Strong One-Switch Utility

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The linear plus exponential utility function has received increasing attention of late as a particularly attractive family for evaluating additive gambles for wealth. In addition to its ability to reflect increasing appreciation for money, risk aversion, and decreasing risk aversion, it is consistent with a risk-return representation in which return is measured by expected value. In this paper we present a new condition, strong one-switch, that characterizes the linear plus exponential family.

1. Introduction
You have been presented with an opportunity to benefit from one of two gambles, A or B. But you do not need to choose either A or B. Instead, one of them will be awarded to you based on the outcome of a coin toss: heads you get A, tails you get B. There is a decision involved, however. Just before the coin is tossed, you are informed that you may increase all of the payoffs of either A or B by a fixed amount, say $1,000.

Of course there is no downside to accepting this offer. Whether you choose to add the $1,000 to A or B, there is a 50-50 chance of receiving the additional money; from an expected value point of view, there is nothing to choose between the two options. But you are risk averse, so you begin to think about whether A or B is more improved by the addition of $1,000 to its outcomes. After due consideration, you conclude that you should add the $1,000 to A.

Suppose the amount to be added had been $2,000, or $10,000. Might your conclusion have been different? Or do you think that a positive increment would always be added to the same gamble in a situation such as this? Assuming you wish to abide by the axioms of expected utility, and that you always prefer more money to less, you will always add the investment to the same gamble, if, and only if, your utility function is linear plus exponential.

In the next section we review previous characterizations of the linear plus exponential utility function. We then present our new condition and our theorem. In §4 we examine some interesting special cases of our condition.

Gambles will be denoted as \( \tilde{x} \) or \( \tilde{y} \) or as A or B. A compound gamble between \( \tilde{x} \) and \( \tilde{y} \) will be denoted by \( (\tilde{x}, \tilde{y}) \), each having a 50-50 chance of being selected. Throughout the paper we assume that utility functions are increasing \( u' > 0 \), risk averse \( u'' < 0 \), and decreasingly risk averse \( u''' < u'' \). While we feel these assumptions are reasonable, their only mathematical role is to rule out the quadratic function, which otherwise would also be a solution, along with the linear plus exponential, in each of our results. The quadratic fails either of two of the conditions we have assumed: when \( b \neq 0 \), \( aw + bw^2 \) is not increasing for all \( w \), nor is it decreasingly risk averse for any \( w \).

2. Risk Consistency
In this section we review previous characterizations of the linear plus exponential utility family. The common thread among them is that the decision maker behaves “consistently,” in a manner that varies by the assumption, in evaluating risk.

For example, suppose you are offered one of two gambles, A or B, and after due consideration, decide
that you prefer A. Just before finalizing your choice, you learn that you have inherited $50,000 from an uncle’s estate. You reconsider your choice between A and B and decide that, in light of being richer, you now prefer B. Could it be that were you to become richer still, by another $50,000, say, you might then switch back to preferring A?

In our experience, most people, including veteran decision analysts, accept that there would be no reason to switch back. The intuition is that the original switch, from A to B, occurred because B was more attractive, but riskier, than A, which explains why the first $50,000 caused the change of mind. Increased wealth led to decreased risk aversion, which meant that B became relatively more attractive. A further increase in wealth, therefore, could only reinforce preference for B over A, eliminating the possibility of a reversal in preference.

Attractive as this intuition may be, it implicitly assumes that the relative riskiness of A and B is independent of wealth. Surprisingly, very few utility functions permit such an interpretation. In introducing the one-switch rule, Bell (1988) showed that only two families of utility functions satisfied both it and the reasonableness conditions (u’ > 0, u’’ < 0, u’u’’ < u’u’’’):

The sum of (two) exponentials (sumex)

\[ u(w) = e^{-aw} - be^{-cw}, \]

and the linear plus exponential

\[ u(w) = aw - be^{-cw}, \]

where a, b, and c are positive constants.

These are the only two families, then, that are consistent with the notion that gambles may be ordered by riskiness and that this rank order is independent of the decision maker’s wealth.

Bell (1988) included two conditions that each narrow the list of acceptable functions only to linear plus exponential. The first (his Proposition 3) added to the one-switch rule the requirement that the decision maker is risk neutral when “extremely rich.” The second (“risk consistency”) required that, whenever a switch in preference occurs due to an increase in wealth, the newly preferred gamble has a higher mean.

In a recent paper, Gelles and Mitchell (1999) provide a characterization of linear plus exponential utility by introducing a property they call “Broadly Decreasing Risk Aversion”: For any two gambles \( \tilde{x} \) and \( \tilde{y} \), with \( E\tilde{x} = E\tilde{y} = 0 \), and defining a conditional risk premium \( \pi(w) \) as follows,

\[ w + \tilde{x} + \tilde{y} \sim w + \tilde{x} - \pi(w), \]

\( \pi \) always decreases with \( w \).

Finally, Bell (1995b) showed that only a utility function in the linear plus exponential family satisfies a “Contextual Uncertainty Condition.” Suppose a person is deciding between gambles A and B but is uncertain about the precise wealth level she actually has. Her wealth level can be expressed as \( w = w_0 + \tilde{z}_1 + \tilde{z}_2 \), where \( w_0 \) is known but \( \tilde{z}_1 \) and \( \tilde{z}_2 \) are two independently distributed “contextual uncertainties.” Though independent of each other, and of A and B, \( \tilde{z}_1 \) and \( \tilde{z}_2 \) share the same probability distribution up to a shift in scale. For example, \( \tilde{z}_1 \) and \( \tilde{z}_2 \) might both be normally distributed, or uniform, or 50-50 gambles. In such cases it is possible to describe one as having a larger spread than the other. (Arbitrary probability distributions cannot be compared in this way.) The contextual uncertainty condition says that if the decision maker is offered an opportunity to resolve one of \( \tilde{z}_1 \) or \( \tilde{z}_2 \) prior to selecting A or B, she should never prefer to resolve the one with the smaller spread.

Other conditions that relate to the “one-switch” set of utility functions have been given by Farquhar and Nakamura (1987), Bell (1995a), and Nakamura (1996).

In this paper we introduce a further characterization of linear plus exponential utility. “Strong One-Switch Utility” extends the domain of the one-switch condition to compound gambles. The new condition is similar in spirit to those of Gelles and Mitchell (1999) and Bell (1995b) in that comparison of riskiness is consistent in the presence of other complicating gambles.

3. The New Condition

The **Strong One-Switch Condition (SOS)**. Let \( \tilde{x} \), \( \tilde{y} \), \( \tilde{a} \), and \( \tilde{b} \) be any independent gambles. If there
exists an amount of money $k^*$ such that $(k^* + \bar{x}, \bar{a}) \sim (k^* + \bar{y}, \bar{b})$, then either $(k + \bar{x}, \bar{a})$ and $(k + \bar{y}, \bar{b})$ are indifferent for all $k$, or one is strictly preferred for $k < k^*$ and the other strictly preferred for $k > k^*$.

The condition is similar in spirit to that of the one-switch rule, which states that as $k$ increases, preference can switch from one of $(k + \bar{x}, \bar{a})$ or $(k + \bar{y}, \bar{b})$ to the other but not back again. Indeed, if $\bar{a} \sim \bar{b}$, then SOS is exactly equivalent to the one-switch rule, for in this case the relative preference of $(k + \bar{x}, \bar{a})$ and $(k + \bar{y}, \bar{b})$ is the same as the relative preference of $(k + \bar{x}, \bar{a})$ and $(k + \bar{y}, \bar{a})$ (by the substitution principle), which is the same as the relative preference of $k + \bar{x}$ and $k + \bar{y}$ (by the cancellation property). Also, and anticipating our theorem, any decision maker who accepts the one-switch rule and who is risk neutral when extremely rich implicitly accepts SOS because both have the same implications for utility.

Though not covered here, we anticipate that the new condition will have advantages over the contextual uncertainty condition when applied to the nonlinear utility theories described in Bell and Fishburn (2000).

To develop some intuition for what is going on, it is useful to consider the function

$$v(k) = Eu(k + \bar{x}) - Eu(k + \bar{y}),$$

where the dependence of $v$ on $\bar{x}$ and $\bar{y}$ is understood in context.

The one-switch rule requires that a graph of $v$ crosses the $x$-axis at most once, but provides no restriction on the shape of $v$ otherwise. As we will show in the proof of Theorem 1, the strong one-switch rule requires that the graph of $v$ crosses any line $v = constant$ at most once. Though SOS may seem like a much more stringent requirement, the intuition for each is the same: that as $k$ increases, the relative preference for $\bar{x}$ versus $\bar{y}$ should shift towards the more risky of $\bar{x}$ and $\bar{y}$. Put mathematically, if $\bar{x}$ is riskier than $\bar{y}$, we would expect $v(k) = Eu(k + \bar{x}) - Eu(k + \bar{y})$ to be increasing for all $k$.

For example, if $\bar{x}$ and $\bar{y}$ are both certain quantities (i.e., $\bar{x} = x$ with probability 1, $\bar{y} = y$ with probability 1), then SOS requires that $v(k) = Eu(k + x) - Eu(k + y)$ be constant or monotonic in $k$. If $x = y$, then $v$ is always zero. If $x \neq y$, then $v$ is strictly monotonic for all $x$ and $y$, if, and only if, $u$ is always risk averse.

**Theorem 1.** A utility function is consistent with the strong one-switch condition if and only if it is linear plus exponential.

**Proof.** In the first part of the proof we show that the linear plus exponential utility function does imply that the $v$ function is either constant or strictly monotonic for all $\bar{x}$ and $\bar{y}$ and therefore satisfies SOS. If $u(w) = aw - be^{-cw}$, then $v(k) = Eu(k + \bar{x}) - Eu(k + \bar{y}) = a(E\bar{x} - E\bar{y}) - be^{-ck}(Ee^{-ck} - Ee^{-cy})$. This is strictly monotonic in $k$ if $bc \neq 0$ and constant otherwise. If $u(w) = aw + bw^2$, then $v(k) = (a + 2kb)(E\bar{x} - E\bar{y}) + b(E(\bar{x}^2) - E(\bar{y}^2))$, which is also monotonic in $k$. The difference between the utilities of $(k + \bar{x}, \bar{a})$ and $(k + \bar{y}, \bar{b})$ varies only with $v(k)$, so SOS is satisfied if $v$ is constant or strictly monotonic.

In the second part of the proof, we show that SOS implies that $v$ is constant or strictly monotonic, which in turn implies $u$ is linear plus exponential (or quadratic). First, we show that if SOS holds, then $v$ must be constant or strictly monotonic. If for $k_1 \neq k_2$ we have $v(k_1) = v(k_2)$, then so long as there exist some $\bar{a}, \bar{b}$, such that $v(k_1) = v(k_2) = Eu(\bar{b}) - Eu(\bar{a})$, then $(k_1 + \bar{x}, \bar{a}) \sim (k_1 + \bar{y}, \bar{b})$ for $i = 1, 2$ and so, by SOS, $v$ must be constant. But such an $\bar{a}, \bar{b}$ do exist, for we may set $\bar{a} = k_1 + \bar{y}$ and $\bar{b} = k_1 + \bar{x}$.

Now we show that $v$ is strictly monotonic or always constant for all $\bar{x}$ and $\bar{y}$ only if $u$ is linear plus exponential (or quadratic). Since $v'(k) = Eu'(k + \bar{x}) - Eu'(k + \bar{y})$, a strictly monotonic or always constant $v$ function has the interpretation that $u'$, if viewed as a utility function in its own right, orders all pairs of gambles $k + \bar{x}$ and $k + \bar{y}$ independently of $k$.

As Pflanzagl (1959) first showed, this condition holds for all $\bar{x}$ and $\bar{y}$ if, and only if, the utility function is linear or exponential. In our case this means that $u'$ is linear or exponential. If $u'$ is linear, then $u$ is quadratic. If $u'$ is exponential, then $u$ is linear plus exponential.

**□**

4. Special Cases

We noted that the one-switch rule is a special case of SOS (when $\bar{a} \sim \bar{b}$), and that if $u$ is risk averse ($u'' < 0$), then SOS always holds for $\bar{x}$ and $\bar{y}$ constant. In this section we present four special cases of SOS that are nevertheless equivalent to it, in the sense that they too imply that $u$ must be linear plus exponential.
SOS1. If for any two amounts $k_1$ and $k_2$ we have $(k_i + \tilde{x}, \tilde{a}) \sim (k_i + \tilde{y}, \tilde{b})$ for $i = 1, 2$, then $(k + \tilde{x}, \tilde{a}) \sim (k + \tilde{y}, \tilde{b})$ for all $k_i < k < k_2$.

This is a special case of SOS because SOS requires that if $(k_i + \tilde{x}, \tilde{a}) \sim (k_i + \tilde{y}, \tilde{b})$ for $i = 1, 2$, then $(k + \tilde{x}, \tilde{a}) \sim (k + \tilde{y}, \tilde{b})$ for all $k_i < k < k_2$.

Proof. Suppose there exists some $\tilde{x}, \tilde{y}, \tilde{a},$ and $\tilde{b}$ such that SOS1 holds but SOS doesn’t. Then $v(k)$ must be constant in a range $k_1 \leq k \leq k_2$, but for some $k_0 < k_1$, say, we have $(k_0 + \tilde{x}, \tilde{a}) < (k_0 + \tilde{y}, \tilde{b})$.

Let $\tilde{z}_1$ and $\tilde{z}_2$ be any two gambles, such that $Eu(k + \tilde{x}) - Eu(k + \tilde{y})$ has a negative slope in the range from $k_0$ to $k_2$. (If $u$ is decreasingly risk averse, then it suffices to let $\tilde{z}_1$ be any positive constant and $\tilde{z}_2 = 0$.)

Denote $(\tilde{x}, \tilde{z}_1)$ by $\tilde{x} = (\tilde{a}(k_1 + \tilde{z}_1))$ by $\tilde{a}$ and $(\tilde{b}, k_1 + \tilde{z}_2)$ by $\tilde{b}$. Using the revised notation, we have, if $\tilde{z}_1$ and $\tilde{z}_2$ are suitably “small,”

\[
\begin{align*}
(k_0 + \tilde{x}, \tilde{a}) &< (k_0 + \tilde{y}, \tilde{b}) & (\tilde{z}_1 \text{ and } \tilde{z}_2 \text{ small}), \\
(k_1 + \tilde{x}, \tilde{a}) &< (k_1 + \tilde{y}, \tilde{b}) & \text{(by construction)}, \\
(k_2 + \tilde{x}, \tilde{a}) &< (k_2 + \tilde{y}, \tilde{b}) & \text{(choice of } \tilde{z}_1, \tilde{z}_2). 
\end{align*}
\]

By continuity of $v$ this sequence implies a contradiction to SOS1. □

A corollary to SOS1 is that, if SOS is false, then there exists a counterexample in which $v$ is strictly nonmonotonic. We use this fact in the proofs of SOS2, SOS3, and SOS4, which follow.

SOS2. A special case of SOS in which $\tilde{a} = \tilde{y}$ and $\tilde{b} = \tilde{x}$. This is the condition described informally in the introductory section of the paper. In this case, of course, $k^* = 0$.

Proof. We will show that if SOS is false, then so too is SOS2. If SOS is false, then there exist $\tilde{x}, \tilde{y}, \tilde{a},$ and $\tilde{b}$ and $k_1 \neq k_2$ such that $(k_i + \tilde{x}, \tilde{a}) \sim (k_i + \tilde{y}, \tilde{b})$ for $i = 1, 2$. But then $((k_1 + \tilde{x}, \tilde{a}), (k_2 + \tilde{y}, \tilde{b})) \sim (k_i + \tilde{y}, \tilde{b})$ which means that $(k_i + \tilde{x}, k_i + \tilde{y}) \sim (k_i + \tilde{x}, k_i + \tilde{y})$. Let $k_1 - k_2 = k^*$ and, without loss, assume $k_2 = 0$ (or change notation, replacing $k_2 + \tilde{x}$ by $\tilde{x}$ and $k_2 + \tilde{y}$ by $\tilde{y}$). So $(k^* + \tilde{x}, \tilde{y}) \sim (\tilde{x}, k^* + \tilde{y})$, and since, trivially, $(0 + \tilde{x}, \tilde{y}) \sim (\tilde{x}, 0 + \tilde{y})$, we have a contradiction to SOS2. □

SOS3. A special case of SOS in which $\tilde{a}$ and $\tilde{b}$ are restricted to being any constants.

Proof. Again, assume SOS is false. Then there exist $\tilde{x}, \tilde{y}, \tilde{a},$ and $\tilde{b}$, and $k_1 \neq k_2$, such that $(k_i + \tilde{x}, \tilde{a}) \sim (k_i + \tilde{y}, \tilde{b})$ for $i = 1, 2$. Let $a$ be the certainty equivalent of $\tilde{a}$, $b$ be the certainty equivalent of $\tilde{b}$, then also $(k_i + \tilde{x}, a) \sim (k_i + \tilde{y}, b)$ for $i = 1, 2$, which contradicts SOS3. Therefore SOS3 implies SOS. □

SOS4. A special case of SOS in which $\tilde{a}$ is the certainty equivalent of $\tilde{y}$ and $\tilde{b}$ is the certainty equivalent of $\tilde{x}$. SOS4 is also a special case of SOS3.

Proof. Combining the proofs of SOS1 and SOS2, we know that if SOS is false, then for some $\tilde{x}$ and $\tilde{y}$ there exists a $k^* \neq 0$ such that $(k^* + \tilde{x}, \tilde{y}) \sim (k^* + \tilde{y}, \tilde{x})$. But then, $(k^* + \tilde{x}, CE(\tilde{y})) \sim (k^* + \tilde{y}, CE(\tilde{x}))$. Since also $(\tilde{x}, CE(\tilde{y})) \sim (\tilde{y}, CE(\tilde{x}))$, we have a contradiction to SOS3. □

As a summary of the last two sections we state:

Theorem 2. The following assumptions are equivalent:

(i) SOS, (ii) SOS1, (iii) SOS2, (iv) SOS3, (v) SOS4, (vi) $Eu(k + \tilde{x}) - Eu(\tilde{y} + k)$ is strictly monotonic or constant in $k$ for all $\tilde{x}, \tilde{y}$, (vii) $u$ is linear plus exponential.

We add that all our results are true if the compound gambles $(\tilde{x}, \tilde{y})$ are interpreted throughout to mean a $p$-chance at $\tilde{x}$ and a $(1-p)$-chance at $\tilde{y}$, for a fixed $p$.

Finally we note that all of our results have parallels in the space of multiplicative gambles, those of the form $w x$ rather than $w + \tilde{x}$. The role of linear plus exponential utility is replaced by that of “log plus power,” $u(w) = a \log w + bw^2$.

References


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