Supplement to "Uncertainty Aversion with Second-Order Utilities and Probabilities" by Robert Nau, *Management Science*, Vol. 52 No. 1, January 2006

Proof of Theorem 1:

Part (a) follows from a double application of the usual argument showing that the independence axiom leads to an additively separable utility representation. (As is well known, the hexagon condition is also needed in the two-dimensional case. Alternatively, the stronger generalized triple cancellation axiom could be used instead of independence.) The details are as follows:

(i) Let $\boldsymbol{w}_j$ denote $(w_{j1}, \ldots, w_{jK})$, the subvector of $\boldsymbol{w}$ consisting of payoffs received in event A_j. Then, by standard arguments (e.g., Debreu 1960 or Wakker 1989), Axiom 2 implies that preferences for $\boldsymbol{w} = (\boldsymbol{w}_1,\ldots, \boldsymbol{w}_J)$ have a representation that is additive over the components $\{\boldsymbol{w}_j\}$, i.e., there exist functions $U_j(\boldsymbol{w}_j)$ such that $\succcurlyeq$ is represented by:

$$U(\boldsymbol{w}) \equiv \sum_{j=1}^{J} U_j(\boldsymbol{w}_j).$$

(ii) Let $\succcurlyeq_j$ denote conditional preferences for $\boldsymbol{w}_j$, i.e., preferences conditional on event A_j as given in Definition 2. Then, by construction, U_j in the preceding representation is an ordinal utility function that represents $\succcurlyeq_j$.

(iii) Axiom 3 implies, again by standard arguments, that for each j, conditional preferences for $\boldsymbol{w}_j$ have a representation that is additive over the components $\{w_{jk}\}$, i.e., there exist functions $\{v_{jk}(w_{jk})\}$ such that $\succcurlyeq_j$ is represented by

$$V_j(\boldsymbol{w}_j) \equiv \sum_{k=1}^{K} v_{jk}(w_{jk}).$$

(iv) Because $U_j(\boldsymbol{w}_j)$ and $V_j(\boldsymbol{w}_j)$ are ordinal utility functions that represent the same preferences, one must be a monotonic transformation of the other, i.e., there exist strictly increasing functions $\{u_j\}$ such that $U_j(\boldsymbol{w}_j) = u_j(V_j(\boldsymbol{w}_j))$, whence $\succcurlyeq$ is represented by:

$$U(\boldsymbol{w}) = \sum_{j=1}^{J} U_j(\boldsymbol{w}_j) = \sum_{j=1}^{J} u_j(V_j(\boldsymbol{w}_j)) = \sum_{j=1}^{J} u_j\Big(\sum_{k=1}^{K} v_{jk}(w_{jk})\Big).$$

Parts (b)-(d) are obtained by some algebra and the risk premium formula from Nau (2003). Parts (e) and (f) follow from the facts that, for all j, $|\,\bar{z}_j\,| = 0$ for a B-spread, $|\,\bar{z}_j\,| > 0$ for an A-spread, and $s_j \geq [>]\ 0$ at all $\boldsymbol{w}$ if u_j is [strictly] concave, while the vector $\boldsymbol{t}\,\boldsymbol{z}^2$ is the same for both spreads.

Proof of Theorem 2:

Axiom 3* implies Axiom 3 when $j=j^*$ and $k=k^*$, so $\succcurlyeq$ is (at least) represented by a utility function of the Model I form. Moreover, a utility function of this form satisfies the cross-state conditions of Axiom 3* if and only if the first-order utility functions $\{v_{jk}\}$ are all related by positive affine transformations. Without loss of generality they can all be written as multiples of a single Bernoulli function v, with unique coefficients $\{q_{jk}\}$ that sum to 1 within each event A_j (with suitable redefinition of the second-order utility functions). This yields a utility function of the form:

$$U(\boldsymbol{w}) = \sum_{j=1}^{J} u_j\left(\sum_{k=1}^{K} q_{jk} v(w_{jk})\right).$$

A function of this form, in turn, satisfies Axiom 4 if and only if the second-order utility functions $\{u_j\}$ are all related by positive affine transformations, hence they can be written as multiples of a Bernouli function u, with unique positive coefficients $\{p_j\}$ that sum to 1.