Comment on “Generating Scenario Trees for Multistage Decision Problems”

Pieter Klaassen
Department of Finance and Financial Sector Management, Vrije Universiteit,
De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands
pklaassen@feweb.vu.nl

In models of decision making under uncertainty, one typically has to approximate the uncertainties by a limited number of discrete outcomes. Høyland and Wallace (2001) formulate a nonlinear programming problem to generate such a limited number of discrete outcomes while satisfying specified statistical properties. They have developed and employed this method for a stochastic multistage asset-allocation problem. When the method is applied to such financial optimization problems under uncertainty, we argue here that it does not suffice to match statistical properties. To obtain realistic outcomes, the (limited) description of the uncertainty in such models should also exclude arbitrage opportunities, and thereby be consistent with financial asset pricing theory. We illustrate that the method proposed by Høyland and Wallace can result in arbitrage opportunities in the scenario tree if only statistical properties are imposed. We show how one can check ex post for the presence of arbitrage opportunities in a scenario tree by checking for the existence of solutions to sets of linear equations. Arbitrage opportunities can also be precluded ex ante in the scenario tree by adding constraints to the nonlinear programming problem of Høyland and Wallace.

(No Arbitrage; Scenario Generation; Multistage Decision Problems; Asset Allocation)

1. Introduction

In models of decision making under uncertainty, one typically has to approximate the uncertainties by a limited number of discrete outcomes. Such a set of discrete outcomes is often generated by sampling from the true or assumed probability distribution. However, the statistical properties of such a sampled set may very well differ from the distribution it aims to approximate. This is especially true if the set is small, which is typically required for stochastic programming problems to maintain computational tractability. Høyland and Wallace (2001) present a method to generate a limited number of discrete outcomes while satisfying specified statistical properties. The method uses nonlinear programming to find a set of outcomes and associated probabilities that minimizes the (quadratic) difference between the implied and the desired statistical properties. As stated in their paper, the motivation for developing the method is the implementation of a stochastic multistage asset-allocation problem. The method then generates a scenario tree containing realizations for the total returns on the available asset classes. In this note we argue that it does not suffice to match statistical properties when considering such asset-allocation problems, but that it is important to make sure that the scenario tree does not contain arbitrage opportunities. We also show how one can detect arbitrage opportunities and avoid these in the method of Høyland and Wallace (2001).

The importance of precluding arbitrage opportunities in scenario trees of asset returns for portfolio optimization problems under uncertainty has been illustrated in Klaassen (1997). If arbitrage opportunities are present, the optimal solution will exploit these to the maximum extent possible. It is unlikely, however, that the arbitrage opportunities will arise in reality, and hence the optimal solution will reflect spurious
profit opportunities. Klaassen (1998) describes how nodes and time periods in a fine-grained scenario tree that is free of arbitrage opportunities can be aggregated while maintaining the arbitrage-free property as well as consistency with observed asset prices. Methods employed in the financial economics literature can be used to arrive at an initial fine-grained scenario tree of possible future asset prices that is free of arbitrage opportunities and consistent with observed asset prices. Gondzio et al. (1999) use the same ideas to aggregate nodes in a scenario tree of stock and option prices when both the stock price and its volatility are stochastic. The aggregated event tree is used in a multistage stochastic programming model that solves for the optimum portfolio composition to hedge an option position. Kouwenberg and Zenios (2001) provide an overview and comparison of several scenario-generation methods for stochastic programming models for asset/liability management.

A drawback of the aggregation methods used in Klaassen (1998) and Gondzio et al. (1999) is that they may alter the statistical properties of asset prices in the event tree. By enforcing the exclusion of arbitrage opportunities within the scenario-generation method of Høyland and Wallace (2001), it will be possible to simultaneously satisfy desired statistical properties, at least approximately.

Section 2 provides an illustration that the scenario-tree-generation method as proposed by Høyland and Wallace (2001) may give rise to arbitrage opportunities in the scenario tree if only statistical properties are imposed. Section 3 describes how one can check ex post for the presence of arbitrage opportunities in a scenario tree. It also shows what constraints could be added to the scenario-tree-generation method to preclude arbitrage opportunities ex ante. Section 4 concludes.

2. Illustration

We consider a simplified one-period version of the asset-allocation problem that is described in Høyland and Wallace (2001). There are four asset classes: cash, bonds, domestic stocks, and international stocks. The scenario tree of interest rate changes and stock returns is generated so that the outcomes in the tree match as closely as possible the specified expected value and standard deviation, as well as a specified correlation structure. Table 1 lists the assumed values for these statistical properties, which are taken from Tables 2 and 3 in Høyland and Wallace (2001).

Thus, there are 14 statistical properties to match in the one-period scenario tree. If we choose to generate three scenarios, there are 14 variables in the optimization program (three realizations for each of the four asset classes, plus two scenario probabilities). Table 2 presents optimal values for the realizations of the cash and bond interest rates and the total returns on the stock asset classes. The expected value and standard deviation for each asset class are matched exactly by the realizations in the scenario tree. The imposed correlations are only matched approximately. In fact, the solver was not able to find any locally optimal solution in which the correlation structure was matched exactly in the three scenarios.

Realizations of the interest rates for cash and bonds are translated to total returns using the following relation (Høyland 1998, §3.2.7):

\[
\text{total return} = c + D \cdot (r_0 - r_{1n}) \cdot (1 + c),
\]

where \(c\) is the coupon rate, \(r_0\) the current interest rate, \(r_{1n}\) the interest rate at time 1 in scenario \(n\) (\(n = 1, 2, 3\)).

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Imposed Statistical Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Expected value</td>
</tr>
<tr>
<td></td>
<td>(%)</td>
</tr>
<tr>
<td>Cash</td>
<td>4.33</td>
</tr>
<tr>
<td>Bonds</td>
<td>5.91</td>
</tr>
<tr>
<td>Domestic stocks</td>
<td>7.61</td>
</tr>
<tr>
<td>International stocks</td>
<td>8.09</td>
</tr>
</tbody>
</table>
and $D$ the duration. The coupon rate $c$ and current interest rate $r_0$ are both set to 4.0% for cash and 5.8% for bonds. The assumed duration $D$ equals 0.25 for cash and 6 for bonds. Table 3 contains the total returns for all asset classes in the locally optimal solution of Table 2.

Now consider the following asset allocation at time 0: $35 short in cash, $61 long in bonds, $141 short in domestic stocks, and $115 long in international stocks. Clearly, the net investment is zero. Furthermore, in each of the three scenarios this portfolio generates a positive cash flow at time 1: $0.21 in scenario 1, $0.11 in scenario 2, and $0.73 in scenario 3. Hence, an arbitrage opportunity exists.

For an investor who prefers more to less, an optimization model that is built upon the above scenario tree will exploit the existing arbitrage opportunity in the tree to the maximum extent possible, potentially leading to unlimited risk- and cost-free gains. Clearly, it is unlikely that such an arbitrage opportunity exists in reality, i.e., under the full distribution of possible returns. The optimal solution will thus be biased towards spurious profit opportunities.

The scenario tree presented in Table 3 is only one example of several locally optimal solutions to the nonlinear optimization program of Høyland and Wallace (2001) that contain arbitrage opportunities. For example, when we chose to generate four instead of three scenarios in the above example, we did not encounter any locally optimal solutions that contained arbitrage opportunities. The tree-generation method as described by Høyland and Wallace does not preclude arbitrage opportunities by itself, however. In practical applications, the number of scenarios to generate, and thereby the precision with which a chosen set of statistical properties can be matched, will have to be traded off against the computational tractability of the resulting optimization model.

3. Detecting and Precluding Arbitrage Opportunities

We will show how arbitrage opportunities can be detected ex post in a scenario tree of asset prices, and what constraints could be added to the nonlinear optimization program of Høyland and Wallace (2001) to preclude arbitrage opportunities ex ante. Let $x_{k,t}$ denote the investment in asset class $k$ at time $t$, and $R_{k,t+1}$ the total return on asset class $k$ ($k = 1, \ldots, K$) between time $t$ and $t+1$ if state of the world $n$ ($n = 1, \ldots, N$) materializes at time $t+1$.

Ingersoll (1987) distinguishes two types of arbitrage opportunities. An arbitrage opportunity of the first type exists if there is an asset allocation $x_t = (x_{1,t}, \ldots, x_{K,t})$ such that:

$$\sum_{k=1}^{K} x_{k,t} = 0,$$

$$\sum_{k=1}^{K} x_{k,t}R_{k,t+1} \geq 0 \text{ for all } n = 1, \ldots, N,$$

$$\sum_{k=1}^{K} x_{k,t}R_{k,t+1} > 0 \text{ for at least one } n \in \{1, \ldots, N\}.$$}

In this case it is thus possible to construct a zero-investment portfolio that has nonnegative payoffs in all states of the world, and a strictly positive payoff in at least one state. This situation was illustrated in the example of the previous section.
An arbitrage opportunity of the second type exists if there is an asset allocation \( x_t = (x_{1,t}, \ldots, x_{K,t}) \) such that

\[
\sum_{k=1}^{K} x_{k,t} < 0, \quad (5)
\]

\[
\sum_{k=1}^{K} x_{k,t}(1 + R_{k,t+1}^n) \geq 0 \quad \text{for all } n = 1, \ldots, N. \quad (6)
\]

This asset allocation thus generates a nonnegative payoff in all future states of the world, while providing an immediate positive cash flow to the investor. In the example of the previous section, it is straightforward to turn the identified arbitrage opportunity of the first type into an arbitrage opportunity of the second type. In general, however, the existence of an arbitrage opportunity of the first type does not imply the existence of an arbitrage opportunity of the second type, or vice versa. We will therefore treat them separately.

If total returns \( R_{k,t+1}^n \) are given for all asset classes \( k = 1, \ldots, K \) and possible states of the world \( n = 1, \ldots, N \), then the following linear program can be solved to detect whether arbitrage opportunities of the first type are present:

\[
\begin{align*}
\text{Max} & \quad \sum_{n=1}^{N} \sum_{k=1}^{K} x_{k,t} R_{k,t+1}^n, \\
\text{subject to} & \quad (2) \text{ and } (3).
\end{align*}
\]

If there is a solution to this linear program with a positive objective value, then there is at least one state of the world at time \( t + 1 \) in which the asset allocation \( x_t \) yields a strictly positive return. Combined with the constraints of the linear program, this implies that an arbitrage opportunity of the first type exists. In that case, the linear program will in fact be unbounded as we can multiply the asset allocation \( x_t \) by an arbitrary positive constant without violating the constraints. By duality theory, this implies that the dual of the above linear program does not have a feasible solution.

This dual program is

\[
\begin{align*}
\text{Min} & \quad 0, \\
\text{subject to} & \quad \sum_{n=1}^{N} \nu_n (1 + R_{k,t+1}^n) = 1 \quad \text{for all } k = 1, \ldots, K, \quad (10) \\
& \quad \nu_n \geq 0 \quad \text{for all } n = 1, \ldots, N, \quad (11)
\end{align*}
\]

with \( \nu_n \) the dual variable associated with constraint (6) for \( n = 1, \ldots, N \). Conversely, if this dual program does have a feasible solution, strong duality implies that for any feasible asset allocation \( x_t \), we must have

\[
\sum_{k=1}^{K} x_{k,t} \geq 0, \quad (12)
\]

and thus no arbitrage opportunity of the first type exists.

To detect whether arbitrage opportunities of the second type are present, the following linear program can be solved:

\[
\begin{align*}
\text{Min} & \quad \sum_{k=1}^{K} x_{k,t}, \\
\text{subject to} & \quad (6).
\end{align*}
\]

If this linear program has a solution with a negative objective value, then there is an arbitrage opportunity of the second type. The linear program will in fact be unbounded as we can multiply the asset allocation \( x_t \) by an arbitrary positive constant without violating the constraints. Hence, by duality theory, the dual of this linear program will not have a feasible solution in this case. This dual program is

\[
\begin{align*}
\text{Max} & \quad 0, \\
\text{subject to} & \quad \sum_{n=1}^{N} \pi_n (1 + R_{k,t+1}^n) = 1 \quad \text{for all } k = 1, \ldots, K, \quad (7) \\
& \quad \pi_n \geq 0 \quad \text{for all } n = 1, \ldots, N, \quad (8)
\end{align*}
\]

where \( \pi_0 \) is the dual variable associated with constraint (2), and \( \pi_n \) the dual variable associated with constraint (3) for \( n = 1, \ldots, N \). Conversely, if this dual program does have a feasible solution, strong duality implies that for any feasible asset allocation \( x_t \), we must have

\[
\sum_{n=1}^{N} \sum_{k=1}^{K} x_{k,t} R_{k,t+1}^n \leq 0, \quad (9)
\]

and thus no arbitrage opportunity of the first type exists.

The existence of solutions to the sets of Equations (7)–(8) and (10)–(11) thus implies that no arbitrage opportunities of the first and second type, respectively, are present. Hence, we can check for the existence of solutions to these equations after we have
KLAASSEN

Comment

generated a scenario tree using the method of Høyland and Wallace (2001). In a multiperiod problem, one has to check for solutions to the sets of linear equations in each node \( n \) at every date \( t \) of the scenario tree before the model horizon. A useful result is that if the set of Equations (10) has a strictly positive solution \( \nu \), then no arbitrage opportunities of either the first or second type are present (see Ingersoll 1987, p. 57). If an arbitrage opportunity is encountered, we can apply the scenario-generation method again (using a different starting point) in the hope that we find an arbitrage-free scenario tree. We may also have to increase the number of scenarios in the tree.

As an alternative, we could add Equations (7)–(8) and (10)–(11) as constraints to the nonlinear optimization program of Høyland and Wallace (2001). We will then preclude arbitrage opportunities of both types in the scenario tree that is generated. As asset returns \( R_{n,t+1} \) are variables in the optimization program of Høyland and Wallace (2001), Equations (7) and (10) represent nonlinear constraints if added to this optimization program. This will therefore complicate the numerical optimization of the nonlinear programming model.

4. Conclusion

We have emphasized the importance of precluding arbitrage opportunities when the scenario-generation method of Høyland and Wallace (2001) is applied to asset allocation problems under uncertainty, as the presence of arbitrage opportunities will unrealistically bias optimal asset allocations. We have shown that arbitrage opportunities can either be detected ex post by checking for solutions to sets of linear equations, or precluded ex ante by adding constraints to the optimization program that is formulated to generate the scenario tree.

References


Accepted by Ravi Jagannathan; received October 20, 2001. This paper was with the authors for 2 revisions.