functions mapping initial net worth to an allocation $\{d_0(w), d_1(w), k^N(w), k^O(w), b(w)\}$, that is, dividends, investment, and debt choices, and a price of old capital q, such that firms maximize the present discounted value of dividends net of equity issuance cost, $\forall w \in \mathcal{W}$, and the market for old capital clears, that is, $\int k^N(w)d\pi(w) = \int k^O(w)d\pi(w)$.

In a stationary equilibrium, the first-order conditions for new and old capital (10) and (11) can be written as investment Euler equations

$$\begin{aligned} 1 &\geq \beta\frac{1}{1+\phi_d}\frac{f_k(k)+(1-\theta)q}{\wp_N} && (13) \\ 1 &\geq \beta\frac{1}{1+\phi_d}\frac{f_k(k)}{q} && (14) \end{aligned}$$

with equality if $k_N > 0$ and $k_O > 0$, respectively, where $k = k_N + k_O$, and we define the down payment per unit on new capital $\wp_N \equiv 1-\beta\theta q$, that is, the price per unit of new capital minus the maximal amount the firm can borrow against the residual value next period, which is determined by the collateral constraint. Analogously, we can define the down payment on old capital as $\wp_O \equiv q$, as the firm cannot borrow against old capital. In the spirit of Jorgenson (1963) we can rewrite (13) and (14) as

$$u_N(w) \equiv u_N + \phi_d \wp_N = 1 - \beta q + \phi_d(1 - \beta\theta q) \quad \geq \quad \beta f_k(k) \tag{15}$$

$$u_O(w) \equiv u_O + \phi_d \wp_O = q + \phi_d q \quad \geq \quad \beta f_k(k), \tag{16}$$

where $u_N(w)$ ($u_O(w)$) is the user cost of new (old) capital to a firm with net worth w. The choice between investment in new and old capital is determined by the trade-off between their user costs if the firm were unconstrained and their down payments.

Combining (13) and (14) we moreover have

$$1 = \beta \frac{1}{(1+\phi_d)} \frac{(1-\theta)q}{\wp_N - \wp_O} + \frac{(\underline{\nu}_N - \underline{\nu}_O)/(1+\phi_d)}{\wp_N - \wp_O}. \tag{17}$$

If $\wp_N \leq \wp_O$, then (17) implies $\underline{\nu}_O > 0$, so no firm would buy old capital, which cannot be true in equilibrium. Therefore, in a stationary equilibrium, $\wp_N > \wp_O$, which means the down payment for new capital exceeds the down payment for old capital; equivalently, $\frac{1}{1+\beta\theta} > q$.

But then (15) and (16) imply that $u_N \leq u_O$, as otherwise there would be no investment in new capital, which is not an equilibrium; equivalently, $q \geq q^{FB}$, that is, the price of old capital in competitive equilibrium weakly exceeds the price in a frictionless economy.

To interpret (17), define $R_O \equiv \frac{(1-\theta)q}{\wp_N - \wp_O}$; this can be interpreted as the shadow interest rate on the additional amount the firm can implicitly borrow by buying old capital instead of new capital. Since $q \geq q^{FB}$, $R_O \geq \beta^{-1}$, that is, borrowing more by buying old capital is costly in equilibrium, and strictly so if $q > q^{FB}$.

Note that in the problem in (6) to (9) the objective is (weakly) concave and the constraint set (with constraints stated as inequality constraints) convex. Hence, the induced value function is weakly concave and, using the envelope condition, the marginal value $1+\phi_d$ weakly decreasing in w. Since $u_O(w) - u_N(w) = u_O - u_N - \phi_d(\wp_N - \wp_O)$, the difference in user costs between old and new capital is increasing in w. Old capital is relatively less costly for more financially constrained firms. This implies that in equilibrium, firms that are sufficiently constrained invest in only old capital, and firms shift to investing in new capital as their net worth increases. We stress that this equilibrium property of our model is consistent with the empirical evidence on capital reallocation (for example, Eisfeldt and Rampini, 2007, and Ma, Murfin, and Pratt, 2020).

In particular, dividend-paying firms have $\phi_d = 0$, so $u_N(w) \leq u_O(w)$, that is, prefer new capital at least weakly. Such firms invest $\overline{k}$ which solves $1 = \beta(f_k(k) + q)$, where $\overline{k} \geq k^{FB}$ with equality iff $q = q^{FB}$. Firms pay dividends if $w \geq \overline{w} = \wp_N \overline{k}$. Firms that are indifferent between new and old capital must have $\beta \frac{1}{1+\phi_d} = R_O^{-1}$ (from (17)) and invest

$\underline{k}$, which solves $1 = R_O^{-1} \frac{f_k(k)+(1-\theta)q}{\wp_N}$, where $\underline{k} \leq k^{FB}$ with equality iff $q = q^{FB}$. Firms are indifferent between new and old capital at the margin if $w \in (\underline{w}_N, \overline{w}_O)$, where $\underline{w}_N = \underline{d}_0 + q\underline{k}$ and $\overline{w}_O = \underline{d}_0 + \wp_N \underline{k}$, $\underline{d}_0 = 0$ if $q = q^{FB}$, and $\underline{d}_0$ solves $1 + \phi_d = \beta R_O$ if $q > q^{FB}$.

We summarize these results in the following proposition:

Proposition 1 (Stationary Competitive Equilibrium Characterization) *A stationary competitive equilibrium is characterized as follows:*

(i) *New capital has a higher down payment than old capital ($\wp_N > \wp_O$), but a (weakly) lower user cost from the perspective of an unconstrained firm ($u_N \leq u_O$).*

(ii) *The price of old capital (weakly) exceeds the price in a frictionless economy ($q \geq q^{FB}$).*

(iii) *If $q > q^{FB}$, there exist thresholds $\underline{w}_N < \overline{w}_O < \overline{w}$ such that: firms with $w \leq \underline{w}_N$ invest only in old capital; firms with $w \in (\underline{w}_N, \overline{w}_O)$ invest $\underline{k}$ and invest in both new and old capital; firms with $w \geq \overline{w}_O$ invest only in new capital; and firms with net worth $w \geq \overline{w}$ pay dividends and invest $\overline{k} > k^{FB} > \underline{k}$. If $q = q^{FB}$, there exists thresholds $\underline{w}_N < \overline{w}_O = \overline{w}$ such that: firms with $w \leq \underline{w}_N$ invest only in old capital; firms with $w \geq \overline{w}_O$ invest k^{FB} and are indifferent between new and old capital at the margin; firms with $w \in (\underline{w}_N, \overline{w}_O)$ invest a strictly positive minimum amount in old capital.*

We now compute a numerical example and use it to illustrate the main properties of the stationary competitive equilibrium. We assume the production function is $f(k) = k^\alpha$ with $\alpha \in (0, 1)$. Net worth is uniformly distributed on $[w_{min}, w_{max}]$. The cost of equity issuance is a power function, $\phi(-d) = \phi_0(-d)^{\phi_1}$ for $d < 0$ and $\phi(-d) = 0$ otherwise. The caption of Figure 1 reports all parameter values used in the example.

The stationary-equilibrium price of old capital associated with this parameterization is $q = 0.511 > q^{FB} = 0.51$. Figure 1 displays the policy functions for new capital (top left), old capital (top right), total capital, that is, the sum of new and old capital (bottom left), and the marginal cost of equity issuance (bottom right), in stationary equilibrium. Consistent with the characterization in Proposition 1, there are three thresholds $\underline{w}_N < \overline{w}^O < \overline{w}$, which we highlight with vertical lines in the figure. Firms with $w < \underline{w}_N$ invest only in old capital. Their total investment increases in net worth, and their marginal cost of equity issuance decreases in net worth. Firms with $\underline{w}_N < w < \overline{w}^O$ invest in both new and old capital, keeping the total investment $\underline{k}$ constant, and issue a common level of equity, resulting in a constant marginal cost of equity issuance. Firms with $\overline{w}^O < w < \overline{w}$ invest only in new capital, while still issuing equity. Firms with $w > \overline{w}$ invest only in new capital and are unconstrained in their investment $\overline{k}$; these firms pay dividends.

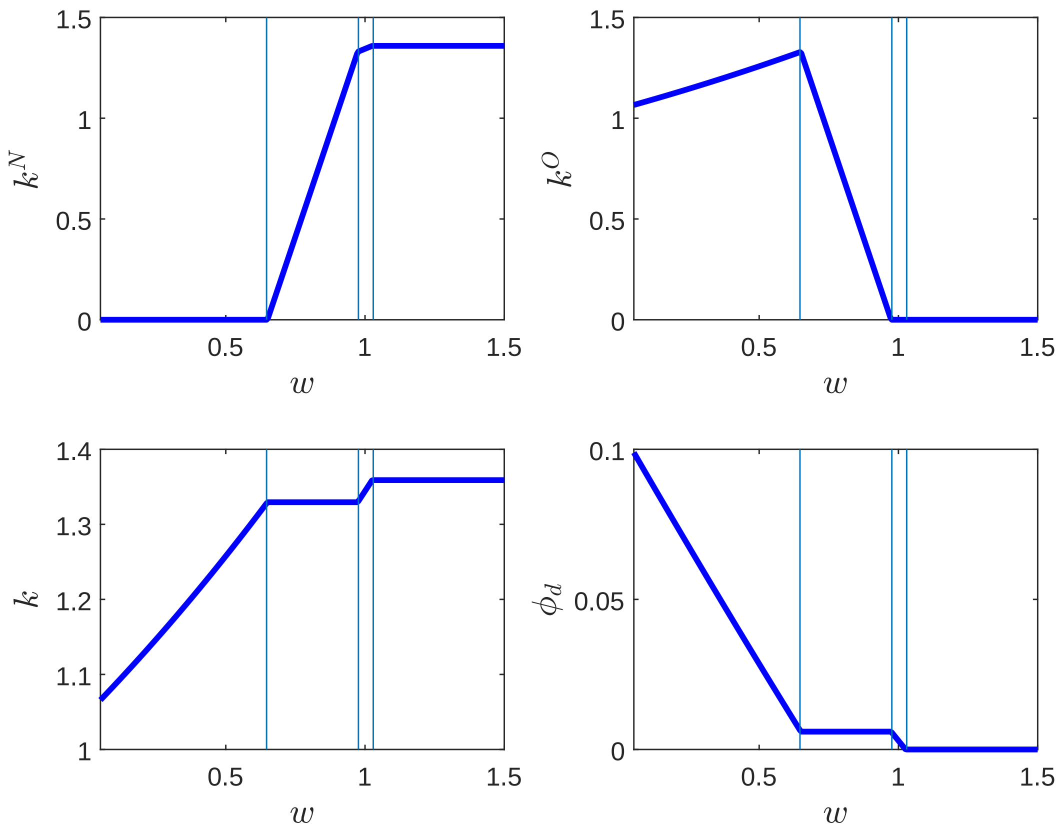

Figure 1: Stationary competitive equilibrium – example. Top left: new capital k^N; top right: old capital k^O; bottom left: total capital k; bottom right: marginal cost of equity issuance ϕ_d. The x-axes report net worth w. The parameter values are: discount rate $\beta = 0.96$; support of net worth distribution $w_{min} = 0.05$ and $w_{max} = 1.5$; curvature of production function $\alpha = 0.6$; collateralizability $\theta = 0.5$; and cost of raising equity parameters $\phi_0 = 0.1$ and $\phi_1 = 2$.

3.4 Constrained (In-)Efficiency

We now characterize the constrained-efficient allocation in this economy, that is, the allocation that arises if a benevolent planner with full commitment makes investment decisions on behalf of firms, subject to the same constraints that are present in the competitive equilibrium. We then use this characterization to analyze the nature of constrained inefficiency in competitive equilibrium.

Given an initial distribution of new and old capital, $k^N_{-1}(w)$ and $k^O_{-1}(w)$, a planner maximizes the present discounted value of aggregate dividends net of costs of equity issuance (or, equivalently, aggregate consumption)

$$\int \left[d_{10}(w) + \sum_{t=0}^{\infty} \beta^t \left(d_{0t}(w) - \phi(-d_{0t}(w)) + \beta d_{1,t+1}(w) \right) \right] d\pi(w), \tag{18}$$

subject to the budget constraints (7) and (8) with multipliers $\beta^t \mu_{0,t}$ and $\beta^{t+1}\mu_{1,t+1}$, the collateral constraint (9) with multiplier $\beta^{t+1}\lambda_t$, the non-negativity constraints on new and old capital with multipliers $\beta^t \underline{\nu}_t^N$ and $\beta^t \underline{\nu}_t^O$, and the market clearing condition for old capital (3) with multiplier $\beta^t \eta_t$.[13]

The first-order conditions with respect new and old capital are

$$
\begin{aligned}
1 + \phi_{d,t} &= \beta \left[f_k(k_t) + q_{t+1} \right] + \beta\theta\lambda_t q_{t+1} + \underline{\nu}_t^N + \beta\eta_{t+1} && (19) \\
q_t(1 + \phi_{d,t}) + \eta_t &= \beta f_k(k_t) + \underline{\nu}_t^O, && (20)
\end{aligned}
$$

and with respect to debt (12). The first-order condition with respect to the price of old capital q_t for $t = 1, 2, ..$ is

$$
\int k_t^O(w) \left(1 + \phi_{d,t}(w)\right) d\pi(w) = \int k_{t-1}^N(w) \left(1 + \theta\lambda_{t-1}(w)\right) d\pi(w). \tag{21}
$$

The left-hand side of equation (21) reports the marginal effect of an increase in q_t on dividends of young firms at t, net of equity issuance costs. The right-hand side reports its marginal effect on the dividends of old firms at t, as well as its effect on collateral constraints at $t-1$. In the absence of financial frictions, we would have $\phi_{d,t}(w) = \lambda_{t-1}(w) = 0$; thus, equation (21) would coincide with the market-clearing condition for old capital (3), and the aggregate welfare effect of a marginal change in q_t would be zero. Thus, using the market clearing condition (3), we can simplify equation (21) to isolate the pecuniary externalities induced by the presence of financial frictions:

$$
\int k_t^O(w)\phi_{d,t}(w) d\pi(w) = \theta \int k_{t-1}^N(w)\lambda_{t-1}(w) d\pi(w). \tag{22}
$$

The left-hand side of equation (22) represents the aggregate distributive externality induced by a marginal increase in the price of old capital q_t: Firms that purchase old capital at t value the additional expenditure they need to incur as the product of the quantity purchased k_t^O and their marginal cost of equity issuance $\phi_{d,t}$.

The right-hand side of equation (22) represents the aggregate collateral externality induced by the same marginal increase in q_t: Firms that purchase new capital at $t-1$ and face a binding collateral constraint are able to borrow against a fraction θ of the additional collateral value, and thus increase their investment; they value this benefit as the product of the additional collateral θk_{t-1}^N and the Lagrange multiplier on their collateral constraint λ_{t-1}.[14]

[13] We explicitly formulate the Lagrangian of this problem in Appendix A.1.

[14] The effect of the *current* price of old capital on *past* collateral constraints implies that the constrained-

Thus, a marginal increase in q_t induces a negative externality on the value of firms that issue equity to purchase old capital at t, and a positive externality on firms that purchase new capital at $t-1$ and are constrained in their borrowing. Equation (22) highlights that these two opposite externalities must offset each other in the constrained-efficient allocation.

Before proceeding to characterize the planning solution, we show that in the stationary competitive equilibrium, the aggregate distributive externality is larger than the aggregate collateral externality, resulting in an equilibrium price of old capital that is higher than the constrained-efficient one. Specifically, we prove that in stationary competitive equilibrium, we have

$$\int k^O(w)\phi_d(w)d\pi(w) > \theta \int k^N(w)\lambda(w)d\pi(w). \tag{23}$$

Let us start by considering the case $q > q^{FB}$. Using the characterization in Proposition 1, we know that $k_N = 0$ for $w < \underline{w}_N$, $k_O = 0$ for $w > \overline{w}_O$, and $\phi_d = 0$ for $w > \overline{w}$, with $\underline{w}_N < \overline{w}_O < \overline{w}$. Firms that are indifferent between new and old capital, that is, firms with $w \in (\underline{w}_N, \overline{w}_O)$, have the same (positive) marginal cost of equity, which we denote by $\overline{\phi}_d$.

As ϕ_d is weakly decreasing in w, no firm purchasing old capital has a marginal value of net worth less than $1+\overline{\phi}_d$, and no firm purchasing new capital has a marginal value of net worth larger than $1+\overline{\phi}_d$. Formally, we have $\phi_d \geq \overline{\phi}_d$ for $w \leq \overline{w}_O$, and $\phi_d \leq \overline{\phi}_d$ for $w \geq \underline{w}_N$.

Furthermore, using the optimality condition for debt (12), $\lambda(w) = \phi_d(w)$ and we can rewrite the right-hand-side of (23) as $\theta \int k^N \phi_d d\pi$. We can then bound the two integrals in (23) as follows:

$$\int k^O(w)\phi_d(w)d\pi(w) = \int^{\overline{w}_O} k^O(w)\phi_d(w)d\pi(w) \geq \overline{\phi}_d \int^{\overline{w}_O} k^O(w)d\pi(w), \tag{24}$$

and

$$\int k^N(w)\phi_d(w)d\pi(w) = \int_{\underline{w}_N}^{\overline{w}} k^N(w)\phi_d(w)d\pi(w) \leq \overline{\phi}_d \int_{\underline{w}_N}^{\overline{w}} k^N(w)d\pi(w). \tag{25}$$

Furthermore, the market-clearing condition for old capital (3), together with the characterization in Proposition 1, implies

$$\int_{\underline{w}_N}^{\overline{w}} k^N(w)d\pi(w) < \int^{\overline{w}_O} k^O(w)d\pi(w), \tag{26}$$

because the left-hand side of (26) is less than the aggregate supply of old capital in station-

efficient plan is time inconsistent. A planner without commitment, such as the one considered by Bianchi and Mendoza (2018), would disregard this effect. However, as we show below, even under our assumption of full commitment, the collateral externality is dominated by the distributive externality.

ary equilibrium, whereas the right-hand side represents aggregate demand for old capital.[15] Combining (24), (25), and (26), we have

$$\int k^O(w)\phi_d(w)d\pi(w) > \int k^N(w)\phi_d(w)d\pi(w)$$

Finally, as $\theta < 1$, we obtain the strict inequality in (23).

Let us now consider the case $q = q^{FB}$. All firms investing in new capital, that is, with $w > \underline{w}_N$, are unconstrained. Thus their marginal cost of equity issuance is zero and we have $\int k^O(w)\phi_d(w)d\pi(w) > 0 = \int k^N(w)\phi_d(w)d\pi(w)$.

Hence, in stationary equilibrium, the aggregate distributive externality is larger than the aggregate collateral externality. By comparing this result with the constrained-efficiency condition for the price of old capital (22), we find that a marginal reduction in the price of old capital has a positive effect on aggregate welfare, implying that the competitive-equilibrium price is too high from the perspective of constrained efficiency.

We summarize this result in the following proposition:

Proposition 2 (Sign of Constrained Inefficiency) *In stationary competitive equilibrium, the aggregate distributive externality is larger than the aggregate collateral externality, that is, $\int k^O(w)\phi_d(w)d\pi(w) > \theta \int k^N(w)\lambda(w)d\pi(w)$. A marginal decrease in the price of old capital induces a positive welfare gain.*

3.5 Implementing First Best

We now characterize the stationary constrained-efficient allocation, and show that it achieves the first-best level of welfare in the stylized model. In stationary equilibrium, the optimality condition for the price of old capital (22) reads

$$\int k^O(w)\phi_d(w)d\pi(w) = \theta \int k^N(w)\lambda(w)d\pi(w).$$

Clearly, an allocation such that all firms pay non-negative dividends, that is, $\phi_d = \lambda = 0$ for all w, satisfies this condition. We now show that the planner induces an allocation that achieves the first-best level of welfare and satisfies all budget and financial constraints, allowing all firms to be unconstrained and produce at the efficient scale k^{FB}. Imposing $\phi_d = 0$ for all w, we can rewrite the optimality conditions (19) and (20) in stationary-

[15]Inequality (26) is strict since we assume a positive mass of firms with $w > \overline{w}$.

equilibrium as follows:

$$1 = \beta\left(f_k(k^{FB}) + q^*\right) + \beta\eta \tag{27}$$

$$q^* + \eta = \beta f_k(k^{FB}), \tag{28}$$

where q^* is the stationary-equilibrium price of old capital in the constrained-efficient plan, and we have restricted attention to an allocation such that $\underline{\nu}^N = \underline{\nu}^O = 0$ for all w.

Let $q^* = \frac{w_{min}}{k^{FB}}$. At this price, firms with the lowest level of initial net worth produce at the efficient scale, by investing entirely in old capital, without issuing equity: $k^O(w_{min}) = k^{FB}$. We need to ensure that the market for old capital clears at this price. To this end, consider the following allocation of new and old capital.

$$k^N(w) = \begin{cases} \frac{w - q^* k^{FB}}{1 - q^*(1+\beta\theta)} & \text{if } w \leq k^{FB}(1-\beta\theta q^*) \\ \overline{k}^N & \text{if } w > k^{FB}(1-\beta\theta q^*) \end{cases} \tag{29}$$

and $k^O(w) = k^{FB} - k^N(w)$, with $\overline{k}^N$ to be determined.

Given this allocation, we can write the market-clearing condition for old capital as follows:

$$\int^{k^{FB}(1-\beta\theta q^*)} \left(k^{FB} - k^N(w)\right) d\pi(w) + \left(1 - \pi\left(k^{FB}(1-\beta\theta q^*)\right)\right)(k^{FB} - \overline{k}^N) = \int^{k^{FB}(1-\beta\theta q^*)} k^N(w) d\pi(w) + \left(1 - \pi\left(k^{FB}(1-\beta\theta q^*)\right)\right)\overline{k}^N, \tag{30}$$

where the left-hand side represents aggregate demand for old capital and the right-hand side represents aggregate demand for new capital, which in stationary equilibrium coincides with the aggregate supply of old capital. Equation (30) can be solved to find the value for $\overline{k}^N$ that guarantees market clearing.[16]

We now present an implementation of the constrained efficient allocation. The planner's allocation can be decentralized as a competitive equilibrium with proportional taxes at rates τ^N and τ^O on new and old capital respectively. These taxes are offset by lump-sum transfers to each firm, in order not to redistribute resources across firms. Tax rates and transfers can be firm specific, that is, are functions of net worth w. With this implementation, the budget constraint of a newborn firm with initial net worth w becomes

[16] A sufficient condition for the solution to be in the interval $\left[0, k^{FB}\right]$ is that there is a sufficiently large mass of firms with initial net worth larger than k^{FB}. If π is uniform, as in our numerical example, $w_{max} \geq k^{FB}$ is sufficient.

$$w + b_t + T_t = d_{0t} + k_t^N(1 + \tau_t^N) + q_t k_t^O(1 + \tau_t^O)$$

with a lump-sum transfer $T_t = \tau_t^N k_t^N + \tau_t^O q_t k_t^O$.

By inspection of equations (27) and (28), we see that the tax rates that implement the first-best stationary equilibrium are

$$\tau^N = -\beta\eta = -\beta(q^{FB} - q^*)$$

and

$$\tau^O = \frac{\eta}{q^*} = \frac{q^{FB}}{q^*} - 1.$$

As $\eta = \beta f'(k^{FB}) - q^* > 0$, that is, old capital is scarce from the perspective of the planner, we have $\tau^N < 0$ and $\tau^O > 0$. The planner internalizes the distributive externalities in the market for old capital and induces a price of old capital sufficiently low that all firms can afford the optimal production scale without incurring equity issuance costs. The optimal policy that supports this allocation is a subsidy on new capital, which increases the future supply of old capital, combined with a tax on old capital, which ensures the first-best production scale is optimal given the low price of old capital required to undo the effects of financial frictions. It might seem counterintuitive that the planner taxes old capital, given the objective to make it cheaper. However, recall that these taxes are rebated in a lump-sum fashion to each agent. Thus, a tax on old capital has only a positive effect on buyers of old capital, that is, constrained firms, because it allows the planner to reduce the price they face. Indeed, the larger the reduction in price required relative to the first-best price q^{FB}, the larger is the optimal tax τ^O. Notice that both tax rates τ^N and τ^O are constant and independent of firms' net worth, whereas they are offset by lump-sum taxes or transfers that vary with firms' net worth, because of heterogeneity in the composition of investment between new and old capital.

We consider again our numerical example and solve for the constrained-efficient allocation. In Figure 2, we illustrate the stationary competitive equilibrium (solid lines) as before, and contrast it with the constrained-efficient allocation rule that supports the first-best outcome described in equation (29) (dashed lines). While total capital (bottom left) is weakly increasing in net worth in competitive equilibrium, inducing inefficient dispersion in marginal products, the constrained-efficient allocation equalizes the scale of production across all firms, increasing aggregate investment and reallocating old capital towards the most constrained firms, without incurring any equity issuance costs (bottom right). The constrained-efficient allocation induces the first-best outcome with equilibrium price $q^* = 0.037$. The tax rates that decentralize this outcome are $\tau^N = -0.454$ and $\tau^O = 12.819$.

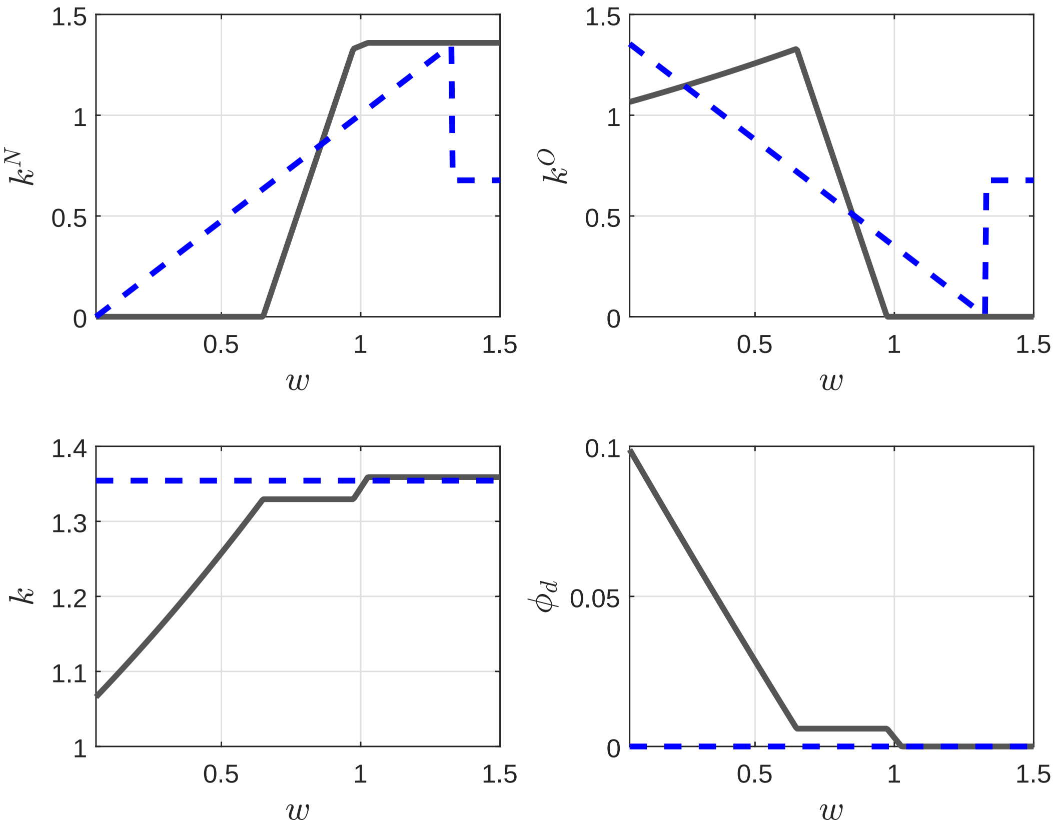

Figure 2: Stationary competitive equilibrium and constrained-efficient allocation – example. Top left: new capital k^N; top right: old capital k^O; bottom left: total capital k; bottom right: marginal cost of equity issuance ϕ_d. The x-axes report net worth w. Solid lines denote the competitive-equilibrium allocation, dashed lines the constrained-efficient allocation. See the caption of Figure 1 for the parameter values.

In Online Appendix OA.1, we consider the case in which the planner faces the additional constraint that purchases of old capital cannot be distorted. The planner makes investment decisions on behalf of firms, taking each firm's Euler equation for old capital as a constraint. Equivalently, the planner chooses proportional taxes on new capital, rebated lump-sum to each firm, but cannot tax old capital. In this case, the planner cannot implement the first-best allocation. However, the optimal policy is still a subsidy on new capital. We also solve for the constrained-efficient allocation in this case in our numerical example.

3.6 Risk-Averse Entrepreneurs

In our stylized model, firms maximize the present discounted value of dividends net of equity issuance cost, and the planner maximizes consumption of an infinitely-lived representative household who consumes aggregate dividends. We now consider the case in which firms are owned by over-lapping generations of risk-averse entrepreneurs, whose individual consumption coincides with dividends from their own firm.

Specifically, entrepreneurs maximize $u(c_{0t}) + \beta u(c_{1,t+1})$, where u is a utility function, with $u_c > 0$, $u_{cc} < 0$, $\lim_{c\to 0} u_c(c) = +\infty$, and entrepreneurial consumption coincides with dividends, which satisfy the budget constraints (7) and (8). We discuss this version of the model in detail in Online Appendix OA.2.

A utilitarian planner maximizes the present discounted value of utilities of all (present and future) entrepreneurs. The (stationary) constrained-efficient price of old capital satisfies the following optimality condition:

$$\int k^O(w) u_c(c_0(w))\, d\pi(w) = \int k^N(w) \left[u_c(c_1(w)) + \theta\lambda(w)\right] d\pi(w),$$

where the left-hand side and the first term in the sum on the right-hand side represent the distributive externalities on buyers and sellers of old capital, respectively, whereas the second term on the right-hand side represents the collateral externality.

In this model, the marginal value of entrepreneurial net worth equals the marginal utility of consumption, which is strictly decreasing in net worth, in contrast to the marginal equity issuance cost in the baseline model, which is equal to a positive constant in the indifference region between new and old capital, and equal to zero for unconstrained firms. Despite this difference, the fact that the marginal utility of consumption is decreasing implies that the planner still wants to induce a lower price of old capital than in competitive equilibrium, in order to redistribute resources toward more financially constrained entrepreneurs, who are net buyers of old capital in equilibrium. Hence, our result on the sign of constrained inefficiency obtains also with risk-averse entrepreneurs. We now state this result formally and prove it in Online Appendix OA.2.[17]

Proposition 3 (Sign of Constrained Inefficiency – Risk-Averse Entrepreneurs) *Assume that in stationary equilibrium $q > q^{FB}$. Then, the aggregate distributive externality exceeds the aggregate collateral externality, that is*

$$\int k^O(w) u_c(c_0(w))\, d\pi(w) > \int k^N(w) \left[u_c(c_1(w)) + \theta\lambda(w)\right] d\pi(w).$$

A marginal decrease in the price of old capital induces a positive welfare gain.

[17] While the proposition focuses on the case $q > q^{FB}$, Online Appendix OA.2 provides a weak condition under which the same result obtains when $q = q^{FB}$.

3.7 Heterogeneity in Productivity

In our baseline model, firms are heterogeneous only in their initial net worth. We now extend this framework to allow for heterogeneity in productivity and show that our main efficiency result obtains in this richer model. At their initial date, firms draw initial net worth w and a level of productivity $s \in \mathcal{S} \equiv \{s_1, ..., s_N\}$ from a joint distribution $\pi(w, s)$. At the production date, firms produce output with production function $y_t = sf(k_t)$. We discuss this model in detail in Online Appendix OA.3.

Allocations in stationary equilibrium are functions of (w, s), and the preference for new vs. old capital is thus tied to both net worth and productivity. Crucially, we show the marginal equity issuance cost is (weakly) increasing in productivity: $\frac{\partial \phi_d(w,s)}{\partial s} \geq 0$. Thus, firms with lower net worth and higher productivity tend to prefer old capital, whereas less financially constrained firms, that is, firms with higher net worth and lower productivity tend to purchase new capital. The market for old capital reallocates capital from less productive to more productive firms.

The (stationary) constrained-efficient price of old capital satisfies the following optimality condition:

$$\int k^O(w, s)\phi_d(w, s)d\pi(w, s) = \theta \int k^N(w, s)\lambda(w, s)d\pi(w, s),$$

where the left-hand side represents the aggregate distributive externality from a marginal change in the price of old capital, and the right-hand side represents the aggregate collateral externality.

In competitive equilibrium, we show that all firms that are indifferent between new and old capital have the same marginal value of net worth, independent of their productivity. This feature allows us to generalize our main efficiency result also to the case with heterogeneous productivity. We now state this result formally and prove it in Online Appendix OA.3.

Proposition 4 (Sign of Constrained Inefficiency – Heterogeneity in Productivity) *In the stationary competitive equilibrium, the aggregate distributive externality exceeds the aggregate collateral externality, that is,*

$$\int k^O(w, s)\phi_d(w, s)d\pi(w, s) > \theta \int k^N(w, s)\lambda(w, s)d\pi(w, s).$$

A marginal decrease in the price of old capital induces a positive welfare gain.

3.8 Firm Life Cycle and Long-Lived Capital

In our stylized model, firms live for two dates and capital is productive for two periods. The assumption that firms live for only one period rules out endogenous net worth dynamics. The assumption that capital is unproductive after two dates rules out the possibility of using old capital as collateral. We now show that our main result on the sign of inefficiency in competitive equilibrium obtains in a more general version of the model in which firms have a stochastic life cycle and capital is long lived.

To this end, we generalize the model in two ways. First, firms follow a stochastic life cycle. Specifically, at each date, with exogenous probability $\rho \in (0,1]$, firms learn that they will die after producing and paying their remaining net worth as a dividend. With probability $1-\rho$, firms continue their activity. Thus, as long as $\rho < 1$, firm net worth evolves endogenously. At each date, a measure ρ of new firms is born with initial net worth drawn from an exogenous distribution $\pi_0(w_0)$. The stationary distribution of net worth $\pi(w)$, however, is an equilibrium object.

Second, capital goods depreciate as follows. For each unit of new capital, a fraction $\delta^N \in (0,1]$ becomes old after production. Old capital depreciates at geometric rate $\delta^O \in (0,1]$ each period. With these assumption, firms can pledge a fraction θ of the resale value of capital next period $(1-\delta^N(1-q_{t+1}))k_t^N + q_{t+1}(1-\delta^O)k_t^O$ as collateral. Hence, both new and old capital serve as collateral.[18]

We analyze this model in Appendix A.2. Here, we state our main result on constrained inefficiency. We introduce the following notation. We denote firm age by $a = 0, 1, ..$ and the mass of age a firms that survive into the next period by $\gamma_a \equiv \rho(1-\rho)^a$. The (stationary) constrained-efficient price of old capital satisfies the following optimality condition:

$$\int \sum_{a=0}^{\infty} \gamma_a \left[k_a^O \phi_{d,a} - \left(\delta^N k_a^N + (1-\delta^O)k_a^O\right)(1-\rho)\phi_{d,a+1}\right] d\pi_0(w_0) = \theta \int \sum_{a=0}^{\infty} \gamma_a \lambda_a \left(\delta^N k_a^N + (1-\delta^O)k_a^O\right) d\pi_0(w_0), \quad (31)$$

where the left-hand side represents the aggregate distributive externality from a marginal change in the price of old capital and the right-hand side the aggregate collateral externality.

Different from the stylized model without firm life cycle, the marginal value of net worth is no longer necessarily constant in the indifference region between new and old capital. Moreover, old capital also serves as collateral, thus inducing a richer set of externalities from the price of old capital. Despite these differences with our baseline case, we can

[18] This environment nests the baseline model, which can be recovered by setting $\rho = \delta^N = \delta^O = 1$.

show that our result on the sign of the constrained inefficiency generalizes also to this environment. The economic intuition is that the more constrained firms are net buyers of old capital; although reducing the price of old capital decreases its collateral value, this effect is dominated by the distributive effect of making old capital cheaper for these firms. We now state this result, which we prove in Appendix A.2, formally.

Proposition 5 (Sign of Constrained Inefficiency – Long-Lived Firms and Capital) *In the stationary competitive equilibrium, the aggregate distributive externality exceeds the aggregate collateral externality, that is, the left-hand side of* (31) *is strictly larger than the right-hand side. A marginal decrease in the price of old capital induces a positive welfare gain.*

All told, we conclude that the distributive externality exceeds the collateral externality in a stationary competitive equilibrium in our model under quite general conditions, including with risk-neutral firms and costly equity issuance, entrepreneurial risk aversion, heterogeneity in productivity, and long-lived firms and capital.

4 Quantitative Model

We now consider a quantitative model of investment and capital reallocation with a stochastic firm life cycle, long-lived capital, and persistent idiosyncratic productivity shocks. In this model, both financial frictions and stochastic productivity are drivers of capital reallocation. We calibrate this model to analyze efficiency quantitatively in Section 5.

4.1 Environment

Time is discrete and infinite. As in the model of Section 3, a representative household with linear utility and discount factor β owns all firms in the economy. In every period, a continuum of measure ρ of firms are born and receive a common initial endowment of output w_0 from the household.[19] Firm i at time t produces output y_{it} combining new and old capital goods $k^N_{i,t-1}$ and $k^O_{i,t-1}$, subject to idiosyncratic productivity shocks s_{it} with the following technology

$$y_{it} = s_{it} f(k_{i,t-1}), \tag{32}$$

with $f_k > 0$, $f_{kk} < 0$, $k_{i,t-1} \equiv g(k^N_{i,t-1}, k^O_{i,t-1})$, where g is a constant returns to scale bundle of new and old capital, with $g_N, g_O > 0$, $g_{NN}, g_{OO} \leq 0$, and subscripts denote first and second

[19]Heterogeneity in net worth arises endogenously because of productivity shocks and net worth accumulation. Thus, for simplicity, we abstract from initial heterogeneity.

partial derivatives with respect to new (N) and old (O) capital, respectively. We assume that new and old capital are imperfect substitutes in the quantitative model, because this is empirically plausible and facilitates the computation by avoiding corner solutions.

As in the model of Section 3.8, firms die with probability ρ at the end of each period. Dying firms produce output and then distribute their new worth as a dividend. We denote age by a and let s^a be a history of realizations of idiosyncratic shocks up to firm age a, with associated exogenous probability $p(s^a)$. The measure of firms of age a that survive and invest to produce in the following period is $\gamma_a = \rho(1-\rho)^a$.

Output can be consumed by the household or transformed into new capital with constant unit marginal cost. Investment requires one period of time to build. A fraction δ^N of each unit of new capital becomes old in the following period. A fraction δ^O of each unit of old capital becomes useless in the following period. Firms can also scrap old capital and recover $\underline{q} \geq 0$ units of output. This assumption is empirically plausible and imposes a lower bound on the price of old capital that the planner can induce. We assume $\underline{q}$ is sufficiently low that no capital is scrapped either in the first-best allocation or in equilibrium.

4.2 Frictionless Economy and First Best

The aggregate resource constraint of the frictionless economy is

$$\sum_{a=0}^{\infty} \gamma_a \sum_{s^{a+1}} p(s^{a+1}) \left[s_{a+1} f(g(k_{t-1}^N(s^a), k_{t-1}^O(s^a))) + (1-\delta^N) k_{t-1}^N(s^a) \right] = C_t + \sum_{a=0}^{\infty} \gamma_a \sum_{s^a} p(s^a) k_t^N(s^a), \tag{33}$$

where the left-hand side is aggregate output and undepreciated new capital, and the right-hand side is consumption of the representative household and aggregate new capital. The evolution of the stock of old capital satisfies

$$\sum_{a=0}^{\infty} \gamma_a \sum_{s^a} p(s^a) \left[\delta^N k_{t-1}^N(s^a) + (1-\delta^O) k_{t-1}^O(s^a) \right] = \sum_{a=0}^{\infty} \gamma_a \sum_{s^a} p(s^a) k_t^O(s^a), \tag{34}$$

where the left-hand side is the sum of depreciated new capital and undepreciated old capital from the previous period, that is, the aggregate supply of old capital, and the right-hand side is the aggregate demand for old capital.

The first-best allocation maximizes the utility of the representative household (1) subject to the resource constraints (33) and (34). The optimality conditions for new and old capital are

$$
\begin{aligned}
1 &= \beta \mathbb{E}_t \left[s_{a+1} f_k(k_t^{FB}(s^a)) g_{N,t}(s^a) + (1 - \delta^N (1 - q_{t+1}^{FB})) \right] && (35) \\
q_t^{FB} &= \beta \mathbb{E}_t \left[s_{a+1} f_k(k_t^{FB}(s^a)) g_{O,t}(s^a) + (1 - \delta^O) q_{t+1}^{FB} \right], && (36)
\end{aligned}
$$

where $\mathbb{E}_t$ denotes the expectation conditional on information at date t, q_t^{FB} denotes the first-best valuation of old capital, and we use shorthand notation $g_{N,t}(s^a)$and $g_{O,t}(s^a)$ to denote the marginal effect of investment in new and old capital on total capital in production, that is, $g_N(k_t^N(s^a), k_t^O(s^a))$ and $g_O(k_t^N(s^a), k_t^O(s^a))$, respectively.

In the stationary First-Best allocation, the efficient value of old capital is

$$
q^{FB} = \frac{1 - \beta(1 - \delta^N)}{1 + \beta\delta^N - \beta(1 - \delta^O)}, \tag{37}
$$

and the allocation of capital satisfies

$$
k^{FB}(s^a) = f_k^{-1} \left(\frac{1 - \beta(1 - \delta^N(1 - q^{FB}))}{\beta \mathbb{E}\left[s_{a+1} \mid s^a \right]} \right). \tag{38}
$$

Unlike in the stylized model, when new and old capital are imperfect substitutes, equations (35) and (36) determine a unique allocation of new and old capital for all firms.

4.3 Financial Frictions and Competitive Equilibrium

We now consider the competitive equilibrium in the presence of financial frictions. As in the stylized model, firms can raise external funds in two ways. First, they can issue equity, subject to a twice differentiable, convex equity issuance cost ϕ. This cost is zero if firms pay a non-negative dividend. Second, they can issue non-contingent debt, subject to a collateral constraint, which specifies that the promised repayment cannot exceed a fraction θ of the total resale value of new and old capital in the following period.

The expected present discounted value of dividends, net of equity issuance costs, of a firm born at time t is

$$
\sum_{a=0}^{\infty} \beta^a \gamma_a \sum_{s^a} p(s^a) \left[(d_{t+a}(s^a) - \phi(-d_{t+a}(s^a))) \right] + \sum_{a=1}^{\infty} \beta^a \gamma_{a-1} \rho \sum_{s^a} p(s^a) w_{t+a}(s^a), \tag{39}
$$

where $d_t(s^a)$ are dividends of continuing firms and $w_t(s^a)$ is net worth, which is paid as a dividend by dying firms. The dividend of a continuing firm satisfies the following budget constraint:

$$
d_t(s^a) = w_t(s^a) + b_t(s^a) - k_t^N(s^a) - q_t k_t^O(s^a), \tag{40}
$$

where q_t is the price of old capital and $b_t(s^a)$ is non-contingent debt, with gross interest rate β^{-1}. Firm net worth evolves as follows. All firms are born with $w_t(s^0) = w_0$. For $a = 1, 2, \ldots$, we have

$$w_t(s^a) = s_a f(k_{t-1}(s^{a-1})) + (1 - \delta^N(1 - q_t))k_{t-1}^N(s^{a-1}) + q_t(1 - \delta^O)k_{t-1}^O(s^{a-1}) - \beta^{-1}b_{t-1}(s^{a-1}) \tag{41}$$

and total capital in production is given by a bundle of new and old capital,

$$k_{t-1}(s^{a-1}) = g(k_{t-1}^N(s^{a-1}), k_{t-1}^O(s^{a-1})). \tag{42}$$

Firms face a collateral constraint, which states that debt cannot exceed a fraction θ of the resale value of new and old capital:

$$\theta\left[(1 - \delta^N(1 - q_{t+1}))k_t^N(s^a) + q_{t+1}(1 - \delta^O)k_t^O(s^a)\right] \geq \beta^{-1}b_t(s^a). \tag{43}$$

The square bracket on the left-hand side of equation (43) reports the value of collateral, which consists of undepreciated new capital, depreciated new capital that is transformed into old capital, and undepreciated old capital.

We denote by $\beta^{t+1}\lambda_t(s^a)$ the multiplier on the collateral constraint and $\phi_{d,t}(s^a)$ the marginal equity issuance cost. The firm optimality conditions for new capital, old capital, and debt, are

$$\begin{aligned} 1 + \phi_{d,t}(s^a) &= \beta\mathbb{E}_t\left[s_{a+1}f_k(k_t(s^a))g_{N,t}(s^a) + (1 - \delta^N(1 - q_{t+1}))\right](1 + (1 - \rho)\phi_{d,t+1}(s^{a+1})) \\ &\quad + \beta\theta\lambda_t(s^a)(1 - \delta^N(1 - q_{t+1})) \qquad (44) \\ q_t(1 + \phi_{d,t}(s^a)) &= \beta\mathbb{E}_t\left[s_{a+1}f_k(k_t(s^a))g_{O,t}(s^a) + (1 - \delta^O)q_{t+1}\right](1 + (1 - \rho)\phi_{d,t+1}(s^{a+1})) \\ &\quad + \beta\theta\lambda_t(s^a)(1 - \delta^O)q_{t+1} \qquad (45) \\ \phi_{d,t}(s^a) &= (1 - \rho)\mathbb{E}_t\phi_{d,t+1}(s^{a+1}) + \lambda_t(s^a). \qquad (46) \end{aligned}$$

We highlight some important differences between these optimality conditions and their counterparts in the stylized model, that is, equations (10), (11), (12). First, productivity is stochastic, implying that both future marginal products and future marginal equity issuance costs are also stochastic. Moreover, we assume that markets are incomplete and firms issue noncontingent debt. Thus, all three optimality conditions (44), (45), and (46) involve the conditional-expectation operator $\mathbb{E}_t$. Second, both new and old capital are long lived, and both serve as collateral. Thus, equation (45) equates the marginal cost of investing in old capital, on the left-hand side, with the marginal benefit, which depends on the future marginal product, as well as the future resale value, and the effect of old capital on the collateral constraint.

In equilibrium, the price of old capital q_t satisfies the market-clearing condition (34).

4.4 Constrained Efficiency

We now consider the problem of a planner who chooses investment in new and old capital, as well as debt, on behalf of individual firms, under the same set of constraints and frictions, but internalizing the effects of these choices on the price of old capital. This problem is equivalent to the problem of a Ramsey planner who chooses firm-specific proportional taxes on new and old capital and rebates them in a lump-sum fashion to each firm.

The planner maximizes the present discounted value of aggregate dividends

$$\sum_{t=0}^{\infty} \beta^t \left[\sum_{a=0}^{\infty} \sum_{s^a} p(s^a) \gamma_a \left[\left(d_t(s^a) - \phi(-d_t(s^a)) \right) \right] + \sum_{a=1}^{\infty} \sum_{s^a} p(s^a) \gamma_{a-1} \rho w_t(s^a) \right] \tag{47}$$

subject to firms' budget constraints, collateral constraints, with multiplier $\beta^{t+1}\lambda_t(s^a)$, and the market-clearing condition (34), with multiplier $\beta^t \eta_t$. Furthermore, the planner must induce a price of old capital that is weakly larger than the scrap value. We denote the multiplier on this constraint by $\beta^t \zeta_t$.

The optimality conditions for new capital, old capital, and debt, are

$$\begin{aligned} 1 + \phi_{d,t}(s^a) &= \beta \mathbb{E}_t \left[s_{a+1} f_k(k_t(s^a)) g_{N,t}(s^a) + (1 - \delta^N (1 - q_{t+1})) \right] (1 + (1-\rho) \phi_{d,t+1}(s^{a+1})) \\ &\quad + \beta \theta \lambda_t(s^a) (1 - \delta^N (1 - q_{t+1})) + \beta \delta^N \eta_{t+1} & (48) \\ q_t(1 + \phi_{d,t}(s^a)) &= \beta \mathbb{E}_t \left[s_{a+1} f_k(k_t(s^a)) g_{O,t}(s^a) + (1 - \delta^O) q_{t+1} \right] (1 + (1-\rho) \phi_{d,t+1}(s^{a+1})) \\ &\quad + \beta \theta (1 - \delta^O) \lambda_t(s^a) q_{t+1} - \eta_t + \beta (1 - \delta^O) \eta_{t+1}, & (49) \end{aligned}$$

and (46). When choosing new and old capital, the planner takes into account the effect of these investment decisions on the resource constraint for old capital, and thus on its price. In particular, an additional unit of new capital leads to δ^N additional units of supply of old capital in the following period. In a similar fashion, demand for old capital draws from the current stock, and adds $(1 - \delta^O)$ units to the future stock. The terms involving the multipliers η_t and η_{t+1} in equations (48) and (49) internalize these effects.

The optimality condition for the price of old capital is

$$\sum_{a=0}^{\infty}\gamma_a\sum_{s^a}p(s^a)k_t^O(s^a)(1+\phi_{d,t}(s^a)) =$$
$$\sum_{a=0}^{\infty}\gamma_a\sum_{s^{a+1}}p(s^{a+1})\left[\delta^N k_{t-1}^N(s^a)+(1-\delta^O)k_{t-1}^O(s^a)\right](1+(1-\rho)\phi_{d,t}(s^{a+1})+\theta\lambda_{t-1}(s^a))+\zeta_t. \tag{50}$$

The sum on the left-hand side of equation (50) represents the marginal cost of increasing the price q_t for firms that purchase old capital. The sum on the right-hand side represents the marginal benefit of increasing net worth for firms that own old capital, as well as the marginal effect of q_t on the borrowing capacity of constrained firms at $t-1$. Thus, as long as the scrappage-value constraint is not binding, the planner sets the net effect of distributive and collateral externalities equal to zero.

We also consider the case in which the planner faces the additional constraint that purchases of old capital cannot be distorted (analogous to the version of the stylized model in Online Appendix OA.1). We assume that the planner can only choose the amount of new capital used in production on behalf of firms, and takes as a constraint their demand for old capital. In the interest of brevity, we relegate the derivation of the optimality conditions to Online Appendix OA.4. Furthermore, in Online Appendix OA.5, we describe our solution method for the stationary constrained-efficient allocation.

5 Calibration and Quantitative Analysis

In this section, we calibrate the model with idiosyncratic productivity shocks. We then provide a quantitative analysis of inefficiency in competitive equilibrium and compare the stationary equilibrium with the constrained-efficient allocation.

5.1 Calibration

We now describe our choices of parameter values, which we report in Table 1. A period in the model coincides with a year, and we thus set $\beta = 0.96$. We make the following assumptions about functional forms. The production function is $f(k) = k^{\alpha}$ with $\alpha \in (0,1)$. We set $\alpha = 0.6$ to reflect a typical value for the capital share in the literature on firm dynamics, adjusted to account for the choice of labor input, which we abstract from modelling.[20]

Firms combine new and old capital in a CES bundle $g(k^N, k^O) = \left[(\sigma^N)^{\frac{1}{\epsilon}}(k^N)^{\frac{\epsilon-1}{\epsilon}} + (1-\sigma^N)^{\frac{1}{\epsilon}}(k^O)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}$. In our stylized model, we assumed perfect substitutability between new and old capital. We use the quantitative model to show that the key insights are robust to a plausible degree of imperfect substitutability. We thus set $\epsilon = 5$ following Lanteri (2018) and $\sigma^N = 0.5$, thereby treating new and old capital symmetrically in production.[21] We further set the depreciation parameters $\delta^N = \delta^O = 0.2$ so that the effective capital depreciation rate, accounting for the transition probability from new to old capital, and the equilibrium price of old capital, is approximately 10%. With these parameter values, the average age of new (old) capital is approximately equal to 4 (9) years.

The cost of equity issuance is a power function, $\phi(-d) = \phi_0(-d)^{\phi_1}$ for $d < 0$ and $\phi(-d) = 0$ otherwise. A high value of ϕ_1 facilitates the computation of the constrained-efficient allocation, especially in the case in which the planner can only distort new investment; thus, we set $\phi_1 = 5$. We then set $\phi_0 = 0.1$, which implies marginal costs of equity in the range of the relevant empirical estimates—for example, Hennessy and Whited (2007) and Catherine, Chaney, Huang, Sraer, and Thesmar (2020). On average, firms in the model face a marginal cost of equity approximately equal to 6% of the issuance (16% conditional on firms that pay negative dividends in equilibrium). We set $\theta = 0.5$, implying that firms can borrow up to half of the resale value of their capital. This baseline value is close to the estimates by Li, Whited, and Wu (2016). Moreover, we report results associated with $\theta = 0$ and $\theta = 0.75$ in Table A1 in the Appendix.[22]

The idiosyncratic productivity shock follows an AR(1) process in logs with persistence parameter χ_s and standard deviation of innovations σ_s. We set $\chi_s = 0.7$ and $\sigma_s = 0.12$, consistent with typical estimates in the literature on investment and reallocation with firm-level productivity shocks (Khan and Thomas, 2013, and Lanteri, 2018). We then discretize this process with a two-state Markov chain using the method of Rouwenhorst (1995). Given this process for the shocks, the standard deviation of firm-level investment rates in competitive equilibrium is equal to 0.29, close to empirical estimates (Cooper and Haltiwanger, 2006). We set $\rho = 0.1$, which approximately matches the average entry (and exit) rate for U.S. firms (Decker, Haltiwanger, Jarmin, and Miranda, 2014).

Newborn firms receive an initial net worth $w_0 = 5$, which corresponds to approximately

[20] With a production function $y = k^{\alpha_k} n^{\alpha_n}$, where n is labor, assuming time to build in capital and flexible labor choice, the effective elasticity of output with respect to capital that is relevant for investment is $\alpha \equiv \frac{\alpha_k}{1-\alpha_n}$. Common values in the investment literature are $\alpha_k \approx 0.25$ and $\alpha_n \approx 0.6$, which support our choice of parameter value.

[21] Edgerton (2011) estimates the elasticity of substitution between new and old capital for several industries and finds values in the range between 1 and 10.

[22] When solving the model for $\theta = 1$, we find that the competitive equilibrium is quantitatively close to the first-best allocation, as only newborn, high-productivity firms are financially constrained.

Table 1: Parameter Values – Calibration

		Parameter	Value
Preferences	Discount rate	β	0.96
Life cycle	Initial net worth	w_0	5
	Death probability	ρ	0.1
Technology	Curvature of production function	α	0.6
	CES elasticity of substitution	ϵ	5
	CES new capital share	σ^N	0.5
	Depreciation of new capital	δ^N	0.2
	Depreciation of old capital	δ^O	0.2
	Scrap value	$\underline{q}$	0.1
	Productivity persistence	χ_s	0.7
	Productivity st. dev. of innovations	σ_s	0.12
Financial constraints	Collateralizability	θ	0.5
	Cost of raising equity parameters	ϕ_0	0.1
		ϕ_1	5

10% of the unconstrained-optimal capital level for high-productivity firms. Under this calibration, our model is broadly consistent with the evidence on the empirical relationship between firm age and capital age reported by Ma, Murfin, and Pratt (2020). They focus on equipment and find that age-0 firms buy machines that are on average 5.5 years old, whereas age-10 firms tend to buy capital that is on average 4 years old. In our model, which encompasses a broader notion of capital (including structures), the corresponding figures are 7.5 and 6.4 years.[23] Also consistent with their empirical findings, the slope of capital age with respect to firm age is steeper for younger firms, which are more financially constrained and thus purchase a larger share of old capital goods in our model.

[23] Our quantitative model features a clear distinction between new and old capital, but does not necessarily distinguish between capital goods of different ages, given the partial depreciation structure with rates δ^N and δ^O. Thus, to compute firm-level capital age in the model, we first compute the *average* age of new capital and the *average* age of old capital, which are $\frac{1-\delta^N}{\delta^N}$ and $\frac{1-\delta^N}{\delta^N} + \frac{1}{\delta^O}$, respectively. Specifically, the average age of new capital is 4 years and the average age of old capital is 9 years in our calibration. Next, we use the optimal portfolio weights on new and old capital for each firm to compute the average capital age for each firm, thereby assuming that the distribution of capital age within new capital and within old capital is homogeneous across firms.

5.2 Quantitative Results

Given our calibration, the stationary competitive-equilibrium price of old capital equals 0.553, whereas the first-best price of old capital equals 0.547. Equilibrium down payments and user costs (from the perspective of unconstrained firms) are

$$\wp_N \equiv 1 - \beta\theta(1 - \delta^N(1 - q)) = 0.563 > \wp_O \equiv q\left[1 - \beta\theta(1 - \delta^O)\right] = 0.340$$

and

$$u_N \equiv 1 - \beta(1 - \delta^N(1 - q)) = 0.126 < u_O \equiv q\left[1 - \beta(1 - \delta^O)\right] = 0.128,$$

respectively.

Figure 3 illustrates the key firm decision rules in the stationary competitive equilibrium (solid) and under the constrained efficient allocation (dashed). We start by discussing the policy functions in competitive equilibrium. Old capital accounts for a larger fraction of the capital operated by firms with lower net worth. As firms grow, they increase the share of new investment goods in their capital bundle. Furthermore, for a given level of net worth, firms with higher productivity are more financially constrained, as indicated by a higher marginal equity issuance cost, and thus choose a higher fraction of old capital goods. Thus, on average, the market for old capital reallocates assets from firms with high net worth and lower productivity to firms with low net worth and high productivity.

We now discuss the constrained-efficient allocation, when the planner chooses both new and old capital. We find that the planner optimally drives the price of old capital down to the scrappage value, thereby fostering capital reallocation toward financially constrained firms, which increase substantially their purchases of old capital and thus their overall productive capacity. The marginal value of net worth of the most constrained firms induced by this allocation is significantly lower than in the competitive equilibrium.

We also compute the tax rates on new and old capital that implement the constrained-efficient allocation as a competitive equilibrium with taxes, rebated to each firm in a lump-sum fashion.[24] On average, the subsidy on new capital equals 8.6% and the tax on old capital equals 103.8%. Consistent with the intuition developed in our analytical results in Section 3.5, a large tax on old capital reflects the fact that the planner achieves a significant reduction of the price of old capital from its competitive-equilibrium value, which exceeds the first-best value, to the lower bound, the scrappage value $\underline{q}$.[25]

The large distortion on old capital motivates us to study a case in which only new-

[24]We illustrate these tax rates in Online Appendix OA.6.

[25]Accordingly, in Section 6.3 we analyze the sensitivity with respect to the scrappage value $\underline{q}$ and find that optimal taxes on old capital are sensitive to this parameter. Specifically, a larger price reduction requires higher taxes on old capital to offset the effect of a low price on firms' optimal production scale.

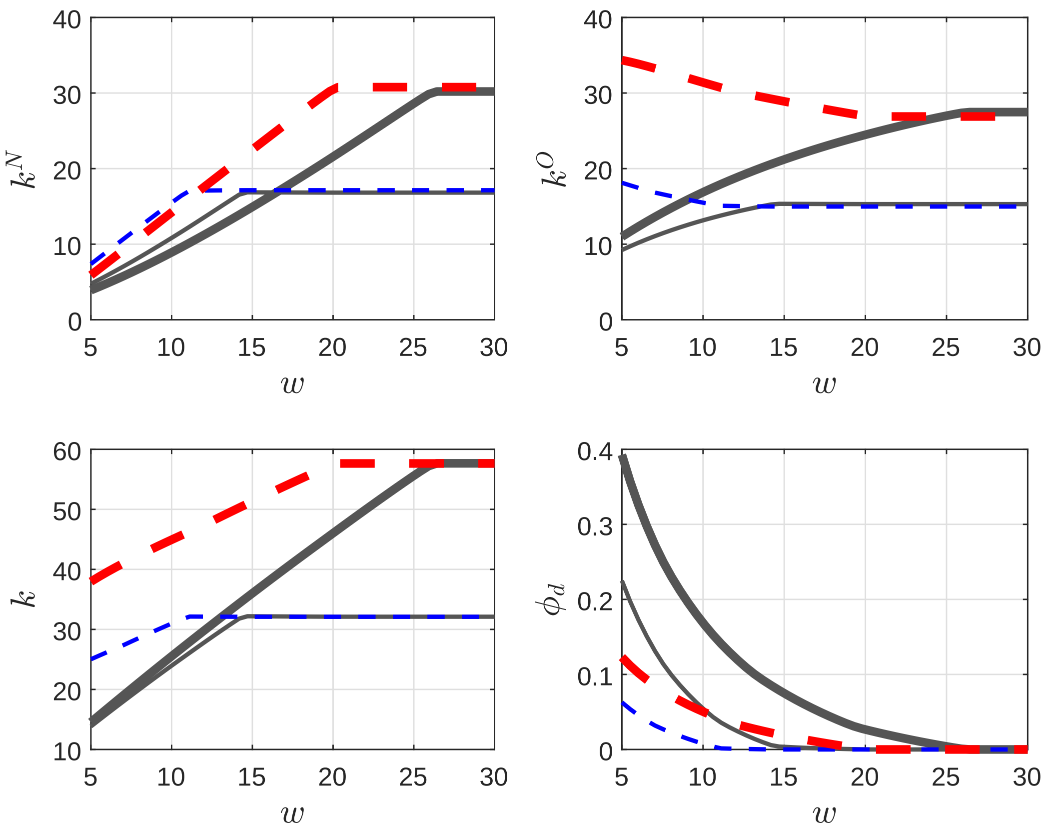

Figure 3: Stationary equilibrium and constrained efficient allocation. Top left: new capital k^N; top right: old capital k^O; bottom left: capital bundle k; bottom right: marginal value of net worth ξ. The x-axes report net worth w. Solid lines denote the competitive-equilibrium allocation, dashed lines the constrained-efficient allocation. Thick lines denote the high productivity realization, thin lines the low realization.

capital taxes—rebated with lump-sum transfers to each firm—are available. We find that the restriction on the instrument set has a significant impact on the optimal allocation. The planner still instructs firms to increase their new investment, thereby decreasing the equilibrium price of old capital. The absence of taxes on old capital, however, implies that a reduction in the price of old capital induces unconstrained firms to also demand old capital and increase their size. Thus, this price reduction is significantly more muted than in the case in which the planner can also tax old capital. For the same reason, the optimal subsidy on new capital is smaller than in the previous case, and equals approximately 0.6%.

Next, we use our quantitative model to measure the pecuniary externalities in the stationary competitive equilibrium. Consistent with our analytical results, we find that the distributive externality dominates the collateral externality. Specifically, in the aggregate, we find that the distributive externality is approximately 2.3 times as large as the collateral

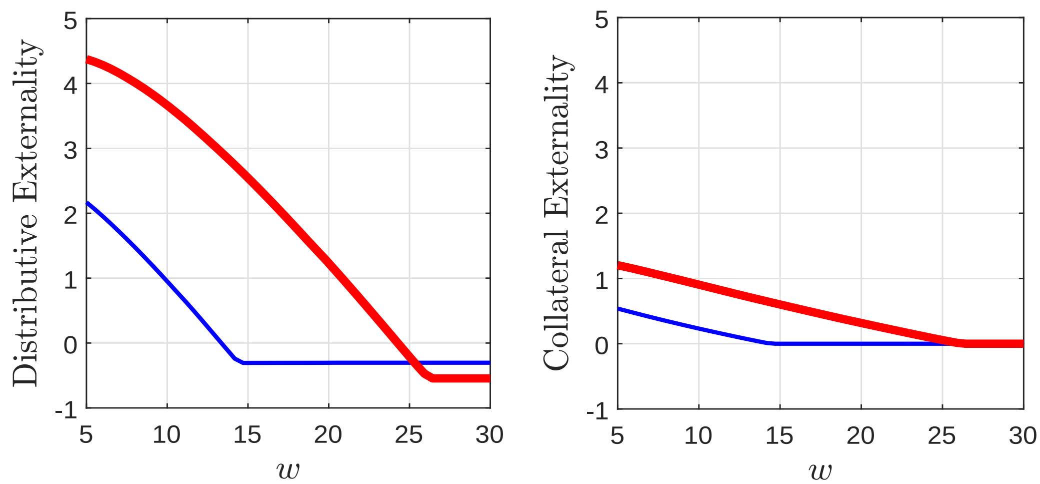

Figure 4: Pecuniary externalities in stationary equilibrium. Left panel: distributive externality; right panel: collateral externality. The x-axes report net worth w. Thick red lines denote the high productivity realization, thin blue lines the low realization. The distributive (collateral) externality is defined as the marginal effect on firm value of a decrease (increase) in the price of old capital. Using recursive notation, for a firm with state variables (w, s), the distributive externality equals $k^O(w,s)(1+\phi_d(w,s)) - \left[\delta^N k^N(w,s) + (1-\delta^O)k^O(w,s)\right](1+(1-\rho)\mathbb{E}\phi_d(w',s'))$, whereas the collateral externality equals $\theta\lambda(w,s)\left[\delta^N k^N(w,s) + (1-\delta^O)k^O(w,s)\right]$.

externality.

In Figure 4, we explore the heterogeneous effects of the pecuniary externalities through the price of old capital, by displaying the cross section of distributive externalities (left panel) and collateral externalities (right panel) as functions of firms' state variables. The distributive externality is defined as the marginal effect on firm value of *decreasing* the price of old capital due to a change in the value of old capital traded. This externality is largest for firms with low net worth and high productivity, because they are net buyers of old capital. As firms' net worth increases, they eventually become net sellers of old capital, and the distributive externality accordingly becomes negative. The collateral externality is defined as the marginal effect on firm value of *increasing* the (future) price of collateral. This externality is also highest for the most financially constrained firms and goes to zero as firms become unconstrained. However, the figure confirms that overall the distributive externality is significantly larger and thus a reduction in the price of old capital is desirable.

In Table 2, we compare the key long-run aggregate outcomes under four alternative allocations: first best; competitive equilibrium; constrained efficiency when the planner chooses both types of capital; constrained efficiency when the planner chooses only new capital. Competitive-equilibrium and constrained-efficient allocations and prices are ex-

Table 2: Quantitative Results

Output, investment, consumption, and the price of used capital for the competitive equilibrium, the constrained efficient allocation, and the constrained efficient allocation without taxes on old capital are expressed as fractions of the corresponding first-best value, reported in parenthesis in the first column.

Variable	First Best	Comp. Eqm.	Constr. Eff.	Constr. Eff. ($\tau^O = 0$)
Output	(9.910)	0.899	0.973	0.921
Investment	(4.497)	0.857	0.962	0.893
Consumption	(5.413)	0.933	0.983	0.943
Price q	(0.547)	1.010	0.184	0.987
Average tax τ^N	0	0	-8.6%	-0.6%
Average tax τ^O	0	0	103.8%	n.a.

pressed as fractions of the corresponding first-best value, which we report in parenthesis in the first column. We find that financial frictions induce an aggregate output loss of approximately 10%, and an aggregate consumption loss of approximately 7%, relative to first best.[26] Notice that aggregate consumption is the relevant measure of welfare, under our assumption of linear utility. When the planner chooses both new and old capital, the constrained-efficient allocation increases output by 8% and consumption by 5% relative to the competitive equilibrium. However, when the planner can only choose new capital, the gains are significantly smaller. In this case, aggregate output increases by slightly more than 2% and aggregate consumption by approximately 1% relative to competitive equilibrium.

6 Additional Analyses and Robustness

This section provides additional analyses of our quantitative model, by considering the transition dynamics, assessing the gains from the reallocation of old capital to more constrained firms, and evaluating the sensitivity of our quantitative results to key parameters.

6.1 Transition Dynamics

Our analysis has focused on stationary equilibria. We now perform an analysis of the transition dynamics associated with the implementation of subsidies on new investment.

[26] For comparison, Catherine, Chaney, Huang, Sraer, and Thesmar (2020) estimate the aggregate output cost of collateral constraints (and costly equity issuance) for US firms to be approximately 7%.

To make this analysis tractable, we consider the undistorted competitive equilibrium as the initial condition and assume that, unexpectedly, all firms face a common, time-invariant tax rate τ^N. We consider a grid of values for τ^N. For each value, we compute the associated final stationary equilibrium, as well as the transition dynamics keeping track of the distribution of firms and clearing the market for old capital in each period. We then find the level of τ^N that maximizes household utility starting from the undistorted competitive equilibrium (that is, the present discounted value of aggregate consumption). Consistent with the key results of the paper, we find that the optimal tax rate on new capital is negative—that is, it is a subsidy on new capital—and approximately equal to -0.3%. We display these results in Online Appendix OA.6.

6.2 Benchmarking the Gains From Reallocation

We have quantified the efficiency gains associated with optimal policy, and compared them to the undistorted competitive equilibrium. In order to provide further perspective on the size of these gains, we now compare our competitive equilibrium to a restricted allocation, in which all firms need to purchase new and old capital goods in the same fixed proportion. This analysis allows us to quantify the gains from reallocating old capital when firms face financial frictions. Specifically, we solve the firm optimization problem adding the constraint $k_t^N(s^a)/k_t^O(s^a) = \delta^O/\delta^N$, which is the aggregate ratio of new to old capital in steady state.[27] We find that removing the restriction induces aggregate welfare gains equal to approximately 0.4 percentage points of permanent consumption. We also find that the size of this gain is somewhat sensitive to the elasticity of substitution between new and old capital. Intuitively, the higher this elasticity, the more efficient it is for financially constrained firms to increase the share of old capital in their investment. For instance, the welfare gain from removing the restriction of constant investment shares increases to 0.8 percentage points of permanent consumption when we set $\epsilon = 10$. Thus, overall, we find that the gains from reallocating old capital in competitive equilibrium are comparable in size to the gains of going from competitive equilibrium to the optimal investment subsidy when the planner cannot distort purchases of old capital, but significantly smaller than the gains we obtain when the planner can choose both new and old capital.

[27]Given this constraint, the market-clearing price is indeterminate. For comparison, we select the equilibrium associated with the same price of used capital as in the unrestricted competitive equilibrium.

6.3 Sensitivity

We now briefly discuss the sensitivity of our quantitative results with respect to changes in three parameters: the degree of collateralizability θ (equal to 0.5 in the baseline calibration); the elasticity of substitution between new and old capital ϵ (equal to 5 in the baseline calibration); and the scrap value $\underline{q}$ (equal to 0.1 in the baseline calibration). We report all results related to this analysis in Table A1 in the Appendix.

We solve the model for $\theta = 0$ (no borrowing) and $\theta = 0.75$. With $\theta = 0$, financial frictions induce substantially larger losses than in our baseline calibration. For instance, competitive-equilibrium output is approximately 20% lower than in the first-best allocation. Moreover, the only pecuniary externality is the distributive externality, contributing to larger gains from the optimal policy of subsidizing investment and reducing the price of old capital. With $\theta = 0.75$, the effects of financial frictions are smaller (competitive-equilibrium output is approximately 5% smaller than under first best), and, accordingly, so are the gains from optimal policy. The distributive externality is 27% larger than the collateral externality in competitive equilibrium. We find, however, that optimal tax rates on new and old capital are quite similar across all values of θ we consider.

Next, we consider different values for ϵ, namely $\epsilon = 1$ and $\epsilon = 10$. We find that our results are quite robust with respect to these changes. The higher the elasticity of substitution, however, the more effective the planner is in allowing constrained firms to produce at a larger scale by using a larger share of old capital, consistent with our theoretical result that first-best welfare can be achieved with perfect substitutability (Section 3.5).

Finally, we consider a lower and a higher scrap value ($\underline{q} = 0.05$ and $\underline{q} = 0.2$) relative to our baseline value ($\underline{q} = 0.1$), to investigate whether this lower bound for the price of old capital, which is a binding constraint for the planner, is important for our results. We find that optimal allocations are similar, irrespective of this change, and, intuitively, welfare gains are larger, the lower the scrap value. We also find that the optimal tax on old capital that supports the constrained-efficient allocation is highly sensitive with respect to this parameter, ranging from approximately 40% when $\underline{q} = 0.2$ to approximately 230% when $\underline{q} = 0.05$.

7 Conclusion

We analyze the constrained-efficient allocation in an equilibrium model of investment and capital reallocation both theoretically and quantitatively. Financial frictions induce pecuniary externalities in the secondary market for capital. Because financially constrained firms tend to be net buyers of old capital, and unconstrained firms tend to sell old capital and replace it with new capital, the competitive-equilibrium price of old capital is inefficiently high. This distributive externality dominates the collateral externality, which would call for increasing the resale price of capital instead, and which is the focus of much of the existing quantitative literature using models with a representative firm. A planner can induce a more efficient allocation by subsidizing new capital, thereby increasing the future supply of old capital and thus alleviating the effects of financial constraints for constrained firms in the future.

Subsidies on new investment are a widely-used policy tool.[28] Despite their popularity, to the best of our knowledge there is scarce theoretical foundation for these policies. Our analysis highlights that new investment induces a positive externality by fostering capital reallocation, thus providing a rationale for investment subsidies. We also show the efficiency gains associated with investment subsidies are tightly linked to equilibrium prices and policy interventions in secondary markets, thus providing a new perspective and guidance on the optimal design of investment incentives.

[28]For instance, in the US, bonus depreciation is a federal budget provision that historically subsidized investment in new equipment. Since 2018, this provision has been extended to include purchases of used capital goods at least until 2023. According to the Joint Committee on Taxation, the estimated cost of this scheme was approximately 63 billion dollars in 2019.

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APPENDIX

This appendix provides additional details and proofs for the stylized model of Section 3.

A.1 Lagrangian for Constrained-Efficient Allocation

In this section, we explicitly formulate the Lagrangian of the problem discussed in Section 3.4. The planner chooses sequences of functions $\left\{d_{0t}(w), d_{1,t+1}(w), k_{t+1}^N(w), k_{t+1}^O(w), b_{t+1}(w)\right\}_{t=0}^{\infty}$ and a sequence of prices $\{q_t\}_{t=0}^{\infty}$, given an initial condition $\left(k_0^N(w), k_0^O(w), b_0(w)\right)$, to maximize the present discounted value of aggregate dividends – or equivalently, aggregate consumption – subject to the sequence of firms' budget constraints when young and old (with multipliers $\mu_{0t}(w)$ and $\mu_{1t+1}(w)$, respectively), collateral constraints (with multiplier $\lambda_t(w)$), non-negativity constraints on new and old capital (with multipliers $\underline{\nu}_t^N(w)$ and $\underline{\nu}_t^O(w)$, respectively), and market-clearing conditions for old capital (with multiplier η_t). We now state the Lagrangian of this problem, dropping the dependence of allocation and distribution on net worth w to simplify notation:

$$\begin{aligned}\mathcal{L} \equiv \sum_{t=0}^{\infty} \beta^t \Bigg\{ &\int \left(d_{0t} - \phi(-d_{0t}) + d_{1t}\right) d\pi - \int \mu_{0t}\left(d_{0t} - w - b_t + k_t^N + q_t k_t^O\right) d\pi \\ &- \int \mu_{1t}\left(d_{1t} - f(k_{t-1}^N + k_{t-1}^O) - q_t k_{t-1}^N + \beta^{-1} b_{t-1}\right) d\pi + \int \lambda_t \left(\beta\theta q_{t+1} k_t^N - b_t\right) d\pi \\ &+ \int \underline{\nu}_t^N k_t^N d\pi + \int \underline{\nu}_t^O k_t^O d\pi - \eta_t \left(\int k_t^O d\pi - \int k_{t-1}^N d\pi\right)\Bigg\}.\end{aligned}$$

A.2 Firm Life Cycle and Long-Lived Capital

In this section, we discuss the model with a stochastic firm life cycle and long-lived capital from Section 3.8 in more detail and we prove Proposition 5.

Competitive Equilibrium with Financial Frictions. The expected present discounted value of dividends, net of equity issuance costs, of a firm born at time t is

$$\sum_{a=0}^{\infty} \beta^a \gamma_a \left[\left(d_{a,t+a} - \phi(-d_{a,t+a})\right)\right] + \sum_{a=1}^{\infty} \beta^a \gamma_{a-1} \rho w_{a,t+a}$$

where d_{at} are dividends of continuing firms of age a at time t and w_{at} is net worth. We leave implicit the dependence of allocations on initial firm net worth w_0 to simplify notation.

The dividend of a continuing firm satisfies the following budget constraint:

$$d_{at} = w_{at} + b_{at} - k_{at}^N - q_t k_{at}^O$$

where k_{at}^N and k_{at}^O are investments in new and old capital, respectively, q_t is the price of old capital, and b_{at} is debt. Firm net worth evolves as follows. For $a > 0$, we have

$$w_{at} = f(k_{a-1,t-1}) + (1 - \delta^N(1 - q_t))k_{a-1,t-1}^N + q_t(1 - \delta^O)k_{a-1,t-1}^O - \beta^{-1}b_{a-1,t-1}$$

where $k_{a-1,t-1} = k_{a-1,t-1}^N + k_{a-1,t-1}^O$ and β^{-1} is the gross interest rate.

Firms face a collateral constraint, which states that debt cannot exceed a fraction θ of the resale value of new and old capital:

$$\theta \left[(1 - \delta^N(1 - q_{t+1}))k_{at}^N + q_{t+1}(1 - \delta^O)k_{at}^O\right] \geq \beta^{-1}b_{at}.$$

Denote the multiplier on the collateral constraint by $\beta^{t+1}\lambda_{at}$ and the marginal equity issuance cost by $\phi_{d,at}$. The firm's optimality conditions for new capital, old capital, and debt, are

$$\begin{aligned}
1 + \phi_{d,at} &= \beta\left[f_k(k_{at}) + (1 - \delta^N(1 - q_{t+1}))\right](1 + (1 - \rho)\phi_{d,a+1,t+1}) \\
&\quad + \beta\theta\lambda_{at}(1 - \delta^N(1 - q_{t+1})) \\
q_t(1 + \phi_{d,at}) &= \beta\left[f_k(k_{at}) + (1 - \delta^O)q_{t+1}\right](1 + (1 - \rho)\phi_{d,a+1,t+1}) \\
&\quad + \beta\theta\lambda_{at}(1 - \delta^O)q_{t+1} \\
\phi_{d,at} &= (1 - \rho)\phi_{d,a+1,t+1} + \lambda_{at}.
\end{aligned}$$

The market-clearing condition for old capital is

$$\int \sum_{a=0}^{\infty} \gamma_a \left[\delta^N k_{a,t-1}^N + (1 - \delta^O)k_{a,t-1}^O\right] d\pi_0(w_0) = \int \sum_{a=0}^{\infty} \gamma_a k_{at}^O d\pi_0(w_0).$$

We define a stationary competitive equilibrium writing the firm problem and the market clearing condition recursively. A *stationary competitive equilibrium* is a set of policy functions mapping net worth to an allocation, that is, dividends, investment, and borrowing choices for continuing firms, $\{d(w), k^N(w), k^O(w), b(w)\}$, a stationary distribution of net worth $\pi(w)$, and a price of old capital q, such that firms maximize the present discounted value of dividends net of equity issuance cost, $\forall w$, the stationary distribution is consistent with firms' policy functions, and the market for old capital clears, that is, $\int k^N(w)d\pi(w) = \int k^O(w)d\pi(w)$. Notice that the stationary distribution of net worth π is an equilibrium object, whereas the distribution of net worth of new firms π_0 is taken as exogenous.

With long-lived capital, we define the down payment per unit of new and old capital as $\wp_N \equiv 1 - \beta\theta(1 - \delta^N(1-q))$ and $\wp_O \equiv q\left(1 - \beta\theta(1-\delta^O)\right)$, respectively, and the user cost of new and old capital to an unconstrained firm as $u_N \equiv 1 - \beta\left(1 - \delta^N(1-q)\right)$ and $u_O \equiv q\left(1 - \beta(1-\delta^O)\right)$, respectively. Analogously to (15) and (16) we define the user cost of new and old capital to a firm with net worth w as

$$u_N(w) \equiv u_N + \phi_d \wp_N = 1 - \beta(1 - \delta^N(1-q)) + \phi_d(1 - \beta\theta(1 - \delta^N(1-q)))$$

and

$$u_O(w) \equiv u_O + \phi_d \wp_O = q(1 - \beta(1-\delta^O) + \phi_d(1 - \beta\theta(1-\delta^O),$$

respectively. The investment Euler equations for new and old capital are

$$1 = \beta\left(\frac{1+(1-\rho)\phi_d'}{1+\phi_d}\right)\frac{f_k(k) + \beta(1-\theta)(1-\delta^N(1-q))}{\wp_N} + \frac{\underline{\nu}_N/(1+\phi_d)}{\wp_N} \qquad \text{(A1)}$$

$$1 = \beta\left(\frac{1+(1-\rho)\phi_d'}{1+\phi_d}\right)\frac{f_k(k) + \beta(1-\theta)(1-\delta^O)}{\wp_O} + \frac{\underline{\nu}_O/(1+\phi_d)}{\wp_O}. \qquad \text{(A2)}$$

Combining these two Euler equations we obtain

$$1 = \beta\left(\frac{1+(1-\rho)\phi_d'}{1+\phi_d}\right)\frac{(1-\theta)((1-\delta^N(1-q)) - q(1-\delta^O))}{\wp_N - \wp_O} + \frac{(\underline{\nu}_N - \underline{\nu}_O)/(1+\phi_d)}{\wp_N - \wp_O}. \quad \text{(A3)}$$

To see that $q < 1$ in a stationary equilibrium, suppose instead that $q \geq 1$; then $u_O > u_N$ and $\wp_O > \wp_N$, implying that old capital would be dominated. To see that $\wp_N > \wp_O$ in a stationary equilibrium, suppose instead that $\wp_N \leq \wp_O$; then (A3) would imply that $\underline{\nu}_O > 0$, that is, no firms would invest in old capital, a contradiction. Note that $\wp_N > \wp_O$ is equivalent to

$$q < \frac{1 - \beta\theta(1-\delta^N)}{1 + \beta\theta\delta^N - \beta\theta(1-\delta^O)} < 1.$$

To see that $u_N \leq u_O$ in a stationary equilibrium, note that otherwise $u_N(w) > u_O(w)$ for all firms, so there would not be any new investment, which cannot be an equilibrium. Further, $u_N \leq u_O$ is equivalent to

$$q \geq q^{FB} \equiv \frac{1 - \beta(1-\delta^N)}{1 + \beta\delta^N - \beta(1-\delta^O)},$$

that is, the price of old capital in competitive equilibrium must be weakly higher than the price of old capital in a frictionless economy.

Let $R_O \equiv \frac{(1-\theta)((1-\delta^N(1-q)) - q(1-\delta^O))}{\wp^N - \wp^O}$. Since $q \geq q^{FB}$, $R^O \geq \beta^{-1}$ (with equality if $q = q^{FB}$). For firms that are indifferent between investing in new and old capital we can write (A3) as

$$1 = \beta \left(\frac{1 + (1-\rho)\phi_d'}{1+\phi_d} \right) R_O.$$

For such firms, we can then write the investment Euler equation for new capital (A1) as

$$1 = R_O^{-1} \frac{f_k(k) + (1-\theta)(1-\delta^N(1-q))}{\wp_N},$$

implying that such firms all invest the same amount $\underline{k}$. (If $q = q^{FB}$, then $\underline{k} = k^{FB}$.)

Constrained Efficiency. The planner maximizes the present discounted value of aggregate dividends net of equity issuance costs

$$\sum_{t=0}^{\infty} \beta^t \int \left[\sum_{a=0}^{\infty} \gamma_a \left[(d_{at} - \phi(-d_{at})) \right] + \sum_{a=1}^{\infty} \gamma_{a-1} \rho w_{at} \right] d\pi_0(w_0) \tag{A4}$$

subject to the transition for net worth, the collateral constraint and the market-clearing condition for old capital, with multiplier η_t.

The optimality condition for the price of old capital is

$$\int \sum_{a=0}^{\infty} \gamma_a k_{at}^O (1 + \phi_{d,at}) d\pi_0(w_0) = \int \sum_{a=0}^{\infty} \gamma_a \left[\delta^N k_{a,t-1}^N + (1-\delta^O) k_{a,t-1}^O \right] (1 + (1-\rho)\phi_{d,a+1,t} + \theta \lambda_{a,t-1}) d\pi_0(w_0). \tag{A5}$$

The summation on the left-hand side of equation (A5) represents the marginal cost of increasing the price q_t for firms that purchase old capital. The summation on the right-hand side represents the marginal benefit of increasing net worth for firms that own old capital, as well as the marginal effect of q_t on the borrowing capacity of constrained firms at $t-1$.

We now prove that in the stationary competitive equilibrium the distributive externality is larger than the collateral externality, that is,

$$\int \sum_{a=0}^{\infty} \gamma_a k_{at}^O (1 + \phi_{d,at}) d\pi_0(w_0) > \int \sum_{a=0}^{\infty} \gamma_a \left[\delta^N k_{a,t-1}^N + (1-\delta^O) k_{a,t-1}^O \right] (1 + (1-\rho)\phi_{d,a+1,t} + \theta \lambda_{a,t-1}) d\pi_0(w_0),$$

or, written recursively,

$$\int k^O(w)(1+\phi_d(w))d\pi(w) > \int \left[\delta^N k^N(w) + (1-\delta^O)k^O(w)\right](1+(1-\rho)\phi_d(w') + \theta\lambda(w))d\pi(w)$$

where w' denotes future net worth associated with current net worth w. Simplifying using the market-clearing condition, we have

$$\int k^O(w)\phi_d(w)d\pi(w) > \int \left[\delta^N k^N(w) + (1-\delta^O)k^O(w)\right]((1-\rho)\phi_d(w') + \theta\lambda(w))d\pi(w).$$

Using the first-order condition for debt to substitute out $\lambda(w)$, we obtain

$$\int k^O(w)\phi_d(w)d\pi(w) > \int \left[\delta^N k^N(w) + (1-\delta^O)k^O(w)\right](\theta\phi_d(w) + (1-\theta)(1-\rho)\phi_d(w'))d\pi(w). \quad \text{(A6)}$$

Notice that ϕ_d is weakly decreasing in net worth. Moreover, $\phi_d(w) \geq \theta\phi_d(w) + (1-\theta)(1-\rho)\phi_d(w')$. Hence, if inequality (A6) holds (weakly) for $\theta = 1$, it holds (strictly) for any $\theta < 1$. Accordingly, we now prove the following inequality:

$$\int k^O(w)\phi_d(w)d\pi(w) \geq \int \left[\delta^N k^N(w) + (1-\delta^O)k^O(w)\right]\phi_d(w)d\pi(w),$$

which, rearranging, we can equivalently express as follows

$$\delta^O \int k^O(w)\phi_d(w)d\pi(w) \geq \delta^N \int k^N(w)\phi_d(w)d\pi(w). \quad \text{(A7)}$$

As no firm invests in old capital above $\overline{w}_O$, market clearing implies:

$$\delta^O \int^{\overline{w}_O} k^O(w)d\pi(w) \geq \delta^N \int_{\underline{w}_N}^{\overline{w}} k^N(w)d\pi(w). \quad \text{(A8)}$$

Furthermore, we can bound the two sides of (A7) as follows. Since $\phi_d(w)$ is weakly decreasing in w,

$$\int^{\underline{w}_N} k^O(w)\phi_d(w)d\pi(w) \geq \phi_d(\underline{w}_N)\int^{\underline{w}_N} k^O(w)d\pi(w). \quad \text{(A9)}$$

In the region of indifference between new and old capital, we apply the following result: $\mathbb{E}[k^O\phi_d] = \mathbb{COV}(k^O,\phi_d) + \mathbb{E}[k^O]\mathbb{E}[\phi_d]$. Since k^O and ϕ_d are both decreasing in w, we have

$\mathbb{COV}(k^O, \phi_d) \geq 0$.[29] Thus,

$$\int_{\underline{w}_N}^{\overline{w}^O} k^O(w)\phi_d(w)d\pi(w) \geq \overline{\phi}_d \int_{\underline{w}_N}^{\overline{w}^O} k^O(w)d\pi(w) \tag{A10}$$

where $\overline{\phi}_d \equiv \int_{\underline{w}_N}^{\overline{w}^O} \phi_d(w)d\pi(w)$. Since $\phi_d(\underline{w}_N) \geq \overline{\phi}_d$, we can combine (A9) and (A10) and get

$$\int^{\overline{w}^O} k^O(w)\phi_d(w)d\pi(w) \geq \overline{\phi}_d \int^{\overline{w}^O} k^O(w)d\pi(w). \tag{A11}$$

Analogously, since k^N is increasing in w, we obtain

$$\int_{\underline{w}_N}^{\overline{w}} k^N(w)\phi_d(w)d\pi(w) \leq \overline{\phi}_d \int_{\underline{w}_N}^{\overline{w}} k^N(w)d\pi(w). \tag{A12}$$

Notice that our characterization of the stationary equilibrium implies $\int_{\overline{w}^O} k^O(w)\phi_d(w)d\pi(w) = \int_{\overline{w}} k^N(w)\phi_d(w)d\pi(w) = 0$. Hence, combining (A8), (A11), and (A12), we get

$$\delta^O \int k^O(w)\phi_d(w)d\pi(w) \geq \delta^N \int k^N(w)\phi_d(w)d\pi(w),$$

which, given $\theta < 1$, proves Proposition 5.

[29]See Schmidt (2003) for a proof of this result.

Table A1: Quantitative Results – Sensitivity Analysis

This table provides the sensitivity analysis of the quantitative results with respect the collateralizability θ (Panel A), elasticity of substitution ϵ (Panel B), and scrap value $\underline{q}$ (Panel C). Output, investment, consumption, and the price of used capital for the competitive equilibrium and constrained efficient allocation are expressed as fractions of the corresponding first-best value, reported in parenthesis the first column of Panel A.

Panel A: Collateralizability θ

		$\theta = 0$		$\theta = 0.75$	
Variable	First Best	Comp. Eqm.	Constr. Eff.	Comp. Eqm.	Constr. Eff.
Output	(9.910)	0.808	0.949	0.949	0.985
Investment	(4.497)	0.736	0.929	0.925	0.978
Consumption	(5.413)	0.865	0.966	0.968	0.991
Price q	(0.547)	1.023	0.181	1.004	0.184
Average tax τ^N	0	0	-8.8%	0	-8.6%
Average tax τ^O	0	0	106.9%	0	102.9%

Panel B: Elasticity of Substitution ϵ

	$\epsilon = 1$		$\epsilon = 10$	
Variable	Comp. Eqm.	Constr. Eff.	Comp. Eqm.	Constr. Eff.
Output	0.894	0.944	0.905	0.985
Investment	0.850	0.919	0.864	0.978
Consumption	0.929	0.964	0.937	0.990
Price q	1.011	0.184	1.010	0.184
Average tax τ^N	0	-8.6%	0	-8.6%
Average tax τ^O	0	103.7%	0	103.3%

Panel C: Scrap Value $\underline{q}$

	$\underline{q} = 0.05$		$\underline{q} = 0.2$	
Variable	Comp. Eqm.	Constr. Eff.	Comp. Eqm.	Constr. Eff.
Output	0.899	0.979	0.899	0.959
Investment	0.857	0.969	0.857	0.942
Consumption	0.933	0.986	0.933	0.974
Price q	1.010	0.092	1.010	0.367
Average tax τ^N	0	-9.6%	0	-6.7%
Average tax τ^O	0	229.9%	0	40.5%

ONLINE APPENDIX – Not for Publication

OA.1 Constrained-Efficient Allocation: No Taxes on Old Capital

In this section, we discuss the problem introduced at the end of Section 3.5. Relative to the problem described in Appendix A.1, we impose the following additional constraint for all firms:

$$q_t(1+\phi_{d,t}) \geq \beta f_k(k_t),$$

which is equivalent to imposing that the planner cannot tax old capital in the implementation with taxes on new and old capital.

The Lagrangian of this problem is

$$\begin{aligned}\mathcal{L} \equiv \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \int \left(d_{0t} - \phi(-d_{0t}) + d_{1t}\right) d\pi - \int \gamma_{0t} \left(d_{0t} - w - b_t + k_t^N + q_t k_t^O\right) d\pi \\ & - \int \gamma_{1t} \left(d_{1t} - f(k_{t-1}^N + k_{t-1}^O) - q_t k_{t-1}^N + \beta^{-1} b_{t-1}\right) d\pi + \int \lambda_t \left(\beta\theta q_{t+1} k_t^N - b_t\right) d\pi \\ + & \int \psi_t \left(q_t(1+\phi_{d,t}) - \beta f_k(k_t^N + k_t^O)\right) d\pi + \int \underline{\nu}_t^N k_t^N d\pi + \int \underline{\nu}_t^O k_t^O d\pi - \eta_t \left(\int k_t^O d\pi - \int k_{t-1}^N d\pi\right) \Bigg\}.\end{aligned}$$

The first-order conditions with respect to new capital, old capital, and debt are

$$1+\phi_{d,t} = \beta\left[f_k(k_t) + q_{t+1}\right] + \beta\theta\lambda_t q_{t+1} + \beta\eta_{t+1} + \psi_t\left[q_t\phi_{dd,t} - \beta f_{kk}(k_t)\right] + \underline{\nu}_t^N \quad \text{(OA1)}$$

$$q_t(1+\phi_{d,t}) = \beta f_k(k_t) - \eta_t + \psi_t\left[q_t^2\phi_{dd,t} - \beta f_{kk}(k_t)\right] + \underline{\nu}_t^O \quad \text{(OA2)}$$

$$1+\phi_{d,t} = 1 + \lambda_t + \psi_t q_t \phi_{dd,t}, \quad \text{(OA3)}$$

and the first-order condition with respect to the price of old capital is

$$\int k_t^O(1+\phi_{d,t})d\pi = \int k_{t-1}^N(1+\theta\lambda_{t-1})d\pi + \int \psi_t(1+\phi_{d,t} + q_t\phi_{dd,t}k_t^O)d\pi. \quad \text{(OA4)}$$

The complementary slackness condition is

$$\psi_t\left[q_t(1+\phi_{d,t}) - \beta f_k(k_t)\right] = 0,$$

and thus for firms that buy a positive amount of old capital, (OA2) implies that

$$\psi_t = \frac{\eta_t}{q_t^2\phi_{dd,t} - \beta f_{kk}(k_t)}.$$

Consider firms that purchase a strictly positive amount of new capital; for such firms, equation (OA1) in stationary equilibrium is

$$1 + \phi_d = \beta \left[f_k(k) + q \right] + \beta\theta\lambda q + \beta\eta + \psi \left[q\phi_{dd} - \beta f_{kk}(k) \right].$$

Compare this optimality condition of the planner to the analogous competitive-equilibrium condition in the presence of taxes on new capital for such firms:

$$(1 + \phi_d)(1 + \tau^N) = \beta \left[f_k(k) + q \right] + \beta\theta\lambda q.$$

Because $\eta \geq 0$ and $\psi \geq 0$, we have $\tau^N \leq 0$. We assume that the planner can also use a tax on debt to induce its desired value for the multiplier λ in competitive equilibrium, consistent with equation (OA3). With a tax on borrowing τ^B, the first-order condition for debt is $(1 + \phi_d)/(1 + \tau^B) = 1 + \lambda$, which by comparing to (OA3) implies that $\tau^B \geq 0$.[30]

We solve for this allocation in our numerical example with parameter values as in the example in Figure 1 and display the policy functions in Figure OA1. We find that the average subsidy on new capital is approximately 1%. The price of old capital is approximately 2% lower than in competitive equilibrium: the planner is severely constrained in reducing the price of old capital, because a low price induces unconstrained firms to also buy old capital, given that these purchases cannot be taxed. This is reflected in the last term on the right-hand-side of equation (OA4), which must be positive, implying that the planner must tolerate a distributive externality that is larger than the collateral externality, and thus cannot achieve the first-best level of welfare.

OA.2 Risk-Averse Entrepreneurs

In this section, we analyze the case from Section 3.6 in which each firm is owned by a risk-averse entrepreneur whose consumption at each date equals the dividend paid by the firm.

Competitive Equilibrium with Financial Frictions. Given their initial net worth w and the price of old capital q_t, entrepreneurs maximize their utility by choosing consumption c_{0t} and $c_{1,t+1}$, new and old capital k_t^N and k_t^O,and borrowing b_t, to solve[31]

$$\max_{\{c_{0t}, c_{1,t+1}, k_t^N, k_t^O, b_t\} \in \mathbb{R}_+^4 \times \mathbb{R}} u\left(c_{0t}\right) + \beta u\left(c_{1,t+1}\right) \tag{OA5}$$

[30]Numerically, in our example this assumption is immaterial for the sign of τ^N, and we obtain a similar subsidy if we impose the absence of taxes on debt.

[31]Because we now interpret dividends as consumption, we require that dividends are non-negative. We could alternatively allow for negative dividends, in which case this model becomes a generalization of our stylized model, which can be obtained as the special case $u(d) \equiv d - \phi(d)$.

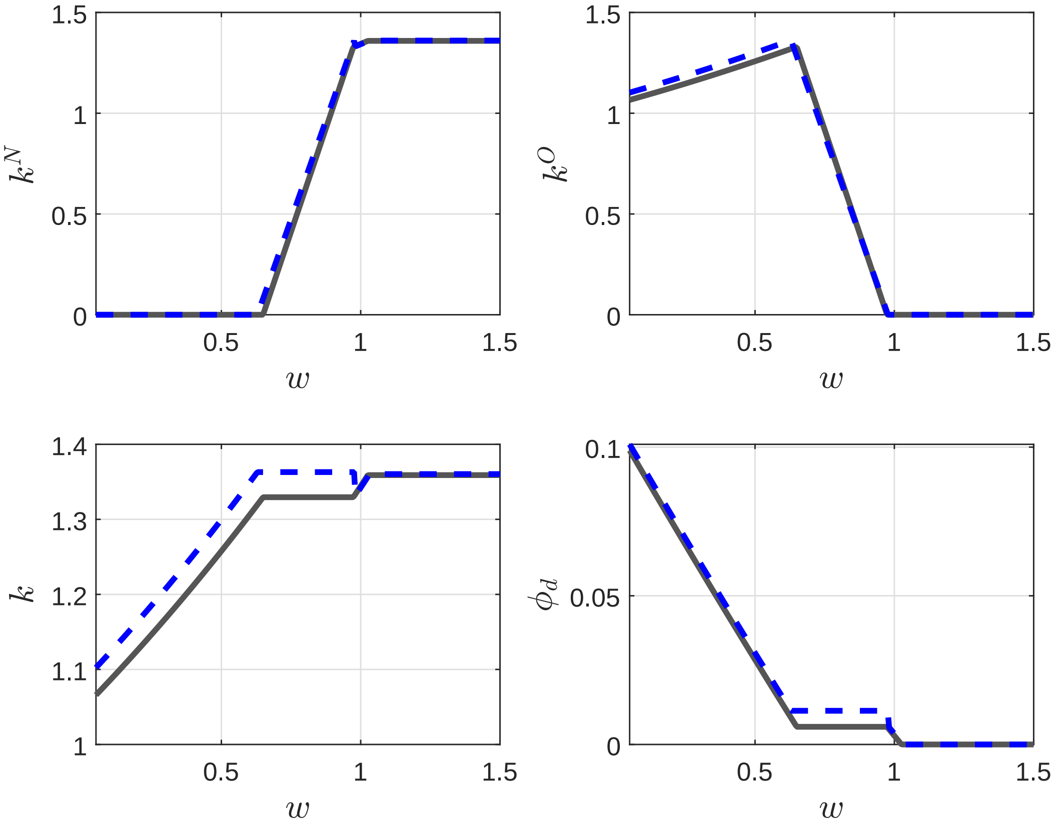

Figure OA1: Stationary competitive equilibrium and constrained-efficient allocation (new-capital taxes only) – example. Top left: new capital k^N; top right: old capital k^O; bottom left: total capital k; bottom right: marginal cost of equity issuance ϕ_d. The x-axes report net worth w. Solid lines denote the competitive-equilibrium allocation, dashed lines the constrained-efficient allocation. See the caption of Figure 1 for the parameter values.

where u is the utility function, with $u_c > 0$, $u_{cc} < 0$, and $\lim_{c\to 0} u_c(c) = +\infty$, subject to the budget constraints for the current and next period,

$$w_{0t} + b_t = c_{0t} + k_t^N + q_t k_t^O \tag{OA6}$$

$$f(k_t^N + k_t^O) + q_{t+1} k_t^N = c_{1,t+1} + \beta^{-1} b_t, \tag{OA7}$$

and the collateral constraint

$$\theta q_{t+1} k_t^N \geq \beta^{-1} b_t. \tag{OA8}$$

Denote the multipliers on the budget constraints by μ_{0t} and $\beta\mu_{1,t+1}$, on the collateral constraint by $\beta\lambda_t$, and on non-negativity constraint for new and old capital by $\underline{\nu}_t^N$ and $\underline{\nu}_t^O$, respectively. The optimal demand for new capital, old capital, and borrowing, as functions of initial net worth w, satisfy the following first-order conditions

$$
\begin{aligned}
u_c(c_{0t}) &= \beta u_c(c_{1,t+1})\left[f_k(k_t)+q_{t+1}\right]+\beta\theta\lambda_t q_{t+1}+\underline{\nu}_t^N && \text{(OA9)}\\
q_t u_c(c_{0t}) &= \beta u_c(c_{1,t+1}) f_k(k_t)+\underline{\nu}_t^O && \text{(OA10)}\\
u_c(c_{0t}) &= u_c(c_{1,t+1})+\lambda_t, && \text{(OA11)}
\end{aligned}
$$

where $k_t = k_t^N + k_t^O$. Moreover, the firm's marginal value of net worth at date t is $\mu_{0,t} = u_c(c_{0t})$.

A *stationary competitive equilibrium* is a set of policy functions mapping initial net worth to an allocation $\{c_0(w), c_1(w), k^N(w), k^O(w), b(w)\}$, that is, consumption, investment, and debt choices, and a price of old capital q, such that entrepreneurs maximize their utility, $\forall w \in \mathcal{W}$, and the market for old capital clears, that is, $\int k^N(w)d\pi(w) = \int k^O(w)d\pi(w)$.

In a stationary equilibrium, the first-order conditions for new and old capital (OA9) and (OA10) can be written as investment Euler equations

$$
\begin{aligned}
1 &\geq \beta\frac{u_c(c_1)}{u_c(c_0)}\frac{\left[f_k(k)+(1-\theta)q\right]}{\wp_N} && \text{(OA12)}\\
1 &\geq \beta\frac{u_c(c_1)}{u_c(c_0)}\frac{f_k(k)}{q}, && \text{(OA13)}
\end{aligned}
$$

with equality if $k^N > 0$ and $k^O > 0$, respectively, where we use the same definition of the down payments as in Section 3.

Using (OA11), we can rewrite (OA12) and (OA13) as

$$
\begin{aligned}
u_N(w) \equiv u_N + \frac{\lambda}{u_c(c_1)}\wp_N = 1-\beta q+\frac{\lambda}{u_c(c_1)}(1-\beta\theta q) &\geq \beta f_k(k)\\
u_O(w) \equiv u_O + \frac{\lambda}{u_c(c_1)}\wp_O = q+\frac{\lambda}{u_c(c_1)}q &\geq \beta f_k(k),
\end{aligned}
$$

where we use the same definitions of the user cost as in Section 3.

Combining (OA12) and (OA13) we moreover have

$$
1 = \beta\frac{u_c(c_1)}{u_c(c_0)}\frac{(1-\theta)q}{\wp_N-\wp_O}+\frac{(\underline{\nu}_N-\underline{\nu}_O)/u_c(c_0)}{\wp_N-\wp_O}. \tag{OA14}
$$

Equation (OA14) implies $\wp_N > \wp_O$. Thus, in equilibrium, $u_N \leq u_O$. Consider first entrepreneurs for which $\lambda = 0$. They invest $\overline{k}$ which solves $1 = \beta\frac{f_k(\overline{k})+(1-\theta)q}{\wp_N}$. Moreover $k^N = \overline{k}$ and $k^O = 0$ if $q > q^{FB}$. Entrepreneurs with $\lambda = 0$ have net worth $w \geq \overline{w}$, where $\overline{w}$ solves $\overline{w} - \wp_N\overline{k} = f(\overline{k}) + (1-\theta)q\overline{k}$.

Entrepreneurs with sufficiently low w strictly prefer old capital, because as $w \to 0$, $f_k(k) \to +\infty$ and therefore $\frac{u_c(c_1)}{u_c(c_0)} \to 0$, and thus equation (OA14) implies $\underline{\nu}_N > 0$. Hence, for sufficiently low w, $k^N = 0$ and $k^O > 0$. Moreover, k^O is strictly increasing in w. To see this, consider $w_+ > w$ and assume $k^O_+ \leq k^O$. Then, $c_{1,+} = f(k^O_+) \leq f(k^O) = c_1$ and $f_k(k^O_+) \geq f_k(k^O)$, whereas $\frac{u_c(c_{1,+})}{u_c(c_{0,+})} > \frac{u_c(c_{1,+})}{u_c(c_0)} \geq \frac{u_c(c_1)}{u_c(c_0)}$, which contradicts equation (OA13).

For w sufficiently close to $\overline{w}$ and $w < \overline{w}$, $k^N > 0$ and $k^O = 0$. Hence (OA12) holds with equality. Moreover, k^N is strictly increasing in w. To see this, consider $w_+ > w$ and assume $k^N_+ \leq k^O$. Then, $f_k(k^N_+) \geq f_k(k^N)$, whereas $c_{1,+} = f(k^N_+) + q(1-\theta)k^N_+ \leq f(k^N) + q(1-\theta)k^N = c_1$ and hence $\frac{u_c(c_{1,+})}{u_c(c_{0,+})} > \frac{u_c(c_{1,+})}{u_c(c_0)} \geq \frac{u_c(c_1)}{u_c(c_0)}$, which contradicts equation (OA12).

Consider now entrepreneurs for which $\underline{\nu}_N = \underline{\nu}_O = 0$. Then, $1 = \beta \frac{u_c(c_1)}{u_c(c_0)} R_O$, where $R_O = \frac{(1-\theta)q}{\wp_N - \wp_O}$. Since c_0 is strictly increasing in w, c_1 is strictly increasing in w. Moreover $1 = R_O^{-1} \frac{[f_k(k)+(1-\theta)q]}{\wp_N} = R_O^{-1} \frac{f_k(k)}{q}$. Hence, $k = \underline{k} \leq \overline{k}$. Since $c_1 = f(\underline{k}) + (1-\theta)qk^N$, k^N is strictly increasing and $k^O = \underline{k} - k^N$ is strictly decreasing in w.

Entrepreneurs who are indifferent between new and old capital have net worth $\underline{w}_N \leq w \leq \overline{w}_O \leq \overline{w}$ and these thresholds are implicitly characterized as follows: $c_0(\underline{w}_N) = \underline{w}_N - q\underline{k}$, $c_1(\underline{w}_N) = f(\underline{k})$, $1 = \beta R_O \frac{u_c(c_1(\underline{w}_N))}{u_c(c_0(\underline{w}_N))}$; $c_0(\overline{w}_O) = \underline{w}_N - \wp_N \underline{k}$, $c_1(\overline{w}_O) = f(\underline{k}) + (1-\theta)q\underline{k}$, $1 = \beta R_O \frac{u_c(c_1(\overline{w}_O))}{u_c(c_0(\overline{w}_O))}$.

Constrained Efficiency. Given an initial distribution of new and old capital, $k^N_{-1}(w)$ and $k^O_{-1}(w)$, a utilitarian planner maximizes the total present discounted value of utility

$$\int \left[u\left(c_{10}(w)\right) + \sum_{t=0}^{\infty} \beta^t \left(u\left(c_{0t}(w)\right) + \beta u\left(c_{1,t+1}(w)\right)\right)\right] d\pi(w),$$

subject to the budget constraints (OA6) and (OA7) with multipliers $\beta^t \mu_{0,t}$ and $\beta^{t+1}\mu_{1,t+1}$, the collateral constraint (OA8) with multiplier $\beta^{t+1}\lambda_t$, the non-negativity constraints on new and old capital with multipliers $\beta^t \underline{\nu}^N_t$ and $\beta^t \underline{\nu}^O_t$, and the market clearing condition for old capital (3) with multiplier $\beta^t \eta_t$.

The first-order condition with respect to the price of old capital q_t for $t = 1, 2, ..$ is

$$\int k^O_t(w) u_c\left(c_{0t}(w)\right) d\pi(w) = \int k^N_{t-1}(w) \left[u_c\left(c_{1t}(w)\right) + \theta \lambda_{t-1}(w)\right] d\pi(w).$$

Thus, in the stationary constrained-efficient allocation, we have

$$\int k^O(w) u_c\left(c_0(w)\right) d\pi(w) = \int k^N(w) \left[u_c\left(c_1(w)\right) + \theta \lambda(w)\right] d\pi(w).$$

We now show that, in stationary competitive equilibrium, the distributive externality is larger than the collateral externality, that is,

$$\int k^O(w) u_c\left(c_0(w)\right) d\pi(w) > \int k^N(w)\left[u_c\left(c_1(w)\right) + \theta\lambda(w)\right] d\pi(w).$$

To do so, it is sufficient to prove that

$$\int k^O(w) u_c\left(c_0(w)\right) d\pi(w) > \int k^N(w) u_c\left(c_0(w)\right) d\pi(w), \tag{OA15}$$

because $u_c\left(c_0(w)\right) = u_c\left(c_1(w)\right) + \lambda(w) \geq u_c\left(c_1(w)\right) + \theta\lambda(w)$.

We can bound the two sides of (OA15) as follows. Let $\overline{u}_c \equiv \int_{\underline{w}_N}^{\overline{w}_O} u_c\left(c_0(w)\right) d\pi(w)$. We have

$$\int k^O u_c(c_0) d\pi = \int^{\overline{w}_O} k^O u_c(c_0) d\pi \geq \overline{u}_c \int^{\overline{w}_O} k^O d\pi, \tag{OA16}$$

because (i) $u_c(c_0) > \overline{u}_c$ for $w < \underline{w}_N$, and (ii) both k^O and $u_c(c_0)$ are strictly decreasing in w for $\underline{w}_N \leq w \leq \overline{w}_O$, thus their covariance is positive, implying $\int_{\underline{w}_N}^{\overline{w}_O} k^O u_c(c_0) d\pi \geq \overline{u}_c \int_{\underline{w}_N}^{\overline{w}_O} k^O d\pi$.[32] Similarly, we have

$$\int k^N u_c(c_0) d\pi = \int_{\underline{w}_N} k^N u_c(c_0) d\pi \leq \overline{u}_c \int_{\underline{w}_N} k^N d\pi, \tag{OA17}$$

because (i) $u_c(c_0) < \overline{u}_c$ for $w > \overline{w}$, and (ii) k^N is strictly increasing in w for $\underline{w}_N \leq w \leq \overline{w}_O$, implying its covariance with $u_c(c_0)$ is negative, and thus $\int_{\underline{w}_N}^{\overline{w}_O} k^N u_c(c_0) d\pi \leq \overline{u}_c \int_{\underline{w}_N}^{\overline{w}_O} k^N d\pi$. Furthermore, notice that at least one of the two inequalities (OA16) and (OA17) is strict, because the distribution of net worth $\pi(w)$ is non-degenerate. Thus, combining (OA16), (OA17), and the market-clearing condition $\int_{\underline{w}_N} k^N d\pi = \int^{\overline{w}_O} k^O d\pi$, we get (OA15). This proves Proposition 3.

We now discuss the case $q = q^{FB}$, which is not included in Proposition 3. We have $\underline{w}_N = \overline{w}_O = \overline{w}$. In this case, the individual choice of new and old capital for unconstrained firms is indeterminate. However, in the aggregate, market clearing implies $\int_{\overline{w}} k^N d\pi(w) > \int_{\overline{w}} k^O d\pi(w)$. Under a weak condition, our result on the sign of constrained inefficiency still obtains. Assume that $k^N > k^O$ for all unconstrained firms.

Then, $u_c(c_0)$ is strictly decreasing in w, we have

$$\int k^O u_c(c_0) d\pi - \int k^N u_c(c_0) d\pi > \int^{\overline{w}} k^O u_c(c_0) d\pi - u_c\left(c_0(\overline{w})\right) \int_{\overline{w}} (k^N - k^O) d\pi \tag{OA18}$$

[32] See Schmidt (2003) for a proof of the sign of the covariance of monotone functions.

Using the market-clearing condition, we can rewrite (OA18) as follows

$$\int k^O u_c(c_0) d\pi - \int k^N u_c(c_0) d\pi > \int^{\overline{w}} k^O u_c(c_0) d\pi - u_c\left(c_0(\overline{w})\right) \int^{\overline{w}} k^O d\pi \qquad \text{(OA19)}$$

Because $u_c(c_0) > u_c\left(c_0(\overline{w})\right)$ for $w \leq \overline{w}$, the right-hand side of (OA19) is positive, and thus also the left-hand side is positive. Hence, (OA15) holds.

OA.3 Heterogeneity in Productivity

In this section, we analyze the model with productivity heterogeneity from Section 3.7 in more detail.

Competitive Equilibrium with Financial Frictions. A firm that draws initial net worth w and productivity s maximizes (6) subject to the budget constraints (7) and

$$sf(k_t^N + k_t^O) + q_{t+1} k_t^N = d_{1,t+1} + \beta^{-1} b_t, \qquad \text{(OA20)}$$

and the collateral constraint (9). Let $v(w,s)$ denote the value function of the firm.

Denote the multipliers on the budget constraints by μ_{0t} and $\beta\mu_{1,t+1}$, on the collateral constraint by $\beta\lambda_t$, and on non-negativity constraint for new and old capital by $\underline{\nu}_t^N$ and $\underline{\nu}_t^O$, respectively. The optimality conditions are

$$\begin{aligned} 1 + \phi_{d,t} &= \beta\left[sf_k(k_t) + q_{t+1}\right] + \beta\theta\lambda_t q_{t+1} + \underline{\nu}_t^N && \text{(OA21)} \\ q_t(1 + \phi_{d,t}) &= \beta s f_k(k_t) + \underline{\nu}_t^O, && \text{(OA22)} \end{aligned}$$

and (12), where $k_t = k_t^N + k_t^O$.

A *stationary competitive equilibrium* is a set of policy functions mapping initial net worth and productivity to an allocation, that is, dividends, investment, and borrowing choices, $\{d_0(w,s), d_1(w,s), k^N(w,s), k^O(w,s), b(w,s)\}$, and a price of old capital q, such that firms maximize the present discounted value of dividends net of equity issuance cost, $\forall (w,s) \in \mathcal{W} \times \mathcal{S}$, and the market for old capital clears, that is, $\int k^N(w,s) d\pi(w,s) = \int k^O(w,s) d\pi(w,s)$.

In a stationary equilibrium, we can rewrite equations (OA21) and (OA22) as

$$\begin{aligned} \wp_N(1 + \phi_d) &= \beta\left[sf_k(k) + (1-\theta)q\right] + \underline{\nu}^N && \text{(OA23)} \\ q(1 + \phi_d) &= \beta s f_k(k) + \underline{\nu}^O && \text{(OA24)} \end{aligned}$$

where $\wp_N = 1 - \beta\theta q$. Following the same arguments we develop in Section 3.3, one can show that $q \geq q^{FB}$. Moreover, for each value of s, there are thresholds $\underline{w}_N(s) \leq \overline{w}_O(s) \leq \overline{w}(s)$ (with strict inequalities if $q > q^{FB}$) such that: firms with $w \leq \underline{w}_N(s)$ invest only in old capital; firms with $w \in (\underline{w}_N(s), \overline{w}_O(s))$ invest $\underline{k}(s)$ and invest in both new and old capital, and firms with $w \geq \overline{w}_O(s)$ invest only in new capital; firms with $w \geq \overline{w}(s)$ pay non-negative dividends and invest $\overline{k}(s) \geq k^{FB}(s) \geq \underline{k}(s)$.

We now show that the marginal equity issuance cost $\phi_d(w,s)$ (or equivalently the marginal value of net worth $v_w(w,s) = 1 + \phi_d(w,s)$) is weakly increasing in s, that is, higher productivity firms are more financially constrained, for a given level of net worth. First, consider firms that pay positive dividends. For these firms, $\phi_d(w,s) = 0$.

Now consider firms with $\underline{\nu}^N > 0$ and $\underline{\nu}^O = 0$; for such firms rewrite equation (OA23) as

$$1 + \phi_d(qk^O - w) = \beta \frac{sf_k(k^O)}{q}$$

where we use $d_0 = w - qk^O$; totally differentiating with respect to s, we obtain $\frac{dk^O}{ds} = \frac{\beta f_k(s) q^{-1}}{q\phi_{dd} - \beta s f_{kk}(k) q^{-1}} > 0$. Thus, d_0 is decreasing in s, which implies that $\phi_d(w,s)$ (and $v_w(w,s)$) is increasing in s.

Next, consider firms with $\underline{\nu}^N = 0$ and $\underline{\nu}^O = 0$. In this case, combining equations (OA23) and (OA24), we can write $1 + \phi_d = \beta R_O$, where $R_O = \frac{(1-\theta)q}{\wp^N - q}$. Thus, all firms that are indifferent between new and old capital issue the same level of equity ($\underline{d}_0$), and feature a constant marginal issuance cost $\overline{\phi}_d$, independent of productivity s. The total investment of such firms satisfies

$$\wp^N R_O = sf_k(\underline{k}(s)) + (1-\theta)q,$$

which implies that $\underline{k}(s)$ is increasing in s. Hence, also the indifference thresholds $\underline{w}_N(s) = \underline{d}_0 + q\underline{k}(s)$ and $\overline{w}_O(s) = \underline{d}_0 + \wp^N \underline{k}(s)$ are increasing in s.

Finally, for firms with $\underline{\nu}^N = 0$ and $\underline{\nu}^O > 0$, rewrite equation (OA23) as

$$1 + \phi_d(\wp_N k^N - w) = \beta \frac{sf_k(k^N) + (1-\theta)q}{\wp_N}.$$

Totally differentiating with respect to s, we obtain $\frac{dk^N}{ds} = \frac{\beta f_k(s) \wp_N^{-1}}{\wp_N \phi_{dd} - \beta s f_{kk}(k) \wp_N^{-1}} > 0$. Thus, d_0 is decreasing in s, which implies that $\phi_d(w,s)$ is increasing in s. We conclude that $\phi_d(w,s)$ (and $v_w(w,s)$) is weakly increasing in productivity s for all firms.

Constrained Efficiency. The planner maximizes

$$\int \left[d_{10}(w,s) + \sum_{t=0}^{\infty} \beta^t \left(d_{0t}(w,s) - \phi(-d_{0t}(w,s)) + \beta d_{1,t+1}(w,s) \right) \right] d\pi(w,s), \tag{OA25}$$

subject to the budget constraints (7) and (OA20), the collateral constraint (9), and the market-clearing condition for old capital. The first-order condition with respect to q_t is

$$\int k_t^O(w,s)\left(1+\phi_{d,t}(w,s)\right)d\pi(w,s) = \int k_{t-1}^N(w,s)\left(1+\theta\lambda_{t-1}(w,s)\right)d\pi(w,s),$$

which, in stationary equilibrium, can be rewritten as follows

$$\int k^O(w,s)\phi_d(w,s)d\pi(w,s) = \theta\int k^N(w,s)\phi_d(w,s)d\pi(w,s), \tag{OA26}$$

where we used the market-clearing condition, as well as the fact that planner optimally sets the marginal equity issuance cost equal to the multiplier on the collateral constraint.

We now show that in stationary competitive equilibrium, the left-hand side of equation (OA26) is larger than the right-hand side, that is, the distributive externality dominates the collateral externality. We can bound the two sides of equation (OA26) as follows. First, notice that the marginal equity issuance cost $\overline{\phi}_d$ is the lower bound for the marginal equity issuance cost of any firms with productivity s purchasing old capital, and the upper bound for the marginal equity issuance cost of any firms with productivity s purchasing new capital. Thus, for any productivity level s, we get

$$\int_w k^O(w,s)\phi_d(w,s)d\pi(w,s) \geq \overline{\phi}_d\int_w k^O(w,s)d\pi(w,s),$$

and

$$\int_w k^N(w,s)\phi_d(w,s)d\pi(w,s) \leq \overline{\phi}_d\int_w k^N(w,s)d\pi(w,s).$$

Next, recall that $\overline{\phi}_d$ is independent of s. Hence, by summing both sides of these two inequalities over productivity levels, we obtain

$$\int k^O(w,s)\phi_d(w,s)d\pi(w,s) \geq \overline{\phi}_d\int k^O(w,s)d\pi(w,s),$$

and

$$\int k^N(w,s)\phi_d(w,s)d\pi(w,s) \leq \overline{\phi}_d\int k^N(w,s)d\pi(w,s).$$

The two bounds reported on the right-hand sides of these inequalities are equal to each other because of market clearing. Thus, $\theta < 1$ implies

$$\int k^O(w,s)\phi_d(w,s)d\pi(w,s) > \theta\int k^N(w,s)\phi_d(w,s)d\pi(w,s),$$

which proves Proposition 4.

OA.4 Additional Optimality Conditions for Quantitative Model

In this section, we provide the optimality conditions for the planner's problem, restricted not to distort investment in old capital, discussed at the end of Section 4.4. We denote by $\psi_t(s^a)$ the multipliers on the old-capital Euler equations, which are now constraints for the planner:

$$q_t(1+\phi_{d,t}(s^a)) = \beta\mathbb{E}_t\left[s_{a+1}f_k(k_t(s^a))g_{O,t}(s^a) + (1-\delta^O)q_{t+1}\right](1+(1-\rho)\phi_{d,t+1}(s^{a+1})) \\ + \beta\theta\lambda_t(s^a)(1-\delta^O)q_{t+1}.$$

In formulating this constraint on the planning problem, we need to substitute a differentiable expression for the Lagrange multiplier on the collateral constraint $\lambda_t(s^a)$. We follow, for instance, Jeanne and Korinek (2019) and use the competitive-equilibrium optimality condition for debt to substitute

$$\lambda_t(s^a) = \phi_{d,t}(s^a) - (1-\rho)\mathbb{E}_t\phi_{d,t+1}(s^{a+1}). \tag{OA27}$$

The planner's optimality condition for new capital is

$$1+\phi_{d,t}(s^a) = \beta\mathbb{E}_t\left[s_{a+1}f_k(k_t(s^a))g_{N,t}(s^a) + (1-\delta^N(1-q_{t+1}))\right](1+(1-\rho)\phi_{d,t+1}(s^{a+1})) \\ + \beta\theta\lambda_t(s^a)(1-\delta^N(1-q_{t+1})) + \beta\delta^N\eta_{t+1} + \psi_t(s^a)\frac{\partial O_t(s^a)}{\partial k_t^N(s^a)}$$

with

$$\frac{\partial O_t(s^a)}{\partial k_t^N(s^a)} = -\beta\mathbb{E}_t s_{a+1}(f_{kk}(k_t(s^a))g_{N,t}(s^a)g_{O,t}(s^a) + f_k(k_t(s^a))g_{NO,t}(s^a))(1+(1-\rho)\phi_{d,t+1}(s^{a+1})) \\ +\phi_{dd,t}(s^a)(q_t - \beta\theta(1-\delta^O)q_{t+1}) + \beta(1-\rho)\mathbb{E}_t\phi_{dd,t+1}(s^{a+1})\left[s_{a+1}f_k(k_t(s^a))g_{N,t}(s^a) + (1-\delta^N(1-q_{t+1}))\right] \\ \times\left[s_{a+1}f_k(k_t(s^a))g_{O,t}(s^a) + (1-\delta^O)(1-\theta)q_{t+1}\right].$$

The optimality condition for old capital is

$$q_t(1+\phi_{d,t}(s^a)) = \beta\mathbb{E}_t\left[s_{a+1}f_k(k_t(s^a))g_{O,t}(s^a) + (1-\delta^O)q_{t+1}\right](1+(1-\rho)\phi_{d,t+1}(s^{a+1})) \\ + \beta\theta(1-\delta^O)\lambda_t(s^a)q_{t+1} - \eta_t + \beta(1-\delta^O)\eta_{t+1} + \psi_t(s^a)\frac{\partial O_t(s^a)}{\partial k_t^O(s^a)}$$

with

$$\frac{\partial O_t(s^a)}{\partial k_t^O(s^a)} = -\beta \mathbb{E}_t s_{a+1}(f_{kk}(k_t(s^a))(g_{O,t}(s^a))^2 + f_k(k_t(s^a))g_{OO,t}(s^a))(1+(1-\rho)\phi_{d,t+1}(s^{a+1}))$$
$$+ \phi_{dd,t}(s^a)q_t(q_t - \beta\theta(1-\delta^O)q_{t+1})$$
$$+ \beta(1-\rho)\mathbb{E}_t\phi_{dd,t+1}(s^{a+1})\left[s_{a+1}f_k(k_t(s^a))g_{O,t}(s^a) + (1-\delta^O)q_{t+1}\right]$$
$$\times \left[s_{a+1}f_k(k_t(s^a))g_{O,t}(s^a) + (1-\delta^O)(1-\theta)q_{t+1}\right].$$

The optimality condition for debt is

$$\phi_{d,t}(s^a) = (1-\rho)\mathbb{E}_t\phi_{d,t+1}(s^{a+1}) + \lambda_t(s^a) - \psi_t(s^a)\frac{\partial O_t(s^a)}{\partial b_t(s^a)}$$

with

$$\frac{\partial O_t(s^a)}{\partial b_t(s^a)} = -\phi_{dd,t}(s^a)(q_t - \beta\theta(1-\delta^O)q_{t+1})$$
$$- (1-\rho)\mathbb{E}_t\phi_{dd,t+1}(s^{a+1})\left[s_{a+1}f_k(k_t(s^a))g_{O,t}(s^a) + (1-\delta^O)(1-\theta)q_{t+1}\right].$$

Notice that this condition implies a difference between the value of this multiplier as perceived by firms (in a competitive equilibrium with taxes, that is, equation (OA27)) and as perceived by the planner. We verify numerically that this deviation is quantitatively unimportant, and we also solve an alternative problem in which the planner does not internalize the effect of debt choices on the Euler equation for old capital (the term $\frac{\partial O_t(s^a)}{\partial b_t(s^a)}$), thus aligning the values of this multiplier for firms and planner, finding similar results.

The constrained-efficient price of old capital satisfies

$$\sum_{a=0}^{\infty}\gamma_a\sum_{s^a}p(s^a)k_t^O(s^a)(1+\phi_{d,t}(s^a)) =$$
$$\sum_{a=0}^{\infty}\gamma_a\sum_{s^{a+1}}p(s^{a+1})\left[\delta^N k_{t-1}^N(s^a) + (1-\delta^O)k_{t-1}^O(s^a)\right](1+(1-\rho)\phi_{d,t}(s^{a+1}) + \theta\lambda_{t-1}(s^a))$$
$$+ \sum_{a=0}^{\infty}\gamma_a\sum_{s^a}p(s^a)(\psi_t(s^a)\frac{\partial O_t(s^a)}{\partial q_t} + \psi_{t-1}(s^a)\beta^{-1}\frac{\partial O_{t-1}(s^a)}{\partial q_t}) + \zeta_t$$

with

$$\frac{\partial O_t(s^a)}{\partial q_t} = 1 + \phi_{d,t}(s^a) + \phi_{dd,t}(s^a)q_t(k_t^O(s^a) - \delta^N k_{t-1}^N(s^{a-1}) - (1-\delta^O)k_{t-1}^O(s^{a-1}))$$

and

$$\frac{\partial O_{t-1}(s^a)}{\partial q_t} = -\beta(1-\delta^O)(1+\theta\phi_{d,t-1}(s^a) + (1-\theta)(1-\rho)\mathbb{E}_{t-1}\phi_{d,t}(s^{a+1}))$$
$$+ \beta(1-\rho)\mathbb{E}_{t-1}\phi_{dd,t}(s^{a+1})(\delta^N k^N_{t-1}(s^a) + (1-\delta^O)k^O_{t-1}(s^a) - k^O_t(s^{a+1}))$$
$$\times \left[s_{a+1} f_k(k_{t-1}(s^a)) g_{O,t-1}(s^a) + (1-\delta^O)(1-\theta)q_t\right].$$

The price of old capital affects the planner's value through its effects on the old-capital Euler equations, which the planner takes as constraints. These additional effects are summarized by the terms $\frac{\partial O_t(s^a)}{\partial q_t}$ and $\frac{\partial O_{t-1}(s^a)}{\partial q_t}$.

OA.5 Solution Method for Quantitative Model

In this section, we discuss the solution method for the quantitative model. We compute the stationary constrained-efficient allocation, in the case in which the planner chooses both new and old capital, using the following iterative procedure:

1. Guess a value for the multiplier on the market clearing condition for old capital η.
   (a) Guess a value for the price of old capital q.
   (b) Solve for the firm policy functions on a grid for net worth w and productivity s, using the optimality conditions (48), (49), and (46) evaluated in stationary equilibrium.
   (c) Obtain the stationary distribution of net worth and productivity by simulating a continuum of firms.
   (d) Check the market-clearing condition (34) and update the guess for the price q accordingly, until convergence.
2. Evaluate the optimality condition for the price of old capital (50) and update the guess for η accordingly, until convergence.

The stationary competitive equilibrium is a special case of steps (a)-(d) with $\eta = 0$.

OA.6 Additional Quantitative Results and Sensitivity

This section provides additional figures related to the quantitative analysis of Sections 5 and 6. Figure OA2 displays the optimal tax rates on new and old capital that implement the constrained-efficient allocation in our calibrated model. Figure OA3 plots the transition dynamics associated with the implementation of the optimal tax rate on new capital (at $t =$

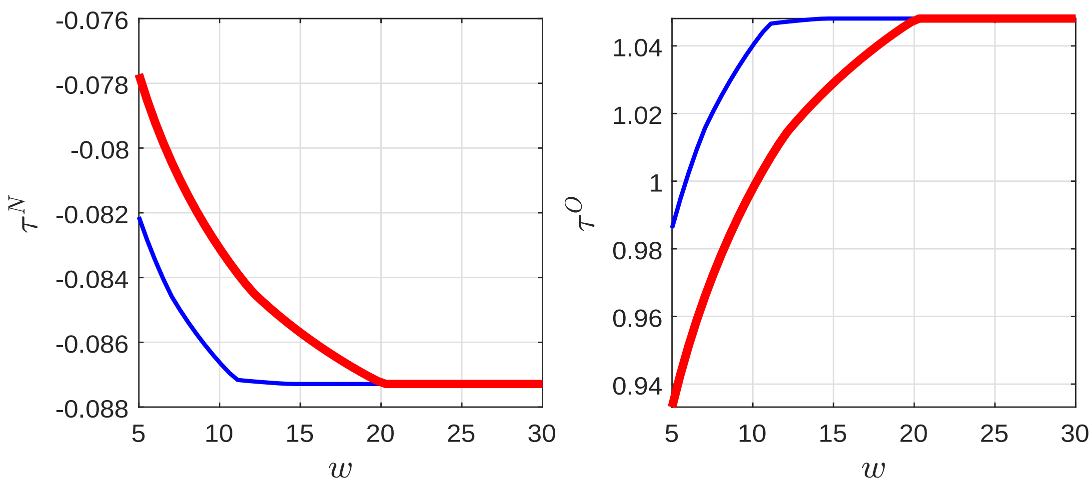

Figure OA2: Optimal tax rates. Left panel: tax rate on new capital (a negative value denotes a subsidy); right panel: tax rate on old capital. The x-axes report net worth w. Thick red lines refer to the high productivity realization; thin blue lines refer to the low realization.

0), common for all firms and constant over time, starting from the competitive equilibrium without policy intervention (at $t = -1$).

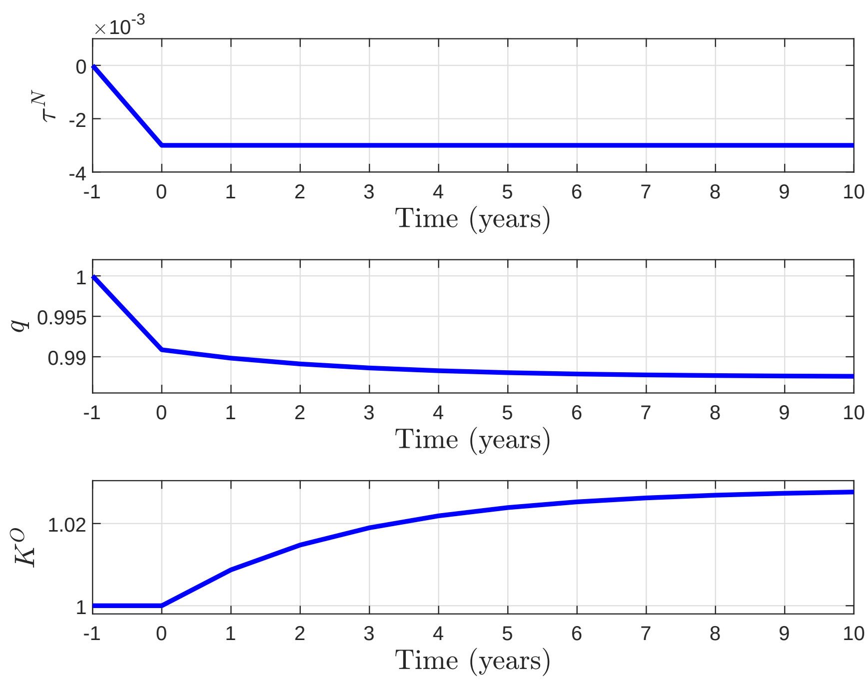

Figure OA3: Equilibrium transition dynamics associated with the optimal constant tax rate τ^N, common for all firms. Top panel: tax rate τ^N; middle panel: price of old capital q_t; bottom panel: aggregate stock of old capital K_t^O.