Arbitrage-Free Correlated Equilibria

Robert F. Nau

The Fuqua School of Business
Duke University
Durham, North Carolina 27708-0120
E-mail: rnau@mail.duke.edu  Fax: 919-684-2818

December 7, 1995
ABSTRACT

A refinement of subjective correlated equilibrium—“arbitrage-free correlated equilibrium”—is derived from the requirement that the outcome of a noncooperative game should not present arbitrage opportunities to an outside observer when the players publicly accept small side-gambles consistent with their beliefs and preferences, regardless of whether they are risk neutral. An arbitrage-free correlated equilibrium is a correlated equilibrium in which the common prior assumption applies to the players’ risk neutral probabilities (products of probabilities and relative marginal utilities for money) rather than their true probabilities. This reformulation of the common prior assumption guarantees that equilibrium expected payoffs for risk averse players are Pareto efficient, a condition not satisfied by completely mixed Nash or objective correlated equilibria.

Key words: correlated equilibrium, joint coherence, arbitrage, common knowledge, common prior assumption, risk neutral probabilities, separation of probability and utility

The author is grateful to Jim Anton, Bob Aumann, Doug Foster, Maarten Janssen, Kevin McCardle, Hervé Moulin, and participants in the Stony Brook Workshop on Knowledge and Game Theory and the Duke-UNC Microtheory Workshop for comments on earlier drafts. This research was supported by the Business Associates Fund and the Hanes Corporation Foundation Fund at the Fuqua School of Business.
“If the players in this game are so smart, why are they playing this silly game? Why don’t they change the rules and play a game where they can do better?” (Remark of Abba Schwartz, cited by Kreps 1990a)

1. Introduction

Suppose that the players in a noncooperative game may make monetary side-gambles with respect to actions of nature and their opponents. What kind of strategic behavior is rational under such conditions, and what equilibrium concepts are appropriate? These questions are important for several reasons. First, side-gambles provide a medium through which players can credibly reveal information about their beliefs and preferences to each other and to outside observers. A game accompanied by side-gambles therefore meets a high operational standard of numerically precise common knowledge of utilities and probabilities, namely “putting one’s money where one’s mouth is.” The concepts of rationality that emerge here provide a normative standard against which rational play under murkier conditions can be judged. Second, game theoretic analysis is increasingly applied directly to the study of strategic behavior in financial markets—environments where many forms of contingent claims are routinely traded and where new financial instruments can be created in response to incentives for efficiency and information. A theory of how ideal games should be played in contingent claim markets is a natural starting point for asking what insights game theory can bring to finance, and vice versa.

To illustrate how side-gambles can be used to reveal the payoff structure and even determine the equilibrium solution of a game, consider the familiar matching-pennies game whose standard payoff matrix is shown in Table 1.

Table 1. Matching pennies

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>(1, -1)</td>
<td>(-1, 1)</td>
</tr>
<tr>
<td>B</td>
<td>(-1, 1)</td>
<td>(1, -1)</td>
</tr>
</tbody>
</table>

-1
Assume that payoffs are in units of money and that the players are risk neutral, so that money payoffs may also be interpreted as utilities. (The risk-averse case and its implications for efficiency will be treated in the next section.) Player 1 (row) faces a choice between two lotteries $u_{1T}$ and $u_{1B}$, whose payoffs as a function of player 2's strategy are $u_{1T} = (+1, -1)$ and $u_{1B} = (-1, +1)$, where the first (second) number in parentheses is the payoff when player 2 plays L (R). Let $E_1(u)$ denote player 1’s expected value for an arbitrary lottery $u$ over player 2’s strategy. Then player 1 will rationally choose T if and only if $E_1(u_{1T}) \geq E_1(u_{1B})$, or equivalently $E_1(u_{1T} - u_{1B}) \geq 0$. Notice that in this case (i.e., the case in which player 1 chooses T), a small side-gamble whose payoffs are proportional to $u_{1T} - u_{1B} = (+2, -2)$ would also be acceptable to player 1, because such a side-gamble would yield a non-negative increase in her total expected payoff. That is, for any small $\alpha > 0$:

$$E_1(u_{1T}) \geq E_1(u_{1B}) \iff E_1(u_{1T} + \alpha(u_{1T} - u_{1B})) \geq E_1(u_{1T}).$$

Now consider an outside observer who finds player 1 offering to accept a gamble proportional to $u_{1T} - u_{1B}$ in the event she plays T. From the observer’s perspective, the strategies of both players are uncertain events, so a gamble between an observer and a player must be represented as a 4-tuple whose elements are the payoffs to be made to the player in the four possible outcomes of the game. The acceptability of the gamble $u_{1T} - u_{1B}$ is conditioned on the event that player 1 plays T: its payoff is therefore zero (i.e., the gamble is called off) if player 1 does not choose T, and otherwise its payoff is +2 or -2 according to whether player 2 chooses L or R. Since the gamble is subject to an arbitrary multiplier, it can be renormalized without loss of generality to a maximum absolute value of 1. The unconditional vector of payoffs from the observer to the player for the renormalized gamble is then:

$$u_{1TB} = (+1, -1, 0, 0), \quad (1a)$$
where the order of the outcomes inside the parentheses is $TL, TR, BL, BR$. (The notation $u_{1TB}$ should be read as “player 1’s relative utility differences between $T$ and $B$, conditional on the former strategy being played.”) Similarly, in the event that player 1 chooses $B$, she presumably finds that $E_1(u_{1B} - u_{1T}) \geq 0$, in which case she would also accept a small gamble proportional to $u_{1B} - u_{1T} = (-2, +2)$. The corresponding (renormalized) unconditional payoff vector is:

$$u_{1BT} = (0, 0, -1, +1),$$  \hspace{1cm} (1b)

where the payoffs are now zeroed-out in the case that $B$ is not played. Player 2, in turn, faces a choice between two lotteries $u_{2L} = (-1, +1)$ and $u_{2R} = (+1, -1)$, where the first (second) number in parentheses is the payoff if player 1 plays $T$ ($B$). Letting $E_2(u)$ denote player 2’s expected value for an arbitrary such lottery $u$ over player 1’s strategy, player 2 will choose $L$ precisely in the case that $E_2(u_{2L} - u_{2R}) \geq 0$, in which case a small gamble proportional to $u_{2L} - u_{2R}$ would also be acceptable to her. Conditioning the latter gamble on the event that player 2 chooses $L$ yields the following renormalized payoff vector for a gamble between player 2 and an observer:

$$u_{2LR} = (-1, 0, +1, 0)$$  \hspace{1cm} (1c)

Finally, the payoff vector for the gamble that would be acceptable to player 2 in the event that she chooses $R$ is:

$$u_{2RL} = (0, +1, 0, -1)$$  \hspace{1cm} (1d)

The side-gambles described above do not change the information structure of the game nor the players’ strategic incentives. If the payoff matrix of the game is common knowledge, as is normally assumed, then the acceptability of the four gambles $\{u_{1TB}, u_{1BT}, u_{2LR}, u_{2RL}\}$ is also naturally common knowledge among risk neutral players. Moreover, the acceptance of any positive multiples of these gambles does not affect
the players’ preferences among their own strategies given their beliefs: the gambles merely amplify existing differences in expected payoffs between chosen and unchosen strategies. Side-gambles constructed in this manner, whose parameters depend only the players’ relative preferences for outcomes and not on their beliefs, are generically known as “preference gambles” (Nau 1992, 1995a).

There is no loss of generality, then, in assuming that the gambles (1a–d) are acceptable to the players—at least when they are risk neutral. Now abstract away the prior assumption of a commonly known game matrix and players who rationally choose strategies so as to maximize their expected payoffs, and never mind the troublesome assumption that rationality itself is common knowledge. Instead, consider only the perspective of an outside observer who finds available to him a set of gambles whose payoffs are pegged to four mutually exclusive and collectively exhaustive events which may (or may not) be under the control of various “players” to whom payments are made. What can the observer infer about the players’ beliefs concerning the outcome, and under what conditions will the observer conclude that they have behaved rationally?

Since the gambles are acceptable in arbitrary non-negative multiples and are in a common currency, the observer may take any non-negative linear combination of them. A minimal *ex ante* standard of rationality is that it should be impossible for the observer to so construct a strictly negative aggregate payoff for the players—a.k.a. an arbitrage opportunity or “Dutch book.” By the well-known theorem of de Finetti (1937, 1974), a collection of acceptable gambles does not admit a Dutch book if and only if there exists at least one probability distribution on the outcome space which assigns non-negative expectation to every gamble.¹ If there is a unique such distribution, and if all the gambles have

---

¹ Actually, an outright Dutch book—*i.e.*, a uniformly negative payoff to the players—can never be constructed from preference gambles derived from the payoff matrix of a game among risk neutral players. This result, which was established by Nau and McCardle (1990) using a Markov chain argument, yields an elementary proof of the existence of correlated equilibria. The Markov chain argument subsequently has been adapted by Myerson (1995) to define the concept of “dual reduction.”
been offered by the same risk neutral individual, we interpret the distribution to represent the subjective beliefs of that individual. If different gambles have been offered by different individuals, the supporting probability distribution can be interpreted to represent their collective or commonly held beliefs. In the case of the matching pennies game, there is in fact a unique distribution assigning non-negative expectation to all four preference gambles, namely the distribution placing probability 1/4 on each outcome. This is also the unique Nash equilibrium distribution of the game, so it turns out that merely by accepting the gambles whose acceptability follows naturally from the payoff structure of the game, the players reveal an apparent collective belief that they are employing the Nash equilibrium concept.

Nau and McCardle (1990) show that, when acceptable preference gambles are derived from a general multi-player noncooperative game in the manner described above, the probability distributions assigning them all non-negative expectation are in general objective correlated equilibria\(^2\) (Aumann 1974, 1987) rather than Nash equilibria. Furthermore, the observer can construct partial Dutch books in which the aggregate payoff to the players is non-positive in all outcomes of the game and strictly negative in any outcome which does not occur with positive probability in some objective correlated equilibrium: the latter outcomes are “jointly incoherent” and should never occur among rational players. Hence, when common knowledge of the rules of the game is defined by the acceptance of preference gambles and common knowledge of Bayesian rationality is defined by the avoidance of jointly incoherent outcomes, the solution concept that emerges is objective correlated equilibrium. This operational approach to defining knowledge and rationality in games deals seamlessly with the transition from individual rationality to strategic rationality, and the results support Aumann’s (1987) contention that objective correlated

\(^2\) An objective correlated equilibrium is a generalization of Nash equilibrium in which randomized strategies of different players may be correlated. Such correlation can be induced by the use of communication devices and/or correlated randomization mechanisms.
equilibrium is the natural expression of Bayesian rationality in noncooperative games—at least among risk neutral players. The same approach has been applied by Nau (1992) to games with incomplete information, where it leads to the concepts of correlated Bayesian equilibrium (Forges 1995) and communication equilibrium (Myerson 1985, Forges 1986).

The purpose of the present paper is to generalize the Nau-McCardle results to the case of players who are not risk neutral—i.e., who may have nonconstant marginal utility for money. The equilibrium concept which emerges here is no longer objective correlated equilibrium, but rather a refinement of subjective correlated equilibrium which will be called “arbitrage-free correlated equilibrium.” In a subjective correlated equilibrium, the players’ true conditional probabilities need not satisfy the Harsanyi doctrine of the common prior distribution. However, in an arbitrage-free correlated equilibrium, the common prior assumption is satisfied by the players’ risk neutral probabilities, which are products of their true probabilities and relative marginal utilities for money. This reformulation of the Harsanyi doctrine reconciles the properties of noncooperative solution concepts (namely, that the players’ reciprocal beliefs should form an appropriate equilibrium) with the properties of efficient allocation in financial markets and syndicates (namely, that the product of probability and relative marginal utility should be equalized across agents; Wilson 1968, Drèze 1970). It also deflects the criticism that that is often leveled at the Harsanyi doctrine, namely that mutually consistent subjective probabilities are unlikely to arise spontaneously in practice (Binmore and Brandenburger 1990, Kreps 1990b, Gul 1992) But mutually consistent risk neutral probabilities are merely the natural result of arbitrage trading (Nau 1995b).

---

3 In his formulation of games of incomplete information, Harsanyi (1967) introduced the assumption that the players' beliefs about each others' types, conditional on their own types, are consistent with a common prior distribution over types, thus obtaining an internally consistent model of an infinite regress of reciprocal beliefs. In a game of complete information, a Nash equilibrium or objective correlated equilibrium also has the property that the players' conditional beliefs about their opponents strategies, given their own choices of strategies, are consistent with a common prior distribution over outcomes of the game.
In addition to requiring a reformulation of the Harsanyi doctrine, non-risk-neutrality introduces two other novel elements into the analysis of games with side-gambles. One is that gambling by non-risk-neutral players partially endogenizes the rules of the game, admitting the possibility that a superior solution might be achieved by tampering with the rules. In extreme cases—e.g. strictly competitive games with completely mixed equilibria—the players may even entirely decouple their actions in this manner. This phenomenon illustrates that the natural decoupling effect of monetary transactions in public markets can sometimes mitigate the need for strategic behavior when the game is accompanied by side-gambles.\(^4\)

The second novel element is that the “true” rules and solution of the game—i.e., the players’ true payoff functions and conditional probabilities—need not be assumed to be directly observable. Instead, the “revealed” payoff functions and probabilities of the players may be distorted by their marginal utilities for money, reflecting the fact that preference measurements which are tied to material rewards generally do not permit a complete separation of probability from utility (Kadane and Winkler 1988; Schervish, Seidenfeld, and Kadane 1990). An analysis of the same phenomenon in the context of single-agent decision analysis is given by Nau (1995a). Despite the fact that common knowledge of “true” utilities is normally considered to be the starting point for game-theoretic analysis—indeed, this was the motivation for von Neumann and Morgenstern’s (1944) axiomatization of cardinal utility—the distortions of probability and utility encountered here merely reshape some familiar solution concepts rather than rendering analysis impossible.

The remainder of the paper is organized as follows. Section 2 presents examples (matching-pennies and battle-of-the-sexes with risk averse players) which illustrate the principles and techniques to be developed later. Section 3 formalizes the assumptions

\(^4\) Strategic behavior does not disappear altogether when players decouple their actions in the game through an accumulation of gambles. Rather, strategic interactions occur at a more primitive level in which the surplus generated by gambling is divided—if it is not first skimmed off by an arbitrageur.
about the “true” game that is being played. Section 4 derives the properties of the “revealed” game that is determined by small monetary gambles acceptable to the players. Section 5 introduces the concept of joint coherence (no arbitrage) as a standard of strategic rationality and proves the main results concerning the properties of arbitrage-free correlated equilibria. Section 6 proves that, under conditions of risk aversion (decreasing utility for money), the revealed game typically looks like an imprecisely specified version of the true game. Section 7 provides a concluding discussion.

2. Games with side-gambles among risk averse players

The effects of risk aversion on information and efficiency will now be illustrated for several well-known games. First, return to the matching-pennies game of Table 1, and suppose that the game is still zero-sum in terms of the original money payoffs but that the players are now risk averse in the sense that their marginal utilities for money are strictly decreasing functions of their utility levels. The acceptance of side-gambles now has the potential to affect the players’ utilities, so henceforth it will be assumed that the game payoffs are “large” ($\pm$1000) while the side-gambles are “small” (on the order of $\pm$1). For purposes of illustration, suppose that the players have identical negative-exponential utility functions $u(x) = 1 - \exp(-x/r)$ with risk tolerances $r = 1000/\ln(2) \approx 1443$, so that $u(1000) = 2.0$ and $u(-1000) = 0.5$. The game is still constant-sum\(^5\) in utilities, and the unique Nash/objective correlated equilibrium still assigns equal probability to all outcomes. The corresponding marginal utility functions $u'(x)$ have the property that $u'(x) \propto 1 - u(x)$, so that each player’s relative marginal utility for money is 4 times higher when she ends up with a payoff of $-1000$ than when she ends up with a payoff of $+1000$. This dependence of marginal utilities for money on the outcome of the game distorts the

---

\(^5\) The important feature of this game is not that it is zero-sum in money or constant-sum in utilities, but rather only that it is strictly competitive—i.e., the players’ payoffs vary oppositely with the outcome.
players’ attempts to articulate their utilities through preference gambles, as will now be shown.

Player 1 now faces a choice between two lotteries whose payoffs in *utility currency* are $u_{1T} = (2.0, 0.5)$ and $u_{1B} = (0.5, 2.0)$, where the numbers in parentheses are utilities received when player 2 plays $L$ or $R$, respectively. Therefore, when she chooses $T$ rather than $B$, she should also be willing to accept a gamble whose payoffs in utility currency are proportional to $u_{1T} - u_{1B} = (+1.5, -1.5) \propto (+1, -1)$, the same as before. However, because money is now worth 4 times as much to player 1 when player 2 plays $R$ rather than $L$, given that player 1 has played $T$, a small *monetary* gamble yielding those relative utilities must have payoffs proportional to $+$1 and $-$0.25, respectively. Hence, the acceptable monetary gamble associated with player 1’s choice of $T$ is now:

$$\hat{u}_{1TB} = (+1.0, -0.25, 0, 0)$$ \hspace{1cm}(2a)$$

where hat-notation is used to indicate that the elements of the vector now correspond to “revealed” utility differences rather than “true” utility differences. (The term $\hat{u}_{1TB}$ should be read as “player 1’s revealed utility differences between strategies $T$ and $B$, conditional on the former strategy being played.” As before, the ordering of the outcomes is $TL, TR, BL, BR$, and the payoffs are zeroed-out in those outcomes where the conditioning event does not occur.) The gamble (2a) is much more favorable to player 1 than the even-odds gamble (1a) she would have accepted had she been risk neutral. In other words, the risk averse player’s preference gamble looks like a hedged version of the risk neutral player’s preference gamble, other things being equal. (This is true in general, as will be proved in Section 6.) The remaining monetary gambles that would be accepted by players 1 and 2 are:

$$\hat{u}_{1BT} = (0, 0, -0.25, +1.0)$$ \hspace{1cm}(2b)$$

$$\hat{u}_{1LR} = (+1.0, 0, -0.25, 0)$$ \hspace{1cm}(2c)$$
\[ \hat{u}_{1TB} = (0, +1.0, 0, -0.25), \]  

(2d)

An observer seeing only the monetary preference gambles (2a–d) would find the players’ apparent collective beliefs about the game’s outcome to be underdetermined: there are many probability distributions which assign these gambles non-negative expectation, unlike the risk-neutral gambles (1a–d). The game is not “solved,” from the observer’s perspective, unless and until the players accept additional gambles reflecting their actual conditional beliefs about each others’ choices of strategies, given their own strategies. Auxiliary gambles of this form are generically referred to as “belief gambles” (Nau 1992, 1995a). If the players are in fact playing the Nash equilibrium strategy, then conditional on player 1 choosing \( T \), she assigns equal probabilities to her opponent playing \( L \) or \( R \). But because her marginal utility for money is 4 times higher when her opponent plays \( R \) rather than \( L \), given that she has played \( T \), she should bet money on her opponent playing \( R \) at odds of 1:4 in this case. In other words, conditional on her own choice of \( T \), she should indifferently accept a gamble in which she wins $0.25 if her opponent plays \( R \) and loses $1 if her opponent plays \( L \). The monetary payoff vector for this belief gamble can be expressed as:

\[ \hat{p}_{1TR} = (-1, +0.25, 0, 0). \]  

(3a)

(Here the notation \( \hat{p}_{1TR} \) should be read as “player 1’s revealed probability for \( R \), conditional on \( T \).”) These payoffs are precisely the negatives of those of the preference gamble \( \hat{u}_{1TB} \) in (2a), but the combination of these two gambles is not unilaterally incoherent for player 1. Rather, they combine to give the appearance that she thinks her opponent is exactly 4 times more likely to play \( R \) than \( L \) when she herself plays \( T \). The monetary payoff vectors for the additional belief gambles that would be acceptable to one Nash player or the other are as follows:

\[ \hat{p}_{1BL} = (0, 0, +0.25, -1), \]  

(3b)

\[ \hat{p}_{1BL} = (0, 0, +0.25, -1), \]  

(3b)

\[ \hat{p}_{1BL} = (0, 0, +0.25, -1), \]  

(3b)

The negatives of the belief gambles (3a–d) are also acceptable belief gambles, but they merely replicate the preference gambles (2a–d) in this case.
\[ \hat{p}_{2LT} = (-1.0, 0, +0.25, 0), \quad (3c) \]
\[ \hat{p}_{2RB} = (0, -1.0, 0, +0.25). \quad (3d) \]

Before proceeding further, the reader may wish to consider whether the acceptance of belief gambles, unlike preference gambles, might somehow violate the noncooperative spirit of the game, restrict the players’ freedom of choice, betray their private information, or introduce new strategic interactions. For example, if player 1 offers to accept a bet reflecting her current beliefs about player 2’s strategy, why shouldn’t player 2 accept the other side of the bet and then alter her strategy? We assert that no such problems arise. First, the players are not entering into bilateral contracts that might allow them to escape from prisoners’ dilemmas or other failures of cooperation. Rather, they are unilaterally offering to gamble with anyone who might be observing, who need not be an opposing player. Second, the parameters of the gambles do not reveal any information which is not otherwise assumed to be common knowledge in a conventional game theoretic analysis, except that when the players are risk averse, the publicly revealed information about their beliefs and preferences is filtered through their marginal utilities for money. (It will be seen later that this merely entails some loss of information concerning the separation of probability from utility.) Finally, and perhaps most importantly, we assume that the observer comes upon the scene only after the players have exhausted whatever incentives they originally may have had to gamble with each other—i.e., the observer finds the players in a stable situation with gambles on the table that are not being revised in response to interplayer transactions. The stability of the public gambles is critical to the assumption that a state of common knowledge exists: if the players were still actively trading with each other, the terms of the gambles presumably would be changing (Nau 1995b). Thus, for example, if one player is observed to assign nontrivial betting probabilities to the actions of her opponent, this must mean that the first player is expressing subjective beliefs.
which she is willing to place on the public record and which the second player does not wish to disturb, and which the first player knows the second player does not wish to disturb, etc..

Returning to the belief gambles of the risk averse Nash players in (3a–d), we now observe a difficulty: these gambles are not supported by any one probability distribution and so give rise to arbitrage opportunities. In particular, a simple sum of the four gambles yields a sure win for anyone simultaneously betting against both players. Thus, risk averse players are irrational as a group if they endorse the Nash equilibrium strategy in the matching-pennies game when their beliefs are backed up by monetary gambles—and the same is clearly true of any completely mixed Nash or objective correlated equilibrium strategy in any strictly competitive game.\footnote{In any completely mixed Nash or objective correlated equilibrium, the players’ true probabilities are mutually consistent. If the game is strictly competitive and the players are risk averse, their marginal utilities will vary oppositely with the outcome of the game. Hence their revealed probabilities, which are proportional to the product of their true probabilities and marginal utilities for money, cannot be mutually consistent in a mixed-strategy equilibrium.}

There are two ways out of this dilemma: the players must either implement a subjective correlated equilibrium, pegging their strategies to events about which they hold genuinely different beliefs, or else they must change the rules of the game through an accumulation of gambles. Wherever an arbitrage opportunity exists for an observer, there is a potential surplus to be divided among the players if they circumvent the arbitrager and gamble directly with each other—which is why they should not stop gambling with each other unless and until they reach a state in which no arbitrage is possible. For the risk-averse players endorsing the Nash strategy in matching-pennies, the sum of player 1’s two belief gambles yields a payoff to her of \(-$1\) if \(TL\) or \(BR\) is played and a payoff of \+$0.25\) if \(BL\) or \(TR\) is played. If player 1 would accept this gamble indifferently, she would be more than willing to accept a dominant gamble paying \(-$1\) if \(TL\) or \(BR\) is played and \+$1\) if \(BL\) or \(TR\) is played. For similar reasons, player 2 would be more than willing to
take the other side of the same gamble, receiving +$1 if $TL$ or $BR$ is played and −$1 if $BL$ or $TR$ is played. In this way, both players would hedge their losses while increasing their expected utilities, transferring more utility into their losing outcomes than would be removed from their winning outcomes.\footnote{This proposed interplayer gamble happens to represent an even division of the surplus in money terms. In general, the division of surplus might depend on strategic bargaining between the players. Thus, some or all of the strategic interactions between the players would be relegated to the preplay stage in which the rules of the game are determined.} Furthermore, as long as they adhered to the Nash equilibrium beliefs (or any other mutually consistent beliefs), they would have incentives to continue gambling until their marginal utilities were equalized across outcomes, which would occur when their utilities were also equalized across outcomes. At this point, their actions would become strategically decoupled: they would no longer care about each other others’ actions and would have no further incentive to randomize. In the process, they would have fulfilled Schwartz’ recommendation, changing the rules of the game so that “they can do better.”

Similar incentives exist even in games which are not strictly competitive, such as the battle-of-the-sexes game whose standard payoff matrix is shown in Table 2.

**Table 2.** Battle of the sexes

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>(2, 1)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>$B$</td>
<td>(0, 0)</td>
<td>(1, 2)</td>
</tr>
</tbody>
</table>

This game has two efficient pure-strategy Nash equilibria (on the diagonal) plus an inefficient mixed-strategy Nash equilibrium in which both players choose their higher-maximum-payoff strategy with probability 2/3. The “fair” solution to this game is usually considered to be the objective correlated equilibrium obtained by 50-50 randomization between the two efficient Nash equilibria—i.e., a coin flip is used to decide whether both players will attend the bullfight or the ballet. However, if the players are risk averse in the
sense that they have diminishing marginal utility for money, they will then find it mutually profitable to bet on the coin flip—or else an arbitrageur will wish to bet with both of them. (It is not actually necessary to observe the coin flip: they can bet on whether the outcome is bullfight-bullfight conditional on it being either bullfight-bullfight or ballet-ballet.) The fair and efficient solution is for the players to exchange outcome-contingent wealth through gambling until their relative marginal utilities are equalized between the two possible outcomes. Of course, they might approximate the same solution by trading contingent claims to other forms of consumption: the person receiving his or her less-desired entertainment could be compensated by getting the choice of a restaurant at which to have dinner beforehand.

3. The model of the true game

To model the phenomena illustrated in the preceding sections, consider a standard, finite $N$-person game of complete information, in which player $n$’s strategy set is $S_n$ and the set of outcomes (joint strategies) is $S = S_1 \times \ldots \times S_N$, with generic elements $s_n$ and $s = (s_1, \ldots, s_N)$. Let $S_{-n}$ denote the joint strategy set of all players other than $n$, with generic element $s_{-n}$, let $s_{-n} k$ denote the outcome in which player $n$ chooses strategy $k \in S_n$ while her opponents choose $s_{-n} \in S_{-n}$, and let $1_{nk}(s)$ denote the indicator on $S$ for the event that $s_n = k$. Let $c_n(s)$ denote the (possibly non-monetary, possibly-multiattribute) consequence that player $n$ receives in outcome $s$ by virtue of participation in the original game, and let $g_n(s)$ denote a real-valued quantity of money received from side-gambles. The players are assumed to be Bayesian rational with respect to gaming and gambling, which is formalized in:

**Assumption 1**: For every $n \in \{1, \ldots, N\}$, player $n$’s beliefs and preferences are represented by conditional probability distributions $p_n$ and utility functions
$U_n(c_n, g_n)$ such that player $n$ will choose strategy $k$ only if it maximizes her conditional expected utility, i.e.: 

$$\sum_{s_{-n} \in S_{-n}} p_n(s_{-n}k)U_n(c_n(s_{-n}k), g_n(s_{-n}k)) \geq \sum_{s_{-n} \in S_{-n}} p_n(s_{-n}k)U_n(c_n(s_{-n}j), g_n(s_{-n}j))$$

for all other strategies $j \in S_n$. (Here, $p_n(s)$ denotes player $n$’s conditional probability for joint strategy $s_{-n} \in S_{-n}$ being chosen by her opponents given that her own choice is $s_n \in S_n$.) Furthermore, for fixed $c$, $U_n(c, g)$ is strictly increasing and differentiable in $g$—i.e., more money is preferred to less, and preferences for money are smooth.

For convenience, let $u_n(s) \equiv U_n(c_n(s), 0)$ denote player $n$’s utility for the consequence received in outcome $s$ in the absence of side-payments. $u_n$ is what is normally called the player’s payoff function in the game, and—for most decision- and game-theoretic purposes—it is unique only up to positive scaling and addition of terms that depend only on $s_{-n}$, not $s_n$. What is uniquely determined, in principle, is the function on $S$ defined by 

$$u_{nkj}(s) \equiv \frac{(u_n(s_{-n}k) - u_n(s_{-n}j))1_{nk}(s)}{\max_{s_{-n}' \in S_{-n}} [u_n(s_{-n}'k) - u_n(s_{-n}'j)]},$$

for every $n$ and for all $j, k \in S_n$. In the event that player $n$ chooses her $k^{th}$ strategy, the quantity $u_{nkj}(s)$ is the difference between the utility she actually received and the utility she would have received by choosing strategy $j$ instead, given that the other players chose $s_{-n}$, normalized (over all $s_{-n} \in S_{-n}$) to a maximum absolute value of 1. (By definition, $u_{nkj}(s) = 0$ if $s_n \neq k$.) The players’ relative utility differences constitute a minimal description of the rules of the “true” game, containing the essential data needed to apply noncooperative solution concepts such as Nash and correlated equilibria, iterated dominance, rationalizability, etc.. Thus, for example, expressions (1a–d) encode the true rules of the game of matching-pennies, whether or not the players are risk averse.
From Assumption 1 it follows that, in the absence of side-payments, player \( n \) will choose strategy \( k \) only if:

\[
\sum_{s \in S} p_n(s) u_{nkj}(s) \geq 0. 
\]  

(4)

Inequality (4) merely re-expresses the condition that, when player \( n \) is observed to choose strategy \( k \), her expected utility for \( k \) must be greater than or equal to her expected utility for any other strategy \( j \), according to her current conditional beliefs. The set of all players’ conditional beliefs \( \{p_n\} \) will be called the (true) solution of the (true) game whose rules are \( \{u_{nkj}\} \). The solution is a subjective correlated equilibrium (Aumann 1974) if every player is Bayesian rational in every outcome—i.e., if (4) holds for every \( n \in \{1, \ldots, N\} \) and every pair of strategies \( k, j \in S_n \). A subjective correlated equilibrium in which the players agree on zero-probability events is called an a posteriori equilibrium.

If, in addition to (4), the conditional distributions of different players are mutually consistent—i.e., if there exists a “common prior” distribution \( \pi \) on \( S \) such that \( p_n(s) = P_\pi(s_n|s_n) \) or else \( P_\pi(s_n) = 0 \) for every \( n \) and \( s \)—then the solution is an objective correlated equilibrium. In this case, (4) can be rewritten in terms of \( \pi \) as

\[
\sum_{s \in S} \pi(s) u_{nkj}(s) \geq 0
\]

If, furthermore, the conditional distributions are independent—i.e., \( p_n(s_nk) = p_n(s_nj) \) for every player \( n \) and pair of strategies \( k \) and \( j \)—the solution is a Nash equilibrium. We take no position a priori as to the reasonableness of any of these solution concepts.

4. The model of the revealed game

An observer of the game cannot necessarily know the players’ actual subjective utilities and probabilities—i.e., the “true” rules and solution of the game. Rather, it can
only be safely assumed that the observer knows those parameters of the game which materially affect the observer, namely the side- gambles publicly accepted by the players. This section characterizes the knowledge that can be obtained in this way, which constitutes the “revealed” game seen by the observer.

When utility for money may be nonlinear, it must be explicitly assumed that incremental gambles are “small” in comparison to game payoffs. To make this notion precise, the concept of “$\epsilon$- acceptability” (acceptability of infinitesimally small multiples of gambles) is introduced, following Nau (1995a):

**DEFINITIONS**: A gamble for player $n$ is a vector whose elements, indexed by $s \in S$, denote quantities of money. The gamble $g$ is $\epsilon$-acceptable to player $n$ given her choice of strategy $k$ if, for every $\epsilon > 0$ there exists $\delta > 0$ such that for any $\alpha$ between 0 and $\delta$ chosen at the discretion of an opponent prior to play of the game, a side payment of $\alpha(g(s) + \epsilon)1_{nk}(s)$ is preferred to a side payment of zero. (Note that the payoff for this conditional gamble depends on the player’s own strategy only to the extent that it is “called off” if $s_n \neq k$.)

Let $v_n(s)$ denote player $n$’s marginal utility for money in outcome $s$. That is,

$$v_n(s) = \frac{\partial U_n(c_n(s), z)}{\partial z} \bigg|_{g=0}$$

Then, if outcome $s$ obtains and player $n$ receives a small monetary side payment equal to $z$, her total utility (to a first-order approximation) is $u_n(s) + zv_n(s)$. Player $n$ is defined to be risk averse if her marginal utility for money is a strictly decreasing function of her game payoff—i.e., if $u_n(s) > u_n(s')$ implies $v_n(s) < v_n(s')$.

Basic properties of $\epsilon$-acceptable gambles which are straightforward consequences of Bayesian rationality (Assumption 1) are summarized in:
**LEMMA 1:** \( g \) is \( \epsilon \)-acceptable to player \( n \) conditional on her choice of strategy \( k \) if and only if it yields non-negative expected marginal utility, i.e.,

\[
\sum_{s \in S} p_n(s)v_n(s)g(s)1_{nk}(s) \geq 0.
\]

**LEMMA 2:** A non-negative linear combination of \( \epsilon \)-acceptable gambles is \( \epsilon \)-acceptable.

**LEMMA 3:** Any gamble which weakly dominates an \( \epsilon \)-acceptable gamble is \( \epsilon \)-acceptable.

The product of probabilities and marginal utilities \( p_n(s)v_n(s) \) appearing in Lemma 1 plays the role of a probability distribution with respect to which gambles are evaluated in a risk neutral fashion—i.e., they are accepted if they yield a non-negative expected payoff. This product is of fundamental importance, so we define, for every \( n \):

\[
\hat{p}_n(s) = \frac{p_n(s)v_n(s)}{\sum_{s'_{-n}, s_n \in S_n} p_n(s'_{-n}s_n)v_n(s'_{-n}s_n)}.
\]

(5)

In other words, \( \hat{p}_n(s) \) is just the product \( p_n(s)v_n(s) \) renormalized to sum to unity over all \( s_{-n} \in S_n \) while \( s_n \) is held fixed. The quantity \( \hat{p}_n(s) \) will be called player \( n \)'s risk neutral probability for \( s_{-n} \) conditional on her choice of strategy \( s_n \).

In conventional game-theoretic analysis, the parameters of the game which are assumed to become common knowledge are the players’ true conditional probabilities \( \{p_n(s)\} \) and their true utility functions \( \{u_n(s)\} \)—or more precisely, their true utility differences \( \{u_{nkj}(s)\} \). The utility differences determine constraints that the true probabilities must satisfy in order for a given strategy to be rationally chosen: it must yield non-negative expected utility differences with respect to alternative strategies as specified in

---

\(^9\) The term “risk neutral probability” has long been used in this context in the finance literature—e.g., Cox and Ross (1976). Other equivalent terms are “risk-adjusted probability” and “utilprob.”
equation (4). Thus, it is usually assumed that the players’ probabilities can be completely separated from their utilities when observing and analyzing the game. However, the revelation mechanism through which probabilities and utilities become commonly, credibly, and separably known is usually not specified.

Here, the acceptance of gambles provides the mechanism through which information about probabilities and utilities is revealed. This revelation process is now formalized in:

**ASSUMPTION 2**: The set of gambles which are \( \epsilon \)-acceptable is common knowledge.

The interpretation of common knowledge adopted here is the operational common knowledge that obtains in a public market with stable prices (Nau 1995b) rather than the set-theoretic definition introduced by Aumann (1976).

When information about beliefs and preferences is filtered through the acceptance\(^{10}\) of gambles, the “revealed” utilities and probabilities of the players become distorted by their marginal utilities for money, as was illustrated in Section 2. A complete separation of probabilities from utilities is therefore impossible. (For discussions of this issue, see Kadane and Winkler 1988, Schervish et al. 1990, Karni and Schmeidler 1993, Nau 1995a). However, a more limited kind of separation holds, namely a distinction between two kinds of acceptable gambles: belief gambles and preference gambles. Belief gambles reveal the the players’ risk neutral probabilities rather than their true probabilities, while preference gambles determine constraints that risk neutral probabilities must satisfy in order for particular strategies to be rationally chosen. The formal characteristics of belief and preference gambles are given in the following lemmas (proofs are given in Nau 1995a):

---

\(^{10}\) Henceforth we shall dispense with the \( \epsilon \) qualifier and simply refer to “acceptable” gambles, with the understanding that this means \( \epsilon \)-acceptable.
**Lemma 4**: For every \( n, k \in S_n \), and \( m \in S_{-n} \), the “belief” gamble defined by:

\[
\hat{p}_{nkm}(s) \equiv (1 - nm(s) - \hat{p}_n(km))1_{nk}(s)
\]  

(6)

is \( \epsilon \)-acceptable to player \( n \) conditional on her choice of strategy \( k \), where \( \hat{p}_n(km) \) stands for \( \hat{p}_n(s) \) evaluated at \( s_n = k \) and \( s_{-n} = m \), \( 1_{-nm} \) denotes the indicator function on \( S \) for the event that \( s_{-n} = m \), and as before \( 1_{nk} \) denotes the indicator function on \( S \) for the event that \( s_n = k \).

Acceptance of the belief gamble in Lemma 4 is equivalent to paying the amount \( \hat{p}_n(km) \) for a lottery ticket which is worth $1 in the event that \( s_{-n} = m \), otherwise zero, all conditional on \( s_n = k \). Thus, in general, the risk neutral probability \( \hat{p}_n(s) \) is player \( n \)’s “revealed” probability for her opponents playing \( s_{-n} \) in the event that she plays \( s_n \), under de Finetti’s (1974) elicitation method in which probabilities are equated with maximum buying prices for lottery tickets. Some of the belief gambles acceptable to the risk averse players in matching-pennies are given in expressions (3a–d).

**Lemma 5**: For all \( n, k, j, \) and \( s \), the “preference” gamble defined by

\[
\hat{u}_{nkj}(s) \equiv \frac{(u_n(s_{-n}k) - u_n(s_{-n}j))1_{nk}(s)/v_n(s)}{\max_{s'_{-n} \in S_{-n}} |u_n(s'_{-n}k) - u_n(s'_{-n}j)|/v_n(s'_{-n}k)},
\]  

(7)

is \( \epsilon \)-acceptable to player \( n \) conditional on her choice of strategy \( k \).

(Notice that \( \hat{u}_{nkj} \propto u_{nkj}/v_n \)— the denominator in (7) is just a normalizing constant.) Thus, \( \hat{u}_{nkj} \) is the vector of player \( n \)’s conditional utility differences between strategy \( k \) and strategy \( j \) divided by the marginal utilities for money that apply when she chooses \( k \). (Recall that conditional risk neutral probabilities are proportional to true probabilities multiplied by the marginal utilities for money that apply when the given strategy is chosen.) Preference gambles are the players’ revealed utility differences, and they constitute the rules of the “revealed” game. The preference gambles accepted by the risk averse players in matching-pennies are given by expressions (2a–d).
In general, the sets of belief gambles and preference gambles overlap—in fact, a rational individual’s belief gambles span the set of all her acceptable gambles, so that her set of preference gambles is necessarily a subset of her set of belief gambles. The importance of separating out the preference gambles, however, is that they depend only on utilities, not in any way on probabilities. Hence it is reasonable to suppose that preference gambles would be the first gambles to be publicly articulated, before the players begin to form beliefs about how their opponents will play, let alone gamble according to those beliefs. In some cases (e.g., the matching pennies game among risk neutral players) it might turn out that the players’ rational beliefs are already tightly constrained by the preference gambles. In other cases, considerable latitude for belief formation might exist. In either case, the players might find mutually beneficial ways to rewrite the rules of the game by gambling with each other during the preplay communication stage. Hypothetical methods for eliciting belief and preference gambles, and distinguishing one from the other, are discussed by Nau (1995a).
5. Joint coherence and arbitrage-free correlated equilibrium

The previous section established that the parameters of the revealed game and its solution consist of preference gambles $\hat{u}_{nkj}(s)$ and risk neutral probabilities $\hat{p}_n(s)$. From the definitions of these terms, it follows that:

$$\sum_{s \in S} p_n(s) u_{nkj}(s) \geq 0 \iff \sum_{s \in S} \hat{p}_n(s) \hat{u}_{nkj}(s) \geq 0.$$  (8)

(The marginal utilities in the numerator of the risk-neutral probability formula (5) and the denominator of the preference gamble formula (7) merely cancel out.) Hence, despite the fact that the parameters of the revealed game do not provide a complete separation of probability from utility, they still contain all the information needed to determine whether a strategy maximizes a player’s conditional expected utility: the expected difference in utility between strategy $k$ and another strategy is non-negative if and only if the corresponding preference gamble has non-negative expectation with respect to the risk neutral probabilities. If all we observe are the players’ acceptable gambles, not their true probabilities and utilities, we can still determine which strategies are Bayesian rational for each player.

More can be said, however, about the relation between acceptable gambles and rational outcomes of the game and about the equilibrium conditions that should connect the beliefs of different players, given their preferences. Namely, rational outcomes of the game are outcomes that are coherent—i.e., which do not admit arbitrage—and reasonable equilibria of the game are those which avoid incoherent outcomes. The remainder of this section formalizes these concepts.

**DEFINITION:** A strategy is *individually coherent* for a player if she has not accepted a gamble yielding strictly negative payoffs when that strategy is played and
non-positive payoffs in all other outcomes. An outcome of the game is *jointly coherent* if the players as a group have not accepted a gamble yielding a strictly negative payoff in that outcome and non-positive payoffs in all outcomes.\textsuperscript{11}

Note that a strategy is jointly coherent only if it is individually coherent for all players, but the converse is not necessarily true.

**THEOREM 1**: Strategy $k$ is individually coherent for player $n$ if and only if her revealed utility differences between strategy $k$ and all other strategies have non-negative risk-neutral expectation (\textit{i.e.}, inequality (8) is satisfied for all other strategies $j \in S_n$).

**Proof**: Apply de Finetti’s theorem to all the gambles accepted by player $n$ conditional on strategy $k$.

We now propose an equilibrium concept which applies to the revealed solution of the revealed game:

**DEFINITION**: The solution of the game is an *arbitrage-free correlated equilibrium* if (8) is satisfied for every $n \in \{1, \ldots, N\}$ and every pair of strategies $k, j \in S_n$, and in addition, the players’ risk neutral probabilities are mutually consistent, \textit{i.e.}, derivable from a common prior distribution.

This is merely our original definition of objective correlated equilibrium with the common prior condition applied to the players’ risk neutral probabilities rather than their true probabilities. The connection between arbitrage-free correlated equilibrium and other familiar equilibrium concepts is given by:

\textsuperscript{11} In view of the “\(\forall \epsilon \exists \delta\)” qualification of acceptability, this means that the ratio of the amount an observer wins under the observed outcome to the maximum amount he might have lost under any other outcome cannot be made arbitrarily large as the scale of the gambles goes to zero.
THEOREM 2: A solution of the game is an arbitrage-free correlated equilibrium if and only if it is an *a posteriori* equilibrium in which the players’ risk neutral probabilities are mutually consistent.

PROOF: (8) holds if and only if (4) holds because $\hat{p}_n(s_{-n}k) \propto p_n(s_{-n}k)v_n(s_{-n}k)$ and $\hat{u}_{nkj} \propto u_{nkj}/v_n$. Mutual consistency of risk neutral probabilities implies that the players agree on the events to which they assign true probabilities of zero. Hence, (8) holds and risk neutral probabilities are mutually consistent (i.e., risk neutral probabilities constitute an arbitrage-free correlated equilibrium) if and only if (4) holds with agreement on zero-probability events (i.e., true beliefs constitute an *a posteriori* equilibrium of the true game) and risk neutral probabilities are mutually consistent. ■

COROLLARY: A Nash or objective correlated equilibrium of a 2-person game is arbitrage-free if it is not completely mixed—i.e., if at least one player plays a pure strategy. A Nash or objective correlated equilibrium of a strictly competitive game among risk averse players is arbitrage-free only if it is not completely mixed.

PROOF: For the “if” part, a Nash or objective correlated equilibrium is in general an *a posteriori* equilibrium. If both players play pure strategies, their true and risk neutral probabilities trivially agree. If strategy $k$ of player $n$ is assigned probability 1 by the other player, then a common prior distribution agreeing with both players’ risk neutral probabilities can be constructed by setting $\pi(s_{-n}k) = \hat{p}_n(s_{-n}k)$ and $\pi(s_{-n}j) = 0$ for all $j \neq k$. For the “only if” part, focus on any 2x2 subgame whose outcomes all receive positive probability, and note that if the true probabilities of the players are mutually consistent within this subgame, then their risk neutral probabilities cannot be, because their marginal utilities vary oppositely. ■

Finally, the normative (and nomenclatorial) justification for arbitrage-free correlated equilibrium is given by:
THEOREM 3: An outcome is jointly coherent if and only if the solution of the game is an arbitrage-free correlated equilibrium in which that outcome has positive probability.\textsuperscript{12}

Proof: Let $G$ denote the matrix whose rows are indexed by $s \in S$ and whose columns are the payoff vectors of acceptable belief and preference gambles. From Lemmas 2 and 3, it follows that the set of acceptable gambles includes all vectors which equal or dominate a vector of the form $G\alpha$, where $\alpha$ is a non-negative vector of weights chosen by an observer. Let $s'$ denote a particular outcome of the game. By linear duality, there does not exist non-negative $\alpha$ such that $G\alpha \leq 0$ and $G\alpha(s') < 0$ if and only if there exists a semi-positive vector $\pi$ such that $\pi(s') > 0$ and $\pi^T G \geq 0$—i.e., a probability distribution which assigns non-negative expectation to every acceptable gamble and positive probability to outcome $s'$. (Here, superscript $T$ denotes transposition.) The assignment of non-negative expectation to the belief gambles means that the players’ risk neutral probabilities must be consistent with $\pi$—i.e., $\hat{p}_{nk}(s_{-n}) = P_\pi(s_{-n}|s_n = k)$ or else $P_\pi(s_n = k) = 0$ for every $n, k$, and $s_{-n}$. The assignment of non-negative expectation by $\pi$ to the preference gambles then implies that the players’ risk neutral probabilities constitute an objective correlated equilibrium of the revealed game, and the condition that $\pi(s') > 0$ means that this outcome has positive probability. \[\square\]

Hence, a posteriori equilibrium is the “true” expression of Bayesian rationality in our setting, but the players’ beliefs must nonetheless satisfy a strong consistency condition, namely a consistency of risk neutral probabilities.

\textsuperscript{12} The statement of this proposition differs from that of Proposition 2 of Nau and McCardle (1990) insofar as it presumes that a solution of the game has been proposed and affirmed by the players—i.e., that belief gambles as well as preference gambles have been articulated. If only preference gambles have been articulated, then an outcome is jointly coherent if and only if there exists an arbitrage-free correlated equilibrium in which it has positive probability.
6. Incomplete revelation of payoff functions under risk aversion

The preceding analysis of the “revealed” game has shown that outcome-dependent marginal utilities for money give rise to discrepancies between true and revealed utility differences. This section studies the qualitative properties of such discrepancies in more detail in order to shed light on the existence and uniqueness of arbitrage-free correlated equilibria.

Let \( \hat{u}_n \) be called a supporting risk neutral payoff function for player \( n \) if, for every \( k, j \in S_n \), there exists \( \alpha \) such that

\[
\hat{u}_{nkj}(s) \geq \alpha(\hat{u}_n(s_{-n}k) - \hat{u}_n(s_{-n}j))1_{nk}(s).
\]

(9)

for every \( s_{-n} \in S_{-n} \). The vector on the LHS is a preference gamble acceptable to player \( n \), whose true payoff function is \( u_n \), while the vector on the RHS is a preference gamble which would be acceptable to a risk neutral player with true payoff function \( \hat{u}_n \). Thus, \( \hat{u}_n \) is called a supporting risk neutral payoff function for player \( n \) if a risk neutral player with payoff function \( \hat{u}_n \) would accept gambles as strong or stronger than those actually accepted by player \( n \).

Let player \( n \)’s revealed utility differences be called complete if there exists a supporting risk neutral payoff function such that (9) holds with equality for all \( k, j, \) and \( s \). This can be true only if \( u_{nkj} = -u_{njk} \) for every \( k \) and \( j \). Consequently, if (9) holds with equality for some risk neutral payoff function, then it holds with equality for every risk neutral payoff function. On the other hand, if there exists a supporting risk neutral payoff function for which at least one of the inequalities (9) is strict, then this must be true of all such functions, in which case the player’s revealed utility differences will be called incomplete. In the latter case, it is as if the player’s preferences for outcomes are represented by a utility function \( \hat{u}_n \) and she is risk neutral, but she has hedged her gambles so that her
true utility differences are not uniquely revealed. This is superficially similar to the situation considered by Aumann (1962), in which the completeness assumption is dropped from the standard assumptions of expected-utility theory.

**THEOREM 4:** The revealed utility differences of a risk averse player are incomplete.

**Proof:** Let the elements of $S$ be labelled from 1 to $|S|$ in increasing order of player $n$’s utility. Thus, $u_n(1) \leq u_n(2) \leq \ldots \leq u_n(|S|)$. By assumption, $v_n$ is a positive and strictly decreasing function of $u_n$, so that $v_n(1) \geq v_n(2) \geq \ldots \geq v_n(|S|)$. Now let $\hat{u}_n$ be defined on $S$ as follows:

$$\hat{u}_n(1) = 0,$$
$$\hat{u}_n(s) = \hat{u}_n(s - 1) + \frac{u_n(s) - u_n(s - 1)}{\frac{1}{2}(v_n(s) + v_n(s - 1))}, \quad s = 2, \ldots, |S|.$$

It follows that for any $s, s' \in S$:

$$\frac{(u_n(s) - u_n(s'))}{v_n(s)} \geq \frac{u_n(s) - u_n(s - 1)}{v_n(s)} + \ldots + \frac{u_n(s' + 1) - u_n(s')}{v_n(s)} = \hat{u}_n(s) - \hat{u}_n(s') \quad (10)$$

with strict inequality if $u_n(s) \neq u_n(s')$. (To see this, note that if $s > s'$ [$s < s'$], then all utility differences are non-negative [non-positive], and the denominators of the LHS terms are all less [greater] than or equal to the denominators of the corresponding RHS terms.) Identifying $s$ and $s'$ with $s_{-n}k$ and $s_{-n}j$, respectively, inequality (10) implies

$$(u_n(s_{-n}k) - u_n(s_{-n}j))/v_{nk}(s_{-n}k) \geq \hat{u}_n(s_{-n}k) - \hat{u}_n(s_{-n}j) \quad \forall s_{-n} \in S_{-n}, \quad (11)$$

which yields (9) when $\alpha$ is chosen so as to undo the effects of normalization on both sides. Thus, $\hat{u}_n$ is a supporting risk neutral payoff function for player $n$. Furthermore, this inequality is strict for any $j, k$, and $s_{-n}$ such that $u_n(s_{-n}k) \neq u_n(s_{-n}j)$. ■
**COROLLARY**: If the consequences of the game for a player are purely monetary and she is risk averse, then her monetary payoff function is a supporting risk neutral payoff function.

**Proof**: Let the player’s utility for the quantity of money $c$ be represented by the utility function $U(c)$, which is concave by the assumption of risk aversion. Let $\hat{u}_n(s) \equiv c_n(s)$, $u_n(s) \equiv U(c_n(s))$, and $v_n(s) \equiv U'(c_n(s))$ where $U'$ is the derivative of $U$. Then inequality (11) can be rewritten as

$$U(c_n(s-nj)) \leq U(c_n(s-nk)) + U'(c_n(s-nk))[c_n(s-nj) - c_n(s-nk)],$$

which is satisfied because the value of the concave function $U(c)$ at $c = c_n(s-nj)$ must lie on or below the tangent to its graph at $c = c_n(s-nk)$.\(^{13}\)

The preceding corollary establishes that, when risk averse players reveal their preferences through gambling, they appear as risk neutral players with incompletely revealed preferences. If, for every $n$, $\hat{u}_n$ is a risk neutral payoff function for player $n$, then a hypothetical game played by risk neutral players with payoff functions $\{\hat{u}_n\}$ will be called a *supporting risk neutral game* for the revealed game. Every supporting risk neutral game has an objective correlated equilibrium (Hart and Schmeidler 1989; Nau and McCardle 1990), and any objective correlated equilibrium of the risk neutral game is also an arbitrage-free correlated equilibrium of the original game, since inequality (9) implies

$$\sum_{s \in S} \hat{p}_n(s)(\hat{u}_n(s-nk) - \hat{u}_n(s-nj))1_{nk}(s) \geq 0 \Rightarrow \sum_{s \in S} \hat{p}_n(s)\hat{u}_{njk}(s) \geq 0.$$

Hence, the existence of arbitrage-free correlated equilibria is guaranteed for games among risk averse players.

Players who are risk seeking (*i.e.*, whose marginal utilities for money increase with the utilities of their payoffs in the game) display opposite qualitative characteristics, as

\(^{13}\) This is the so-called “subgradient inequality” (Rockafellar 1970).
might be expected. In particular, they accept preference gambles which are stronger (less favorable to themselves) than those that would be accepted by risk neutral players. In the matching-pennies game, this means that the preference gambles accepted by risk seeking players would in general be incoherent: both players would be willing to pay for the privilege of transferring wealth from their losing outcomes to their winning outcomes, creating arbitrage opportunities. In other words, they would be willing to pay a premium to an arbitrageur merely to allow them to increase stakes in the zero-sum game. (Indeed, risk seeking players would prefer to play matching-pennies for infinite stakes.) In games without unique mixed-strategy equilibria, such as battle-of-the-sexes, risk seeking behavior generally shrinks the set of correlated equilibria of the revealed game but not necessarily to the point of incoherence.

In summary, the true game gives rise to a revealed game (unique up to the choice of monetary currency), and to every revealed game among risk averse players there corresponds a (not necessarily unique) risk neutral game. The former is an incomplete image of the latter, whence every objective correlated equilibrium of the latter is also an arbitrage-free correlated equilibrium of the former. In a game with purely monetary consequences, the monetary payoff functions of the players define a supporting risk neutral game. If the players’ revealed beliefs constitute an arbitrage-free correlated equilibrium of the revealed game, then their true beliefs constitute an \textit{a posteriori} equilibrium of the true game, and the same outcomes receive positive probability in both cases.

7. Discussion

Game theory is increasingly applied to the prediction of behavior in competitive markets, fulfilling the agenda laid down by von Neumann and Morgenstern (1944). Most often, the Nash equilibrium concept and its refinements are invoked: it is assumed that agents have common knowledge of each others’ utilities (\textit{i.e.}, the rules of the true game),
and it is predicted that they will construct beliefs about the game’s outcome which are
not only mutually consistent but also stochastically independent, and perhaps also sym-
meteric, robust against mistakes or empty threats, etc. But, unless homogeneous marginal
utilities are also assumed, the Nash concept is seemingly at odds with classical concerns
about Pareto efficiency in exchange economies or modern concerns about arbitrage in cap-
ital markets. (The departure of game-theoretic analysis from the “arbitrage intuition”
of financial markets is observed by Ross 1987.) But should rational agents tolerate com-
mon knowledge of inefficiency or arbitrage opportunities? The necessary condition for effi-
ciency or no-arbitrage is precisely a consistency of the players’ risk neutral probabilities—
i.e., the products of their probabilities and relative marginal utilities—rather than their
true probabilities.

This paper reconciles strategic behavior with efficient allocation by considering how
the rules of the game and the beliefs of the players might come to be common knowledge,
showing that this can be achieved in principle through the public acceptance of small side-
gambles, which are special cases of transactions in contingent claims markets. This leads
to a natural reformulation of familiar equilibrium concepts in terms of risk neutral prob-
abilities. To be sure, the proposed market is stylized and somewhat unusual: claims are
contingent on events directly under control of the players. This type of market admittedly
does not exist and is even hard to imagine in many game-theory applications, which only
raises the question: do agents often possess the numerically precise common knowledge of
each others’ probabilities and utilities that game-theoretic solution concepts presuppose?

We have observed that, when communication among players is mediated by mone-
tary gambles, they can end up strategically decoupling their actions—which, of course, is
one of the important functions of money in an economy. The transfer of utility through
monetary side payments is important to the theory of cooperative games (c.f. Shubik
but it has largely been peripheral to noncooperative theory. This paper shows that monetary side payments can enter noncooperative game theory in a fundamental way as a medium of measurement and communication, prompting a reevaluation of standard assumptions about the nature of equilibria and the consistency of beliefs.

References


