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NOTES

BLAU'S DILEMMA REVISITED*

ROBERT F. NAU
Fuqua School of Business, Duke University, Durham, North Carolina 27706

The issue of equivalence between chance-constrained programming problems (CCPP's) and Bayesian utility-maximization problems (BUMP's), and the anomalous evaluation of information in CCPP's, are re-examined in light of a recent paper by Jagannathan and the ensuing exchange between Jagannathan and LaValle. Difficulties in "explaining" value of information within the framework of chance-constrained programming are illustrated in a numerical example due to Jagannathan.
(CHANCE-CONSTRAINED PROGRAMMING; INFORMATION EVALUATION; DECISION ANALYSIS)

The debate between the detractors and defenders of chance-constrained programming (CCP) is, at bottom, a clash between Bayesian and non-Bayesian world views. Defenders of CCP (Charnes and Cooper 1963) have identified themselves with other critics of the Bayesian paradigm of perfectly rational economic man (e.g., Simon 1956). Critics of CCP (Hogan *et al.* 1981) have compared it to other paradigms (e.g., the Neyman-Pearson lemma) that are defective from a Bayesian viewpoint. Not surprisingly, these criticisms have often been perceived to miss the mark by their intended victims, and the argument is far from being settled, as the foregoing exchange between LaValle and Jagannathan indicates. Jagannathan maintains that a meaningful equivalence can be established between certain classes of chance-constrained programming problems (CCPP's) and Bayesian utility-maximization problems (BUMP's) for purposes of information-evaluation in decision-making situations, despite repudiations of this notion by Hogan *et al.* (1981) and LaValle (1986a, b). This note will attempt to penetrate some of the "murkiness of the issues" that still seems to linger.

LaValle's (1987) response is provoked by the following remark of Jagannathan, and the analysis that follows it: "The main point of the critique by Hogan *et al.* (1981) revolves around the concept of EVPI which they believe cannot be adequately explained by chance-constrained programming. The Bayesian framework of the present paper enables us to derive expression[s] for EVSI and EVPI." Jagannathan considers a problem in which a decision, x, is sought to maximize an objective, $z(x)$, while attempting to meet goals $\{\Sigma\ a_{ij}x_j \geq \xi_i\}$, where $\{\xi_i\}$ are random variables. The CCP formulation of this problem is to establish "risk levels" $\{\alpha_i\}$, which are lower bounds on the probabilities of meeting the goals, yielding the canonical CCPP:

$$\max_{x \in S} z(x) = \sum_j c_j x_j \qquad \text{s.t.} \qquad P(\sum_j a_{ij}x_j \geq \xi_i) \geq \alpha_i, \qquad i = 1, \cdots m.$$

The probability distribution of ξ_i is of known form and indexed by an uncertain parameter, θ_i. When the distribution of the latter, denoted G_i, can be updated through the collection of sample information, it is of interest to measure the value of such information.

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The basic Bayesian position is that the chance-constraint approach to weighing the objective against the other goals is an irrational response to risk that generally misrepresents the value of information: the decision-maker should incorporate his preferences for all relevant attributes of decision consequences into a single utility function, and solve a BUMP instead. Jagannathan, however, shows that with "suitably defined α-risks," which he denotes as $\{\alpha_i^*(G_i)\}$, the above CCPP has the same solution as a certain BUMP, and suggests that the desirable information-evaluation properties of the latter should therefore be attributed to the former. In Jagannathan's "equivalent" BUMP, the arguments of the utility function are the objective value, $z(x)$, and the goal-attainment probabilities, $P(\Sigma\ a_{ij}x_j \geq \xi_i)$, of the CCPP. In a more conventional Bayesian model, *amounts* rather than probabilities of deviations from goals would be expected to appear as arguments in the utility function.

LaValle's objection is that Jagannathan's formulation, though interesting in its use of probabilities as utility attributes, does not circumvent the "dilemmas in the transition" from BUMP's to CCPP's that were identified by Blau (1974) and Hogan *et al.* (1981). In particular: (i) "suitable" risk levels for the CCPP can only be found by explicitly solving the BUMP, negating any computational advantages of the CCP formulation; (ii) if the risk levels are fixed at values determined by the prior distributions, the solutions sets of the CCPP and the BUMP generally will not remain equivalent when the distributions are updated; (iii) even if the solutions did remain equivalent as the state of information changed, the CCPP still would not correctly measure the value of information, since its objective function does not reflect the decision-maker's utilities for realized levels of goal attainment. For example, a CCPP may yield *negative* values for EVSI and/or EVPI, a phenomenon now known as "Blau's dilemma." Details of the examples used by Blau (1974) and Hogan *et al.* (1981) were criticized by Charnes and Cooper (1975, 1983), but a more rigorous theoretical analysis of the anomalous behavior of CCPP's has been provided by LaValle (1986a, b).

LaValle also objects to Jagannathan's equivocal treatment of "utility:" is it linear in the probabilities or isn't it? Jagannathan's pivotal result for the equivalency argument (Theorem 8) requires the linearity assumption—as does LaValle's (1986b) proof of the general non-Bayesability of CCPP's with fixed risk levels. But the appearance of the linearity assumption only in the statement of the theorem suggests that it is made for mere technical convenience, and that the von Neumann-Morgenstern axioms of expected utility (which imply linearity) are in general not assumed. LaValle's point is that if the von Neumann-Morgenstern axiomatic framework (or its equivalent) is assumed, then the use of *probabilities* as arguments in the utility function brings those arguments within the purview of the linearity-in-probability requirement. But if vNM utility is *not* assumed, it would be of interest to know what assumptions *are* made about the decision-maker's preferences. On what basis do we assume the existence of a utility function with specified properties and adopt the principle of maximizing expected utility? (Alternatives to vNM utility are a lively subject of current research, as surveyed by Bell and Farquhar 1986.)

In his counter-response (1987), Jagannathan is very explicit about wishing to allow nonlinearity, and in rejecting the need to have any sort of axiomatic basis (vNM or otherwise) for expected-utility decision analysis. His view is apparently that any monotonic function may be adopted as a utility function on purely "operational" grounds. Yet, absent any standards for rational choice under uncertainty, it is hard to advise the decision-maker on how to assess the form or parameters of this utility function, and it is not clear in what sense maximizing its expectation will yield an "optimal" decision or an appropriate valuation of information.

Still, Jagannathan's model seems to hold promise in several respects. The anomalies uncovered by Blau (1974), Hogan *et al.* (1981), and LaValle (1986a, b, 1987) all occurred in CCPP's in which the risk levels were assumed to be fixed; i.e., not to vary with the sample information. Jagannathan's notation "$\{\alpha_i^*(G_i)\}$," which explicitly shows the risk levels as functions of the probability distributions, is suggestive of another possibility: that risk levels might be updated with the probability distributions *precisely* so as to maintain solution-set equivalence between the BUMP and the CCPP upon receipt of sample information. Jagannathan makes clear in his counter-response (1987) that this is what he intended. This is an interesting innovation in the CCP model, which might be hoped to improve its information-evaluation behavior. It might also be hoped that the *rule* for determining risk levels $\{\alpha_i^*\}$ from probability distributions $\{G_i\}$, derived from the expected-utility model, would turn out to be simple and intuitive enough to stand on its own merits and provide insight on the interpretation of risk levels in CCPP's. It will be shown below, however, that information-evaluation dilemmas persist even with updated risk levels, and that the rule by which risk levels are determined is by no means intuitive.

First, it must be noted that the BUMP-derived risk levels, $\{\alpha_i^*(G_i)\}$, actually depend not only on the distributions $\{G_i\}$, but also on parameters of the decision-maker's utility function that are exogenous to the original CCPP formulation. A multiattribute utility function is typically expressed as a *weighted* sum or product of univariate utility functions of its separate arguments. The weight parameters serve to correct for different scales of measurement and to quantify the tradeoffs that the decision-maker would be willing to make among attributes. Jagannathan's utility function, whose arguments are the objective value and the goal-attainment probabilities of the CCPP, must have such a set of weights. Indeed, the example of a nonlinear utility function given at the end of Jagannathan's (1987) response to LaValle explicitly shows weights for deviations from goal-attainment probabilities as $\{w_1, \ldots, w_m\}$. Similarly, his original numerical example uses the utility function: $U(z, \alpha) = z + 4\alpha$. The "4" is a weight parameter whose origin is not explained; presumably it has been directly assessed by the decision-maker. While Jagannathan's analysis does not mention the assessment of the utility weights, the solution of the BUMP and the values it yields for EVSI and EVPI depend strongly on it, for therein the decision-maker must quantify the tradeoffs he is willing to make between the objective, z, and the degrees of attainment of his other goals. The need to quantify such tradeoffs in order to reach a decision is made explicit in the expected-utility formulation of the problem, but obscured in the CCP formulation.

These issues are well illustrated by a closer analysis of Jagannathan's own numerical example (1985, p. 107). Unlike the examples presented by Blau (1974), Hogan *et al.* (1981), and LaValle (1987), it is offered as a *positive* example of information-evaluation in CCPP's, and presumably has not been deliberately contrived to put CCP in an unflattering light. Jagannathan's example begins with the following CCPP:

$$\begin{aligned} &\text{Maximize} \quad && z(x_1, x_2) = 3x_1 + 2x_2 \\ &\text{Subject to} \quad && P(x_1 + 6x_2 \geq \xi) \geq \alpha, \\ &x_1 + x_2 \leq 2, \quad && x_1 \geq 0, \qquad x_2 \geq 0, \end{aligned}$$

where ξ takes on values in the interval (0, 12). The distribution of ξ is assumed to be a mixture of uniform (0, 4) and uniform (4, 12) distributions, with mixing parameter θ. The prior distribution of θ is Beta (1, 1), which is effectively Uniform (0, 1). The sample information consists of observations of n independent random variables drawn from the same distribution as ξ, and the sufficient statistic is the number, s, of observed values in the interval (0, 4). The only relevant feature of the distribution for θ is its mean, which will be generically denoted p, having prior value $p_0 = \frac{1}{2}$ and posterior value $p_n = (1 + s)/(2 + n)$. The optimal solution to the CCPP satisfies $x_1^* + x_2^* = 2$, with x_2^* determined from α and p by:

$$x_2^*(\alpha, p) \equiv \begin{cases} 0 & \text{if} \quad 0 \leq \alpha \leq p/2, \\ -\frac{2}{5} + 4\alpha/5p & \text{if} \quad p/2 < \alpha \leq p, \\ \frac{2}{5} + 8(\alpha - p)/5(1-p) & \text{if} \quad p < \alpha \leq 1. \end{cases} \tag{1}$$

Note that, for any p strictly between 0 and 1, x_2^* can be made to assume any value in [0, 2] by a suitable choice of α.

Jagannathan then proposes a BUMP having a utility function of the form $U(z, \alpha) = z + w\alpha$, where $z = 3x_1 + 2x_2$, $\alpha = P(x_1 + 6x_2 \geq \xi)$, and $w = 4$. This utility function is monotonic increasing in both x_1 and x_2, so its maximization under the constraints $x_1 + x_2 \leq 2$, $x_1 \geq 0$, and $x_2 \geq 0$ will always lead to a solution satisfying $x_1^* + x_2^* = 2$, as in the CCPP. With the substitution $x_1 = 2 - x_2$, the utility function reduces to a piecewise linear function of x_2 on [0, 2] with a single kink at $x_2 = \frac{2}{5}$, so that if this problem has a unique optimal solution, x_2^* can assume only one of three possible values, namely $x_2^* = 0$, $x_2^* = \frac{2}{5}$, or $x_2^* = 2$. (A nonunique solution, consisting of an interval between two of these points, is also possible.) In general, the choice among these possible solutions depends on both p and w. In particular, if $w > 2.4$, then the optimal solution is:

$$x_2^*(p, w) \equiv \begin{cases} 2 & \text{if} \quad p < p^*(w), \\ \text{any value in } [\frac{2}{5}, 2] & \text{if} \quad p = p^*(w), \\ \frac{2}{5} & \text{if} \quad p > p^*(w), \end{cases} \tag{2}$$

where $p^*(w) = 1 - (8/(5w))$. (If $w < 2.4$, then $x_2^* = 0$ is also possible for some values of p, and there are two relevant threshold probabilities in the decision rule.)

With $n = 5$ and $w = 4$ as assumed by Jagannathan, we have $p^*(w) = \frac{3}{5}$; the prior optimal solution is $x_2^* = 2$; and the posterior solution changes to $x_2 = \frac{2}{5}$ if and only if $(1 + s)/7 > \frac{3}{5}$, which occurs when $s \geq 4$. Jagannathan notes that EVSI for the BUMP in this case is equal to 0.104. However, since the reason for assuming $w = 4$ in the utility function has not been given, we should ask whether EVSI is robust against changes in w. Indeed it is not! In fact, any finite change in w will produce a finite change in EVSI. E.g., if $w = 3.9$ is assumed, then EVSI is equal to 0.108; and $w = 3$ yields EVSI = 0.151.

Now consider the "suitably defined" risk level that is implied for the CCPP. As noted earlier, it depends on parameters of both the decision-maker's probability distribution and his utility function—in this case p and w—and will therefore be written as $\alpha^*(p, w)$. By inspection of equations (1) and (2), the solutions of the CCPP and BUMP coincide when:

$$\alpha^*(p, w) \equiv \begin{cases} p & \text{if} \quad p \geq p^*(w), \\ 1 & \text{if} \quad p < p^*(w), \end{cases}$$

where, as before, $p^*(w) = 1 - (8/(5w))$. (A slight technical problem arises when p is exactly $p^*(w)$: the CCPP then has the unique solution $x_2 = \frac{2}{5}$, whereas the BUMP admits any value of x_2 in the interval $[\frac{2}{5}, 2]$. This situation does not arise, however, in Jagannathan's sampling example with $n = 5$.)

Although the above rule $\alpha^*(p, w)$ is simple in form, it is hardly intuitive: it is discontinuous and nonmonotonic in p. Also, for a significant range of values of p (namely $p < p^*(w)$), the risk level is unity, in which case the chance constraint is not a chance constraint at all. And, for values of p such that the risk level is less than unity (namely $p \geq p^*(w)$), it varies with p in precisely such a way as to hold the solution constant, again trivializing the chance constraint.

Jagannathan reports that EVSI and EVPI for the BUMP are 0.104 and 0.333, respectively, when $w = 4$. But, are these values "explained by chance-constrained programming" in any meaningful sense? It has already been noted that they depend on the choice of w, which does not appear explicitly in the original CCPP formulation of the problem. Still, let us ask whether the same (or at least comparable) values can be extracted from the CCPP once w is given and the rule $\alpha^*(p, w)$ is known. The natural way to measure the value of information in the CCPP is in terms of the expected change in its own optimal objective value. Now, the "prior" solution of the CCPP in this example is $x_2^*(\alpha^*(p_0, w), p_0)$, where $x_2^*(\alpha, p)$ is given by (1), and by definition it coincides with the prior solution of the BUMP. However, there are several ways in which a "posterior" solution might be obtained. One possibility is to use a fixed risk level (namely $\alpha^*(p_0, w)$) on the RHS of the chance constraint while updating the distribution on the LHS. This is the approach used in the examples considered by Blau (1974) and Hogan *et al.* (1981), and in LaValle's (1986a) "posterior imposition" analysis. The posterior solution in this case is $x_2^*(\alpha^*(p_0, w), p_n)$, which generally differs from the posterior solution of the BUMP. Jagannathan has introduced another possibility, noted above, in which the risk level is updated along with the distribution, yielding a posterior solution $x_2^*(\alpha^*(p_n, w), p_n)$ that is also optimal for the BUMP. (A third approach, "prior imposition" of the chance constraints, is considered by LaValle 1986a.)

The results of calculating EVSI and EVPI for the BUMP, the CCPP-with-fixed-α^*, and the CCPP-with-updated-α^* are shown below, both for the cases $w = 4$ and $w = 3$:

		BUMP	CCPP (fixed α^*)	CCPP (updated α^*)
$w = 4$	EVSI	0.104	0	0.300
	EVPI	0.333	0	0.640
$w = 3$	EVSI	0.151	−0.790	−0.800
	EVPI	0.327	−0.123	−0.747

Note that the CCPP with *fixed* α^* yields *zeroes* for EVSI and EVPI when $w = 4$. In this case, $p^*(w) = \frac{3}{5}$, so we have $p^*(w) > p_0$, and the common prior solution is $x_2^* = 2$, which requires $\alpha^* = 1$. If the risk level is fixed at this value, the solution cannot change upon receipt of information, hence the information has no value. The CCPP with *updated* α^* yields values that are well-behaved when $w = 4$, but nonetheless different from those of the BUMP. With $w = 3$, however, both CCPP's exhibit Blau's dilemma: "ignorance is bliss." In this case, $p^*(w) = \frac{7}{15}$, and because $p^*(w) < p_0$, the prior risk level is now $\alpha^* = p_0 = \frac{1}{2}$, and the common prior solution is $x_2 = \frac{2}{5}$. For the CCPP with updated α^*, sample information can at best leave the solution the same and at worst force it to change to $x_2^* = 2$, which has a lower objective value. (Recall that $x_1^* + x_2^* = 2$, whence $z^* = 6 - x_2^*$.) For the CCPP with fixed α^*, a spread of posterior solutions from $x_2^* = 0.0667$ to $x_2^* = 1.0667$ is possible, but the *expected* posterior value of x_2^* is greater than $\frac{2}{5}$, yielding a decrease in expected objective value.

In summary, the solution of Jagannathan's BUMP and the values it yields for EVSI and EVPI are sensitive to the choice of certain weight parameters in the utility function. These do not appear explicitly in the CCP formulation of the decision problem, and we are given no guidelines on how they are to be assessed, nor (in the nonlinear case) do we have recourse to the well-known assessment methods of von Neumann-Morgenstern utility. Furthermore, no matter how risk levels in the CCPP are tied to the solution of the BUMP, the CCPP yields misleading and sometimes negative results for EVSI and EVPI if these are computed in terms of the expected change in its own objective value. Finally, the rule for adjusting risk levels in order to achieve solution-set equivalence between the CCPP and the BUMP has the effect of trivializing the chance constraints. Whatever the merits of Jagannathan's expected-utility model, it provides no vindication of chance-constrained-programming with respect to the information-evaluation dilemmas identified by Blau (1974), Hogan *et al.* (1981), and LaValle (1986a, b).

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