Risks For the Long Run: Estimation and Inference*

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First Draft: June 2006
Current Draft: September 2007

Abstract

In this paper we empirically evaluate the ability of the long-run risks model to explain asset returns. Exploiting asset pricing Euler equations we develop methods for estimating the long-run risks model, and show that it can successfully account for the market, value, and size sorted returns at reasonable values of risk aversion and the intertemporal elasticity of substitution. Our empirical evidence highlights the importance of low-frequency movements and time-varying uncertainty in economic growth for understanding risk-return tradeoffs in financial markets.

*We thank Andy Abel, George Constantinides, Lars Hansen, John Heaton, Tom Sargent, Martin Schneider, Annette Vissing-Jorgensen, seminar participants at University of California-Berkeley, Copenhagen School of Business, Duke University, Norwegian School of Management, University of Chicago, Wharton and conference participants at the Summer Econometric Society Meetings, the NBER Asset-Pricing meeting, and CIRANO Financial Econometrics Conference in Montreal for useful comments. Yaron thanks support from the Rodney White Center at the Wharton School.

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1 Introduction

Asset market data ought to contain valuable information about investors' behavior (see Cochrane and Hansen (1992)). It should also contain information about the sources of risks that concern investors and hence drive asset markets. Asset market features, such as the low real returns on bills, the equity premium, large returns on value (high book-to-market) stocks relative to growth (low book-to-market) stocks, among others, provide an important market laboratory to simultaneously learn about sources of risks and preferences of investors. A challenging task is to account for these asset market facts with well identified risk sources and plausible investor behavior. Bansal and Yaron (2004) develop a long-run risks (LRR) asset pricing model and show that it can account for the risk free rate, equity premium and volatility puzzles. Further, they suggest that the same long-run risks in consumption should empirically account for a rich cross-section of asset returns with reasonable risk preferences. In this paper, we develop methods and empirically evaluate the ability of the LRR model to account for asset market data using Euler equation based estimation methods.

An elegant approach to evaluate the empirical plausibility of an asset pricing model, developed in Hansen and Singleton (1982), is to exploit its asset pricing Euler equations using the Generalized Method of Moment (GMM) estimation technique. This approach provides a convenient way to impose the model restrictions on asset payoffs and learn about investor behavior. *A priori* it is not entirely clear how to proceed with this estimation as the intertemporal marginal rate of substitution in the LRR model, based on the Krpes and Porteus (1978), Epstein and Zin (1989) and Weil (1989) recursive preferences, incorporates the return on the consumption asset, which is not directly observed by the econometrician. In this paper we present methods for estimating models with these recursive preferences using Euler equations and a GMM estimator.

To make estimation feasible in the LRR model we exploit the dynamics of aggregate consumption growth and the model's Euler restrictions to solve for the unobserved return on the claim over the future consumption steam. The LRR model has three risk sources in the aggregate consumption dynamics: (i) high frequency or short-run risks in consumption, (ii) low frequency or long-run movements in consumption, and (iii) fluctuations in consumption uncertainty, i.e., consumption volatility risk. We derive expressions for the intertemporal marginal rate of substitution (IMRS) in terms of these risk sources for a wide range of risk aversion and intertemporal substitution parameters. We document that our methods for characterizing the model's pricing kernel are very accurate. Earlier work by Epstein and Zin (1991) also pursues the strategy of exploiting the Euler equation-GMM method for estimation; however, they assume that the return on the consumption asset coincides with the observed value-weighted market return. This premise, we show, can distort the estimated preferences and lead to false rejections of the model.

More recently several other papers explore the ability of long-run risks to account for asset market data. Bansal, Dittmar, and Lundblad (2005), Hansen, Heaton, and Li (2005) show that long-run risks in cash flows are an important risk source in accounting for asset returns. Bekaert, Engstrom, and Xing (2005), Bansal, Gallant, and Tauchen (2005), Kiku (2006), Malloy, Moskowitz, and Vissing-Jorgensen (2004), and Lettau and Ludvigson (2005), also explore implication of LRR for asset returns. However, these papers, unlike the focus of this paper, do not evaluate the empirical plausibility of the LRR model from the perspective of the Euler equation-GMM based estimation approach for a rich cross-section of assets.

Exploiting the estimation methods we develop, we find considerable empirical support for the LRR model at plausible preference configurations. Our evidence suggests that the investor concerns about long-run risks and economic uncertainty are empirically important for understanding asset returns. More specifically, our Euler equation-GMM based estimation of the LRR model shows that: (i) the long-run risk component is highly persistent and displays fluctuations that are longer than those associated with business cycles, and is economically and statistically significantly predictable by theoretically motivated variables, (ii) in the cross section, assets with large mean returns (e.g., value and small assets) are more sensitive to innovations in the long-run risk variable and news about economic uncertainty, (iii) the long-run risks component accounts for most of the risk premia, (iv) the model is not rejected by the overidentifying restrictions and can account for considerable portion of the observed risk premia. In the annual data, the estimated risk aversion is in excess of fifteen while estimates for the IES are less than one. However, as discussed below, after accounting for time averaging and finite sample effects, values of risk aversion and IES are closer to 10 and 2, respectively.

Time averaging and finite sample effects play an important role in interpreting our estimates and evidence. Time averaging arises when the decision interval of the agent and the frequency with which an econometrician observes consumption data do not coincide. In the context of the LRR model, if consumption data is observed only on a coarser interval (e.g., annual), while the decision interval of the agent is on a finer interval (e.g., monthly), then the estimates of the IES will be lower than their true values, and typically less than one. This effect is important as much of the earlier work focuses on the region of high risk aversion and low IES (e.g., see Campbell (1999) and Hall (1988) for estimation-based evidence, and Kandel and Stambaugh (1991) for calibration). Finite sample effects seem to lead to estimates of risk aversion that are too large relative to their true values. Our evidence indicates that much of the earlier evidence and the associated views regarding low values of IES and high risk aversion could be an artifact of time averaging and finite sample effects in estimation.

In sum, the evidence in this paper shows that the long-run risks model is quite capable in quantitatively pricing the time series and cross section of returns, and doing so with plausible parameter estimates. These parameter estimates can be quite difficult to precisely estimate using annual data. They tend to produce a somewhat misspecified model that leads to preference parameter estimates that are biased towards what is often found in the literature.

The paper continues as follows: Section 2 presents the model and its testable restrictions. Section 3 presents the data, while Section 4 provides the results of our empirical analysis. Section 5 presents Monte-Carlo evidence regarding time averaging and finite sample effects. Section 6 provides SMM based estimates, and Section 7 discusses the implication of incorrectly using the market return in the pricing kernel. Section 8 provides concluding remarks.

2 Model

In this section we specify the long-run risks model based on Bansal and Yaron (2004). The underlying environment is one with complete markets and a representative agent has Epstein and Zin (1989) type preferences, which allow for a separation of risk aversion and the elasticity of intertemporal substitution. Specifically, the agent maximizes her life-time utility, which is defined recursively as,

$$V_t = \left[(1-\delta) C_t^{\frac{1-\gamma}{\theta}} + \delta \Big(E_t \big[V_{t+1}^{1-\gamma} \big] \Big)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}}, \tag{1}$$

where C_t is consumption at time t, $0 < \delta < 1$ reflects the agent's time preferences, γ is the coefficient of risk aversion, $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$, and ψ is the elasticity of intertemporal substitution (IES). Utility maximization is subject to the budget constraint,

$$W_{t+1} = (W_t - C_t) R_{c,t+1} \,, \tag{2}$$

where W_t is the wealth of the agent, and $R_{c,t}$ is the return on all invested wealth.

Consumption growth has the following dynamics:

$$\begin{aligned} \Delta c_{t+1} &= \mu_c + x_t + \sigma_t \eta_{t+1} \\ x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1} \\ \sigma_{t+1}^2 &= \bar{\sigma}^2 + \nu(\sigma_t^2 - \bar{\sigma}^2) + \sigma_w w_{t+1} \,, \end{aligned} \tag{3}$$

where Δc_{t+1} is the growth rate of log consumption. The conditional expectation of consumption growth is given by $\mu_c + x_t$, where x_t is a small but persistent component that captures long-run risks in consumption growth. The parameter ρ determines the persistence in the conditional mean of consumption growth. For parsimony, as in Bansal and Yaron (2004), we have a common time-varying volatility in consumption, which, as shown in their paper, leads to time-varying risk premia. The unconditional variance of consumption is $\bar{\sigma}^2$ and ν governs the persistence of the volatility process.

It is easily shown that, for any asset j, the first-order condition yields the following asset pricing Euler condition,

$$E_t \left[\exp \left(m_{t+1} + r_{j,t+1} \right) \right] = 1 \,, \tag{4}$$

where m_{t+1} is the log of the intertemporal marginal rate of substitution (IMRS), and $r_{j,t+1}$ is the log of the gross return on asset j.

2.1 The Long-Run Risks Model's IMRS

For these preferences, the log of the IMRS, m_{t+1}, is

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1} \,, \tag{5}$$

where $r_{c,t+1}$ is the continuous return on the consumption asset, which is endogenous to the model. Thus, in order to characterize the intertemporal marginal rate of substitution, one needs to solve for the unobservable return on the consumption claim. To solve for $r_{c,t+1}$, we use the dynamics of the consumption growth and the log-linear approximation for the continuous return, namely,

$$r_{c,t+1} = \kappa_0 + \kappa_1 z_{t+1} + \Delta c_{t+1} - z_t \,, \tag{6}$$

where $z_t = \log(P_t/C_t)$ is log price-consumption ratio (i.e., the valuation ratio corresponding to a claim that pays aggregate consumption), and κ's are constants of log-linearization,

$$\kappa_1 = \frac{\exp(\bar{z})}{1+\exp(\bar{z})} \tag{7}$$

$$\kappa_0 = \log(1+\exp(\bar{z})) - \kappa_1 \bar{z} \,, \tag{8}$$

$\bar{z}$ here denotes the mean of the log price-consumption ratio.

To derive the time series for $r_{c,t+1}$ we require a solution for log price-consumption ratio, which we conjecture follows,

$$z_t = A_0 + A_1 x_t + A_2 \sigma_t^2 \,. \tag{9}$$

The solution coefficients A's depend on all the preference parameters and the parameters that govern the dynamics of consumption growth. For notational ease, let $Y_t' = [1 \quad x_t \quad \sigma_t^2]$ be the vector of the state variables , and $\mathbf{A}' = [A_0 \quad A_1 \quad A_2]$. Then the solution for $z_t = \mathbf{A}' Y_t$, where[1]

$$\mathbf{A}' = \left[A_0 \qquad \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho} \qquad -\frac{(\gamma-1)(1-\frac{1}{\psi})}{2\ (1-\kappa_1\nu)} \left[1 + \left(\frac{\kappa_1 \varphi_e}{1-\kappa_1\rho} \right)^2 \right] \right] . \tag{10}$$

As discussed in Bansal and Yaron (2004) the elasticities of the price-consumption ratio with respect to the expected growth component, x_t, and volatility, σ_t, depend on the preference configuration. In particular, for the elasticity A_1 to be positive, the IES parameter has to be greater than one. Moreover, for the price-consumption ratio to exhibit a negative response to an increase in economic uncertainty, the IES again has to be larger than one given that risk aversion is greater than one.

Note that the derived solutions depend on the approximating constants, κ_0 and κ_1, which,

[1] The expressions for A_0 and Γ_0 in equation (14) below are given in Appendix A.1.

in their turn, depend on the endogenous mean of the price-consumption ratio, $\bar{z}$. In order to solve for $\bar{z}$, we first substitute expressions for κ's (equations (7) and (8)) into the expressions for A's and solve for the mean of the price-consumption ratio. Specifically, $\bar{z}$ can be found numerically by solving a fixed-point problem,

$$\bar{z} = \mathbf{A}(\bar{z})'\bar{Y} \,, \tag{11}$$

where the dependence of A's on $\bar{z}$ is given above, and $\bar{Y}$ is the mean of the state vector Y. This is quite easy to implement in practice. The endogeneity of $\bar{z}$ has been also emphasized in Campbell and Koo (1997).

Given the solution for z_t, the IMRS can be stated in terms of the state variables and innovations,

$$m_{t+1} = \mathbf{\Gamma}' Y_t - \mathbf{\Lambda}' \zeta_{t+1} \,, \tag{12}$$

where the three sources of risks are

$$\zeta'_{t+1} = \begin{bmatrix} \sigma_t \eta_{t+1} & \sigma_t e_{t+1} & \sigma_w w_{t+1} \end{bmatrix} \,, \tag{13}$$

and the three dimensional vectors $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ are given by,

$$\mathbf{\Gamma}' = \begin{bmatrix} \Gamma_0 & -\frac{1}{\psi} & -(\gamma-1)(\gamma-\frac{1}{\psi})\frac{1}{2}\left[1+(\frac{\kappa_1\varphi_e}{1-\kappa_1\rho})^2\right] \end{bmatrix} \,, \tag{14}$$

$$\mathbf{\Lambda}' = \begin{bmatrix} \gamma & (\gamma-\frac{1}{\psi})\frac{\kappa_1\varphi_e}{1-\kappa_1\rho} & -(\gamma-1)(\gamma-\frac{1}{\psi})\frac{\kappa_1}{2\,(1-\kappa_1\nu)}\left[1+(\frac{\kappa_1\varphi_e}{1-\kappa_1\rho})^2\right] \end{bmatrix}. \tag{15}$$

Note that the stochastic discount factor in equation (12) is exact up to an approximation error emanating from the linearization around the theoretical value of average price-consumption ratio. We find that this approximation error is quite small and does not materially affect our empirical results that follow. Appendix A.5 provides a detailed discussion of the magnitude of the approximation error and a comparison of the above log-linear solution with a solution based on numerical methods.

Given the expression for the IMRS in equation (12), it follows that the risk premium on any asset j is,

$$E_t[r_{j,t+1} - r_{f,t} + 0.5\sigma^2_{t,r}] = \sum_{i=\{\eta,e,w\}} \beta_{i,j}\lambda_i\sigma^2_{i,t} \,, \tag{16}$$

where $\beta_{i,j}$ is the beta with respect to the i^{th} risk source in ζ_{t+1} for asset j, and λ_i is the i^{th} entry of the vector of market prices of risks, $\mathbf{\Lambda}$.

2.2 Special Case: IES=1

Intertemporal elasticity of substitution is a critical parameter in the LRR model. Many papers entertain the case, in which IES is set at one (e.g., Giovannini and Weil (1989), Tallarini (2000), Hansen, Heaton, and Li (2005), Hansen and Sargent (2006)). This provides analytical convenience in certain situations. Note that our estimation methodology nests the case of IES=1 in a continuous fashion (details are given in Appendix A.3). Namely, the IMRS components as given in (12) adjust in a continuous way as one takes the limit of the IES parameter to one.[2] That is,

$$\lim_{\psi \to 1} \kappa_1 = \delta \quad \lim_{\psi \to 1} \Gamma' = \Gamma'(\psi = 1, \kappa_1 = \delta) \quad \lim_{\psi \to 1} \Lambda' = \Lambda'(\psi = 1, \kappa_1 = \delta)\,. \tag{17}$$

The discussion above highlights the fact the generalized pricing kernel (12) does not confine an econometrician to a prespecified value of the IES. That is, in estimation the IES is a free parameter.

2.3 Recovering the State Variables

Estimation of the model requires time-series of the latent states, x_t and σ_t^2. As asset prices depend on the state variables, the latter two can be recovered from available financial data. In particular, x_t and σ_t^2 can be extracted from the observed risk-free rate and the price-dividend ratio for the aggregate market portfolio. Similar to the solution for the price-consumption ratio, the risk-free rate and the market price-dividend ratio are linear functions of x_t and σ_t (the exact expressions are provided in Appendix A.2) and, thus, are natural candidates for recovering the states.

More specifically, the long-run risk component, x_t, can be identified by regressing consumption growth on the risk-free rate and the market price-dividend ratio. Subsequently,

[2] Evaluating the pricing kernel (12) under the above restrictions gives exactly the same solution as in Giovannini and Weil (1989), Tallarini (2000) and Hansen, Heaton, and Li (2005).

the volatility component, σ_t^2, can be extracted by projecting the squared consumption residual on the same set of observables. To summarize, one can recover the state variables using the following system,

$$\begin{aligned} \Delta c_{t+1} &= b_x' Y_t + \sigma_t \eta_{t+1} \,, & (18) \\ \sigma_t^2 \equiv E_t[(\Delta c_{t+1} - b_x' Y_{t+1})^2] &= b_\sigma' Y_t \,, & (19) \end{aligned}$$

where $Y_t = (1 \;\; z_{m,t} \;\; r_{f,t})'$. The state variables, $x_t \equiv b_x' Y_t$ and $\sigma_t^2 \equiv b_\sigma' Y_t$. Given specification in equation (3), the long-run and volatility innovations, e_{t+1} and w_{t+1}, can be recovered by fitting independent AR(1) dynamics to the extracted state processes. The short-run consumption risk, η_{t+1}, is identified by the above projection in equation (18). Thus, all the components needed to construct the IMRS in equation (12) are fully recoverable.

3 Data

In this paper, we use data on consumption and asset prices for the time period from 1930 till 2002. We take the view that this sample better represents the overall variation in asset and macro economic data. Importantly, the long span of the data helps in achieving more reliable statistical inference. We work with the data sampled on an annual frequency as they are less prone to measurement errors that arise from seasonalities and other measurement problems highlighted in Wilcox (1992).

In our empirical tests, we employ portfolios with opposite size and book-to-market characteristics that are known to have provided investors with quite different premia over the years. In addition, our asset menu comprises the aggregate stock market portfolio and a proxy of a risk-less asset. The construction of portfolios is standard (see Fama and French (1993)). In particular, for the size sort, we allocate individual firms across 10 portfolios according to their market capitalization at the end of June of each year. Book-to-market deciles are likewise re-sorted at the end of June by ranking all the firms into 10 portfolios using their book-to-market values as of the end of the previous calendar year. NYSE breakpoints are used in both sorts. For each portfolio, including the aggregate market, we construct value-weighted monthly returns as well as per-share price and dividend series as in Campbell and

Shiller (1988), Bansal, Dittmar, and Lundblad (2005), and Hansen, Heaton, and Li (2005). Monthly data are then time-aggregated to an annual frequency and converted to real using the personal consumption deflator. Table I provides descriptive statics for returns, dividend growth rates and price-dividend ratios for the five portfolios of interest — small and large (i.e., firms in the top and bottom market capitalization deciles), growth and value (firms with the lowest and highest book-to-market ratios, respectively), and the aggregate stock market. The first column illustrates the well-known size and value premia. Over the sample period, small stocks have outperformed large firms by about 9%; the spread in returns on value and growth firms has averaged 6.4%. Both high book-to-market and small firms have experienced higher growth rate of dividends and have been much more volatile than their corresponding counterparts. The bottom line of the table reports the mean and the standard deviation of the risk-free rate. The real interest rate is constructed by subtracting the 12-month expected inflation from the annualized yield on the 3-month Treasury bill taken from the CRSP treasury files.

Finally, we take seasonally adjusted per-capita data on real consumption and gross domestic product (GDP) from the NIPA tables available on the Bureau of Economic Analysis website. Aggregate consumption is defined as consumer expenditures on non-durables and services. Growth rates are constructed by taking the first difference of the corresponding log series.

4 Empirical Findings

Our empirical work is based on the annual data described above. As is common practice, we assume that the decision interval of the agent and the sampling interval of the data observed by an econometrician coincide.

4.1 Evidence on Consumption Growth and Uncertainty

As outlined above, we extract the expected consumption growth and the volatility component from the observed time-series of the market price-dividend ratio and the risk-free rate (see equations (18) and (19)). Given the dynamics of x_t^a, $\sigma_t^{2,a}$ and the shocks η_{t+1}^a, e_{t+1}^a and

w^a_{t+1}, where we now use superscript a to explicitly denote the fact that we are dealing with annual data, the pricing kernel in (12) can be computed for any configuration of preference parameters.[3]

Table II provides evidence on the consumption dynamics, the extracted long-run component, x^a_t, and the conditional volatility of consumption, $\sigma^{2,a}_t$. First, note that consumption growth is highly predictable with an adjusted R^2 of 35%. This evidence clearly shows that the data are far from an *i.i.d.* view for consumption growth. The slope coefficients for the long-run piece, x^a_t, are very well estimated. The coefficients for $\sigma^{2,a}_t$ are not significant at conventional levels, most likely due to the difficulty in detecting stochastic volatility components in the presence of time-aggregation (see Drost and Nijman (1993)). Nonetheless, the signs of these coefficients are consistent with the model implications discussed in section 2.1 (e.g., negative coefficient on the price-dividend ratio). The estimate of an AR(1) parameter of the long-run risk process, ρ^a, is 0.78, and ν^a is estimated at 0.81. The correlation between the short-run innovation in consumption growth, η^a_t, and the innovations in the long-run risk variable, e^a_t, is small, of about 0.15 — hence, we treat the correlation between them as zero. The time-series dynamics of the extracted state variables are presented in Figures 1 and 2: Figure 1 plots the conditional mean of consumption growth along with its realized values, Figure 2 shows the estimate of the volatility component. Notice that the conditional volatility exhibits a pronounced variation across time and a considerable decline in the 90's.

To assess the magnitude and duration of short and long run risks, Figure 3 provides the accumulated impulse response of consumption growth to a one standard deviation shock in x^a_t – the long-run risk shock, as well as the response to the short-run shock, η^a_t. As the figure shows, the impact of the expected growth shock is economically large suggesting that these shocks significantly alter growth expectations. Furthermore, the expected growth shocks have long lasting effects on consumption growth — quite distinct from typical business cycle risks.

[3]To absorb any residual persistence in consumption growth, we allow for an MA(1) error structure in the consumption growth dynamics. Subsequently, we appropriately adjust the model solutions, in particular, the solution for the IMRS, as described in Appendix A.4. Overall, our empirical findings are robust to the inclusion of the MA(1) term. Note that persistence in annual growth news, η^a_{t+1}, may arise due to time averaging of consumption data – this issue is discussed in greater detail in Section 5 below.

4.2 Returns and Betas

Before formally estimating the model, we provide preliminary evidence on the risk-return tradeoff by linking mean returns and betas as described in equation (16). To derive asset betas, we first estimate the expected return for a given asset, $\bar{r}^a_{j,t+1}$, by projecting the asset return on the extracted state variables, x^a_t and $\sigma^{2,a}_t$. The asset's betas are then measured by the covariation of the innovation in return, $u^a_{r,j,t+1} = r^a_{j,t+1} - \bar{r}^a_{j,t+1}$, and the corresponding consumption shock, η^a_{t+1}, e^a_{t+1} or w^a_{t+1}.[4]

The cross-sectional risk-return relation is characterized by a regression of mean returns on the three betas, i.e.,

$$\bar{R}^a_j = \lambda^a_0 + \lambda^a_\eta \beta^a_{\eta,j} + \lambda^a_e \beta^a_{e,j} + \lambda^a_w \beta^a_{w,j} + \epsilon^a_j \, . \tag{20}$$

To expand the degrees of freedom, in the regression above we employ the full asset menu consisting of 10 size and 10 book-to-market sorted portfolios, plus the aggregate stock market. We find that, together, the three betas explain about 84% of the cross-sectional variation in mean returns. It is worthy to note that the predictive ability in the cross-section is foremost attributed to the long-run and volatility betas. Table III provides the betas for the five returns: small and large, growth and value, and the aggregate market portfolio. As the table shows, there is a clear link between assets' average returns and their exposures to long-run risks measured by β^a_e. For example, value and growth portfolios have the long-run risk beta of 22.9 and 17.23, respectively, reflecting the value premium. Similarly, the dispersion in long-run risk betas of small and large stocks (28.1 v.s. 15.4) captures the well-known effect of size on average returns. The volatility betas also significantly contribute to the spread in expected returns. In particular, small and value stocks seem to be much more sensitive to fluctuations in economic uncertainty relative to portfolios of large and growth stocks. Consistent with the model predictions, the volatility betas all have a negative sign. As the price of the volatility risk is also negative, the cross-sectional dispersion in volatility betas accounts for a significant portion of variation in risk premia across assets. For comparison,

[4]Specifically, the long-run consumption beta is defined $\beta^a_{e,j} \equiv \dfrac{\mathrm{Cov}(e^a_{t+1}, u^a_{r,j,t+1})}{\mathrm{Var}(e^a_{t+1})}$, the exposure of asset returns to transient risks in consumption is constructed as $\beta^a_{\eta,j} \equiv \dfrac{\mathrm{Cov}(\eta^a_{t+1}, u^a_{r,j,t+1})}{\mathrm{Var}(\eta^a_{t+1})}$, finally, the exposure to volatility (economic uncertainty) risk is computed as $\beta^a_{w,j} \equiv \dfrac{\mathrm{Cov}(w^a_{t+1}, u^a_{r,j,t+1})}{\mathrm{Var}(w^a_{t+1})}$.

the last column of Table III provides the standard CCAPM betas, β^a_{ccapm}. It is evident that the traditional betas fall short in capturing differences in mean returns, confirming the well-documented failure of the CCAPM.

4.3 Euler Equation Estimation Evidence

This section provides the main results of our paper. The vector of structural parameters we seek to estimate includes risk aversion and IES. We estimate preference parameters by exploiting an annual version of the Euler equation (4) for the six asset returns: the risk-free rate, market return, value and growth, large and small firm portfolios. Table IV provides the estimates of the structural parameters of the LRR model for two configurations of the IMRS: one that excludes the volatility innovations (Panel A), and the one that accounts for time-varying economic uncertainty (Panel B). In both cases, we use the identity weighting matrix. Each panel reports estimates of investors' preferences, namely, IES and risk aversion (ψ and γ), where we pre-set the time discount rate, δ, to 0.987. In addition, we provide the average pricing error for each asset in the cross-section, and the J-test statistics for overidentifying restrictions.

The results are quite illuminating. For the case, in which the IMRS includes the volatility channel (Panel B), risk aversion is estimated at just above 15 (with the HAC standard error of about 4), and the IES is estimated at 0.4 (with the HAC standard error of 0.5). The results in Panel A, where volatility is not accounted for, are quite similar: the estimate of risk aversion is around 16, and that of IES is 0.5. In both cases, the standard errors for the IES and risk aversion are quite large. Nevertheless, the model prices assets quite well. Formally, the model is not rejected as the overidentifying restrictions have p-values of above 0.5 in Panel A, and 0.4 in Panel B. Furthermore, the pricing errors are quite small. The largest pricing error is only 0.034 for the return on the portfolio of small stocks (see Panel B). Moreover, the t-statistics of the pricing errors of the size and value premia are both statistically insignificant. Figure 4 plots the model-based equity premium at the point estimates. As can be seen, the model-implied premium declines considerably over the latter part of the sample.

Table V presents the level of the risk-free rate and risk premia implied by the estimated LRR model. For comparison, the last two columns of the table report the corresponding moments implied by the power utility for two commonly used configurations of risk aversion of 4 and 40. It is evident that the standard time-separable preferences have considerable difficulty in pricing the cross-section of assets. Not only the pricing errors are large, the premium on value-minus-growth and small-minus-large portfolios have the wrong sign, clearly speaking to the poor performance of the power utility specification.

Table VI provides a decomposition of the assets' conditional risk premia into the three risk sources of the long-run risks model. Based on the GMM estimates of Table IV, compensations for various consumption risks are determined by asset betas and the corresponding prices of risks. The table demonstrates that the long-run risk component accounts for the lion share, about 50%, of risk premia, while the short-run and volatility risks account about equally for the remaining part of the risk premia.

It is worth noting that our results are robust to alternative use of instruments. In addition to the estimation discussed above, we have analyzed a more elaborate system of instruments to capture potentially important variations in conditional moments. We find that adding lagged consumption growth, risk free rate or market return as instruments leads to very similar estimates, in particular, fairly large estimates of risk aversion (of above fifteen) and low estimates of the IES.

4.4 Robustness

The above empirical evidence builds on using the risk-free and the price-dividend ratio as forecasting variables – a choice naturally motivated by the LRR model. To check the robustness of our empirical evidence and address concerns regarding potential measurement errors in the data, we also entertain a richer set of predictive variables. In predicting consumption growth to extract x_t^a, we use a two-year moving average of lagged consumption growth, the log of consumption to GDP ratio, price-dividend ratio of the aggregate market portfolio, the short interest rate, and the default premium.

Consistent with our earlier evidence in Table II, consumption growth, in the augmented VAR system, is highly predictable with an adjusted R^2 of 37%. The persistence parameter of expected consumption growth, ρ^a, is quite large and is estimated at 0.67. The accumulated impulse response to a long-run shock at 30 years is about 3. These results provide additional evidence that the long-run component of consumption growth is distinct from business cycle fluctuations.[5]

Consistent with our earlier evidence in Table II, consumption growth, in the augmented VAR system, is highly predictable with an adjusted R^2 of 37%. The implied persistence parameter of the growth component, ρ^a, is quite large of about 0.67. Consequently, news about expected consumption growth persist far into the future. Quantitatively, the accumulated impulse response of consumption growth to a unit shock in the long-run risk component reaches 3 at the 30 year horizon. These results, once again, highlight the low frequency dynamics of consumption growth which are distinct from business cycle fluctuations.[6]

Table VII reports the GMM estimation evidence based on the larger predictive system. Similar to the two-variable set-up, the estimate of risk aversion is quite large, while the estimate of the IES is below one. As before, the model generates relatively low pricing errors and is not rejected by the overidentifying restrictions. Overall, we find our GMM evidence to be robust to alternative empirical specifications for consumption growth.

In sum, our empirical evidence shows that once the return to wealth is appropriately accounted for, the long-run risks model goes a long way in explaining both the time-series and cross-sectional variation in returns. A concerning open issue is the fact that the IES estimate is well below one and the estimate of risk aversion is quite large. This parameter configuration in simulations of the LRR model is likely to produce counterfactual asset pricing features. This, consequently, raises an important question of how to interpret the empirical evidence documented in this paper and in earlier works. In the remaining sections we address this issue by highlighting the effects of time averaging and finite sample size of the data at hand.

[5] One potential important issue is whether the long-run response of consumption growth, based on this larger system and its AR(1) univariate representation for x_t^a, is similar to that obtained from a VAR system. Indeed we find that the two responses are virtually the same.

[6] For this predictive system, one potential important issue is whether the long-run response of consumption growth based on its AR(1) univariate representation for x_t^a is similar to that obtained from the VAR. Indeed, we find that the two responses are virtually the same.

5 Decision Interval and Time Averaging

As discussed above, while there is considerable empirical support for the long-run risks model, the magnitudes of the preference parameter estimates are hard to interpret from the perspective of the model. First, the low value of the IES, implies that asset valuations rise with higher economic uncertainty; earlier evidence (see Bansal and Yaron (2004), Bansal, Khatchatrian, and Yaron (2005)) shows that this implication is economically and empirically implausible. Second, the magnitude of the risk aversion estimate, of about 15, can be viewed as fairly large. It is also worth noting that the high risk aversion and low IES estimates are a fairly common empirical finding using alternative estimation procedures (e.g., Campbell (1999) and Hall (1988)). In this section, we show that the magnitude of preference parameter estimates is driven by time averaging and finite sample biases. Specifically, we write down a *monthly* LRR model of the type specified in equation (3) and show that time averaging to *annual* data leads to downward bias in the estimated IES and an upward bias in the estimated risk aversion, while maintaining the ability of the model to price assets. In particular, we show that even if the population value of risk aversion is low and that of the IES is larger than one, the finite-sample GMM applied to time-averaged data will produce estimates, as in the observed sample, of high risk aversion and low IES. Therefore, the simulation evidence suggest that the bias corrected magnitude of risk aversion and IES are about 10 and 2 respectively — parameter configuration that is consistent with the economic implications of the recursive-preferences based models.

5.1 Time Averaging and Finite Sample Effects on Estimation

In this section we examine the effects the misalignment of the sampling frequency with the agent's decision interval (time averaging) and finite sample biases on the economic plausibility of the LRR model and the interpretation of structural parameters.[7] In particular, we assess how these issues affect the estimates reported in the previous section. We calibrate a monthly version of the LRR model, time aggregate the data to construct simulated annual variables, and apply the same GMM estimation methodology used earlier for the observed data.

[7] The role of time averaging in dynamic models has been emphasized in the past; Hansen and Sargent (1983) highlight its effect in the framework of adaptive expectations models, while Heaton (1995) analyzes this issue in the context of an asset pricing model with time-nonseparable preferences. In this paper, we show the importance of this issue in the context of the long-run risks model.

Time averaging, in our context, arises since the sampling frequency of the data is different from the decision interval of the agent. Specifically, in simulations that follow we assume that the decision interval of the agent is monthly while the sampling frequency of the data used for estimation is annual. Time averaging therefore emanate from replacing monthly consumption by annual consumption $C_t^a = \sum_{j=0}^{11} C_{t-j}$, where C_{t-j} corresponds to month j consumption in year t. Consequently the estimates of monthly x_t and σ_t^2 are replaced by their annual counterparts x_t^a and $\sigma_t^{2,a}$.

It can be shown (see Appendix A.6) that in the presence of time averaging an econometrician will (i) be able to recover the appropriate state variables x_t^a and the corresponding persistence parameter ρ^a, which will be just the monthly ρ raised to the twelve power, (ii) will *not* be able to recover the true short-run shocks, η^a, and long-run e_t^a shocks. Similar results hold in extracting the volatility component. Thus, an econometrician using only annual data, while the decision interval is monthly, will not be able to characterize the model's IMRS and, hence, estimate the parameters consistently. The message from this is simple but important. Time averaging can have critical effects for deducing model parameters. Note, that these potential effects would be absent in a model that focuses on $i.i.d$ consumption growth, in which case the long-run component is absent, and time averaging essentially does not effect the estimation of model parameters.

In addition to the time averaging effects, another channel that may distort preference parameter estimates is finite sample bias. For example, it is well known (see Kendall (1954)) that the persistence parameter in a standard AR(1) process is biased downwards. Since the market price of long-run risk critically depends on the persistence parameter of expected consumption growth (ρ^a), any downward bias in this parameter can have significant effect on the model implied risk premia and estimated preference parameters. We provide, via simulations, a decomposition of the bias arising solely due to time averaging and that due to finite sample effects.

5.2 Consumption, Dividends, and Asset Returns

In order to simulate the model, we need in addition to the consumption dynamics already given in (3), to specify the dynamics of asset dividends. We assume that for each asset j

dividend growth, $\Delta d_{j,t+1}$, follows,

$$\Delta d_{j,t+1} = \mu_{d,j} + \phi_j x_t + \pi_j \sigma_t \eta_{t+1} + \varphi_j \sigma_t u_{d,j,t+1}. \quad (21)$$

We assume that all shocks are $i.i.d$ normal and are orthogonal to each other, but allow for cross-sectional correlations in dividend news, $u_{d,j,t+1}$. Dividends have a levered exposure to the persistent component in consumption, x_t, which is captured by the parameter ϕ_j. In addition, we allow the $i.i.d$ consumption shock η_{t+1} to influence the dividend process, and thus serve as an additional source of risk premia. The magnitude of this influence is governed by the parameter π_j.[8] The dynamics of asset dividends are similar to those in Bansal and Yaron (2004), Bansal, Dittmar, and Lundblad (2005), and Kiku (2006).[9]

The model, as discussed above, is assumed to have a monthly decision interval. We set up the following simulation experiment. First we simulate from the monthly consumption and dividend dynamics specified in equations (3) and (21). We then construct time series of annual consumption and dividends, C_t^a and D_t^a, respectively. Specifically, we simulate the model with 876 months, which result in 73 annual observations, as in our data set. We repeat this procedure 500 times. Panel A in Tables VIII and IX present the parameters governing the consumption and dividend dynamics respectively. Throughout we use a risk aversion parameter of 10 and an IES value of 2.

Panel B of Table VIII provides the Monte-Carlo evidence regarding the annual time series of consumption growth. The columns under the 'model' heading in Table VIII provide the median and standard deviation of key moments of annual consumption growth across simulations. As the table shows, the model successfully matches the mean, standard deviation, and autocorrelation of annual consumption growth.

Panel B of Table IX provides information regarding the mean, standard deviation of dividend growth and its correlation with consumption growth for the five assets we consider for the data and the model. The model's statistics, again, are based on the median and standard deviation across the 500 Monte-Carlo simulations. By and large, the model's output matches quite well with the data. For example, the correlations between consumption and

[8] This type of specification is isomorphic to one in which $\pi_j = 0$ but the correlation between η_{t+1} and $u_{d,j,t+1}$ is non-zero.

[9] Note that as we are dealing with dividend per-share, specification (21) does not impose cointegration between consumption and dividends as in Bansal and Yaron (2006).

dividend growth are essentially indistinguishable from their point estimates in the data. The data and the model's mean dividend growth rates are all within standard error of each other. More importantly, the relative ranking of mean growth rates across the five assets is maintained. The volatility of simulated dividend growth for the market, growth and large portfolios is well within one standard error of the data. For the small and value portfolios, the model implied median volatility is somewhat smaller than their data counterpart. Given that these two portfolios' extreme volatilities are driven by few data points, we chose to be more conservative with respect to these volatility numbers, while ensuring the model generates average returns that are comparable to what is observed in the data. Table X provides the data and model predictions for the mean and volatility of the return, the level of the price dividend ratio for each portfolio, as well as the level and volatility of the risk free rate. Again, the model replicates well all of these statistics. The data is well within one standard error of the model estimates. In particular, note that the model is able to generate the size and value premium as also highlighted in Kiku (2006).

5.3 Finite Sample Biases and Time Averaging effects on Estimation

Equipped with plausible model-generated data for returns, we apply GMM to the annual simulated data using test equation (4) in an analogous fashion to the estimation procedure we used for the observed data. Table XI provides the distribution of $\bar{R}^2$s of consumption growth projection, the persistence of the long-run component of consumption growth (ρ^a), risk aversion and IES estimates, as well as the J-stat and p-values across the 500 simulations. This table highlights several important implications. First, the medians of risk aversion and IES estimates across simulations are quite close to those estimated directly from the annual data in Table IV. Noteworthy is the fact that the median risk aversion is quite large (14) and there is apparent dispersion and right skewness in the estimated risk aversion parameter. The median estimate for the IES is less than 1. Recall, these simulations are based on a model in which risk aversion is 10 and the IES is 2. Further, note that the model is not rejected by the overidentifying restrictions. In fact the median J-stat is quite close to that estimated in the data — the two are well within the 90% confidence interval from each other.[10] In all, Table XI remarkably replicates the results in Tables IV, providing additional support in favor of the model proposed and estimated. A natural question to ask is why the IES is estimated with a downward bias while risk aversion is estimated with an upward bias? The answer hinges on time averaging and finite sample effects.

To isolate the effects of time averaging from that of having a finite sample we estimate the model using a very long annualized sample. The results of this experiment are given under the heading Pop-Values in Table XI. The population values for $\bar{R}^2$ and ρ^a are somewhat larger than the data and the median of the finite sample distribution. Thus, finite sample contributes to a small reduction in the persistence of the long-run component. Importantly, the IES estimate is still downward biased below 1, while risk aversion at 10.11 is very close to its true value. Thus, by simply looking at the long sample, one can conclude that time averaging is quantitatively important for the downward bias in IES, but it is the finite sample effects that generate the upward bias in risk aversion. Finally, note that based on the population p-value and J-stat, the misspecification induced by the time-averaging is sharply detected. In sum, the results in Table XI provide evidence that time averaging and finite sample effects jointly contribute to the observed biases in the preference parameter estimates.

6 SMM based Model Estimation

One procedure to account for the time averaging effects is to rely on the SMM estimation procedure as in Duffie and Singleton (1993). This simulation based approach requires that one models the monthly consumption and dividend dynamics for all the assets. Given these inputs, the model implications for asset returns can be derived using the results of section 5.2. This allows to compute the annual data counterpart from a model in which the decision interval is monthly. Furthermore, using long simulated draws from the model one can evaluate the implications of finite sample biases.

Table XII uses the consumption and dividend dynamics used in Tables VIII and IX effectively as first stage estimates of the cashflow dynamics and then uses SMM to estimate

[10] In simulations, not reported here, for the case of power utility, the model is rejected and generates counter-factual risk premia as in Table V. In this respect, the results of imposing the power utility restriction are similar to those estimated in the data.

the preference parameters. Risk aversion is estimated at 10.23 with a standard error of 2, while the IES is estimated at 2.43 with a standard error of 1.3. These SMM estimates are very close to the parameter configuration used earlier in section 5.2, and demonstrate that an IES of 2 and risk aversion of 10 are plausible from the perspective of this model. The fact the SMM procedure naturally takes into account time averaging and that the IES estimate is now larger than one, indicates that time averaging play an important role in the downward bias in estimation of the IES. Finally, note that the pricing errors are small and are insignificantly different from zero. Furthermore, the model's overidentifying restrictions are not rejected at conventional levels. In all, the SMM approach provides a useful method of recovering the underlying preference parameters in the presence of time averaging effects.

7 Market versus Consumption Return

Epstein and Zin (1991) pursue a GMM estimation approach but in evaluating the pricing kernel in equation (5) replace the return on the consumption claim $r_{c,t+1}$ with the observed value weighted NYSE stock market return. In Table XIII we use our simulated data to estimate the model with a pricing kernel based on the market return. The estimated risk aversion in this case is quite low. This, to a large extent, is due to the large volatility induced into the pricing kernel by the volatile market return. Finally, and most importantly, the table shows that the model's overidentifying restrictions are overwhelmingly rejected. Finite sample experiments also lead to vast rejections of the model. The implication of this experiment is that deriving the appropriate return on consumption is critical for appropriately assessing the LRR model.

8 Conclusions

This paper develops methods for estimating the long-run risks model in Bansal and Yaron (2004). In particular, the paper shows how to estimate the short-run, long-run and volatility risk components in aggregate consumption and utilize these to construct the unobservable return to aggregate wealth — a key input in estimating models with Krpes and Porteus (1978), Epstein and Zin (1989)-Weil (1989) preferences.

Empirically we find that the long-run risks model is able to successfully capture the time-series and cross-sectional variation in returns. The model accounts for the low risk free rate, and the level of the market, value, and size premia. The model also generates the volatility of the market return and the risk free rate, as well as the returns of several portfolios and their price-dividend ratios.

We provide evidence that estimates of risk aversion and the IES are particularly susceptible to time averaging and finite sample biases. These biases provide a potential explanation for the large risk aversion and small intertemporal elasticity of substitution estimates reported in earlier papers. Correcting for these biases we find that the magnitude of risk aversion is modest and the level of the intertemporal elasticity of substitution is larger than one. At these values for preference parameters the market price of long-run risks is high relative to that of the short-run and volatility risks. This evidence highlights that long-run risks as opposed to short run risks are important for understanding asset prices.

Appendix

A.1 Consumption Claim

To derive asset prices we use the IMRS together with consumption and dividend dynamics given in (3) and (21). The Euler condition in equation (4) implies that any asset j in this economy should satisfy the following pricing restriction,

$$E_t\Big[\exp\big(\theta\ln\delta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{c,t+1} + r_{j,t+1}\big)\Big] = 1\ , \tag{A-1}$$

where $r_{j,t+1} \equiv \log(R_{j,t+1})$ and $r_{c,t+1}$ is the log return on wealth. Notice that the solution to (A-1) depends on time-series properties of the unobservable return r_c. Therefore, we first substitute $r_{j,t+1} = r_{c,t+1}$ and solve for the return on the aggregate consumption claim; after that, we present the solution for the return on a dividend-paying asset.

We start by conjecturing that the logarithm of the price to consumption ratio follows, $z_t = A_0 + A_1 x_t + A_2\sigma_t^2$. Armed with the endogenous variable z_t, we plug the approximation $r_{c,t+1} = \kappa_0 + \Delta c_{t+1} + \kappa_1 z_{t+1} - z_t$ into the Euler equation above. The solution coefficients, A's, can now be easily derived by collecting the terms on the corresponding state variables. In particular,

$$\begin{aligned}
A_0 &= \frac{1}{1-\kappa_1}\Big[\log\delta + \kappa_0 + \Big(1-\frac{1}{\psi}\Big)\mu_c + \kappa_1 A_2(1-\nu)\bar{\sigma}^2 + \frac{\theta}{2}\Big(\kappa_1 A_2\sigma_w\Big)^2\Big] \\
A_1 &= \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho} \\
A_2 &= -\frac{(\gamma-1)(1-\frac{1}{\psi})}{2\ (1-\kappa_1\nu)}\Big[1+\Big(\frac{\kappa_1\varphi_e}{1-\kappa_1\rho}\Big)^2\Big]
\end{aligned} \tag{A-2}$$

For more details, see the the appendix in Bansal and Yaron (2004).

Notice that the derived solutions depend on the approximating constants, κ_0 and κ_1, which, in their turn, depend on the unknown mean of the price to consumption ratio, $\bar{z}$. In order to solve for the price of the consumption asset, we first substitute expressions for κ's (equations (7) and (8)) into the expressions for A's and solve for the mean of the price to consumption ratio. Specifically, $\bar{z}$ can be found by numerically solving a fixed-point problem:

$$\bar{z} = A_0(\bar{z}) + A_2(\bar{z})\bar{\sigma}^2 \ ,$$

where the dependence of A's on $\bar{z}$ is given above.

The solution for the price-consumption ratio, z_t, allows us to write the pricing kernel as a function of the state variables and the model parameters,

$$m_{t+1} = \Gamma_0 + \Gamma_1 x_x + \Gamma_2 \sigma_t^2 - \lambda_\eta \sigma_t \eta_{t+1} - \lambda_e \sigma_t e_{t+1} - \lambda_w \sigma_w w_{t+1} \ , \tag{A-3}$$

where

$$\begin{aligned} \Gamma_0 &= \log\delta - \frac{1}{\psi}\mu_c - 0.5\,\theta(\theta-1)\big(\kappa_1 A_2 \sigma_w\big)^2 \\ \Gamma_1 &= -\frac{1}{\psi} \\ \Gamma_2 &= (\theta-1)(\kappa_1\nu - 1)A_2 \end{aligned} \tag{A-4}$$

and

$$\begin{aligned} \lambda_\eta &= \gamma \\ \lambda_e &= (1-\theta)\kappa_1 A_1 \varphi_e \ = \ \big(\gamma - \frac{1}{\psi}\big)\frac{\kappa_1\varphi_e}{1-\kappa_1\rho} \\ \lambda_w &= (1-\theta)\kappa_1 A_2 \ = \ -(\gamma-1)\big(\gamma - \frac{1}{\psi}\big)\frac{0.5\,\kappa_1}{1-\kappa_1\nu}\Big[1 + \Big(\frac{\kappa_1\varphi_e}{1-\kappa_1\rho}\Big)^2\Big] \end{aligned} \tag{A-5}$$

Note that λ's represent market prices of transient (η_{t+1}), long-run (e_{t+1}) and volatility (w_{t+1}) risks respectively. For more detailed discussion see Bansal and Yaron (2004).

A.2 Dividend Paying Assets

The solution coefficients for the valuation ratio of a dividend-paying asset j can be derived in a similar fashion as for the consumption asset. In particular, the price-dividend ratio for a claim to dividends, $z_{j,t} = A_{0,j} + A_{1,j} x_t + A_{2,j} \sigma_t^2$, where

$$
\begin{aligned}
A_{0,j} &= \frac{1}{1-\kappa_{1,j}} \left[\Gamma_0 + \kappa_{0,j} + \mu_{d,j} + \kappa_{1,j} A_{2,j} (1-\nu) \bar{\sigma}^2 + \frac{1}{2} \Big(\kappa_{1,j} A_{2,j} - \lambda_w \Big)^2 \sigma_w^2 \right] \\
A_{1,j} &= \frac{\phi_j - \frac{1}{\psi}}{1 - \kappa_{1,j} \rho} \qquad \text{(A-6)} \\
A_{2,j} &= \frac{1}{1-\kappa_{1,j}\nu} \left[\Gamma_2 + \frac{1}{2} \Big((\pi_j - \lambda_\eta)^2 + (\kappa_{1,j} A_{1,j} \varphi_e - \lambda_\epsilon)^2 \Big) \right]
\end{aligned}
$$

It follows then that the innovation into the asset return is given by,

$$
r_{j,t+1} - E_t[r_{j,t+1}] = \varphi_j \sigma_t u_{j,t+1} + \beta_{\eta,j} \sigma_t \eta_{t+1} + \beta_{e,j} \sigma_t e_{t+1} + \beta_{w,j} \sigma_w w_{t+1} \ , \qquad \text{(A-7)}
$$

where the asset's betas are defined as,

$$
\begin{aligned}
\beta_{\eta,j} &= \pi_j \\
\beta_{e,j} &= \kappa_{1,j} A_{1,j} \varphi_e \\
\beta_{w,j} &= \kappa_{1,j} A_{2,j}
\end{aligned}
$$

The risk premium for any asset is determined by the covariation of the return innovation with the innovation into the pricing kernel. Thus, the risk premium for $r_{j,t+1}$ is equal to the product of the asset's exposures to systematic risks and the corresponding risk prices,

$$
\begin{aligned}
E_t[r_{j,t+1} - r_{f,t}] + 0.5 \sigma_{t,r_j}^2 &= -Cov_t \Big(m_{t+1} - E_t(m_{t+1}), r_{j,t+1} - E_t(r_{j,t+1}) \Big) \\
&= \lambda_\eta \sigma_t^2 \beta_{\eta,j} + \lambda_e \sigma_t^2 \beta_{e,j} + \lambda_w \sigma_w^2 \beta_{w,j}
\end{aligned}
$$

A.3 IES=1

When $\psi = 1$, the log of the IMRS is given in terms of the value function normalized by consumption, $vc_t = \log(V_t/C_t)$,

$$m_{t+1} = \log\delta - \gamma\Delta c_{t+1} + (1-\gamma)vc_{t+1} - \frac{1-\gamma}{\delta}vc_t$$

Conjecturing that $vc_t = B_0 + B_1 x_t + B_2\sigma_t^2$ and using the evolution of vc_t:

$$vc_t = \frac{\delta}{1-\gamma}\log E_t\Big[\exp\{(1-\gamma)(vc_{t+1} + \Delta c_{t+1}\}\Big],$$

the solution coefficients are given by,

$$B_0 = \frac{\delta}{1-\delta}\left[\mu + B_2(1-\nu)\bar{\sigma}^2 + \frac{1}{2}(1-\gamma)(B_2\sigma_w)^2\right]$$

$$B_1 = \frac{\delta}{1-\delta\rho} \tag{A-8}$$

$$B_2 = -(\gamma-1)\frac{0.5\,\delta}{1-\delta\nu}\left[1+\Big(\frac{\delta\varphi_e}{1-\delta\rho}\Big)^2\right] \tag{A-9}$$

As above, the pricing kernel can be expressed in terms of underlying preference parameters, state variables and systematic shocks,

$$m_{t+1} = \Gamma_0 + \Gamma_1 x_t + \Gamma_2\sigma_t^2 - \lambda_\eta\sigma_t\eta_{t+1} - \lambda_\epsilon\sigma_t\epsilon_{t+1} - \lambda_w\sigma_w w_{t+1} \tag{A-10}$$

where:

$$\Gamma_0 = \log\delta - \mu - 0.5\,(1-\gamma)^2(B_2\sigma_w)^2 \tag{A-11}$$

$$\Gamma_1 = -1$$

$$\Gamma_2 = -\frac{(\gamma-1)^2}{2}\left[1+\Big(\frac{\delta\varphi_e}{1-\delta\rho}\Big)^2\right]$$

and

$$
\begin{aligned}
\lambda_\eta &= \gamma \\
\lambda_\epsilon &= (\gamma-1)\frac{\delta\varphi_e}{1-\delta\rho} \\
\lambda_w &= -(\gamma-1)^2\frac{0.5\,\delta}{1-\delta\rho}\left[1+\left(\frac{\delta\varphi_e}{1-\delta\rho}\right)^2\right]
\end{aligned}
\tag{A-12}
$$

Finally, note that in the IES=1 case, the wealth to consumption ratio is constant, namely, $\frac{W_t}{C_t}=\frac{1}{1-\delta}$. The price to consumption ratio, therefore, is equal $\frac{P_t}{C_t}=exp(\bar{z})=\frac{\delta}{1-\delta}$. Consequently, the parameter of the log-approximation of the log-wealth return,

$$
\kappa_1 = \frac{\exp(\bar{z})}{1+\exp(\bar{z})} = \frac{\frac{\delta}{1-\delta}}{1+\frac{\delta}{1-\delta}} = \delta.
$$

Plugging $\kappa_1=\delta$ and $\psi=1$ into equations (A-3), (A-4) and (A-5), yields exactly equation (A-10), (A-11) and (A-12). It then follows that

$$
\lim_{\psi\to 1}\kappa_1=\delta \quad \lim_{\psi\to 1}\Gamma'=\Gamma'(\psi=1,\kappa_1=\delta) \quad \lim_{\psi\to 1}\Lambda'=\Lambda'(\psi=1,\kappa_1=\delta) \tag{A-13}
$$

A.4 Generalized Consumption Dynamics

To absorb any short-run persistence induced by time averaging of consumption data, we allow for an MA innovation structure when modelling the dynamics of annual growth rates. Specifically,

$$
\Delta c_{t+1} = \mu_c + x_t + \alpha\,\sigma_{t-1}\eta_t + \sigma_t\eta_{t+1} \tag{A-14}
$$

The evolution of x_t and σ_t remains the same (see equation (3)). Notice that in this specification, predictable variations in consumption growth are driven by the long-run risk component, x_t, as well as past consumption innovation, $\sigma_{t-1}\eta_t$. Hence, the intertemporal marginal rate of substitution and asset valuations will reflect information in both. In particular, the log of the price to consumption ratio will follow, $z_t = A_0 + A_1x_t + A_2\sigma_t^2 +$

$A_3\sigma_{t-1}\eta_t$, where:

$$
\begin{aligned}
A_0 &= \frac{1}{1-\kappa_1}\Big[\log\delta + \kappa_0 + \Big(1-\frac{1}{\psi}\Big)\mu_c + \kappa_1 A_2(1-\nu)\bar{\sigma}^2 + \frac{\theta}{2}\Big(\kappa_1 A_2\sigma_w\Big)^2\Big] \\
A_1 &= \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho} \\
A_2 &= \frac{0.5\,\theta}{1-\kappa_1\nu}\Big[\Big(1-\frac{1}{\psi}+\kappa_1 A_3\Big)^2 + \big(\kappa_1 A_1\varphi_e\big)^2\Big] \\
A_3 &= \alpha\Big(1-\frac{1}{\psi}\Big)
\end{aligned}
\tag{A-15}
$$

The log of the IMRS can now be expressed as,

$$
m_{t+1} = \Gamma_0 + \Gamma_1 x_t + \Gamma_2\sigma_t^2 + \Gamma_3\sigma_{t-1}\eta_t - \lambda_\eta\sigma_t\eta_{t+1} - \lambda_\epsilon\sigma_t\epsilon_{t+1} - \lambda_w\sigma_w w_{t+1}\ , \tag{A-16}
$$

where the loadings (Γ's) and risk prices (λ's) are given by,

$$
\begin{aligned}
\Gamma_0 &= \log\delta - \frac{1}{\psi}\mu_c - 0.5\,\theta(\theta-1)\big(\kappa_1 A_2\sigma_w\big)^2 \\
\Gamma_1 &= -\frac{1}{\psi} \\
\Gamma_2 &= (\theta-1)(\kappa_1\nu-1)A_2 \\
\Gamma_3 &= -\frac{\alpha}{\psi} \\
\lambda_\eta &= \Big(\gamma-\frac{1}{\psi}\Big)\kappa_1\alpha + \gamma \\
\lambda_e &= (1-\theta)\kappa_1 A_1\varphi_e = \big(\gamma-\frac{1}{\psi}\big)\frac{\kappa_1\varphi_e}{1-\kappa_1\rho} \\
\lambda_w &= (1-\theta)\kappa_1 A_2 = -(\gamma-1)\big(\gamma-\frac{1}{\psi}\big)\frac{0.5\,\kappa_1}{1-\kappa_1\nu}\Big[1+\Big(\frac{\kappa_1\varphi_e}{1-\kappa_1\rho}\Big)^2\Big]
\end{aligned}
\tag{A-17}
$$

It is straightforward to derive analogues solutions for the case of IES=1; for brevity, they are not presented here.

A.5 Pricing Kernel Approximation Error

In our empirical work, we rely on the approximate analytical solutions of the model presented above. In this section, we evaluate the accuracy of the log-linear approximation by comparing the approximate analytical solution for the price to consumption ratio to its numerical counterpart. The magnitude of the approximation error allows us to assess the reliability of the log-linear solution for the stochastic discount factor, and consequently, model implications based on the log-linear approximation.

Notice that the value function in the Epstein-Zin preferences is given by,

$$V_t = (1-\delta)^{\frac{\psi}{\psi-1}} W_t (W_t/C_t)^{\frac{1}{\psi-1}}, \tag{A-18}$$

i.e., the life-time utility of the agent, normalized by the level of either consumption or wealth, is proportional to the wealth to consumption ratio. Hence, the solution to the wealth-consumption ratio (or, alternatively, price to consumption) based on the log-linearization of the wealth return in equation (6) determines the dynamics of the value function. Recall also that the evolution of the IMRS (see equation (5)), through the return on wealth, depends on the valuation of the consumption claim. Thus, the log-linear solution for the IMRS, as well, hinges on the accuracy of the log-linear approximation of the price-consumption ratio.

Our numerical solutions are based on the approach proposed by Tauchen and Hussey (1991). This method relies on a discrete representation of the conditional density of the state variables, x and σ^2, which allows us to solve the pricing equation by approximating the integral in (4) with a finite sum using the Gauss-Hermite quadrature. Note that the resulting solutions are subject to a discretization error. In order to minimize the error and ensure the high quality of the benchmark numerical solutions, we use a sufficiently large number of grid points in the quadrature rule.[11] In addition, in this exercise we shut-off the channel of time-varying consumption volatility. Aside from this restriction, we evaluate and compare numerical and log-linear analytical solutions using the same parametrization of consumption growth dynamics that we employ in our simulation experiment (see Section 5.2 and Panel A of Table VIII. Table XIV presents the mean level of the price-consumption

[11] Specifically, we discretize the dynamics of the expected growth component, x_t, using a 100-point rule. We find that increasing the number of grid points leads to virtually identical numerical solutions.

ratio and its volatility for various combinations of risk aversion and IES; the time-discount preference paprameter δ is set at 0.9989.

Overall, we find approximate analytical and numerical solutions to be remarkably close to each other. In particular, for risk aversion of 10 and IES of 2, the mean and the volatility of the log price to consumption ratio implied by the log-linear approximation are 4.716 and 0.0321. Numerical solutions yield 4.724 and 0.0318, respectively.[12] The approximation error, expressed as a percentage of the corresponding numerical value, is about 0.17% for the mean and 0.86% for the standard deviation of the log price-consumption ratio. As the elasticity of intertemporal substitution decreases to 0.5, the percentage error falls to about 0.02% for $\bar{z}$ and 0.42% for σ_z. Although the accuracy of the log-linearization slightly deteriorates as the magnitude of risk aversion increases, deviations between analytical and numerical solutions remain quite small. For example, holding IES at 2 and varying risk aversion between 5 and 15 results in 0.03%–0.51% error band for the mean and 0.17%–2.17% for the standard deviation of the log price to consumption ratio.

As discussed above, the dynamics of the price to consumption ratio has a direct bearing on the time-series properties of the IMRS. The fairly small approximation error in the price-consumption ratio that we document guarantees the accuracy of the pricing implications based on the log-linear solutions. Indeed, we find that approximate analytical and numerical solutions deliver very similar quantitative implications along all dimensions of the model, including levels and variances of the risk-free rate, price-dividend ratios, returns on consumption and dividend claims, and the pricing kernel.[13] This evidence confirms that empirical findings presented in the paper are robust to the log-linearization of the model.

A.6 Mapping between Monthly and Annual Dynamics

Let C^a_{12t} denote the level of the agent's consumption in year t. In what follows, we count time in calendar months; thus, subscript $12t$ refers to December of year t. Superscript a labels annual quantities measured over the past 12 months. Given the above notation, the

[12] All the numbers reported in this section are in monthly terms.

[13] Available upon request, the detailed evidence is not reported here for brevity.

growth rate of annual consumption from year t to $t+1$ is defined as,

$$\Delta c^a_{12(t+1)} \equiv \log \frac{C^a_{12(t+1)}}{C^a_{12t}} = \log \frac{\sum_{j=0}^{11} C_{12(t+1)-j}}{\sum_{j=0}^{11} C_{12t-j}}\,, \tag{A-19}$$

where $C_{12(t+1)-j}$ is consumption over $\{12(t+1)-j\}$–month. Specifically, $C_{12(t+1)-0}$ is consumption in December of year $t+1$, $C_{12(t+1)-1}$ is the November consumption, and so on.

It can be shown that annual growth in (A-19) can be approximated by a weighted sum of monthly consumption growth rates. First, factor our C_{12t} and rewrite the numerator in the expression above as,

$$\log \sum_{j=0}^{11} C_{12(t+1)-j} = \log C_{12t} + \log \sum_{j=0}^{11} \frac{C_{12(t+1)-j}}{C_{12t}} \tag{A-20}$$

Notice that each term in the summation on the right-hand side can be presented as a product of monthly growth rates, i.e., $\dfrac{C_{12(t+1)-j}}{C_{12t}} = \prod_{k=j}^{11} \dfrac{C_{12(t+1)-k}}{C_{12(t+1)-k-1}}$. Consequently,

$$\log \sum_{j=0}^{11} C_{12(t+1)-j} \;\approx\; \log C_{12t} + \log 12 + \sum_{j=0}^{11} \frac{j+1}{12} \Delta c_{12(t+1)-j} \tag{A-21}$$

Applying a similar approximation to the denominator in (A-19), we can now express annual growth as,

$$\Delta c^a_{12(t+1)} \approx \sum_{j=0}^{22} \tau_j \, \Delta c_{12(t+1)-j} \tag{A-22}$$

where

$$\tau_j = \begin{cases} \dfrac{j+1}{12}\,, & j < 12 \\ \dfrac{23-j}{12}\,, & j \geq 12 \end{cases} \tag{A-23}$$

Equation (A-22) provides a mapping between monthly and annual growth rates. It follows that growth in annual consumption can be approximated by a weighted average of the current and past monthly growth rates with weights taking on a $\wedge$-shape.

The derived approximation allows us to describe the time-series dynamics of annual consumption growth in terms of the underlying monthly model as specified in equation (3). In particular,

$$\begin{aligned} \Delta c^a_{12(t+1)} &\approx \phi^a_c \, x^a_{12(t+1)-23} + \eta^a_{12(t+1)} \\ x^a_{12(t+1)} &= \rho^{12} \, x^a_{12t} + \varphi_e e^a_{12(t+1)} \end{aligned} \tag{A-24}$$

where the loading of annual growth on the long-run risk component and composite innovations are given by,

$$\begin{aligned} \phi^a_c &= \sum_{j=0}^{22} \tau_j \, \rho^j \\ \eta^a_{12(t+1)} &= \sum_{j=0}^{22} \tau_j \left[\sigma_{12(t+1)-j-1} \eta_{12(t+1)-j} + \varphi_e \sum_{k=1}^{22-j} \rho^{k-1} \sigma_{12(t+1)-j-1-k} e_{12(t+1)-j-k} \right] \\ e^a_{12(t+1)} &= \sum_{j=0}^{11} \rho^j \, \sigma_{12(t+1)-j-1} e_{12(t+1)-j} \end{aligned} \tag{A-25}$$

Note that the time subscript in (A-24) is measured in yearly increments (i.e., $t = 0, 1, 2 \ldots$), correspondingly, all the variables progress on an annual basis.

Equations (A-24) and (A-25) yield several important observations. First, the long-run risk component sampled on an annual frequency coincides with its December counterpart, $x^a_{12t} \equiv x_{12t}$. Thus, annual observations on x, can be extracted by regressing annual consumption growth onto the past price-dividend ratio and the risk-free rate. Similar result holds for the volatility component. Given that $\sigma^{2,a}_{12t} \equiv \sigma^2_{12t}$ and $E_{12(t+1)-23}[(\eta^a_{12(t+1)})^2] = \varphi^a_c \sigma^{2,a}_{12(t+1)-23}$, consumption volatility as of end of December can be identified from an annual projection of the squared consumption residual onto the market price-dividend ratio and the riskless bond. The innovations that characterize the true annual pricing kernel, however, cannot be recovered from annual regressions.

Further, consistent with the result documented in Working (1960), time averaging introduces an MA(1) error structure in consumption growth dynamics. As follows from (A-25), η^a_{12t} exhibits a non-zero first-order serial correlation. The evolution of annual consumption growth, therefore, can be equivalently described as,

$$\Delta c^a_{12(t+1)} \approx \phi^a_c \, x^a_{12(t+1)-23} \, + \, \tilde{\eta}^a_{12(t+1)} + \omega \tilde{\eta}^a_{12t} \, , \tag{A-26}$$

where $\text{Cov}(\tilde{\eta}^a_{12t}, \tilde{\eta}^a_{12(t+k)}) = 0$ for $k \neq 0$. Another side effect of time averaging is distortion of the innovation correlation structure. Note that even if underlying monthly shocks η and e are uncorrelated, synthesized innovations in annual consumption growth and the long-run risk component, η^a and e^a, will display a non-trivial covariation.

Since all the data we rely on are sampled on an annual frequency, in our empirical work we exploit a slightly modified specification for consumption growth. We consider the following dynamics,

$$\Delta c^a_{12(t+1)} \approx \phi^{a*}_c \, x^a_{12(t+1)-12} \, + \, \tilde{\eta}^{a*}_{12(t+1)} + \omega^* \tilde{\eta}^{a*}_{12t} \, . \tag{A-27}$$

All the implications of time averaging discussed above apply to (A-27) as well. In particular, it can be shown that $\text{Cov}(\tilde{\eta}^{a^*}_{12t}, \tilde{\eta}^{a^*}_{12(t+k)}) = 0$ for $k \neq 0$.

References

Bansal, Ravi, Robert F. Dittmar, and Christian Lundblad, 2005, Consumption, dividends, and the cross-section of equity returns, *Journal of Finance* 60, 1639–1672.

Bansal, Ravi, Ron Gallant, and George Tauchen, 2005, Rational Pessimism, Rational Exuberance, and Markets for Macro Risks, Working paper, Duke University.

Bansal, Ravi, Varoujan Khatchatrian, and Amir Yaron, 2005, Interpretable asset markets?, *European Economic Review* 49, 531–560.

Bansal, Ravi, and Amir Yaron, 2004, Risks for the long run: A potential resolution of asset pricing puzzles, *Journal of Finance* 59, 1481–1509.

Bansal, Ravi, and Amir Yaron, 2006, The asset pricing-macro nexus and return-cash flow predictability, Working paper, The Wharton School, University of Pennsylvania.

Bekaert, Geert, Eric Engstrom, and Yuhang Xing, 2005, Risk, Uncertainty and Asset Prices, Working paper, .

Campbell, John, and Hyeng Keun Koo, 1997, A comparison of a numerical and approximate analytical solutions to an intertemporal consumption choice problem, *Journal of Economic Dynamics and Control* 21, 273–295.

Campbell, John, and Robert Shiller, 1988, Stock Prices, Earnings, and Expected Dividends, *Journal of Finance* 43, 661–676.

Campbell, John Y., 1999, *Asset prices, Consumption and the Business Cycle* Elsevier Science pp. 1231–1303 Handbook of Macroeconomics, Volume 1, John B. Taylor and Michael Woodford, editors.

Cochrane, John, and Lars Hansen, 1992, Asset Pricing Explorations for Macroeconomics, *NBER Macro Annual* pp. 115–165.

Drost, Feike C., and Theo Nijman, 1993, temporal Aggregation of GARCH processes, *Econometrica* 61, 909927.

Duffie, Darrell, and Kenneth Singleton, 1993, Simulated Methods Estimation of Markov Models of Asset Prices, *Econometrica* 61, 929–952.

Epstein, Larry G., and Stanley E. Zin, 1989, Substitution, risk aversion, and the intertemporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* 57, 937–969.

Epstein, Larry G., and Stanley E. Zin, 1991, Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis, *Journal of Political Economy* 99, 263–286.

Fama, Eugene, and Kenneth French, 1993, Common risk factors in the returns on stocks and bonds, *Journal of Financial Economics* 33, 3–56.

Giovannini, Alberto, and Philippe Weil, 1989, Risk Aversion and Intertemporal Substitution in the Capital Asset Pricing Model, Working paper, NBER Working Paper No. 2824.

Hall, Robert, 1988, Intertemporal Substitution in Consumption, *Journal of Political Economy* 96, 339–357.

Hansen, Lars, John Heaton, and Nan Li, 2005, Consumption strikes back?, Working paper, University of Chicago.

Hansen, Lars, and Thomas Sargent, 2006, Fragile Beliefs and the Price of Model Uncertainty, Working paper, .

Hansen, Lars Peter, and Thomas Sargent, 1983, Aggregation Over Time and the Inverse Optimal Predictor Problem for Adaptive Expectations in Continuous Time, *International Economic Review*.

Hansen, Lars Peter, and Kenneth Singleton, 1982, Generalized instrumental variables estimation of nonlinear rational expectations models, *Econometrica* 63, 767–804.

Heaton, John, 1995, An Empirical Investigation of Asset Pricing with Temporally Dependent Preference Specifications, *Econometrica* 63, 681–717.

Kandel, Eugene, and Robert Stambaugh, 1991, Asset returns and intertemporal preferences, *Journal of Monetary Economics* 27, 39–71.

Kendall, M, 1954, Note on the Bias in Estimation of Autocorrelation, *Biometrica* 44, 270–272.

Kiku, Dana, 2006, Long Run Risks and the Value Premium Puzzle, Working paper, Duke University.

Krpes, David, and Evan Porteus, 1978, Temporal Resolution of Uncertainty and Dynamic Choice Theory, *Econometrica* 46, 185–200.

Lettau, Martin, and Sydney Ludvigson, 2005, Euler Equation Errors, Working paper, New York University.

Malloy, Christopher, Tobias Moskowitz, and Annette Vissing-Jorgensen, 2004, Long-Run stockholder consumption risk and asset returns, Working paper, University of Chicago.

Tallarini, Thomas, 2000, Risk Sensitive Real Business Cycles, *Journal of Monetary Economics* 45(3), 507–532.

Tauchen, George, and Robert Hussey, 1991, Quadrature-Based Methods for Obtaining Approximate Solutions to Nonlinear Asset Pricing Models, *Econometrica* 59, 371–96.

Weil, Philippe, 1989, The Equity Premium Puzzle and the Risk-free Rate Puzzle, *Journal of Monetary Economics* 24, 401–421.

Wilcox, David, 1992, The construction of the U.S. consumption data: Some facts and their implications for empirical work, *American Economic Review* 82, 922–941.

Working, Holbrook, 1960, Note on the Correlation of First Differences of Averages in a Random Chain, *Econometrica* 28, 916–918.

Table I
Summary Statistics

	Returns		Div Growth		Log(P/D)	
	Mean	StdDev	Mean	StdDev	Mean	StdDev
Size Portfolios						
Small	0.166	0.40	0.066	0.27	4.07	0.62
Large	0.076	0.19	0.003	0.11	3.30	0.43
B/M Portfolios						
Growth	0.070	0.22	-0.003	0.16	3.71	0.62
Value	0.134	0.33	0.047	0.29	3.42	0.68
Market	0.083	0.20	0.007	0.11	3.33	0.46
Risk-free Rate	0.008	0.01				

Table I presents descriptive statistics for returns, dividend growth rates and logarithms of price-dividend ratios of size and book-to-market sorted portfolios, and the aggregate stock market. Small and large portfolios represent firms in the top and bottom market capitalization deciles, growth and value correspond to the lowest and highest book-to-market decile. Returns are value-weighted, dividends and price-dividend ratios are constructed on the per-share basis, growth rates are measured by taking the first difference of the logarithm of dividend series. The bottom line reports the mean and the standard deviation of the annualized yield on the 3-month Treasuary bill. All asset data are real, sampled on an annual frequency and cover the period from 1930 to 2002.

Table II

Consumption Growth Dynamics

	Cons Growth		Variance	
Regressor	Estimate	t-stat	Estimate($\times 100$)	t-stat
Log(P/D)	0.025	3.10	-0.027	-1.28
Short Interest Rate	-0.213	-2.04	0.151	0.74
MA(1)	0.536	4.91		
	$\bar{R}^2 = 35\%$		$\bar{R}^2 = 2.8\%$	

Table II presents predictability evidence for consumption growth. The second column reports estimated projection coefficients on lagged price-dividend ratio of the aggregate market portfolio and the risk-free rate, and the estimated MA(1) parameter. The corresponding t-statistics are calculated using the Newey-West variance-covariance estimator with 4 lags. The data employed in the regression are annual and span the period from 1930 to 2002.

Table III

Consumption Betas

	Mean Ret	β^a_η	β^a_e	β^a_w	β^a_{ccapm}
Small	0.166	4.82	28.05	-3222.7	0.71
Large	0.076	2.49	15.36	-1830.6	0.69
Growth	0.070	2.62	17.23	-2021.5	0.82
Value	0.134	3.41	22.91	-2746.7	0.14
Market	0.083	2.69	16.74	-1984.8	0.59

Table III presents mean returns and consumption betas for firms in the lowest and highest deciles of size and book-to-market sorted portfolios — small and large, and growth and value, respectively, as well as the aggregate stock market. Consumption betas are calculated as the covariation between consumption news and innovations in asset returns scaled by the variance of the corresponding consumption shock. β^a_η represents the return exposure to transient shocks in consumption, β^a_e and β^a_w measure risks related to fluctuations in the long-run growth and consumption uncertainty. Long-run and discount-rate risks are extracted by fitting AR(1) processes to the estimated expected growth and volatility components. The frequency of the data is annual, the sample covers the period from 1930 to 2002.

Table IV

Estimation Evidence: Long-Run Risks Model

	Panel A: No Vol		**Panel B**: With Vol	
Parameters	Estimate	SE	Estimate	SE
RA	16.33	5.96	15.12	4.08
IES	0.47	1.20	0.37	0.50
Assets	PrError	t-stat	PrError	t-stat
Small	0.041	0.33	0.034	0.34
Large	-0.016	-0.17	-0.015	-0.19
Growth	-0.027	-0.28	-0.026	-0.32
Value	0.015	0.14	0.011	0.12
Market	-0.014	-0.15	-0.014	-0.17
Risk-Free	0.006	0.04	0.011	0.10
Small-Large	0.057	1.38	0.049	1.31
Value-Growth	0.042	1.46	0.037	1.28
J-stat	3.44		4.30	
P-value	0.49		0.37	

Table IV presents GMM estimates of the long run risks model: the risk aversion parameter (RA) and the elasticity of intertemporal substitution (IES). The two panels differ in their treatment of the volatility channel: it is ignored in Panel A, while being incorporated in Panel B. In both cases, moments are weighted using the identity matrix. The asset menu consists of firms with small and large market capitalization, low and high book-to-market ratio (growth and value, respectively), aggregate stock market and the risk-free rate. Average pricing errors and their t-statistics are presented for each asset. The bottom two lines report J-statistics for overidentifying restrictions and the corresponding p-values. The data employed in the estimation are annual and cover the period from 1930 to 2002.

Table V

Implied Risk Premia and Risk-free Rate

		Data	Long-Run Risks	$CRRA_{RA=4}$	$CRRA_{RA=40}$
Risk Premia	Small	0.158	0.139	-0.002	-0.100
	Large	0.068	0.093	0.000	0.029
	Growth	0.062	0.099	0.001	0.050
	Value	0.126	0.131	-0.003	-0.050
	Market	0.075	0.100	0.000	0.020
Risk-Free Rate		0.008	0.003	0.097	0.153

Table V presents model-implied unconditional risk premia for the five portfolios of assets and the mean of the risk-free rate. The first column reports corresponding moments in the data. The asset data are real and span the period from 1930 to 2002.

Table VI

Model-Implied Conditional Risk Premia Decomposition

	Short-Run	Long-Run	Volatility	Total
Small	0.028	0.083	0.026	0.137
Large	0.015	0.045	0.015	0.075
Growth	0.016	0.051	0.016	0.083
Value	0.020	0.068	0.022	0.110
Market	0.016	0.049	0.016	0.081

Table VI presents the decomposition of conditional risk premia implied by the long-run risks model. Compensations for various consumption risks are determined by asset betas and the corresponding prices of risks. Specifically, the short-run risk premium is computed as $\lambda_\eta^a \bar{\sigma}_\eta^{2,a} \beta_\eta^a$, compensations for long-run and volatility risks correspond to $\lambda_e^a \bar{\sigma}_e^{2,a} \beta_e^a$ and $\lambda_w^a \sigma_w^{2,a} \beta_w^a$, respectively. Risk prices are based on the GMM estimates in Table IV. The data employed in the estimation are annual and cover the period from 1930 to 2002.

Table VII
Robustness Evidence

GMM Evidence

Parameters	Estimate	SE
RA	27.70	8.67
IES	0.59	2.57
Assets	PrError	t-stat
Small	0.038	0.30
Large	-0.020	-0.18
Growth	-0.032	-0.29
Value	0.021	0.18
Market	-0.018	-0.16
Risk-Free	0.012	0.09
J-stat	5.60	
P-value	0.23	

Table VII reports GMM estimates of long run risks model: the risk aversion parameter (RA) and the elasticity of intertemporal substitution (IES). The asset menu consists of firms with small and large market capitalization, low and high book-to-market ratio (growth and value, respectively), aggregate stock market and the risk-free rate. Average pricing errors and their t-statistics are presented for each asset. The bottom two lines report J-statistic for overidentifying restrictions and the corresponding p-value. The data employed in the estimation are annual and cover the period from 1930 to 2002.

Table VIII
Consumption Growth Dynamics

Panel A: Calibration of Monthly Consumption Growth

μ_c	ρ	φ_e	$\bar{\sigma}$	ν	σ_w
0.0015	0.982	0.042	0.0054	0.98	0.0000068

Panel B: Dynamics of Annual Consumption Growth

Statistic	— **Data** —		— **Model** —	
$E[\Delta c]$	1.96	(0.34)	1.83	(0.62)
$\sigma(\Delta c)$	2.21	(0.38)	2.27	(0.37)
$AC(1)$	0.44	(0.13)	0.47	(0.12)

Panel A of Table VIII summarizes the calibration of parameters that govern the dynamics of monthly consumption growth:

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_t + \sigma_t \eta_{t+1} \\
x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1} \\
\sigma^2_{t+1} &= \bar{\sigma}^2 + \nu(\sigma^2_t - \bar{\sigma}^2) + \sigma_w w_{t+1}
\end{aligned}$$

Panel B reports the mean, the volatility and the first-order autocorrelation of annual consumption growth. "Data" column presents summary statistics of observed per-capita consumption of non-durables and services over the period from 1930 till 2002. Numbers in parentheses are robust standard errors calculated using the Newey-West variance-covariance estimator with 4 lags. The entries reported in "Model" column are based on 500 simulated samples, each with 876 months, time-aggregated to 73 annual observations. Model-implied statistics represent the median and the standard deviation (in parentheses) of the corresponding statistics across simulations.

Table IX

Dividend Growth Dynamics

Panel A: Calibration of Monthly Dividend Growth Rates

Asset	μ_d	ϕ	π	φ
Small	0.0058	5.4	1.5	7.3
Large	0.0015	2.3	3.3	5.7
Growth	0.0015	2.0	3.6	7.1
Value	0.0040	4.4	1.9	5.2
Market	0.0015	2.3	3.8	5.4

Panel B: Dynamics of Annual Dividend Growth Rates

Asset	Statistic	— **Data** —		— **Model** —	
Small	$E[\Delta d]$	6.57	(4.15)	6.63	(3.71)
	$\sigma(\Delta d)$	27.2	(4.19)	15.4	(2.15)
	$Corr(\Delta c, \Delta d)$	0.44	(0.09)	0.45	(0.14)
Large	$E[\Delta d]$	0.34	(1.09)	1.87	(2.05)
	$\sigma(\Delta d)$	10.5	(1.69)	12.4	(1.60)
	$Corr(\Delta c, \Delta d)$	0.50	(0.15)	0.54	(0.11)
Growth	$E[\Delta d]$	-0.26	(1.65)	1.77	(2.41)
	$\sigma(\Delta d)$	16.3	(1.94)	14.4	(1.80)
	$Corr(\Delta c, \Delta d)$	0.47	(0.10)	0.47	(0.11)
Value	$E[\Delta d]$	4.67	(3.40)	4.74	(2.81)
	$\sigma(\Delta d)$	28.6	(3.61)	11.8	(1.78)
	$Corr(\Delta c, \Delta d)$	0.51	(0.07)	0.57	(0.12)
Market	$E[\Delta d]$	0.74	(1.18)	1.87	(2.02)
	$\sigma(\Delta d)$	11.0	(1.92)	12.3	(1.63)
	$Corr(\Delta c, \Delta d)$	0.60	(0.14)	0.60	(0.10)

Panel A of Table IX presents the calibration of monthly dividend growth rates for the cross-section of assets:

$$\Delta d_{j,t+1} \quad = \quad \mu_{d_j} + \phi_j x_t + \pi_j \sigma_t \eta_{t+1} + \varphi_j \sigma_t u_{d,j,t+1}$$

The asset menu comprises small and large market capitalization firms, growth and value portfolios that represent low and high book-to-market firms respectively, and the aggregate stock market. Panel B reports the mean and the volatility of dividend growth rates, as well as their correlation with consumption growth. "Data" column presents summary statistics of the per-share dividend series observed over 1930-2002 time period. Numbers in parentheses are robust standard errors calculated using the Newey-West variance-covariance estimator with 4 lags. The entries reported in "Model" column are based on 500 simulated samples, each with 876 months, time-aggregated to 73 annual observations. Model-implied statistics represent the median and the standard deviation (in parentheses) of the corresponding statistics across simulations.

Table X

Asset Pricing Implications

Asset	Statistic	**— Data —**		**— Model —**	
Small	$E(R)$	16.60	(4.18)	15.40	(4.32)
	$\sigma(R)$	40.4	(3.84)	35.7	(6.48)
	$E(pd)$	4.07	(0.15)	3.59	(0.15)
Large	$E(R)$	7.58	(2.19)	7.90	(2.20)
	$\sigma(R)$	19.1	(1.79)	19.2	(2.74)
	$E(pd)$	3.30	(0.10)	3.22	(0.05)
Growth	$E(R)$	7.01	(2.40)	6.87	(2.59)
	$\sigma(R)$	21.6	(1.89)	21.1	(3.11)
	$E(pd)$	3.71	(0.15)	3.48	(0.05)
Value	$E(R)$	13.37	(3.03)	12.58	(3.05)
	$\sigma(R)$	33.1	(3.89)	26.4	(4.45)
	$E(pd)$	3.42	(0.15)	3.18	(0.11)
Market	$E(R)$	8.27	(2.10)	8.10	(2.17)
	$\sigma(R)$	20.1	(1.88)	19.6	(2.75)
	$E(pd)$	3.33	(0.11)	3.15	(0.05)
Risk-Free Rate	$E(R)$	0.76	(0.27)	1.08	(0.34)
	$\sigma(R)$	1.12	(0.22)	0.87	(0.18)

Table X presents asset pricing moments for five equity portfolios and the risk-free rate. Small and large are portfolios of firms with low and high market capitalization, growth and value correspond to the top and the bottom book-to-market deciles. $E(R)$, $\sigma(R)$ and $E(pd)$ denote expected returns, return volatilities and means of log price-dividend ratios respectively. "Data" column presents summary statistics of the observed annual data that span the period from 1930 to 2002. Numbers in parentheses are robust standard errors calculated using the Newey-West variance-covariance estimator with 4 lags. The entries reported in "Model" column are based on 500 simulated samples, each with 876 months, time-aggregated to 73 annual observations. Model-implied statistics represent the median and the standard deviation (in parentheses) of the corresponding statistics across simulations.

Table XI

Simulation Evidence: Long Run Risks Model

	Data	Pop Values	5%	50%	95%
ρ^a	0.78	0.81	0.63	0.77	0.87
$\bar{R}^2$	0.34	0.47	0.25	0.43	0.65
RA	15.12	10.11	5.42	13.60	25.16
IES	0.37	0.74	0.29	0.62	3.51
J-stat	4.30	11.84	0.91	3.96	7.80
P-value	0.37	0.01	0.10	0.41	0.92

Table XI presents population values and monte-carlo distributions of various parameters of interest: persistence in the extracted long-run component (ρ^a), predictability of consumption growth ($\bar{R}^2$), GMM estimates of the long-run risks model, J-statistic for overidentifying restrictions and the corresponding p-value. RA and IES denote risk aversion and the elasticity of intertemporal substitution, respectively. Population values are based on a simulated sample with 20,000 annual observations. Percentile cutoffs are obtained by simulating 500 samples, each with 876 months, time-aggregated to 73 annual observations. The asset menu consists of firms with small and large market capitalization, low and high book-to-market ratio, aggregate stock market and the risk-free rate. For comparison, the first column reports data estimates.

Table XII

SMM Evidence

Parameters	Estimate	SE
RA	10.28	1.99
IES	2.43	1.33
Assets	PrError	SE
Small	0.02	3.64
Large	-0.49	1.34
Growth	-0.07	1.82
Value	-0.46	1.24
Market	-0.17	1.01
Risk-Free	0.00	0.15
J-stat	4.10	
P-value	0.39	

Table XII presents the SMM estimates of the IES and risk aversion (RA), the pricing errors of the assets used in estimation, the J-statistic for overidentifying restrictions, and the corresponding p-value. The parameters governing the consumption and dividend dynamics are those given in Tables VIII and IX. The data employed in the estimation are annual and cover the period from 1930 to 2002.

Table XIII

Simulation Evidence: Pricing Kernel based on Market Return

	Long Sample
RA	1.78
IES	1.71
P-value	0.00

Table XIII presents GMM estimates of the long-run risks model, and the p-value of J-statistic of the overidentifying restrictions, for a pricing kernel, in which the return on consumption, $r_{c,t}$, is replaced with the market return, $r_{m,t}$. The model and estimation are based on monthly frequency. RA and IES denote risk aversion and the elasticity of intertemporal substitution respectively. The asset menu comprises firms with small and large market capitalization, low and high book-to-market ratio, aggregate stock market and the risk-free rate. The entries are based on a sample with 120,000 monthly observations.

Table XIV

Approximation Error

Panel A: Approximate Analytical Solutions

		Mean log(P/C)			Vol log(P/C)		
		IES			IES		
		0.5	**1.5**	**2**	**0.5**	**1.5**	**2**
	5	3.592	4.754	5.058	0.059	0.021	0.032
RA	**10**	3.789	4.572	4.716	0.060	0.021	0.032
	15	4.055	4.421	4.470	0.062	0.021	0.032

Panel B: Numerical Solutions

		Mean log(P/C)			Vol log(P/C)		
		IES			IES		
		0.5	**1.5**	**2**	**0.5**	**1.5**	**2**
	5	3.594	4.755	5.060	0.059	0.021	0.032
RA	**10**	3.788	4.576	4.724	0.060	0.021	0.032
	15	4.033	4.436	4.493	0.061	0.021	0.031

Panel C: Approximation Error (as a % of numerical values)

		Mean log(P/C)			Vol log(P/C)		
		IES			IES		
		0.5	**1.5**	**2**	**0.5**	**1.5**	**2**
	5	0.05	0.01	0.03	0.04	-0.16	-0.17
RA	**10**	-0.02	0.10	0.17	-0.42	-0.83	-0.86
	15	-0.54	0.32	0.51	-1.84	-2.16	-2.17

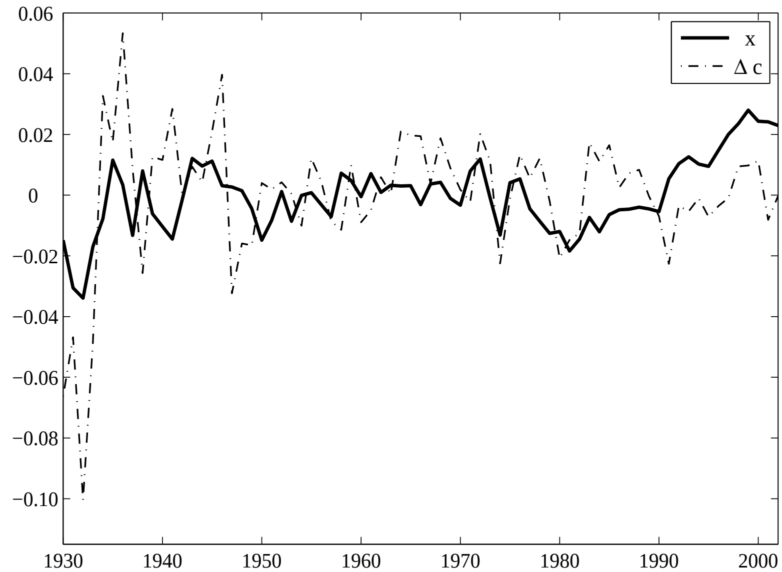

Figure 1. Realized and Expected Growth of Consumption

Figure 1 plots time series of realized (dash-dotted line) and expected (thick line) growth in consumption. Consumption is defined as the per-capita expenditure on non-durables and services. The expected consumption growth is constructed according to the predictability evidence presented in Table II. The data are real, sampled on an annual frequency and cover the period from 1930 to 2002.

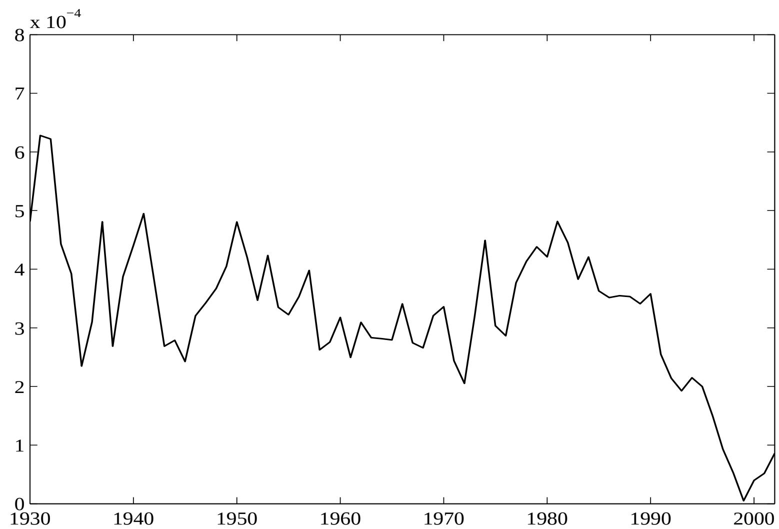

Figure 2. Conditional Volatility of Consumption Growth

Figure 2 plots time series of the extracted volatility component of consumption growth. Consumption is defined as the per-capita expenditure on non-durables and services; data are real, sampled on an annual frequency and cover the period from 1930 to 2002.

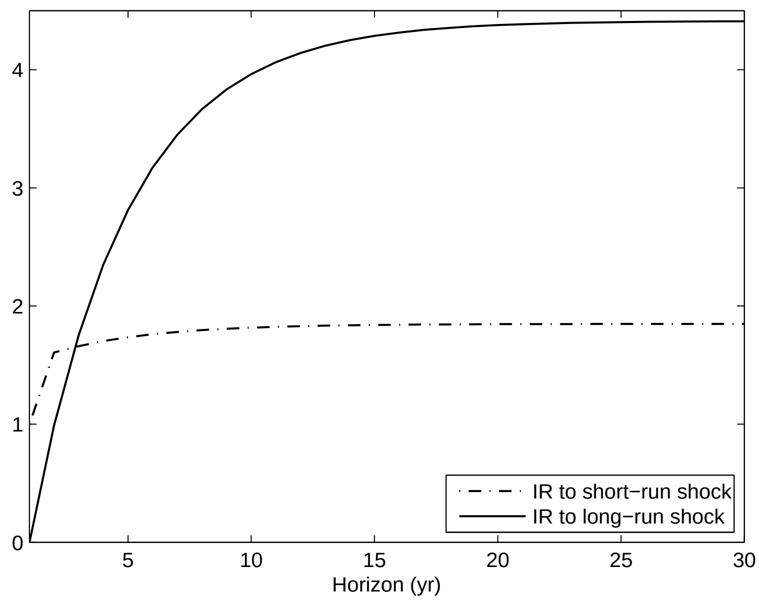

Figure 3. Accumulated Impulse Response of Consumption

Figure 3 plots the accumulated impulse response of consumption to a short-run shock, η_t^a (dash line) and to a long-run shock, e_t^a. These Impulse response functions are based on the estimated system (18).

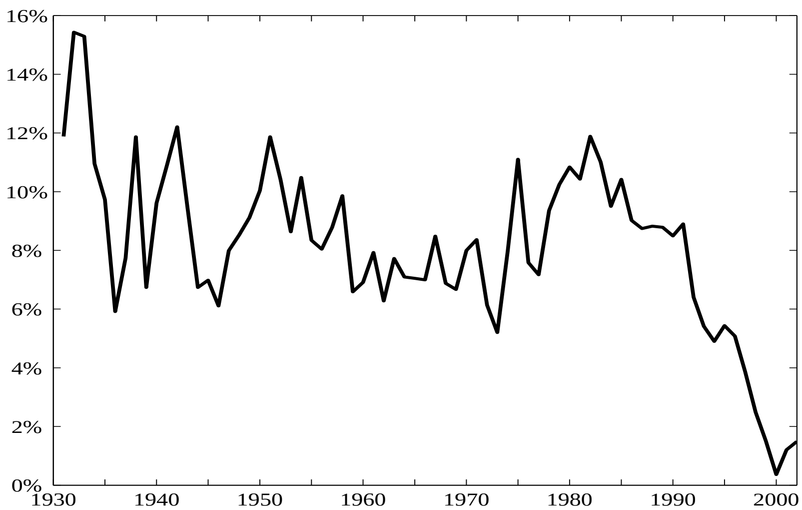

Figure 4. Conditional Equity Premium

Figure 4 plots time series of the conditional equity premium. The risk premium on the aggregate market portfolio is constructed using the GMM estimates of the long-run risks model. The data employed in estimation are real, sampled on an annual frequency and cover the period from 1930 to 2002.