Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles *

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Abstract

We model consumption and dividend growth rates as containing (i) a small long-run predictable component and (ii) fluctuating economic uncertainty. The magnitudes of the predictable variation and changing volatility in growth rates are quite small. These growth rate dynamics, for which we provide empirical support, in conjunction with plausible parameter configurations of the Epstein and Zin (1989) preferences can explain key observed asset markets phenomena. Our model captures the intuition that financial markets dislike economic uncertainty and that economic growth prospects are important for asset prices. We show that the model can justify the observed equity premium, the risk free rate, and the ex-post volatilities of the market return, real risk free rate, and the price-dividend ratio. As in the data, the model also implies that dividend yields predict returns and that market return volatility is stochastic.

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1 Introduction

Several key aspects of asset market data pose a serious challenge to economic models. Many economic models find it difficult to justify the 6% equity premium and the low risk free rate (Mehra and Prescott (1985), Weil (1989), and Hansen and Jagannathan (1991)). In addition, the risk premia on equities are time-varying and move countercyclically (see Fama and French (1989), Campbell and Shiller (1988a), and Cochrane (1992)). Further, price-dividend ratios predict long horizon equity returns (see Campbell and Shiller (1988b)).

The literature on volatility tests highlights the difficulty in justifying the observed market volatility of 19% per annum (see Shiller (1981) and LeRoy and Porter (1981)). Additionally, the conditional variance of the market return, as shown by Bollerslev, Engle, and Wooldridge (1988), fluctuates across time and is very persistent. Further, as documented in this paper, realized volatility of consumption growth rates and price-dividend ratios are significantly negatively correlated at long leads and lags. This new evidence highlights an important link between economic uncertainty and asset prices.

We present a model that helps explain the above features of asset market data. There are two main ingredients in the model. First, we rely on the standard Epstein and Zin (1989) preferences, which allow for a separation between the intertemporal elasticity of substitution (IES) and risk aversion, and consequently permits both parameters to be simultaneously larger than one. Second, we model consumption and dividend growth rates as containing (i) a small persistent expected growth rate component, and (ii) fluctuating volatility – which captures time-varying economic uncertainty. We show that this specification for consumption and dividends is consistent with observed annual consumption and dividend data. Our model captures the (Wall Street) intuition that financial markets dislike economic uncertainty, and fluctuating growth prospects are important for asset valuations (e.g., see Graham and Dodd (1934)). In our economy, agents demand large equity risk premia as they fear that a reduction in economic growth prospects or a rise in economic uncertainty will lower asset prices. Our results show that risks related to varying growth prospects and fluctuating economic uncertainty can indeed quantitatively justify many of the observed features of asset market data. In a recent paper Hall (2001) also underscores the importance of low frequency movements in growth rates for interpreting asset prices.

What is the importance of persistence in the growth prospects, i.e., the expected growth rate of the economy? In a partial equilibrium model, Barsky and DeLong (1993) show that persistence in expected dividend growth rates is an important source for volatility of price-dividend ratios. In particular, they show that the dividend elasticity of equity prices increases as the persistence of expected dividend growth rates rise. In our equilibrium model
the degree of persistence in expected growth rate news not only affects the volatility of the price-dividend ratio (as in Barsky and DeLong (1993)), but it also critically determines the risk-premium on the asset. That is, news regarding future expected growth rates leads to large reactions in the price-dividend ratio and the ex-post equity return; these reactions positively co-vary with the consumption (more precisely with the IMRS) of the representative agent, and hence lead to large equity risk-premia. Not surprisingly, the dividend elasticity of asset prices and the risk premia on assets rise as the degree of permanence of expected dividend growth rates increases. We formalize this intuition in section 2 using a general equilibrium model that incorporates fluctuations in growth prospects, but to build economic intuition, it abstracts from time-varying economic uncertainty.

The above model is one in which all risk premia and Sharpe Ratios are constant. To allow for time varying risk-premia and Sharpe Ratios we augment the above model to incorporate fluctuations in economic uncertainty. We model varying economic uncertainty as changes in the conditional variance of future growth rates. The specific model we use for this is a standard GARCH(1,1) model (see Bollerslev (1986)). What does this accomplish? The inclusion of changing economic uncertainty directly affects the price-dividend ratios; a rise in economic uncertainty leads to a fall in asset prices. Further, in our economy, fluctuations in economic uncertainty carry a positive risk premia. Quantitatively, about half of the volatility of price-dividend ratios in the model can be attributed to variation in expected growth rates and the remaining to variation in economic uncertainty (or expected return). This is distinct from models where growth rates are i.i.d, and consequently, all the variation in price-dividend ratio is attributed to changing cost of capital.

A core issue is whether observed consumption and dividend data is consistent with a persistent expected growth rate process and with fluctuating economic uncertainty. For these channels to have a significant quantitative impact on the equilibrium risk premium and volatility of asset prices, the persistence in expected growth rate has to be quite large, close to 0.98.\textsuperscript{1} At first brush this might seem implausible given the low autocorrelations observed in the consumption and dividend growth rate data. However, this is not true. Shephard and Harvey (1990), and Barsky and DeLong (1993) show that in finite samples, a growth rate process that contains a small persistent expected growth rate component plus a volatile i.i.d process, cannot be distinguished from a purely i.i.d process. Further, Shephard and Harvey (1990) show that the evidence will be stacked against finding the small persistent component in favor of the i.i.d process. In other words, it is quite difficult, in finite samples, for an econometrician to distinguish between an i.i.d growth rate process and a process where

\textsuperscript{1}Barsky and DeLong (1993) choose a value of one. Our choice ensures that the growth rate process is stationary.
the fluctuations in expected growth rates are persistent but quite small (i.e., small variance). However, the asset pricing implications of allowing for such a process are very different from those of assuming that ex-post growth rates are i.i.d. Given the well recognized difficulties in discriminating across alternative descriptions of the data, Anderson, Hansen, and Sargent (2000) argue for modeling economic agents as making robust economic decisions.

We provide direct empirical evidence which motivates our channel of fluctuating economic uncertainty. We find that there is an intimate link between realized volatility of consumption growth rates and observed price-dividend ratios. We document that in annual data, price-dividend ratios are significantly correlated at long leads and lags with the ex-post consumption volatility (i.e., absolute value of consumption residuals). Further, the variance ratios of realized consumption volatility increase up to 15 years; this provides additional support in favor of fluctuating economic uncertainty in aggregate consumption data.\(^2\)

In terms of preferences, all our results are based on a risk aversion of under 10 and an IES that is somewhat larger than 1. There is considerable debate about what are reasonable magnitudes for these parameters. Mehra and Prescott (1985) argue that risk aversion of 10 and below seems reasonable. The magnitude of the IES is also a subject of considerable controversy. As discussed below, our estimates of the IES are consistent with the findings of Hansen and Singleton (1982) and many other authors. In addition, we show that the presence of fluctuating economic uncertainty leads to a serious downward bias in estimating the IES when using the regression approach used in Hall (1988). This bias may help interpret the small estimates of the IES in Hall (1988).

There is a voluminous literature that addresses the aforementioned asset market anomalies. Notable examples, Abel (1990), Abel (1999), Bansal and Coleman (1997), Barberis, Huang, and Santos (2001), Campbell (1996), Campbell and Cochrane (1999), Cecchetti, Lam, and Mark (1990), Chapman (2001), Constantinides (1990), Constantinides and Duffie (1996), Hansen, Sargent, and Tallarini (1999), Heaton (1995), Heaton and Lucas (1996), and Kandel and Stambaugh (1991) address various aspects of the asset market anomalies discussed above. The approaches taken to address these asset market phenomena include transaction costs, incomplete markets, and time-non-separable preferences. This paper, in contrast, relies on the channels associated with fluctuating expected growth rates and economic uncertainty to quantitatively explain key aspects of asset markets.

In summary, our model captures many key aspects of asset markets data and is consistent with annual consumption and dividend data. It captures the intuition that a rise in economic uncertainty lowers asset valuations and a rise in expected economic growth raises asset

\(^2\)Note that if residuals of consumption growth are i.i.d then the absolute value of these residuals will not be predictable and the variance ratios will be flat across different horizons.
valuations. Asset prices fluctuate a lot in response to small but long lasting innovations in expected growth rates and economic uncertainty. The remainder of the paper is organized as follows. In section 2 we formalize this intuition and present the economics behind our model. The data are described in section 3. In section 4 we provide the model’s quantitative results. The last section provides concluding comments.

2 An Economic Model for Asset Markets

We consider a representative agent with the Epstein and Zin (1989) - Weil (1989) recursive preferences. For these preferences, Epstein and Zin (1989) show that the asset pricing restrictions for gross return $R_{i,t+1}$ satisfy:

$$E_t[\delta^\theta G_{t+1} R_{a,t+1}^{(1-\theta)} R_{i,t+1}] = 1,$$  \hspace{1cm} (1)

where $G_{t+1}$ is the aggregate gross growth rate of consumption, and $R_a$ is the gross return on an asset paying off aggregate consumption, $0 < \delta < 1$ is the time discount factor, $\theta \equiv \frac{1-\gamma}{1-\psi}$, where $\gamma \geq 0$ is the risk-aversion (sensitivity) parameter, and $\psi \geq 0$ is the intertemporal elasticity of substitution. The sign of $\theta$ is determined by the magnitudes of the risk-aversion and the elasticity of substitution.

We distinguish between the unobservable return on a claim to aggregate consumption, $R_{a,t+1}$, and the observable return on the market portfolio $R_{m,t+1}$; the latter is the return on the aggregate dividend claim. As in Campbell (1996), we model aggregate consumption and aggregate dividends as two separate processes; the agent is implicitly assumed to have access to labor income.

Although we ultimately solve our model numerically, we demonstrate the mechanisms working in our model via approximate analytical solutions. To derive analytical solutions for the model we use the standard approximations derived in Campbell and Shiller (1988a), and Campbell (1993),

$$r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}$$  \hspace{1cm} (2)

where lower case letters refer to logs, so that $r_{a,t+1} = \log(R_{a,t+1})$ is the continuous return, $z_t = \log(P_t/C_t)$ is the log price-consumption ratio, and $\kappa_0$ and $\kappa_1$ are approximating constants that both depend only on the average level of $z$.

Analogously, $r_{m,t+1}$ and $z_{m,t}$

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3In particular if $\psi > 1$ and $\gamma > 1$ then $\theta$ will be negative. Note that when $\theta = 1$, that is $\gamma = (1/\psi)$, the above recursive preferences collapse to the standard case of expected utility.

4Note that $\kappa_1 = \exp(\bar{z})/(1 + \exp(\bar{z}))$. $\kappa_1$ is approximately 0.997, which is consistent with the magnitude of $\bar{z}$ in our sample and with magnitudes used in Campbell and Shiller (1988a).
correspond to the market return and its log price-dividend ratio.

The logarithm of the Intertemporal Marginal Rate of Substitution (IMRS) is

\[ m_{t+1} = \theta \log \delta - \frac{\theta}{\psi}g_{t+1} + (\theta - 1)r_{a,t+1}. \]

It follows that the innovation in \( m_{t+1} \) is driven by the innovations in \( g_{t+1} \) and \( r_{a,t+1} \). Covariation with the innovation in \( m_{t+1} \) determines the risk premium for any asset. When \( \theta \) equals one, the above IMRS collapses to the usual case of power utility. To present the intuition of our model in a simple manner we first discuss the simpler case (Case I) in which there are fluctuations only in the expected growth rates. Subsequently, we present the complete model (Case II) which also includes fluctuating economic uncertainty.

### 2.1 Case I: Fluctuating Expected Growth Rates Only

We first solve for the consumption return \( r_{a,t+1} \), as this determines the pricing kernel and consequently risk premia on the market portfolio, \( r_{m,t+1} \), as well as all other assets. To do so we first specify the dynamics for consumption and dividend growth rates. We model consumption and dividend growth rates, \( g_{t+1} \) and \( g_{d,t+1} \) respectively, as containing a small persistent predictable component \( x_t \) – which determines the conditional expectation of consumption growth,

\[
\begin{align*}
    x_{t+1} &= \rho x_t + \varphi_x \sigma_e t + e_{t+1} \\
    g_{t+1} &= \mu + x_t + \sigma \eta_{t+1} \\
    g_{d,t+1} &= \mu_d + \phi x_t + \varphi_d \sigma u_{t+1}
\end{align*}
\]

and all three shocks, \( e_{t+1}, u_{t+1} \) and \( \eta_{t+1} \) are mutually independent. The above growth rate dynamics are also utilized by Campbell (1999), Cecchetti, Lam, and Mark (1993), and Wachter (2002) to model consumption growth rate. \( \phi > 1 \) and \( \varphi_d > 1 \) are two additional parameters that allow us to calibrate the overall volatility of dividends (which in the data is significantly larger than that of consumption) and its correlation with consumption. The parameter \( \phi \) can be interpreted as in Abel (1999) as the leverage ratio on expected consumption growth.\(^5\)

The parameter \( \rho \) determines the persistence of the expected growth rate process. First, note that when \( \varphi_e = 0 \), the processes \( g_t \) and \( g_{d,t+1} \) are \( i.i.d. \). Second, if \( e_{t+1} = \eta_{t+1} \), the

\(^5\) This characterization of the data implies that dividends and consumption are not co-integrated. We pursue this approach, as Campbell and Cochrane (1999), due to its simplicity.
process for consumption is the ARMA(1,1) used in Bansal and Yaron (2000). Additionally, if \( \varphi_e = \rho \) then consumption growth as in Mehra and Prescott (1985) is an AR(1) process. In general \( \epsilon_{t+1} \) and \( \eta_{t+1} \) can have arbitrary correlation but to maintain parsimony in terms of parameters we set this correlation to zero.

Since \( g \) and \( g_d \) are exogenous processes, a solution for the log price-consumption ratio \( z_t \) and the log price-dividend ratio \( z_{m,t} \) leads to a complete characterization of the returns \( r_{a,t+1} \) and \( r_{m,t+1} \) (using equation (2)). The relevant state variable for deriving the solution for \( z_t \) and \( z_{m,t} \) is the expected growth rate of consumption \( x_t \). Exploiting the Euler equation (1), the solution for the log price-consumption \( z_t \), has the form \( z_t = A_0 + A_1 x_t \). An analogous expression holds for the log price-dividend ratio \( z_{m,t} \). Details of both derivations are provided in the appendix.

The solution coefficients for the effect of expected growth rate \( x_t \) on the price-consumption ratio, \( A_1 \), and the price-dividend ratio, \( A_{1,m} \), respectively are

\[
A_1 = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho} \quad A_{1,m} = \frac{\phi - \frac{1}{\psi}}{1 - \kappa_{1,m} \rho}.
\]

(4)

It immediately follows that \( A_1 \) is positive if IES, \( \psi \), is greater than one. In this case the intertemporal substitution effect dominates the wealth effect. In response to higher expected growth (higher expected rates of return), agents respond by buying more assets and consequently wealth to consumption ratio rises. In the standard power utility model, the need to have risk aversion larger than one also implies that \( \psi < 1 \) and hence \( A_1 \) is negative. Consequently, the wealth effect dominates the substitution effect.\(^6\) Finally, the positivity of \( A_1 \) captures the usual intuition of the textbook Gordon Growth formula that higher expected growth, all else equal, should increase the valuation. In addition, note that \( A_{1,m} > A_1 \) when \( \phi > 1 \); consequently, expected growth rate news leads to a larger reaction in the price of the dividend claim than in the price of the consumption claim.

Substituting the equilibrium return for \( r_{a,t+1} \) into the IMRS, it is straightforward to show that the innovation to the pricing kernel is (see equation (15) in the appendix)

\[
m_{t+1} - E_t(m_{t+1}) = \left[ -\frac{\theta}{\psi} + \theta - 1 \right]\sigma \eta_{t+1} - (1 - \theta)[\kappa_1 (1 - \frac{1}{\psi}) \frac{\varphi_e}{1 - \kappa_1 \rho}]\sigma e_{t+1} \\
= \lambda_{m,e} \sigma \eta_{t+1} - \lambda_{m,e} \sigma e_{t+1}
\]

(5)

\( \lambda_{m,e} \) and \( \lambda_{m,n} \) capture the pricing kernel’s exposure to the expected growth rate and the

\(^6\)An alternative interpretation with power utility model is that higher expected growth rates increase the risk free rate to an extent that discounting dominates the effects of higher expected growth rates. This leads to a fall in asset prices.
independent consumption shocks, $\eta_{t+1}$. The key observation is that the exposure to expected growth rate shocks $\lambda_{m,e}$ rises as the permanence parameter $\rho$ rises. The conditional volatility of the pricing kernel is constant as all risk sources have constant conditional variances.

As asset returns and the pricing kernel in this model economy are conditionally log-normal, the continuous risk premium on any asset $i$, $E_t[r_{i,t+1} - r_{f,t}] = -\text{cov}_t(m_{t+1}, r_{i,t+1}) - 0.5 \sigma^2_{r_{i,t}}$. Given the solutions for $A_1$ and $A_{1,m}$, it is straightforward to derive the equity premium on the market portfolio (see section 6.1.3 in the appendix),

$$E(r_{m,t+1} - r_{f,t}) = \beta_{m,e} \lambda_{m,e} \sigma^2 - 0.5 \text{Var}(r_{m,t}), \quad (6)$$

where $\beta_{m,e} \equiv [\kappa_1, m (\phi - \frac{1}{\psi}) \frac{\varphi}{1 - \kappa_1, m \rho}]$ and $\text{Var}_t(r_{m,t+1}) = [\beta^2_{m,e} + \varphi^2_d] \sigma^2$. The exposure of the market return to expected growth rate news is $\beta_{m,e}$. The expression for this reveals that a rise in $\rho$ simultaneously increases $\beta_{m,e}$ and the market price of expected growth rate risk $\lambda_{m,e}$. Consequently, the asset's risk premium also rises. We show that similar arguments imply that the volatility of the market return is also magnified by a rise in the permanence of expected growth rate news (see equation (22) in the appendix).

Due to our assumption of a constant $\sigma$, the conditional risk premium on the market portfolio in (6) is constant, and so is its conditional volatility. Hence, the ratio of the two, namely the Sharpe Ratio is also constant. In order to address issues that pertain to time-varying risk premia and predictability of risk premia, we augment our model in the next section and introduce time-varying economic uncertainty.

### 2.2 Case II: Incorporating Fluctuating Economic Uncertainty

We model fluctuating economic uncertainty as time-varying volatility of consumption growth. We assume (as in Bollerslev, Engle, and Wooldridge (1988)) that this volatility follows a GARCH(1,1) process. We modify the system (3) to:

$$
\begin{align*}
x_{t+1} &= \rho x_t + \varphi \sigma_t e_{t+1} \\
g_{t+1} &= \mu + x_t + \sigma_t \eta_{t+1} \\
g_{d,t+1} &= \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1} \\
\sigma^2_{t+1} &= \sigma^2 + \nu \left( \sigma^2_t - \sigma^2 \right) + \sigma_w w_{t+1} \\
\end{align*}
$$

$$
e_{t+1}, u_{t+1}, \eta_{t+1}, w_{t+1} \sim \text{N.i.i.d}(0,1),
$$

where $\sigma_{t+1}$ represents the time varying economic uncertainty incorporated in consumption growth rate and $\sigma^2$ is its unconditional mean. To maintain parsimony, we assume that the
shocks are uncorrelated, and allow for only one source of economic uncertainty to affect consumption and dividends.

The relevant state variables in solving for the equilibrium price-consumption (and price-dividend) ratio are now \( x_t \) and \( \sigma_t^2 \). Thus, the approximate solution for the price consumption ratio is \( z_t = A_0 + A_1 x_t + A_2 \sigma_t^2 \). The solution for \( A_1 \) is unchanged (equation (4)). The solution coefficient \( A_2 \) for measuring the sensitivity of price-consumption ratios to volatility fluctuations is

\[
A_2 = \frac{0.5\left( (\theta - \frac{\theta}{\psi})^2 + (\theta A_1 \kappa_1 \varphi_e)^2 \right)}{\theta(1 - \kappa_1 \nu_1)}.
\]  

(8)

An analogous coefficient for the price-dividend ratio, \( A_{2,m} \), is derived in the appendix and has similar a form. There are two relevant observations. First, if IES, and risk aversion are larger than one, then \( \theta \) is negative, and a rise in volatility lowers the price-consumption ratio. Similarly, an increase in economic uncertainty raises risk premia and lowers the market price-dividend ratio. Second, an increase in the permanence of volatility shocks, that is \( \nu_1 \), magnifies the effects of volatility shocks on valuation ratios as changes in economic uncertainty are perceived as being long lasting.

Note that since the price-consumption ratio is affected by volatility shocks so is the return \( r_{a,t+1} \). Consequently, the pricing kernel (IMRS) is also affected by volatility shocks. Specifically, the innovation in the pricing kernel is now:

\[
m_{t+1} - E_t(m_{t+1}) = \lambda_{m,n} \sigma_t \eta_{t+1} - \lambda_{m,e} \sigma_t e_{t+1} - \lambda_{m,w} \sigma_w w_{t+1},
\]  

(9)

where \( \lambda_{m,w} \equiv (1 - \theta) A_2 \kappa_1 \), while \( \lambda_{m,n} \) and \( \lambda_{m,e} \) are defined in equation (5). This expression is the same as in the earlier model (see equation (5)), except for the inclusion of the risk term \( w_{t+1} \) which captures the effects of shocks to consumption volatility. Clearly, in the special case of power utility, where \( \theta = 1 \), volatility innovations are not reflected in the innovation of the pricing kernel.\(^7\)

The equation for the equity premium will now have two sources of systematic risk. The first one is the innovation to the expected growth rate which will now be time-varying. In addition, with the Epstein-Zin preferences, the time-varying volatility will be priced as well.

\(^7\)Recall that in our specification the conditional volatility and expected growth rate processes are independent. With power utility, the volatility shocks will not be reflected in the innovations of the IMRS. With Epstein and Zin (1989) preferences, in spite of this independence, volatility shocks influence the innovations in the pricing IMRS.
The equity premium in the presence of time-varying economic uncertainty is therefore

\[ E_t(r_{m,t+1} - r_{f,t}) = \beta_{m,e}\lambda_{m,e}\sigma_t^2 + \beta_{m,w}\lambda_{m,w}\sigma_w^2 - 0.5 \text{Var}_t(r_{m,t+1}) \]  

(10)

where \( \beta_{m,w} \equiv \kappa_{1,m}A_{2,m} \) and \( \text{Var}_t(r_{m,t+1}) = \{\beta_{m}\sigma_t^2 + \varphi_d\sigma_t^2 + \beta_{m,w}\sigma_w^2\} \).

It is important to note that now the risk premium is time-varying as \( \sigma_t \) fluctuates. Further, \( \lambda_{m,w} \) is the market compensation for volatility risk associated with \( w_{t+1} \) – that is, volatility risk carries systematic risk compensation. Given the above expressions, the ratio of the conditional risk premium to the conditional volatility of the market portfolio fluctuates with \( \sigma_t \), and hence the Sharpe Ratio is time-varying. Further, note that the maximal Sharpe Ratio in this model economy, which approximately equals the conditional volatility of the pricing kernel innovation (equation (5)), is also time-varying as \( \sigma_t \) fluctuates. For further discussion on the specialization of the risk premia under expected utility see Bansal and Yaron (2000).

The first order effects on the level of the risk free rate (see equation (23) in the appendix) are the rate of time preference and the average consumption growth rate, divided by the IES. Increasing the IES keeps the level low. In addition, the variance of the risk-free rate is primarily determined by the volatility of expected consumption growth rate and the IES. Increasing the IES lowers the volatility of the risk free rate.

### 3 Data: Expected Growth & Fluctuating Uncertainty

To derive asset market implications of the model described in (7), we calibrate the model at the monthly frequency such that its time aggregated annual growth rates of consumption and dividends match salient features of their annual counterparts in the data, and at the same time allow the model to reproduce many observed asset pricing features.\(^8\) We follow Campbell and Cochrane (1999) and Kandel and Stambaugh (1991), and calibrate the model so the decision interval of the agent is monthly but the targeted data to be matched is annual. Throughout the paper we use the following set of parameters:

\(^8\)In the numerical solutions we use standard projection methods as in Judd (1998).
Table 1: Model Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Growth Rate Dynamics</strong></td>
<td></td>
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<tr>
<td>Mean consumption growth</td>
<td>$\mu$</td>
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<tr>
<td>Persistence of expected growth rate</td>
<td>$\rho$</td>
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</tr>
<tr>
<td>Conditional volatility of expected consumption</td>
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<tr>
<td>Conditional volatility of consumption</td>
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<tr>
<td>'Levered' dividend</td>
<td>$\phi$</td>
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<tr>
<td>Conditional volatility of dividends</td>
<td>$\varphi_d$</td>
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<tr>
<td><strong>Fluctuating Economic Uncertainty</strong></td>
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<tr>
<td>Persistence of volatility</td>
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<tr>
<td>Conditional volatility of economic uncertainty</td>
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<td><strong>Preferences</strong></td>
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<tr>
<td>Intertemporal Elasticity of Substitution</td>
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<tr>
<td>Rate of time preference</td>
<td>$\delta$</td>
<td>0.998</td>
</tr>
</tbody>
</table>

Model parameters pertain to the process in (7). Where relevant the values are annualized, e.g., $\mu \times 1200$, $\sigma \times \sqrt{12} \times 100$, $\rho^{12}$, etc. In addition, $\mu_d = \mu$.

3.1 Parameter Choices

Our choices of the time series and preference parameters are designed to simultaneously match observed growth rate data and asset market data. In order to isolate the economic effects of persistent expected growth rates from those of fluctuating economic uncertainty, we report our results first for Case I where fluctuating economic uncertainty has been shut off ($\sigma_w$ is set to zero), and then consider the model specification where both channels are operational.

Barsky and DeLong (1993) rely on a persistence parameter $\rho$ equal to one. Our choice of monthly persistence of 0.98 (0.784 annually) is motivated to ensure that expected growth rates are stationary and permit the possibility of considerable magnification effects on the dividend elasticity of equity prices and equity risk premia. Our choice of $\varphi_e$ and $\sigma$ is motivated to ensure that we match the unconditional variance and the autocorrelation function of annual consumption growth. Our choice of $\phi$ is very similar to that in Abel (1999) and captures the “levered” nature of dividends. This choice in conjunction with the
chosen value of $\varphi_d$ allows us to match the unconditional variance of dividend growth and its annual correlation with consumption. For the model in which we allow for fluctuating economic uncertainty we additionally need to calibrate $\sigma_w$ and $\nu_1$. These are chosen to match the unconditional variance and the persistence of squared residuals of annual consumption growth. The annualized standard deviation of the conditional volatility of consumption growth is very small (only .022% relative to the overall consumption volatility of 2.9%). However, volatility shocks are very long lasting as captured by $\nu_1$. Consequently, as documented below, fluctuations in volatility will explain a significant portion of the variation of price-dividend ratios.

3.1.1 Persistence in Expected Growth Rates

Since our model emphasizes the long horizon implications of the predictable component $x_t$, we first demonstrate that our proposed process for consumption is consistent with annual consumption data along a variety of dimensions. We use BEA data on real per capita annual consumption growth of non-durables and services for the period of 1929 to 1998. This is the longest single data source of consumption data. Dividends and the value weighted market return data are taken from CRSP. All nominal quantities are deflated using the CPI. To facilitate comparisons between the model, which is calibrated to a monthly decision interval, and the annual data we time aggregate our monthly model and report its annual statistics. As there is considerable evidence of small sample biases in estimating autoregression coefficients and Variance Ratios (see Hurwicz (1950), Ansley and Newbold (1980), and Campbell and Mankiw (1987)), we report statistics based on 1000 Monte-Carlo experiments each with 840 monthly observations — each experiment corresponding to the 70 annual observations available in our data set. Increasing the size of the Monte-Carlo makes little difference to the results.

The real per-capita consumption growth mean is about 1.8% and its standard deviation is about 2.9%. The volatility is somewhat lower for our sample than for the period considered in Mehra and Prescott (1985), Kandel and Stambaugh (1991), and Abel (1999). Table 2 shows that in the data consumption growth has a large first autocorrelation coefficient and a small second order one. The standard errors on these auto-correlations are sizeable. One simple way to view the long horizon properties of the model is to use variance ratios (see Cochrane (1989)). In the data the variance ratios first rise significantly and at about 7 years out start to decline. The standard errors on these variance ratios are quite substantial.

The model implies a first autocorrelation that is similar to that in the data. The second, fifth and tenth autocorrelations are also similar and are well within one standard error of the data. It is important to note that the predictable component in consumption, $x_t$, is
only a small component of the overall variance of consumption. In particular, the $R^2$ for the consumption process is about 5% (on a monthly basis). The model’s variance ratios mimic the pattern in the data. The point estimates are slightly larger than the data, but they are well within one standard error of the data. As in the data, the model’s variance ratios also eventually decline. In the simpler case (Case I), the volatility of consumption and dividend growth rates is less than the counterparts in the data. However, in the augmented model that includes fluctuating economic uncertainty, the volatility of consumption and dividend growth is higher and closely matches that in the data. In addition, the correlation of dividends with consumption of about 0.3 is similar across the models and is somewhat lower, but within one standard error of its estimate in the data. Other than the larger volatilities of consumption and dividends in Case II, the two cases produce similar annual time series dynamics. Overall, the message of Table 2 is that allowing for a persistent predictable component in consumption is largely consistent with the data.

Table 2: Annualized Time Averaged Growth Rates

<table>
<thead>
<tr>
<th>Variable</th>
<th>Data</th>
<th>Model Case I</th>
<th>Model Case II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma(g)$</td>
<td>2.93 (0.69)</td>
<td>2.62</td>
<td>2.88</td>
</tr>
<tr>
<td>$AC(1)$</td>
<td>0.49 (0.33)</td>
<td>0.48</td>
<td>0.48</td>
</tr>
<tr>
<td>$AC(2)$</td>
<td>0.15 (0.41)</td>
<td>0.24</td>
<td>0.25</td>
</tr>
<tr>
<td>$AC(5)$</td>
<td>-0.08 (0.44)</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>$AC(10)$</td>
<td>-0.12 (0.41)</td>
<td>-0.05</td>
<td>-0.05</td>
</tr>
<tr>
<td>$VR(2)$</td>
<td>1.61 (0.10)</td>
<td>1.41</td>
<td>1.42</td>
</tr>
<tr>
<td>$VR(5)$</td>
<td>2.01 (0.75)</td>
<td>2.24</td>
<td>2.25</td>
</tr>
<tr>
<td>$VR(10)$</td>
<td>1.57 (1.68)</td>
<td>2.71</td>
<td>2.73</td>
</tr>
<tr>
<td>$VR(15)$</td>
<td>1.08 (2.21)</td>
<td>2.21</td>
<td>2.23</td>
</tr>
<tr>
<td>$\sigma(g_d)$</td>
<td>11.39 (1.22)</td>
<td>8.52</td>
<td>11.17</td>
</tr>
<tr>
<td>$corr(g, g_d)$</td>
<td>0.55 (0.35)</td>
<td>0.37</td>
<td>0.32</td>
</tr>
</tbody>
</table>

Case I pertains to the model with fluctuations in expected growth rate but without time varying uncertainty, i.e., $\sigma_w = 0$. Case II adds fluctuating economic uncertainty. The model parameters are given in Table 1. The statistics for the data are based on annual observations from 1929 to 1998. Consumption is real non-durables and services (BEA); dividends are from the CRSP Value Weighted return. $AC(j)$ is the $j^{th}$ autocorrelation, $VR(j)$ is the $j^{th}$ variance-ratio. Standard errors are Newey and West (1987) corrected using 10 lags. The
statistics for the model are based on 1000 simulations each with 840 monthly observations that are time aggregated.

It is often argued that consumption growth is close to being \( i.i.d \). As shown in Table 2 the consumption dynamics which contain a persistent but small predictable component are also largely consistent with the data. As discussed earlier, Shephard and Harvey (1990) and Barsky and DeLong (1993) show that, in finite samples, discrimination across the \( i.i.d \) growth rate model and the one considered above is extremely difficult. While the financial market data is hard to interpret from the perspective of the \( i.i.d \) dynamics, it is, as shown below, interpretable from the perspective of the growth rate dynamics considered above.

### 3.1.2 Fluctuating Economic Uncertainty

It is well recognized in the literature that it is difficult to detect high frequency GARCH effects once the data is time aggregated (see Nelson (1991), Drost and Nijman (1993)). Nevertheless, there is evidence in support of fluctuating economic uncertainty in annual data.

In Panel A of Table 3 we show that the Variance Ratios of the absolute value of residuals from regressing current consumption growth on five lags increase gradually out to 10 years. This suggests slow moving predictable variation in this measure of realized volatility.

| Horizon | Panel A: Variance Ratios | Panel B: Predicting \( |\epsilon_{g^a,t+j}| \) |
|---------|--------------------------|---------------------------------|
|         | VR(\( j \)) | S.E. | B(\( j \)) | S.E. | \( R^2 \) |
| 2       | 0.95 | (0.38) | -0.11 | (0.04) | 0.06 |
| 5       | 1.26 | (1.09) | -0.10 | (0.05) | 0.04 |
| 10      | 1.75 | (2.46) | -0.08 | (0.08) | 0.03 |

The entries in Panel A are the Variance Ratios for \( |\epsilon_{g^a,t}| \) which is the absolute value of the residual from regressing \( g_{t}^a = \sum_{j=1}^{5} A_{j} g_{t-j}^a + \epsilon_{g^a,t} \), and \( g_{t}^a \) denotes annual consumption growth rate. Panel B provides regression results of \( |\epsilon_{g^a,t+j}| = \alpha + B(j)(p_{t} - d_{t}) + v_{t+j} \), and \( j \) indicates the forecast horizon in years. The statistics are based on annual observations from 1929 to 1998 of real non-durables and services consumption (BEA); The price-dividend ratio is based on CRSP Value Weighted Return. Standard errors are Newey and West (1987) corrected using 10 lags.

In panel B of Table 3 we provide evidence which shows that future realized consumption volatility is predicted by current price-dividend ratios. The current price-dividend ratio predicts future realized volatility with negative coefficients, with robust t-statistics around
2, and $R^2$s around 5%. If consumption volatility was not time-varying, the slope coefficient on the time-varying price-dividend ratio would indeed be zero. As suggested by our theoretical model, this evidence indicates that information regarding persistent fluctuations in economic uncertainty is contained in asset prices. Overall, the evidence in Table 3 lends support to the view that the conditional volatility of consumption is time-varying. Figure 1 also displays the variation in realized consumption volatility as well as NBER business cycle recessions. Finally, note that in higher frequency data there is considerable support for time-varying volatility. In particular, Bollerslev and Hodrick (1995) and Bansal and Yaron (2000) show that there is time-varying volatility in monthly dividend data. We also find evidence for time-varying volatility in quarterly consumption data (which is not reported here).

Given the evidence above, the value of $\nu_1$, the parameter governing the persistence of conditional volatility, allows the model to capture the slow moving fluctuations in economic uncertainty. This choice permits fluctuating uncertainty to have a non-trivial impact on risk-premia and asset valuations.

3.1.3 Preferences

Our chosen preference parameters satisfy economic considerations. In particular $\delta < 1$, and the risk aversion parameter $\gamma$ is 9.5. Using non-parametric bounds on asset prices, Hansen and Jagannathan (1991), and Cochrane and Hansen (1992) document that large risk aversion helps in justifying observed risk premia. Mehra and Prescott (1985) argue that a reasonable upper bound for risk aversion is around 10. In this sense, our choice for risk-aversion is reasonable. The magnitude for IES that we have chosen is 1.5. Hansen and Singleton (1982), and Attanasio and Webber (1989) estimate the IES to be well in excess of 1.5. More recently, Vissing-Jorgensen (2001), and Guvenen (2001) also argue that the IES is well over one. However, Hall (1988) and Campbell (1999) estimate the IES to be well below one. Their results are based on a model without fluctuating economic uncertainty. Below we show that the exclusion of time varying economic uncertainty leads to a very serious downward bias in the estimates of the IES, if it is estimated as in Hall (1988). Given this, and the wide range of estimates in the literature, we think our value of 1.5 for the IES is quite plausible.

Given the preference parameters and the dynamics of consumption and dividends, we now proceed to analyze the asset pricing implications of the model.

4 Asset Pricing Implications

Table 4 provides statistics for the equity premium, market return volatility and the mean and volatility of the risk free rate. Column 2 and 3 provide the statistics and their respective
standard errors for our sample data. Column 4 provides the model’s corresponding statistics for Case I where fluctuations in economic uncertainty have been shut off, while column 5 allows for their presence.

### Table 4: Asset Pricing Implications

<table>
<thead>
<tr>
<th>Variable</th>
<th>Data</th>
<th>Model</th>
<th>Case I</th>
<th>Case II</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>S.E.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Returns</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(E[r_m - r_f])</td>
<td>6.33 (2.15)</td>
<td>4.05</td>
<td>6.27</td>
<td></td>
</tr>
<tr>
<td>(E[r_f])</td>
<td>0.86 (0.42)</td>
<td>1.34</td>
<td>1.12</td>
<td></td>
</tr>
<tr>
<td>(\sigma(r_m))</td>
<td>19.42 (3.07)</td>
<td>14.21</td>
<td>18.56</td>
<td></td>
</tr>
<tr>
<td>(\sigma(r_f))</td>
<td>0.97 (0.28)</td>
<td>0.61</td>
<td>0.88</td>
<td></td>
</tr>
<tr>
<td>Price Dividend</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(E[\exp(p - d)])</td>
<td>26.56 (2.53)</td>
<td>23.71</td>
<td>21.35</td>
<td></td>
</tr>
<tr>
<td>(\sigma(p - d))</td>
<td>0.29 (0.04)</td>
<td>0.15</td>
<td>0.24</td>
<td></td>
</tr>
<tr>
<td>(AC1(p - d))</td>
<td>0.81 (0.09)</td>
<td>0.82</td>
<td>0.81</td>
<td></td>
</tr>
<tr>
<td>(AC2(p - d))</td>
<td>0.64 (0.15)</td>
<td>0.66</td>
<td>0.67</td>
<td></td>
</tr>
</tbody>
</table>

Model entries are population values based on the process in (7) with parameter values given in Table 1. Standard errors are Newey and West (1987) corrected using 10 lags. Case I pertains to the model with fluctuations in expected growth rate but without time varying uncertainty, i.e., \(\sigma_w = 0\).

We start our discussion by concentrating on Case II, the model that incorporates both fluctuations in expected growth rates as well as fluctuations in economic uncertainty. As can be seen in the last column in Table 4, this model matches the unconditional equity premium, the mean of the risk free rate, as well as the volatilities of the market return and the risk free rate very well. Further note that the model essentially duplicates the volatility and persistence of observed log price-dividend ratio. The long lasting effects of the persistence in expected growth rates as well as the permanence in volatility innovations allows the model to replicate these aspects of the data.

In column 4 we report the asset market implications for the model without fluctuating economic uncertainty (that is we set \(\sigma_w = 0\)). Recall that the rest of the model parameters are identical to the case discussed above. Not surprisingly the model’s equity return volatility and premium are somewhat smaller. The comparison of these magnitudes to those in column 5 gives a sense of the role played by fluctuating economic uncertainty. In particular, about 2

\(^9\)Note that the volatility of the price-dividend ratio when the data sample does not include the last 6 years is only 0.26.
percentage points of the market risk premium (the difference between the equity premium in column 5 and column 4) can be attributed to risks emanating from fluctuating volatility. In spite the simplicity of the model without fluctuating economic uncertainty, it is still capable of quantitatively capturing key features of asset market data.

Finally, it is worth noting that there are some important trade-offs between various parameters of the models. For example, even with lower unconditional variance of consumption growth, the model can still reproduce many of the asset pricing features, if the persistence parameter $\rho$, and/or the risk aversion parameter $\gamma$ were to increase. Additionally, increasing the IES parameter helps to some extent increase the market volatility and equity premium.

Weil (1989), and Kandel and Stambaugh (1991) also explore the implications of the Epstein and Zin (1989) preferences for asset market data. However, these papers find it difficult to quantitatively explain the aforementioned asset market features. A natural question to ask is why do we succeed in capturing these asset market features with Epstein and Zin (1989) preferences? Weil (1989) uses i.i.d consumption growth rates. With i.i.d consumption and dividend growth rates the risks associated with fluctuating expected growth and economic uncertainty are absent. Consequently, the model has great difficulty in explaining the asset market data.\footnote{It is immediate to show that when consumption and dividend growth are i.i.d, that is $\varphi_e = 0$, the premium on the consumption return reduces to $E_t(r_{a,t+1} - r_{f,t}) = \gamma \sigma^2$. At our parameter values this yields an annualized premium of only 0.61%. Moreover, the equity premium on the market return $E_t(r_{m,t+1} - r_{f,t})$ is zero since $\eta_t$ and $u_t$ are independent. These results, again, underscore the need for persistence in consumption and dividend growth rates.}

Kandel and Stambaugh (1991) consider a model in which there is predictable variation in consumption growth rates and volatility. However, the persistence in the expected growth and conditional volatility in their specification is not large enough to permit significant response of asset prices to news regarding expected consumption growth and news regarding consumption volatility. In contrast, our specification of high persistence in expected growth rates, along with an IES larger than one, leads to positive elasticities of asset prices to news regarding expected growth rate (as in Barsky and DeLong (1993)). Similarly, the high persistence in the conditional variance of consumption ensures a large negative elasticity of asset prices with regard to news about fluctuating uncertainty. Both the high persistence and an IES greater than one are important for the success of our model to capture key aspects of asset market data.

To further see these implications consider equation (6),

$$E(r_{m,t+1} - r_{f,t}) = \beta_{m,e}\lambda_{m,e}\sigma^2 - 0.5\text{Var}(r_{m,t})$$
where, \( \beta_{m,e} \equiv [\kappa_{1,m}(\phi - \frac{1}{\psi}) \frac{\varphi_e}{1 - \kappa_{1,m}\rho}] \) is the asset return’s \( \beta \) with respect to news about expected consumption growth rate, and \( \lambda_{m,e} \equiv (\frac{\gamma - \frac{1}{\psi}}{1 - \frac{1}{\psi}})[\kappa_1(1 - \frac{1}{\psi}) \frac{\varphi_e}{1 - \kappa_1\rho}] \) is the associated market price of risk. With IES larger than one, an increase in \( \rho \), simultaneously increases the asset’s \( \beta_{m,e} \) and the market price of risk \( \lambda_{m,e} \), and consequently increases the risk premium. A parallel argument also holds with respect to the persistence of the conditional volatility of consumption, \( \nu_1 \) (as discussed in section 2.2).

In addition, Kandel and Stambaugh (1991) primarily focus on the case in which the IES is close to zero. At very low values of IES (\( \psi \)), \( \lambda_{m,e} \) and \( \beta_{m,e} \) are negative. This may still imply a sizeable equity premium. However, this parameter configuration, leads to counterfactually high levels of the risk free rate and/or its volatility even with large risk aversion. In contrast, our IES, which is bigger than one ensures that the level and volatility of the risk free rate are low and comparable to those in the data. This feature, in conjunction with modest risk aversion and large persistence, allows the model to match the equity premium and volatility of asset returns along the various other data dimensions discussed below.

Our model underscores the importance of persistent components in growth rates and varying economic uncertainty. This view is also consistent with the work of Alvarez and Jermann (1999) who show that the economic costs associated with relatively low frequency fluctuations in consumption are large.

### 4.1 Additional Asset Pricing Implications

The main additional asset pricing implications that we focus on relate to predictability of returns and fluctuations in Sharpe Ratios. As noted earlier, in the model where we shut-off fluctuating economic uncertainty (Case I), both risk-premia and Sharpe Ratios are constant—hence, this simple specification cannot address issues regarding predictability of risk premia. The model where we incorporate fluctuating economic uncertainty (Case II) does permit risk premia to fluctuate, and henceforth, we focus, almost entirely, on this model specification.

#### 4.1.1 Variability of the Pricing Kernel

The maximal Sharpe ratio, as shown in Hansen and Jagannathan (1991), is determined by the conditional volatility of the pricing kernel. This maximal Sharpe ratio for our model is the volatility of the pricing kernel innovation defined in equation (9), that is \( [Var_t(m_{t+1} - E_t(m_{t+1}))]^{1/2} \), where

\[
Var_t(m_{t+1} - E_t(m_{t+1})) = \lambda_{m,\eta}^2 \sigma_t^2 + \lambda_{m,e}^2 \sigma_t^2 + \lambda_{m,w}^2 \sigma_w^2 
\]
Note that the maximal Sharpe ratio, and the risk premia on assets rise as $\sigma_t$ rises.\textsuperscript{11} This means that during periods of high economic uncertainty risk premia will rise. As discussed earlier, the model has small fluctuations in the real risk-free rate as well — hence the vertex of the mean-variance frontier will move due to these fluctuations. In addition, since the maximal Sharpe ratio fluctuates, the slope of the tangent rays emanating from the risk-free rate will also fluctuate.

In Table 5, we provide a sense of the overall magnitude and the contributions of different shocks to the volatility of the pricing kernel. First, note that the maximal Sharpe ratio reported below, 0.62, is quite large. The maximal Sharpe ratio that would be obtained with \textit{i.i.d} growth rates is $\gamma \ast \sigma$ and with our parameter configuration would equal 0.28. Consequently, it is the Epstein and Zin preferences and the departure from \textit{i.i.d} growth rates that are responsible for this larger maximal Sharpe ratio. Second, the important sources of risk are shocks to the expected growth rate ($e_{t+1}$), followed by that of fluctuating economic uncertainty ($w_{t+1}$). Recall that $\lambda_{m,e}$ increases in the parameter $\rho$, that is, larger permanence of expected growth rate news translates to a larger contribution to systematic risk from this source. Further, the variance of fluctuating volatility news, i.e., $w_{t+1}$, is tiny. However, as this news is quite long lasting, it also yields a non-trivial contribution to the innovations in the pricing kernel, and consequently to the risk-premia of the aggregate dividend claim. Finally, the variance of high-frequency consumption news, $\eta_{t+1}$, is relatively large, but based on the relative size of $\lambda_{m,\eta}$, it follows that this risk-source contributes little to systematic risk. Indeed, by making the simplifying assumption that innovations in aggregate dividend growth are independent of $\eta_{t+1}$, we have in essence zeroed out its effect on the risk-premia on the aggregate dividend claim. Incorporating this risk, given its minimal contribution to the pricing kernel innovation, makes little to no difference to our results regarding the risk-premium on the market portfolio.

\begin{table}[h]
\centering
\caption{Decomposing Variation in the Pricing Kernel}
\begin{tabular}{cccc}
\hline
Volatility of & Independent & Expected & Fluctuating Economic \\
Pricing Kernel & Consumption & Growth Rate & Uncertainty \\
\hline
$[\Var(m_{t+1})]^{1/2}$ & $\frac{\lambda_{m,\eta}^2 \sigma_{t+1}^2}{\Var(m_{t+1})}$ & $\frac{\lambda_{m,e}^2 \sigma_{t+1}^2}{\Var(m_{t+1})}$ & $\frac{\lambda_{m,w}^2 \sigma_{w}^2}{\Var(m_{t+1})}$ \\
0.62 & 9\% & 56\% & 35\% \\
\hline
\end{tabular}
\end{table}

\textsuperscript{11}As in Campbell and Cochrane (1999), given the normality of the growth rate dynamics, the maximal Sharpe ratio is simply given by the standard deviation of the log pricing kernel.
Entries are the relative contributions by shocks to the volatility of the pricing kernel given in equation (9). The volatility of the maximal Sharpe ratio is annualized in order to make it comparable to the Sharpe ratio on annualized returns.

4.1.2 Predictability

Dividend yields seem to predict multi-horizon returns. A rise in the current dividend yield predicts a rise in future expected returns. Our model performs quite well in capturing this feature of the data. However, it is important to recognize that these predictability results are quite sensitive to changing samples, estimation techniques, and data sets (see Hodrick (1992), Goyal and Welch (1999), Ang and Bekaert (2001) for recent discussions of this topic). Further, most dimensions of the evidence related to predictability (be it growth rates or returns) are estimated with considerable sampling error, as highlighted by Hodrick (1992).12 Underscoring this issue, in our sample from 1929-1998 the market-price to dividend ratio has trended up considerably. This in conjunction with the rather high persistence in the price-dividend ratio suggests that considerable caution should be exercised in interpreting the evidence regarding predictability based on price-dividend ratios.

In Panel A of Table 6 we report the predictability regressions of future excess returns for horizons of 1, 3, and 5 years for our sample data. In column 4 we report the corresponding evidence from the perspective of the model. The model captures the positive relationship between expected returns and dividend yields. The absolute value of the slope coefficients and the corresponding $R^2$s rise with the return horizon, as in the data. The predictive slope coefficients and the $R^2$s in the model are somewhat lower than those in the data; however, the model’s slope coefficients are within one standard error of the estimated coefficients in the data.

---

12Hodrick (1992) shows that the predictability results are sensitive to whether one regresses sums of future returns on current price dividend ratios or alternatively future returns on sums of lagged price-dividend ratios. The latter form seems to display much less predictability than the former. In addition, he shows that for a given method, the slope coefficients, the standard errors, and $R^2$ are very sensitive to whether one uses pre-1952 or post 1952 sample.
Table 6: Predictability Regression

<table>
<thead>
<tr>
<th>Variable</th>
<th>Panel A: Excess Returns</th>
<th>Panel B: Growth Rates</th>
<th>Panel C: Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data</td>
<td>S.E.</td>
<td>Model</td>
</tr>
<tr>
<td>$B(1)$</td>
<td>-0.08</td>
<td>(0.07)</td>
<td>-0.16</td>
</tr>
<tr>
<td>$B(3)$</td>
<td>-0.37</td>
<td>(0.16)</td>
<td>-0.46</td>
</tr>
<tr>
<td>$B(5)$</td>
<td>-0.66</td>
<td>(0.21)</td>
<td>-0.65</td>
</tr>
<tr>
<td>$R^2(1)$</td>
<td>0.02</td>
<td>(0.04)</td>
<td>0.02</td>
</tr>
<tr>
<td>$R^2(3)$</td>
<td>0.19</td>
<td>(0.13)</td>
<td>0.08</td>
</tr>
<tr>
<td>$R^2(5)$</td>
<td>0.37</td>
<td>(0.15)</td>
<td>0.17</td>
</tr>
</tbody>
</table>

This table provides predictability regressions of future excess returns, growth rates, and realized volatility of consumption. The entries in Panel A correspond to regressing $r_{t+1} + r_{t+2} + \ldots r_{t+j} = \alpha(j) + B(j) \log(P_t/D_t) + v_{t+j}$, where $r_{t+1}$ is the excess return, and $j$ denotes the forecast horizon in years. The entries in Panel B, correspond to regressing $g_{t+1} + g_{t+2} + \ldots g_{t+j} = \alpha(j) + B(j) \log(P_t/D_t) + v_{t+j}$, and $g^a$ is annualized consumption growth. The entries in Panel C correspond to $\log(P_{t+j}/D_{t+j}) = \alpha(j) + B(j) |\epsilon_{g^a,t}| + v_{t+j}$, where $|\epsilon_{g^a,t}|$ is the realized volatility of consumption – see details in Table 3. Model parameters are based on the process in (7), with parameter values given in Table 1. The entries for the model are based on 1000 simulations each with 840 monthly observations that are time aggregated. Standard errors are Newey and West (1987) corrected using 10 lags.

The connection between growth rates and price-dividend ratios is also an important dimension. In many consumption based models, such as those with habits (see Abel (1990), and Campbell and Cochrane (1999)), there is an intimate link between price-dividend ratios and lagged consumption growth rates. In our model such an intimate link exists as well; expected growth rates of consumption are important in determining price-dividend ratios. In Panel B of Table 6 we provide regression results where the dependent variable is the sum of annual consumption growth rates. In the data it seems that price-dividend ratios have little predictive power particularly at longer horizons. The slope coefficients and $R^2$s of these regressions are quite low in data and the model as well. Further, in the data and the model the $R^2$s decline with horizon (even after 5 years). Given the role that expected consumption growth plays in our model one could have suspected that price-dividend ratios would strongly predict future consumption growth rates. However, as shown above this is not the case. The $R^2$s, in the model, are relatively small due to two reasons. First, price-dividend ratios are determined by expected growth rates, and the variation in expected growth rates is quite small.\(^\text{13}\) Second, price-dividend ratios are also affected by independent movements in consumption.

\(^\text{13}\)Recall that the monthly $R^2$ for consumption dynamics is less than 5% — i.e., the variation in consumption growth rate is dominated by the independent and unpredictable component $\eta_t$. 

20
economic uncertainty which lowers the ability of the price-dividend ratios to predict future growth rates. In all, the model as in the data does not imply that growth rates at long horizons are predicted by price-dividend ratios in any economically sizeable manner.\textsuperscript{14}

In Panel C of Table 6 we report how well current realized consumption volatility predicts future price-dividend ratios. First, note that in the data there is strong evidence for this relationship. The regression coefficients for predicting future price-dividend ratios with current volatility for one, three and five years are all negative, have robust t-statistics that are well above 2, and $R^2$'s of about 10%. The model produces similar results which are within one standard error of the data. Taken together with the results in Panel B of Table 3 the evidence is consistent with the economics of the model; fluctuating economic uncertainty, captured via realized consumption volatility, predicts future price-dividend ratios and is predicted by lagged price-dividend ratios. Bansal, Khatchatrian, and Yaron (2002) show that this new evidence is robust across many samples, frequencies, and is consistently found in many developed economies.

How much of the variation in the price-dividend ratio is from growth rates and what part is due to variation in expected returns? We follow Campbell and Cochrane (1999) and Cochrane (1992) and decompose the variance of price-dividend ratios to covariances of log ($P/D$) ratios with future subsequent returns and with future dividend growth rates.

\textbf{Table 7 : Variance Decomposition of Price-Dividend Ratios}

<table>
<thead>
<tr>
<th>Source</th>
<th>Data</th>
<th>S.E.</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dividends</td>
<td>-6%</td>
<td>(31%)</td>
<td>58%</td>
</tr>
<tr>
<td>Returns</td>
<td>108%</td>
<td>(42%)</td>
<td>54%</td>
</tr>
</tbody>
</table>

Table entries are the percentage of $\text{var}(p - d)$ accounted for by dividend growth and returns:

$$100 \times \sum_{j=1}^{15} \Omega_j \frac{\text{cov}(p_t - d_t, x_{t+j})}{\text{var}(p_t - d_t)}$$

where $x = -r$ and $g_d$ respectively, and $\Omega = 1/(1 + E(r))$. Model parameters are based on the process in (7), with parameter values given in Table 1. The entries for the model are based on 1000 simulations each with 840 monthly observations that are time aggregated. The annualized returns and growth rates are used with every 12th price-dividend ratio to represent an annual time series. The entries do not add to 100% due to the finite truncation. Standard errors are Newey and West (1987) corrected using 15 lags.

\textsuperscript{14}Our model can be easily modified to further lower the predictability of growth rates. Consider an augmented model (as in Cecchetti, Lam, and Mark (1993)) that allows for additional predictable movements in dividend growth rates which are unrelated to consumption. This will not affect the risk free rate and the risk premia in the model. However, this predictable component affects movements in price-dividend ratios and further lowers the ability of price-dividend ratios to predict future consumption growth rates.
The point estimates in Table 7 indicate that the model’s covariances between future subsequent growth rates and current price-dividend ratios are somewhat too large. However, the standard errors of the point estimates of this decomposition are very large. Our model produces ratios that are within standard error bands of these ratios in the data. Finally, an additional aspect of the data is the relatively low correlation between consumption growth and equity returns. In the data this contemporaneous correlation is very small at the monthly frequency and is about 0.20 at the annual frequency. Our model produces comparable magnitudes with correlations of 0.03 and 0.14 for the monthly and annual frequencies respectively. In summary, our model is capable of reproducing many of the predictability aspects discussed in Tables 6 and 7.

**Term Premia**

An important feature of the data is that the term premium on nominal bonds, the average one period excess return on an \( n \) period discount bond, is small. This implies that the equity premium is not driven by a large term premium. It is worth noting that our model’s equity premium is not driven by a large positive term-premium. The term premium (which in our model is on real bonds) is in fact small and slightly negative. The explicit formulas for the real term structure and the term premia are presented in Bansal and Yaron (2000). This implication of our model is consistent with the evidence provided in Evans (1998) who documents that for inflation indexed bonds in the U.K. (1983-1995) the term premia is significantly negative (less than -2% at 1 year horizon) while the term premia for nominal bonds is very slightly positive. The main point, however, is that the large equity premium in the model is not a by-product of a large positive term premium.

**Conditional Volatility of returns**

There is a large literature which documents that market return volatility displays a GARCH(1,1) pattern with fairly persistent volatility shocks (see Bollerslev (1986)). Note that this feature of the data is easily reproduced in our model. The market volatility process, as described in equation (17) in the appendix, is a linear function of the conditional variance of the consumption growth rate process \( \sigma_t \). As the conditional variance of the consumption growth rate process is an AR(1) process, it follows that the market volatility inherits this property. Note that the coefficient on the conditional variance of the consumption process in market volatility is quite large, which magnifies the conditional variance of the market portfolio. The persistence in the market volatility also coincides with the persistence on the volatility of the consumption growth rate process. In the monthly market return data
this persistence parameter is about 0.986 (see Bollerslev, Engle, and Wooldridge (1988)) and in the model it equals \( \nu_1 \), 0.987. As consumption volatility is high during recessions this implies that the market volatility also rises during recessions. Also note that during periods of high consumption volatility, that is during recessions, the equity premium also rises. This is consistent with the evidence provided in Fama and French (1989).

4.1.3 Bias in Estimating IES

As in Hall (1988), the IES is typically measured by the slope coefficient from regressing date \( t + 1 \) consumption growth rate on the date \( t \) risk-free rate. This projection would indeed recover the IES if there was no fluctuating uncertainty that affected the risk-free rate. However, the risk-free rate in our model fluctuates due to both changing expected growth rate and independent fluctuations in the volatility of consumption. Thus, the above projection is misspecified and creates a downward bias. This bias is quite significant as inside our model, where the value of IES is set at 1.5, Hall’s (1988) regression would estimate the IES parameter to be 0.62. Our model is a simple one and there may be alternative instrumental variable approaches to undo this bias. However, we view this result of the downward bias as suggestive of the difficulties in accurately pinning down the IES. As discussed earlier in section 3.1.3, several papers report an estimated IES which is well over one. This evidence, along with the potential downward bias in estimating IES makes our choice of larger than one IES quite reasonable.\(^{15}\)

5 Conclusions

In this paper we explore the idea that news about growth rates and economic uncertainty continuously alters perceptions regarding long-term expected growth rates and economic uncertainty, and that this feature is important for explaining various asset market phenomena. If news about consumption and dividends has a non trivial impact on long-term expected growth rates or economic uncertainty, then asset prices will be fairly sensitive to small growth rate and economic uncertainty shocks.

We provide empirical support for the view that the observed aggregate consumption-dividend growth process contains a small persistent expected growth rate and conditional

\(^{15}\)Epstein and Zin (1991) also report low values of IES. However, in their estimation they do not impose the theoretical restriction that the return of aggregate consumption determines the pricing kernel. They choose to proxy this unobserved return with the return to the observed market portfolio. The two returns, however, have different means and are imperfectly correlated. Further, due to fluctuating economic uncertainty the consumption to wealth ratio and the price-dividend ratio are not perfectly correlated either. Consequently, their evidence does not directly relate to the approach we pursue.
volatility component. We document that the interaction between such growth rate dynamics, in conjunction with Epstein and Zin (1989)-Weil (1989) preferences can indeed explain many outstanding asset market puzzles. In particular, we show that such a model is capable of justifying the observed magnitudes of the equity premium, the low risk free rate and the volatility of market return, dividend-yield, and the risk free rate. In addition, the model is also capable of justifying the predictive relation between dividend yields and returns, and the well documented GARCH-type stochastic volatility in ex-post equity returns. In our model approximately half of the variability in equity prices is due to fluctuation in expected growth rates and the remaining is due to fluctuations in costs of capital.
References


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Figure 1
Realized Consumption Volatility and NBER Business Cycles

The figure depicts realized consumption volatility based on $|\epsilon_{g^n,t}|$, the absolute value of the residual from regressing $g^n_t = \sum_{j=1}^{5} A_j g^n_{t-j} + \epsilon_{g^n,t}$, and $g^n_t$ denotes annual consumption growth rate. The solid (dashed) bar lines represent the start (end) of NBER business recessions.
6 Appendix-A

The consumption and dividend process given in (7) is

\[
\begin{align*}
    g_{t+1} &= \mu + x_t + \sigma_t \eta_{t+1} \\
    x_{t+1} &= \rho x_t + \varphi \sigma_t e_{t+1} \\
    \sigma^2_{t+1} &= \sigma^2 + \nu_1 (\sigma^2 - \sigma^2) + w_{t+1} \\
    g_{d,t+1} &= \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1}
\end{align*}
\]

(11)

The IMRS (Intertemporal Marginal Rate of Substitution) for this economy

\[
\ln M_{t+1} = \theta \ln \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1)r_{a,t+1}.
\]

We derive asset prices using this IMRS and the standard asset pricing condition \(E_t[M_{t+1}R_{i,t+1}] = 1\), so that

\[
E_t[\exp(\theta \ln \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1)r_{a,t+1} + r_{i,t+1})] = 1
\]

(12)

for any asset \(r_{i,t+1} \equiv \log(R_{i,t+1})\). We first start by solving the special case where \(r_{i,t+1} = r_{a,t+1}\) - the return on the consumption portfolio, and then solve for market return \(r_{m,t+1}\), and the risk free rate \(r_f\).

6.1 The return on consumption portfolio, \(R_0\)

We conjecture that the log price-consumption ratio follows, \(z_t = A_0 + A_1 x_t + A_2 \sigma_t^2\). Armed with the endogenous variable \(z_t\) we substitute the approximation \(r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}\) into the Euler equation (12).

Since \(g\), \(x\) and \(\sigma_t^2\) are conditionally normal, \(r_{a,t+1}\) and \(\ln M_{t+1}\) are also normal. Exploiting the normality of \(r_{a,t+1}\) and \(\ln M_{t+1}\), we can write down the Euler equation (12) in terms of the state variables \(x_t\) and \(\sigma_t\). As the Euler condition has to hold for all values of the state variables, it follows that all terms involving \(x_t\) must satisfy the following:

\[-\frac{\theta}{\psi} x_t + \theta [\kappa_1 A_1 \rho x_t - A_1 x_t + x_t] = 0.
\]

It immediately follows that,

\[A_1 = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho}
\]

which is (4) in the main text.

Similarly, collecting all the \(\sigma_t^2\) terms leads to the solution for \(A_2\),

\[\theta [\kappa_1 \nu_1 A_2 \sigma_t^2 - A_2 \sigma_t^2] + \frac{1}{2} (\theta - \frac{\theta}{\psi})^2 + (\theta A_1 \kappa_1 \varphi e^2) \sigma_t^2 = 0,
\]

which implies that

\[A_2 = \frac{0.5[(\theta - \frac{\theta}{\psi})^2 + (\theta A_1\kappa_1 \varphi e^2)]}{\theta (1 - \kappa_1 \nu_1)},
\]

the solution given in (8).

Given the solution above for \(z_t\) it is possible to derive the innovation to the return \(r_a\) as a function of the evolution of the state variables and the parameters of the model.

\[r_{a,t+1} - E_t(r_{a,t+1}) = \sigma_t \eta_{t+1} + B \sigma_t e_{t+1} + A_2 \kappa_1 \sigma_w w_{t+1},
\]

(13)

where \(B = \kappa_1 A_1 \varphi = \kappa_1 \frac{\varphi}{1-\kappa_1 \rho}(1 - \frac{1}{\psi})\). Further it follows that the conditional variance of \(r_{a,t+1}\) is

\[Var_t(r_{a,t+1}) = (1 + B^2) \sigma_t^2 + (A_2 \kappa_1)^2 \sigma_w^2.
\]

(14)
6.1.1 IMRSs

Now substituting for \( r_{a,t+1} \) and the dynamics of \( g_{t+1} \), we can re-write the IMRS in terms of the state variables — referring to this as the pricing kernel. Suppressing all the constants in the pricing kernel,

\[
m_{t+1} = \ln M_{t+1} = \theta \ln \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1}
\]

\[
E_t[m_{t+1}] = m_0 - \frac{x_t}{\psi} + A_2 (\kappa_1 \nu_1 - 1) (\theta - 1) \sigma_t^2
\]

\[
m_{t+1} - E_t(m_{t+1}) = (-\frac{\theta}{\psi} + \theta - 1) \sigma_t \eta_{t+1} + (\theta - 1)(A_1 \kappa_1 \varphi_e) \sigma_t e_{t+1} + (\theta - 1) A_2 \kappa_1 \sigma_w w_{t+1}
\]

\[
= \lambda_{m,\eta} \sigma_t \eta_{t+1} - \lambda_{m,e} \sigma_t e_{t+1} - \lambda_{m,w} \sigma_w w_{t+1}
\]

where \( \lambda_{m,\eta} \equiv \{-\frac{\theta}{\psi} + (\theta - 1)\}, \lambda_{m,e} \equiv (1 - \theta) B, \lambda_{m,w} \equiv (1 - \theta) A_2 \kappa_1, \) and \( B \) and \( A_2 \) are defined above. Note that the \( \lambda \)'s represent the market price of risk for each source of risk, namely \( \eta_{t+1}, \epsilon_{t+1}, \) and \( w_{t+1}. \)

6.1.2 Risk Premia for \( r_{a,t+1} \)

The risk premium for any asset is determined by the conditional covariance between the return and \( m_{t+1}. \) Thus the risk premium for \( r_{a,t+1} \) is equal to

\[
E_t[r_{a,t+1} - r_{f,t}] = -\text{cov}_t[m_{t+1} - E_t(m_{t+1}), r_{a,t+1} - E_t(r_{a,t+1})] - 0.5 \text{var}_t(r_{a,t+1})
\]

Exploiting the innovations in (13) and (15) it follows that,

\[
E_t[r_{a,t+1} - r_{f,t}] = -\lambda_{m,\eta} \sigma_t^2 + \lambda_{m,e} B \sigma_t^2 + \kappa_1 A_2 \lambda_{m,w} \sigma_w^2 - 0.5 \text{var}_t(r_{a,t+1})
\]

where \( \text{Var}_t(r_{a,t+1}) \) is defined in equation (14).

6.1.3 Equity Premium and Market Return Volatility

The risk premium for any asset is determined by the conditional covariance between the return and \( m_{t+1}. \) Thus the risk premium for the market portfolio \( r_{m,t+1} \) is equal to

\[
E_t[r_{m,t+1} - r_{f,t}] = -\text{cov}_t[m_{t+1} - E_t(m_{t+1}), r_{m,t+1} - E_t(r_{m,t+1})] - 0.5 \text{var}_t(r_{m,t+1})
\]

Equation (15) already provides the innovation in \( m_{t+1}. \) We now proceed to derive the innovation in the market return. The price-dividend ratio for the claim on dividends is

\[
z_{m,t} = A_{0,m} + A_1,m x_t + A_2,m \sigma_t^2
\]

Consequently, the market return is

\[
r_{m,t+1} = g_{d,t+1} + \kappa_1 A_1,m x_{t+1} - A_1,m x_t + \kappa_1 A_2,m \sigma_t^2 - A_2,m \sigma_t^2
\]

Exploiting the Euler condition \( E_t[\exp(m_{t+1} + r_{m,t+1})] = 1, \) and collecting all the \( x_t \) terms we find that

\[-\frac{x}{\psi} + x \kappa_1 A_1,m \rho - A_1,m x + \phi x = 0,\]

which implies that

\[
A_1,m = \frac{\phi}{1 - \kappa_1,m \rho}
\]

It follows that

\[
r_{m,t+1} = g_{d,t+1} + \kappa_1 A_1,m x_{t+1} - A_1,m x_t + \kappa_1 A_2,m \sigma_t^2 - A_2,m \sigma_t^2
\]

\[
r_{m,t+1} - E_t(r_{m,t+1}) = \varphi_d \sigma_t u_{t+1} + \kappa_1 A_1,m \varphi_e \sigma_t \epsilon_{t+1} + \kappa_1 A_2,m \sigma_w w_{t+1}
\]

\[
= \varphi_d \sigma_t u_{t+1} + \beta_{m,e} \sigma_t \epsilon_{t+1} + \beta_{m,w} \sigma_w w_{t+1},
\]

where \( \beta_{m,e} \equiv \kappa_1 A_1,m \varphi_e, \) and \( \beta_{m,w} \equiv \kappa_1 A_2,m. \)
It follows that
\[ \text{Var}_t(r_{m,t+1}) = (\beta_{m,e}^2 + \phi_2^2) \sigma_t^2 + \beta_{m,w}^2 \sigma_w^2. \]

The solution for \( A_{2,m} \) follows from exploiting the asset pricing condition,
\[ \exp\{E_t(m_{t+1}) + E_t(r_{m,t+1}) + 0.5 \text{Var}_t(m_{t+1} + r_{m,t+1})\} = 1, \]
and collecting all \( \sigma_t \) terms. Note that
\[ \text{Var}_t(m_{t+1} + r_{m,t+1}) = \text{Var}_t[\lambda_{m,e} \sigma_t \eta_{t+1} - \lambda_{m,w} \sigma_w w_{t+1} - \lambda_{m,e} \sigma_t \epsilon_{t+1} + 2 \sigma_t \epsilon_{t+1} + \nu + \alpha + \epsilon_{t+1} + 0.5 \text{Var}_t(m_{t+1})] = [\lambda_{m,e}^2 + (-\lambda_{m,e} + \beta_{m,e})^2 + 2 \phi_0^2] \sigma_t^2 + [\lambda_{m,w}^2 + \beta_{m,w}^2] \sigma_w^2 \]
where \( \lambda_{m,e} \equiv [\lambda_{m,e}^2 + (-\lambda_{m,e} + \beta_{m,e})^2 + 2 \phi_2^2]. \) Collect now all the \( \sigma_t^2 \) terms in equation (18). Note that \( \sigma_t \) also appears in \( E_t(r_{m,t+1}) \) as well as \( E_t(m_{t+1}) \). This leads to the following restriction,
\[ (\theta - 1)A_2(\lambda_{1,m} - 1) + A_{2,m}(\lambda_{1,m} - 1) + \frac{H_m}{2} = 0 \]
which implies that
\[ A_{2,m} = \frac{(1 - \theta)A_2(1 - \lambda_{1,m}) + 0.5H_m}{(1 - \lambda_{1,m})} \]

We now derive the expression for the equity premium.
\[ E_t(r_{m,t+1} - r_f,t) = \beta_{m,e} \lambda_{m,e} \sigma_t^2 + \beta_{m,w} \lambda_{m,w} \sigma_w^2 - 0.5 \text{Var}_t(r_{m,t+1}), \]
where \( \text{Var}_t(r_{m,t+1}) \) is defined in equation (17).

To derive the unconditional variance of the market return, note that
\[ r_{m,t+1} - E(r_{m,t+1}) = -x_t \psi + \beta_{m,e} \sigma_t e_{t+1} + \varphi_d \sigma_t u_{t+1} + A_{2,m}(\nu_1 \lambda_{1,m} - 1)[\sigma_t^2 - E(\sigma_t^2)] + \beta_{m,w} \sigma_w w_{t+1} \]
\[ = -x_t \psi + \beta_{m,e} \sigma_t e_{t+1} + \varphi_d \sigma_t u_{t+1} + A_{2,m}(\nu_1 \lambda_{1,m} - 1)[\sigma_t^2 - E(\sigma_t^2)] + \beta_{m,w} \sigma_w w_{t+1}. \]

Hence, the unconditional variance is
\[ \text{Var}(r_m) = \frac{\sigma_t^2}{\psi^2} + [\beta_{m,e}^2 + \phi_2^2] \sigma^2 + [A_{2,m}(\nu_1 \lambda_{1,m} - 1)^2] \text{Var}(\sigma_t^2) + \beta_{m,w}^2 \sigma_w^2. \]

The unconditional variance of the \( z_{m,t} \)—the price-dividend ratio for the market portfolios
\[ \text{Var}(z_{m,t}) = A_{1,m}^2 \text{Var}(x_t) + A_{2,m}^2 \text{Var}(\sigma_t^2) \]

6.2 The Risk Free Rate and its Volatility

To derive the risk free rate start with (12) and plug in the risk-free rate for \( r_i \)
\[ r_{f,t} = -\theta \log(\delta) + \frac{\theta}{\psi} E_t[g_{t+1}] + (1 - \theta) E_t[r_{a,t+1}] - \frac{1}{2} \text{Var}_t[\frac{\theta}{\psi} g_{t+1} + (1 - \theta) r_{a,t+1}], \]
subtract \( (1 - \theta)r_{f,t} \) from both sides and divide by \( \theta \), where it is assumed that \( \theta \neq 0 \). It then follows that,
\[ r_{f,t} = -\log(\delta) + \frac{1}{\psi} E_t[g_{t+1}] + \frac{(1 - \theta)}{\theta} E_t[r_{a,t+1} - r_i] - \frac{1}{2\theta} \text{Var}_t[\frac{\theta}{\psi} g_{t+1} + (1 - \theta) r_{a,t+1}] \]
To solve the above expression note that $V ar_t(\frac{\theta}{\psi} g_{t+1} + (1 - \theta) r_{a,t+1}) \equiv V ar_t(m_{t+1})$. Thus,

$$V ar_t(m_{t+1}) = (\lambda_{m,\eta}^2 + \lambda_{m,e}^2) \sigma_t^2 + \lambda_{m,w}^2 \sigma_w^2,$$

(24)

Further, if the innovation in the growth rate process is homoskedastic, the above expression simplifies as $\sigma_w^2 = 0$. The unconditional mean of $r_{f,t}$ is derived by substituting the expression for the risk premium for $r_{a,t+1}$ given in (16) and (24) into (23). This substitution yields:

$$E(r_{f,t}) = -\log(\delta) + \frac{1}{\psi} E(\phi) + \frac{(1 - \theta)}{\theta} E[r_{a,t+1} - r_t] - \frac{1}{2\theta} [(\lambda_{m,\eta}^2 + \lambda_{m,e}^2) E[\sigma_t^2] + \lambda_{m,w}^2 \sigma_w^2]$$

where note that $E[\sigma_t^2] = V ar(\eta)$.

The unconditional variance of $r_{f,t}$ is,

$$V ar(r_{f,t}) = (\frac{1}{\psi})^2 V ar(x_t) + \left(\frac{1 - \theta}{\theta} Q_1 - Q_2 \frac{1}{2\theta}\right)^2 V ar(\sigma_t^2),$$

(25)

where $Q_2 = (\lambda_{m,\eta}^2 + \lambda_{m,e}^2)$, and $Q_1 = (-\lambda_{m,\eta} + (1 - \theta) B^2 - 0.5(1 + B^2))$, where $B$ is defined above. Note that $Q_1$ determines the time varying portion of the risk-premium on $r_{a,t+1}$. For all practical purposes the variance of the risk free rate is driven by the first term.