DUKE THE FUQUA SCHOOL OF BUSINESS
FUQUA
School of Business

BA 513/STA 234: Ph.D. Seminar on Choice Theory
Professor Robert Nau
Spring Semester 2008

Notes for class #5: a bestiary of solution concepts for noncooperative games (revised February 14, 2008)

Primary readings:

1a. "Game Theory" by Robert Aumann, from *The New Palgrave,* 1987

1b. "A survey of applicable game theory" by Robert Gibbons, *Journal of Economic Perspectives*, 1997

1c. "Solution concepts for noncooperative games" by David Kreps, chapter 12 of *A Course in Microeconomic Theory*, 1990

Notes and guide to the readings:

> bestiary \'bes(h)-che,er-e [ML *bestiarium*, fr. L, neut. of *bestiarius* of beasts, fr. *bestia*]: a medieval allegorical or moralizing work on the appearance and habits of real or imaginary animals

Game theory is at the heart of the modern rational choice revolution. The revolution was launched in the 1940's by von Neumann and Morgenstern, who proposed to make game theory the new mathematical foundation of economics. Along the way, they axiomatized and rehabilitated the concept of cardinal expected utility, which turned out to be epochal in its own right. But their main goal was to develop a mathematical theory that would describe the behavior of a small number of rational individuals: the problem of "2, 3, 4,… bodies." By most measures, their vision has been fulfilled. Game theory is "rampant" in economics (in Gibbons' words) and has provided the wedge with which rational choice theory has penetrated other social-scientific disciplines such as political science, sociology, philosophy, and management science. This week's readings provide an introduction to the basic solution concepts and orthodoxy of noncooperative game theory.

The historical survey by Aumann (1987) portrays game theory as a rich tapestry that has been woven over many decades, dating back to the early 1900's. Aumann emphasizes the many interesting contributions to *mathematics* that have emerged from game theory. He is careful not to take sides in the debates that have swept through the field, nor to dwell on the disappointments or retrenchments. He describes a steady march of progress, a harmonious ensemble (rather than a cacaphony) of competing ideas. Aumann chronicles the development of the theory of two-person zero sum games in the 1920's and the introduction of the concept of stable sets in the 1940's (both of which were due to von Neumann), the noncooperative equilibrium concept and bargaining model introduced by Nash in 1950-51, the rise of cooperative game theory in the 50's and '60's, the explosion of interest in the study of repeated games (particularly the prisoner's dilemma), the resurgence of noncooperative game theory in the '70's and '80's (the era of refinements of Nash equilibrium), and the more recent fascination of game theorists with biological models of strategic interaction.

Aumann is also careful not to claim that game theory makes *predictions*: it just provides a set of instruments for studying interactive decisions from different perspectives, a formal *language* in which to describe various outcomes that might occur in a game of strategy. The section of his paper on "solution concepts" (pp. 10-11) makes this clear:

> "Given a game, what outcome may be expected? Most of game theory is, in one way or another, directed at this question. In the case of two-person zero-sum games, a clear answer is provided: the unique individually rationally outcome [i.e., the minimax solution]. But in almost all other cases, there is no unique answer. There are different criteria, approaches, points of view, and they yield different answers.
>
> A *solution concept* is a function (or correspondence) that associates outcomes, or sets of outcomes, with games. .... for example, the strategic [Nash] equilibrium and its variants for strategic form games, and the core, the von Neumann-Morgenstern stable sets, the Shapley value and the nucleolus for coalitional games. Each represents a different point of view.
>
> What will 'really' happen? Which solution concept is 'right'? None of them; they are indicators, not predictions."

Nevertheless, in most applied game theory, a solution concept is invoked precisely so as to generate a strong prediction!

A sizable chunk of Aumann's article is concerned with developments in *cooperative* game theory, which occupied center stage in the 1950's and 1960's. The highlight of this period was the discovery of the "core equivalence theorem," which states that as the number of agents in a *market game* tends to infinity, the *core* of the game shrinks to the set of competitive equilibrium allocations, as conjectured by Edgeworth 70 years earlier. This result seemed to fulfill von Neumann and Morgenstern's dream of a game-theoretic bridge between single-agent decision theory and competitive market theory. (We will see later that there is another, and much simpler, way to build the same bridge, namely by appealing to the arbitrage principle.) However, cooperative game theory was deflated in the late '60's by the discovery that even simple games sometimes have empty cores. Many researchers began to take a renewed interest in the noncooperative concepts that had been pioneered by Nash and extended to incomplete-information games by Harsanyi in 1967. Apart from the question of whether it always yields solutions, noncooperative game theory makes lower informational demands on the players than does cooperative theory, and intuitively it seems more suitable as a foundation for models of decentralized economic behavior.

In the 1970's, many researchers began to seek *refinements* (i.e., strengthenings) of the concept of Nash equilibrium in order to obtain tighter predictions, following Selten's pioneering work on "subgame perfect" and "trembling hand perfect" equilibrium. Others (including Aumann) proposed *coarsenings* of Nash equilibrium, which had more thoroughly "Bayesian" axiomatic foundations or which were otherwise more realistic in their assumptions about the cognitive abilities of game players. In the late '70's and early '80's, refinements and coarsenings of Nash equilibrium were proliferating as fast as axiomatic theories of non-expected utility. For a time it seemed as though the discovery of a new concept that would uniquely solve every noncooperative game might be just around the corner. (Indeed, Harsanyi and Selten developed a solution method called the "tracing procedure" that claimed to achieve this goal.) But the refinements-and-coarsenings movement ultimately failed to converge, and interest among game theorists began to shift in other directions, toward biological analogies of strategic equilibrium and toward phenomena of learning and adaptation in games.

It is a shame that Aumann was not at least a co-recipient of the 1994 Nobel prize in economics that was first awarded for contributions to game theory, which went to Nash, Harsanyi, and Selten instead. (http://www.nobel.se/economics/laureates/1994/index.html). Aumann has done more than anyone else to make game theory the robust institution that it is today, and he has made some of the deepest contributions to the mathematical and conceptual foundations of the theory, which we will study next week. Aumann introduced a more general and more "correct" solution concept for noncooperative games—namely correlated equilibrium—although the concepts introduced by Nash, Harsanyi, and Selten are more widely used. Happily, Aumann was finally awarded the Nobel prize (along with Thomas Schelling) in 2005.
(http://nobelprize.org/economics/laureates/2005/).

The paper by Gibbons (1997) makes it appear that game theory is almost completely cut-and-dried. There are four major classes of games, (static and dynamic, each with or without complete information), and each has its own received solution concept: Nash equilibrium for static games of complete information, backward induction and subgame perfect Nash equilibrium for dynamic games of complete information, Bayesian Nash equilibrium for static games of incomplete information, and perfect Bayesian equilibrium for dynamic games of incomplete information. Gibbons states:

> "This outline may seem to suggest that game theory invokes a brand new equilibrium concept for each class of games, but one theme of this paper is that these equilibrium concepts are very closely linked. As we consider progressively richer games, we progressively strengthen the equilibrium concept to rule out implausible equilibria in the richer games that would survive if we applied equilibrium concepts suitable for simpler games. In each case, the stronger equilibrium concept differs from the weaker concept only for the richer games, not for the simpler games."

Note the euphemism "implausible." Ruling out implausible equilibria really means just trying to cut down on the combinatorial explosion of rational solutions that occurs as the game gets more complex. Although each strengthening of the solution concept does rule out some implausible equilibria, it typically throws out a good many plausible ones as well, as we will see.

Nevertheless, Gibbons gives a very accessible introduction to the solution concepts that are currently most popular.

Kreps writes with refreshing candor about fundamental issues in game theory. This chapter from his microeconomics textbook (1990) provides a much more thorough (and more technical) introduction to the major solution concepts for noncooperative games than does Gibbons. The chapter begins with a table of 15 games that are used as the basis for subsequent discussions. Kreps asks the reader to ponder the games and come up with his or her own solutions before proceeding. *Do this: ask yourself how you would play each of the games in figure 2.1, and what predictions you would make for your opponent.* (The same games are revisited later in the chapter. We will discuss these examples in class and compare notes.)

Kreps is a good deal more skeptical than Gibbons about the authority of the various solution concepts, particularly the varieties of Nash equilibrium. He writes:

> "In the great majority of the applications of noncooperative game theory to economics, the mode of analysis is equilibrium analysis. And in many of these analyses, the analyst identifies a Nash equilibrium (and sometimes more than one) and proclaims it (them?) as "the solution." I wish to stress that this practice is sloppy at best and probably a good deal worse. ...it is clear that having the answer 'Nash equilibrium' is pretty thin gruel if what we are after is a way to solve games."

The rules of the game

Aumann's survey article begins with the observation: " 'Interactive Decision Theory' would perhaps be a more descriptive name for the discipline usually called Game Theory." But game theory circumscribes the "interaction" between two or more players in a curious way. On the front end, it begins with the assumption that the *rules of the game*—i.e., the payoff matrices containing the players' utilities for outcomes—are already written and are *commonly known*.[1] Usually the question of how or why the rules were written in that way or how they came to be commonly known is outside the scope of the model, and there is little or no provision for communication between players. On the back end, game theory does not directly address the question of how the players form subjective beliefs about the actions of their opponents: belief formation is addressed only indirectly through the invocation of solution concepts. The theory mainly focuses on the question of how the solution of a game is "objectively" determined (or at least constrained) by the rules which have already been laid down and the solution concept that has been invoked. Even when the issue of "learning" or "convergence to equilibrium" is raised,

[1] Actually, the players' *absolute* utility payoffs are not assumed to be commonly known—they are not uniquely determined by preferences anyway. (Recall that von-Neumann Morgenstern utilities are unique only up to positive affine scaling.) Rather, what noncooperative game theory assumes to be commonly known are the *relative differences in utilities* between every pair of strategies of every player. We will return to this point later in our discussion of the prisoner's dilemma game and yet again in our discussion of correlated and arbitrage-free equilibria.

it is usually in the context of a fixed game that is played over and over again without communication.

Think for a moment about what it means for the rules of a game to be common knowledge, in light of our earlier discussion of subjective expected utility and more general theories of state-dependent-utility and non-expected utility. A rational decision maker's preferences may or may not conform to all the axioms of SEU theory, and even if they do, it may difficult or impossible for an *observer* to uniquely determine the decision maker's probabilities and utilities from a finite number of feasible measurements. Nevertheless, game theory heroically starts with the assumption that each player is able to construct an SEU representation of her own decision problem *and everyone else's decision problem*, and the SEU representation that A attributes to B agrees with the SEU representation that B constructs for herself. The SEU representation specifies the states and consequences that are relevant for decision making, including such subtleties as the "constant" consequences that are needed to uniquely separate utilities from probabilities, and it also requires the contemplation of counterfactual acts (arbitrary mappings of states to consequences). Furthermore, it is not enough for this to happen: it must also be *common knowledge* that it has happened. Although we are all familiar with situations under which facts may be considered to be common knowledge (namely, situations in which we all observe each other observing the same thing at the same time, or when we all know that we have read or watched the same recorded accounts of the same events), it is actually rather tricky to make this notion precise. Aumann's famous paper on "agreeing to disagree," which we will discuss next week, addresses that problem.

We are not yet done. In games of incomplete information, where uncertainty exists about exogenous states of nature as well as endogenous moves of human players, it is usually assumed (following Harsanyi) that the players' beliefs about states are consistent with a *common prior distribution*. That is, each player is imagined to have several possible *types* (e.g., different possible utility functions and/or different states of information with respect to exogenous events), and each player's probability distribution over the types of the other players, given her own type, is obtained by performing Bayesian updating on a common prior distribution over types. A Nash equilibrium of an incomplete-information game with a common prior distribution is therefore called a *Bayesian Nash equilibrium*. The common prior distribution, like the rest of the rules of the game, is simply "given."

We are still not done. If the solution of the game is not trivial or obvious, it is necessary for the players to invoke the same solution concept—i.e., the same species of equilibrium—and for this, too, to be common knowledge. And finally, if the solution concept does not yield a unique solution (as it often does not), it is necessary for the players to tacitly agree on additional constraints that should be imposed in order to coordinate on a particular equilibrium.

Of course, no one seriously argues that the steps described above actually take place: it is merely "as if" they take place. *Somehow* it is as if the players know each other's utilities, they share the same prior probabilities for exogenous states of nature, and they agree on the solution concept to be used and the particular equilibrium to be selected.

At bottom, game theory (like all of rational choice theory) merely assumes that the behavior of different players is in some ways *consistent*: it would seem pointless and unfair to prescribe behavior that is self-contradictory. The axioms of expected utility or subjective expected utility provide a standard of *intra-agent* consistency of behavior, and the common knowledge assumptions and solution concepts of game theory provide additional standards of *inter-agent* consistency. Notice, however, that the assumptions of inter-agent consistency are rather different in kind from those of intra-agent consistency. As we have seen, decision theorists take great pains to express the axioms of individual rationality in terms of "primal" behavior, namely *preferences* among *acts*. From those axioms, it then follows as a theorem that there is a "dual" representation of rational behavior in terms of probability distributions and utility functions with respect to which an agent seemingly optimizes. This can viewed as a kind of mind-body duality: the body makes choices that reveal preferences, while the mind harbors probabilities and utilities. But the "axioms" of game theory refer directly to the *dual* representation—the probabilities and utilities—rather than the primal behavior. We have seen that even the concepts of preferences and acts are rather slippery and that the preference axioms are not always descriptively valid. Our grip is even less firm when assumptions are imposed on unobservable mental constructs whose very reality and concreteness depend on the validity of the preference axioms!

The prisoner's no-brainer

Discussions of game theory often begin with the prisoner's dilemma game (and for critics of the theory they sometimes end there). The prisoner's dilemma is a 2×2 game with a payoff matrix such as the following:

	Left	Right
Top	1, 1	11, 0
Bottom	0, 11	10, 10

At first glance, it appears that the players would want to choose Bottom-Right and get the socially efficient outcome of 10 units[2] each. On closer inspection, that solution is seen to be unstable: if Row really thought that Column would play Right, then she would be better off by playing Top (thereby getting 11 instead of 10); and similarly if Column thought that Row would play Bottom, then she should play Left. This game is technically a no-brainer: it is **dominance solvable**. Top is a dominant strategy for Row and Left is a dominant strategy for Column. The

[2] In the payoff matrix of a noncooperative game, the payoffs are theoretically supposed to be expressed in units of personal utility rather than money, so the scaling of each player's payoffs is completely arbitrary. Thus, if both players have possible payoffs of 0, 1, and 10, it does not mean the two players get the "same" payoff in any objective sense when the outcome of the game is (1, 1) or (10, 10). Rather, it just means their relative utilities are the same—i.e., the utility difference between 10 and 0 is ten times as large as the difference between 1 and 0 for both players. For concreteness, though, payoffs are often loosely interpreted as amounts of money—as von Neumann intended when he axiomatized expected utility.

idea that one should choose a dominant strategy over a dominated strategy is a seemingly uncontroversial requirement of individual rationality, hence we should expect the outcome of the game to be Top-Left: the players will settle for (1, 1) instead of (10, 10).

The lesson that is usually drawn from the prisoner's dilemma is that **individually rational behavior is not always socially rational.** This observation has spawned a vast literature on the general subject of "social dilemmas." It is also a prelude to what we will find as we study the various solution concepts that are used to solve more complicated games. It will often be the case that a more refined (i.e., "more highly rational") solution concept forces the players to accept a worse outcome than a less refined concept. Thus, more refined solution concepts do not necessarily yield better payoffs to the players: sometimes it is better to be (at least a little bit) irrational.

There is also a methodological issue here concerning the peculiar way in which noncooperative game theory frames an interactive decision problem: *it ignores some critical information about the players' preferences*. Suppose that the two participants in the prisoner's dilemma game initially know only their own payoffs but not the payoffs of the other player. Imagine that they begin to reveal their private information to each other via the following conversation:

> **Dialog 1:**
>
> ROW: Regardless of whether you play Left or Right, I will be $1 better off if I play *Top* rather than Bottom.
>
> COLUMN: Regardless of whether you play Top or Bottom, I will be $1 better off if I play *Left* rather than Right.

Given *only* this information, there is no dilemma concerning how the game should be played. Everyone should expect Row to play Top and Column to play Left, which appears to yield the best-case outcome for both players. But wait—there is more information that could be revealed! The players could continue the conversation like this:

> **Dialog 2:**
>
> ROW: Regardless of whether I play Top or Bottom, I will be $10 better off if you play *Right* rather than Left.
>
> COLUMN: Regardless of whether I play Left or Right, I will be $10 better off if you play *Bottom* rather than Top.

If *both* these dialogs were to occur, there would be a real dilemma. The players' preferences for their own actions would be seen to be in conflict with their stronger preferences for their opponents' actions, and their individually rational behavior would be revealed to be socially irrational. But *Dialog 2 is forbidden in noncooperative game theory*—at least in the context of a single-play game. The "language" of noncooperative game theory is incapable of expressing the idea that one player has preferences concerning the actions of her opponent, because it rules out the possibility of binding multilateral contracts or any other reciprocal influence. Notice that the statements in Dialog 2 refer to payments that one player might be willing to make in return for a reciprocal move by another player—i.e., they refer to multilateral commitments—whereas the statements in Dialog 1 do not. (The statements in Dialog 1 are special cases of unilateral offers to accept gambles, as we will see when we get to the topic of arbitrage-free equilibria.) The only "talk" that is permitted under noncooperative rules of play is "cheap talk," which can never persuade players to choose dominated strategies or otherwise trade off their own petty interests against the larger interests of society. The prisoners obviously could break out of the dilemma if they could sign a binding contract to play the socially optimal strategy, but that's another department: cooperative game theory.

Technically, the meaning of "noncooperative" play is that the players only consider the consequences for themselves of variations in their own strategies, *taking the strategies of their opponents as fixed*. Thus, solution concepts of noncooperative game theory use only a subset of the information contained in the players' preferences: they only look at the *relative differences in utilities between pairs of strategies of a single player*. If you perturb the payoffs of the original game by adding to each player's payoff function an arbitrary function of the *other* players' strategy choices, you get a game that is "best-response equivalent" to the original game, because the added payoffs will cancel out when differences in utilities between two strategies of a given player are evaluated.

To illustrate this idea, let the prisoner's dilemma game be perturbed by adding 10 to Row's payoffs if Column plays Left and subtracting 10 from Row's payoffs if Column plays right, and similarly for Column's payoffs. Then you get a game with the following payoff matrix:

	Left	Right
Top	11, 11	1, 10
Bottom	10, 1	0, 0

There is no dilemma here—it is a real no-brainer. Top-Left is the only rational solution no matter how you look at it. But this game is best-response equivalent to the prisoner's dilemma! When you compute the differences in Row's payoffs between her Top and Bottom strategies, you lose the information that she is $10 better off in the first game if Right is played, or $10 better off in the second game if Left is played. The lost information is precisely what was expressed in Dialog 2—precisely what we are required to ignore under noncooperative rules of engagement.

By the way, the trick that was used here—namely adding to each player's payoffs an arbitrary function of the others players' strategy choices—can be used for dramatic effect to make particular solutions look especially good or especially bad in nearly any game, without changing the set of equilibrium solutions. This is why "refinements" do not necessarily yield attractive solutions: if a refinement selects one out of several equilibria in the original game, it is often possible to jigger the payoffs so that the equilibria are unchanged but the more refined equilibrium yields worse all-around payoffs than some less refined equilibrium. Therefore, you should beware of "anecdotal" evidence in which a few well-chosen examples are used to showcase a particular solution concept, for better or worse.

A cynical view of the preceding result is that it exposes the relative poverty of noncooperative game theory as a language for studying human interaction: the language is simply deaf to a lot of important preference information. Indeed, the inability of noncooperative game theory to provide a satisfactory resolution of the one-shot prisoner's dilemma is widely viewed as a failure of the theory by its detractors. (On several occasions when I've told colleagues in other fields that I'm working on solution concepts for games, their knee-jerk reaction has been "first tell me: how does your theory solve the prisoner's dilemma?") In reply, it can be argued that social dilemmas are real problems that do arise when contracting is impossible, social norms fail, or communication otherwise breaks down. It can also be argued that the language becomes enriched when *repeated play* is considered, giving the players time to learn about each other's idiosyncracies and to punish each other for socially irresponsible behavior. Over the years, thousands of papers have been written on the subject of the repeated prisoner's dilemma. But the theory of repeated games has its own dilemmas—such as the "folk theorem" which says that nearly any behavior can be supported in an equilibrium of an infinitely repeated game—and in any case, many important strategic decisions are faced just once.

Dominance solvability and backward induction

The prisoner's dilemma illustrates the "coarsest" and least controversial solution concept for noncooperative games, namely **dominance solvability** in strictly dominant strategies. The fact that the solution of that game is far from uncontroversial is a sign of the stormy weather ahead. Here are some brief comments on the other solution concepts that will be discussed later. More detail is provided in the articles by Gibbons and Kreps—or any book on game theory.

In looking for ways to solve a game, one usually begins by not only deleting strictly dominated strategies from consideration, but also by deleting any strategies which become strictly dominated after other strictly dominated strategies are eliminated. This is called **iterated** or **successive strict dominance**. Consider the following game:

	Left	Right
Top	2, 2	1, 1
Bottom	3, 1	0, 0

Here, Row does not initially have a dominant strategy, but Column does: Left strictly dominates Right for Column, and after Right is deleted, Bottom is (trivially) strictly dominant for Row, hence the solution is Bottom-Left. Iterated strict dominance generally yields sensible results within the limits of noncooperative rules of engagement. In fact, if the players do not comply with iterated strict dominance, they are vulnerable to arbitrage in the same sense that any individual choosing a dominated strategy would be. (More about this later.)

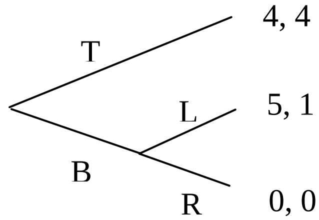
Many theorists argue that **weakly dominated** strategies should also be eliminated. Consider the following example:

	Left	Right
Top	4, 4	4, 4
Bottom	5, 1	0, 0

Here Right is a weakly dominated strategy: it is no better than Left in all cases, and it is strictly worse if Bottom is played. After eliminating Right as an option for Column, we are left with Bottom as the obvious move for Row, hence the solution is Bottom-Left. Are you convinced that this is the unique rational solution to the game? I am not. Notice that the weakly dominated strategy was eliminated on behalf of Column, who then gets the short end of the deal when Row plays Bottom. You might expect Column to say—"hey, wait, *I* don't care about weak dominance—I'm going to play Right and you had better play Top!" The standard objection to this ploy is that Column's "threat" to play Right would not be considered credible by Row, since it would be painful to carry out. Column might huff and puff about her intention to play Right, but when it came time to move, she would hedge her risks by playing Left, taking Row's move (whatever it might be) as "given" at that point. Or would she? If Column is able to convince Row of her resoluteness, then we might not be surprised to see Top-Right as the result. In fact, Top is optimal for Row if she thinks there is even a 20% chance that Column will play Right. Of course, on the other hand, it is conceivable that Row would seize the advantage by declaring "I'm going to play Bottom in any case—and I dare you to carry out your threat," which might lead to a result of Bottom-Left. Both Top-Right and Bottom-Left are pure-strategy equilibria of this game. Depending on the real or hypothetical conversation through which the players construct their reciprocal beliefs, you might expect to get one or the other. .

The preceding example illustrates the least controversial application of weak dominance, in which only a single round of elimination is used. Some theorists go further and recommend the use of **iterated weak dominance**, but this leads to horrors. The solution of the game may then depend on the order in which strategies are eliminated, and even more bizarrely, the solution can be affected by the *addition* of dominated strategies such as the option for one player to "burn money." A famous example of this was given by Eric von Damme: in the battle of sexes game (introduced below), if one player is given the option to burn or not burn money in conjunction with her original move, she can force a unique solution of the game in which in she gets the superior payoff, if the solution method is iterated weak dominance.

Iterated weak dominance is related to the solution concept of **backward induction** which is applicable to extensive-form games of perfect information. Solving a game by backward induction is just like rolling back a decision tree: starting at the end of the game tree, each player chooses the branch that leads to the better outcome for him or her, and they work their way back to the front. In principle, the game of chess can be solved this way. For example, the preceding example is actually the normal-form representation of the following extensive-form game:



When the game is presented in this form, the argument that BL should be played is generally considered even more compelling: in the event that the node is reached in which the second player has a move to make, her best option is to play L rather than R. Knowing this, the first player can safely choose B. But if the entire structure of the game is common knowledge, then the second player must know that she can get a better deal for herself by somehow persuading the first player that she resolutely intends to play R. We can spin various yarns about how this might be accomplished.

Another way to frame the question of whether TR is a rational solution of the preceding game(s) is to take the viewpoint of an outside observer. If he observes that the result of the game is the payoff pair (4, 4) rather than (5, 1), should he conclude that the players have acted irrationally? Would he be able to exploit them in the sense of being able to earn an arbitrage profit? We will see later that he cannot: no-ex-post-arbitrage does not necessarily require backward induction or iterated weak dominance to be used.

To paraphrase Glenn Shafer (in his "Savage Revisited" article), the question raised by the preceding discussion is whether we should deny the players the freedom to exercise their imagination and their will in pursuit of feasible goals. If it is possible for the players to construct mutually consistent beliefs leading to a particularly desirable outcome—say, the payoff pair (4, 4) in the last example—should they be prohibited from doing so? The more refined solution concepts for games establish such prohibitions, not in the interests of the players but in the interest of the modeler who would like to obtain tighter predictions.

To be fair, most theorists are careful not to push the solution concepts to extremes in particular cases. For example, it is well known that backward induction can lead to absurdities when many too many rounds of it are applied, as illustrated by the famous "centipede game" introduced by Robert Rosenthal. (See the discussion of it in Kreps' chapter.) So, most theorists admit that deviations from the theory are appropriate in some situations.

Nash equilibrium

Nash equilibrium is the most over-used and abused concept in game theory: it is to game theory what R-squared is to statistics. Many authors defend Nash equilibrium with the following argument: "if there were an obvious way to play the game—i.e., a uniquely correct solution in pure strategies—then it would have to be a Nash equilibrium." But the same statement applies equally well to other solution concepts that are coarser than Nash equibrium, such as iterative deletion of strictly dominated strategies, correlated equilibrium, and rationalizability. The defining property of a *Nash* equilibrium is what it says about the case in which players use "mixed" (i.e., randomized) strategies rather than "pure" (i.e., deterministic) strategies: a Nash equilibrium is a profile of pure *or independently mixed* strategies from which no player would have an incentive to deviate unilaterally. Thus, if the players use mixed strategies (or if, for whatever reason, some players are uncertain about the strategies being played by others), the probability distributions are required to be *independent between players* in order for the solution to be a Nash equilibrium. The classic example of a mixed strategy is "bluffing" in poker, i.e., sometimes betting as though you have a good hand when you don't. Other examples are abundant in competitive sports and warfare: baseball pitchers randomly vary their pitch selection to keep batters off guard, football teams randomize their play-calling to confuse the defense, fighter planes make random movements to avoid getting shot down, and so on.

Randomized (or otherwise uncertain) strategies certainly do arise in nature, and often for good reasons. That is one of the elementary lessons of the study of games. But… why assume that randomized strategies must always be probabilistically *independent*? A cynical answer is that independence is assumed merely because Nash discovered a beautiful and powerful existence proof, based on the Kakutani fixed-point theorem, for the case in which mixed strategies are independent. (Indeed, Nash is credited with ushering fixed-point methods into economics more generally.) Modelers naturally gravitate toward the strongest equilibrium concept for which existence can be proved, and this existence result is so powerful that it applies under very general conditions (technically, to any game in which the players' best-reply correspondences are "upper-hemicontinuous," which includes all finite games and some infinite ones). But a more conventional answer is that independence is assumed because it seemingly epitomizes the spirit of "noncooperative" play. The players are imagined to be alone in their cubicles in the psychology lab, cut off from the world, at the instant they make their moves. If they use randomization under such conditions, they must do it independently of everyone else. This line of defense ignores the fact that a good deal of implicit or explicit communication between players must take place in order for them to arrive at a state of common knowledge concerning the game they are playing and concerning the solution concept they are mutually adopting (when there is not a unique, obvious solution). It also ignores the fact that, under a subjective interpretation of probability, events are almost never *probabilistically* independent, even when they are physically independent, as de Finetti pointed out when he introduced the important concept of exchangeability.

Last but not least, the standard defense ignores the fact that the "obvious" solution of the game, when one exists, may involve *correlated* mixed strategies: even two-player games can have correlated strategies that are better for both players than any Nash strategy. (A 2-player example is given in my 1990 paper with McCardle. A 3-player example is given below.) Against this last point, Nash equilibrium is sometimes defended by the argument that it is technically "without loss of generality" because an equilibrium in correlated strategies can always be implemented as a Nash equilibrium of a larger game that includes a communication or correlation device. The players then "independently" decide to do whatever the device tells them to do. This defense is technically correct, but it gets the horse and cart completely backward: it can be argued (and will be argued by me) that *correlated equilibrium* is the more fundamental concept—because it rests directly on the same no-arbitrage standard of rationality and the same separating-hyperplane argument that underpin most of the other key results in choice theory—while Nash equilibrium is merely an interesting special case. In any event, Nash equilibrium is hardly ever mentioned in connection with correlating devices: it or one of its refinements is usually used to deliberately exclude the possibility of correlation so as to cut down on the size of the set of equilibria that would otherwise be possible, regardless of whether correlation would be beneficial.

Some of the issues surrounding Nash equilibria are illustrated by the coordination game known as "battle of the sexes" (Luce and Raiffa 1957), which has the following payoff matrix:

	Ballet	Boxing
Ballet	2, 1	–1, –1
Boxing	–1, –1	1, 2

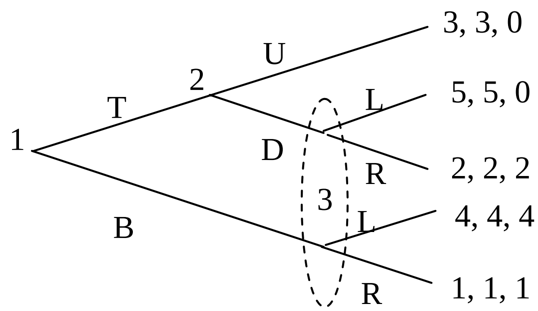
The story line behind this game goes something like this: Alice (the row player) wants to go to the ballet tonight, while Bob (the column player) would prefer to attend the boxing match, but they both strongly prefer to go to somewhere together rather than separately. The game has two pure Nash equilibria: ballet-ballet and boxing-boxing. It also has a "completely mixed" Nash equilibrium in which Alice goes to the ballet with probability 3/5 and Bob *independently* goes to the boxing match with probability 3/5, which makes it more than likely (a probability of 13/25) that they will split up for the evening. What makes the latter state of affairs an equilibrium is that Bob thinks it is just likely enough that Alice will randomly choose the ballet that he doesn't care what he chooses for himself, so he decides to randomize his own choice in just such a way as to make Alice not care about her choice either, and Alice does the same with respect to Bob. The completely mixed Nash equilibrium is a rather dismal solution, as is usually the case in games that have multiple equilibria. Any completely mixed Nash solution remains an equilibrium of the game if the players try to *minimize* their expected utilities—i.e., to maximize their unhappiness. Of course, the solution of the battle-of-sexes game that most persons naturally think of is for Alice and Bob to *flip a coin* in order to choose between ballet and boxing, but that would be a correlated equilibrium, not a Nash equilibrium.[3]

Refinements of Nash equilibrium

One of the cardinal virtues of Nash equilibria is that they always exist in finite games. A shortcoming—at least from the perspective of a modeler who wants to make sharp predictions—is that more than one of them may exist in any particular game. In the worst case, the number of Nash equilibria can grow exponentially with the size of the game. A great deal of effort has been expended on the search for refinements of the Nash concept which would eliminate some of the

[3] A variant of the separate-cubicle story is sometimes used to argue that the mixed-strategy Nash equilibrium is the *only* rational solution of the game: Alice and Bob have gone their separate ways during the day without having decided where to meet in the evening, so they are unable to implement either of the pure-strategy equilibria, let alone the correlated equilibrium. But this begs the question: was it rational for them to not make better plans in advance if they understood the rules of the game?

equilibria—ideally the more "implausible" ones. This line of work was pioneered by Nobel laureate Reinhard Selten, who developed the concepts of **subgame perfect** and **trembling-hand perfect** Nash equilibria. Subgame perfection is a refinement that applies to extensive form games—essentially it is a backward-induction application of the Nash equilibrium concept. Under subgame perfection, the solution in every proper subtree of the game must be a Nash equilibrium. The concept of subgame perfection does not always "bite," since not all games have proper subtrees. The concept of trembling-hand perfect equilibrium is stronger than subgame perfect equilibrium—in fact, it is one of the strongest refinements—and it applies to normal and extensive-form games, with or without subtrees. (The qualifier "trembling-hand" is usually used only in applications to extensive-form games, where it is applied to the *agent normal form* of the game, but I will use the term more broadly here.) Trembling-hand perfection requires that each player's strategies should be robust against mistakes that might be made, with very small probabilities, by other players: each player is assumed to contemplate the possibility that her opponent's hand will "tremble" at the instant he makes his move, leading to a mistake. Trembling-hand perfection obviously eliminates weakly dominated strategies, since a weakly dominated strategy can only be justified by the hypothesis that one's opponent will play some strategies with probability zero. If every player has trembling hands, all strategies have positive probability. The following 3-player game example was used by Selten (and also appears as Figure 12.12(a) in Kreps):



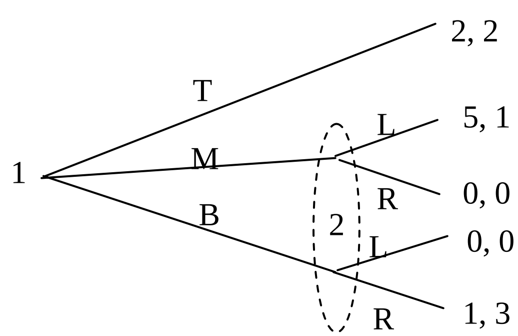
Here, player 1 chooses T or B, player 2 chooses U or D, and player 3 chooses L or R. The dashed ellipse indicates an "information set" of player 3: she does not know which of the two nodes has been reached at the time she makes her move. This game has two pure Nash equilibria: TUR which yields (3, 3, 0) and BUL which yields (4, 4, 4). The "implausible" equilibrium here is supposedly the one that yields (4, 4, 4), because U—which is played "off the equilibrium path"—is not robust against trembling hands in the context of BL. If player 2 thinks there is any non-zero chance that player 1 will play T, then she would be better off to defect to D, in which case 3 should play R, in which case player 1 *should* play T, but then 2 would want to play U after all, leading to TUR. Are you convinced? A striking feature of this example is that the "plausible" equilibrium is strictly payoff-dominated by the implausible one. Should the players be forbidden to pursue the dominant solution? If the outcome of the game is the payoff triple (4, 4, 4), should we consider the players irrational? If communication is possible, I can imagine a scenario in which player 2 sizes up the situation and declares (cheaply) "I am going to play U" and then walks off, leaving 1 and 3 to work out the implications. Or, even if communication is impossible, the strategies that yield the outcome (4, 4, 4) might be considered "focal" by all the players. They might also misjudge each other and make a mistake in trying achieve that result, but if the story of trembling hands is taken seriously, an occasional mistake is inevitable, and perhaps it should not dominate their thinking.

Of course, by the same trick we used in the prisoner's dilemma game, we can jigger the payoffs of the preceding game to eliminate the social inefficiency without changing the set of equilibria. For example, if we change (3, 3, 0) to (3, 3, 10), which doesn't affect player 3's comparisons between L and R, the trembling-hand-perfect TUR equilibrium is moved out to the efficient frontier.

Trembling-hand perfection has an intuitive quality of robustness, but it is rather ad hoc from a decision-theoretic perspective: it does not fully specify what the players actually *believe* at the moment of choice. The refinements that are most commonly used nowadays, **sequential equilibrium** and (what is almost the same thing) **perfect Bayesian equilibrium**, have a more Bayesian flavor. A solution of the game consists not only of a set of prescribed actions for each player at each decision node but also a set of *beliefs* supporting that action at that node. Beliefs are required to be updated by Bayes' rule wherever possible, and conditional beliefs are specified even at nodes that are off the equilibrium path, which are expected to be reached with probability zero. Details are given in the articles by Gibbons and Kreps.

Alas, a difficulty with notions of "sequential" rationality, including subgame perfection and sequential equilibrium, is that they can yield different solutions for two fully equivalent forms of the same game. For example, if one player has a three-branch decision node—say, Top/Middle/Bottom—the solution of the game can be changed by splitting it into two consecutive nodes, in which the first choice is between Top/Not-Top and the Not-Top branch then leads to a choice between Middle and Bottom, even though logically it is the same decision. Kohlberg and Mertens (1986) point out that no solution concept which prescribes a single strategy profile can possibly meet all the desiderata for refinements that have been proposed (including robustness against the node-splitting problem just mentioned), and they argue for a set-valued solution concept known as **strategic stability**. (I, too, feel that a set-valued solution concept is appropriate, but rather on the grounds that the players should be granted as much flexibility in the construction of their reciprocal beliefs as is consistent with the rationality principle of no ex post arbitrage—more about this next class.)

Kohlberg and Mertens also give an example of a game that illustrates yet-another mode of game-theoretic reasoning: **forward induction**. The game can be drawn like this:



Here, player 1 chooses T, M, or B and player 2 chooses L or R without knowing player 1's move. There are two pure Nash equilibria, TR leading to (2, 2) and ML leading to (5, 1). The first equilibrium is sequentially rational, because it is possible to specify beliefs of player 2 that justify her choice of R over L given that her information set is reached, namely, she should think there is at least a 25% chance that B has been played rather than M. These beliefs could be justified, for example, on the hypothesis that any move by player 1 other than T must be a random mistake, and hence M and B are equally likely in the event of not-T. Yet… why should player 1 ever choose B—it is strongly dominated by T! We can argue that ML ought to be the solution on the following grounds: if player 2 gets the opportunity to move, she should assume that player 1 has chosen M rather than B, angling for the payoff of 5 rather than 2. In other words, player 2 should reason about the "signal" that player 1 has sent in the event that she plays something other than T. This type of reasoning is called forward induction because it focuses on what has happened before rather than what is going to happen afterward.

A final word on refinements is given by Kreps (in chapter 13 of his book):

> "So, what is the bottom line for refinements of Nash equilibrium? The philosophy espoused here can be paraphrased as follows: The bottom line is that there is no bottom line. In refining Nash equilibrium, one is speculating about what is supposed to happen after there is evidence that the going theory is incorrect. Depending on why you (and the players involved) think one sees deviations from a priori likely play and what this portends about future play, one supports or diminishes the relevance of particular formal refinements. Since the appropriate story is apt to be specific to the context (and, especially, to depend on why one thinks there is a "solution" in the first place), it seems fruitless to try to choose among refinements in the abstract."

The real rules of the game

The rules of a game in normal form are typically presented in the form of a payoff matrix, in which the values in the cells are the utility payoffs to the players, as in the following generic 2×2 example:

	Left	Right
Top	a, a'	b, b'
Bottom	c, c'	d, d'

The rules are assumed to be common knowledge. Thus, it might seem as if both players in the game are supposed to know that the utility payoffs of player 1 are $\{a, b, c, d\}$ while those of player 2 are $\{a', b', c', d'\}$. But, not so fast! As we saw in the analysis of the prisoner's dilemma and non-dilemma games above, the preference information that is used in noncooperative analysis does not consist of *absolute* utility payoffs, but rather only the *relative differences* in utility between strategies of a given player, taking the strategies of the other players as fixed. The latter information is summarized by the following matrix, which will henceforth be denoted by G:

	TL	TR	BL	BR
1TB	$a - c$	$b - d$		
1BT			$c - a$	$d - b$
2LR	$a' - b'$		$c' - d'$	
2RL		$b' - a'$		$d' - c'$

The matrix G has one column for every pure-strategy outcome of the game and one row for every possible comparison between a "chosen" strategy and an "alternative" strategy of every player. For this 2×2 game, the first row of G, which is labeled 1TB, represents the scenario in which player 1 chooses T over the alternative strategy B, and the values in the cells are the differences in utility payoffs between T and B, as a function of player 2's choice (namely, L or R). Notice that the utility differences are computed only for the outcomes in which the "chosen" strategy is played (in this case, T). The blank cells are interpreted as zeroes. Similarly, the second row, labeled 1BT, shows the differences in utility between strategies B and T, stored in the cells corresponding to outcomes where B is played. Of course, these are just the negatives of values in the first row, shifted over into different columns, so there is a slight amount of redundancy in the data. (The reason for this structure will become apparent later.)

Claim: The matrix G represents the "real" rules of the game for purposes of noncooperative analysis.

For a general n-player game, the rules matrix G is constructed as follows

- The rows of G are indexed by ijk, where i = player number, j = a strategy of player i, and k = an alternative strategy of player i. The number of rows is therefore $\sum_{i=1}^{n} |S_i| \times (|S_i| - 1)$, i.e., one row for each combination of a strategy and an alternative strategy for each player.

- Columns of G are indexed by $s = (s_1, \ldots, s_n)$, i.e., pure-strategy outcomes of the game. The number of columns is therefore $|S|$, i.e., one column for every possible outcome.

- The element of G in row ijk and column s is determined by

  $$g_{ijk}(s) = (u_i(s) - u_i(k, s_{-i}))1_{jk},$$

  where 1_{jk} is equal to 1 if $j=k$ and equal to 0 otherwise.

Thus, G consists of vectors of *conditional utility differences* between strategies of individual players, holding the strategies of other players fixed. The ijk^{th} row of G is the vector of utility differences between strategies j and k of player i, *conditioned* on the event that i chooses strategy j. Its element $g_{ijk}(s)$ is equal to zero if player i does not play strategy j in the joint strategy s, otherwise it is equal to the difference in utility between strategies j and k of player i when the other players play s_{-i}. For a 3×3 game, G has 12 rows and 9 columns. For a 4×4 game, G has 24 rows and 16 columns. For a 2×2×2 game, G has 6 rows and 8 columns, and so on.

The deeper reasoning behind the designation of G as the "real" rules of the game will be presented in the following lecture. In particular, it will be shown how G might come to be common knowledge and why it is central to the definition of strategically rational behavior. For the time being, note that G contains the "dialog 1" information that is used to determine noncooperative equilibria and refinements, minus the "dialog 2" information that determines social efficiency. In particular, G contains precisely the data needed to apply the following noncooperative solution concepts:

Nash equilibrium:
An independent strategy profile $\sigma^* = \sigma_1^* \times \ldots \times \sigma_n^*$ is a Nash equilibrium if and only if $G\sigma^* \geq \mathbf{0}$.

Proof: if s_i^* is a strategy of player i that has positive probability in σ_i^*, then $G\sigma^* \geq \mathbf{0}$ implies $u_i(s_i^*, \sigma_{-i}^*) - u_i(s_i, \sigma_{-i}^*) \geq 0$ for any other strategy s_i, which is equivalent to the usual definition. (Here $u_i(\sigma_i^*)$ is shorthand for the expectation of $u_i(s)$ with respect to the joint distribution σ^*, etc.)

Correlated equilibrium:
A probability distribution σ^* on outcomes of the game (not necessarily independent between players) is a correlated equilibrium distribution if and only if $G\sigma^* \geq \mathbf{0}$.

Perfect equilibrium (Fudenberg & Tirole definition 8.5C, following Myerson):
A strategy profile σ^ε is an ε-perfect equilibrium if it is completely mixed (i.e., all strategies have strictly positive probability) and $\sigma_i^\varepsilon(j) < \varepsilon$ for any strategy j of player i for which $[G\sigma^\varepsilon]_{ijk} < 0$ for some k. In other words, strategy j of player i must have probability less than ε if it is worse than some other strategy k. A perfect equilibrium is any limit of ε-perfect equilibria σ^ε for some sequence of positive ε's converging to zero.

How to find equilibria

Given a game and a solution concept, it is natural to ask how one goes about *finding* an equilibrium solution—especially if we are to imagine that economic agents routinely do this on their own. The simplest equilibria to compute are solutions of *two-person zero-sum games*. Indeed, the first major breakthrough in the theory of games was von Neumann's proof of the "minimax theorem" in 1928. The minimax theorem states that in a zero-sum game, problem of maximizing the minimum gain for one player has the same optimal solution as the problem of minimizing the maximum loss for the other player. Let A denote the $m \times n$ payoff matrix of the game, expressed in terms of gains to player 1, where m is the number of strategies available to player 1 and n is the number of strategies available to player 2. Thus, the element in the i^{th} row and j^{th} column of A, denoted A_{ij}, is the gain to player 1 (and the loss to player 2) when 1 plays her i^{th} strategy and 2 plays her j^{th} strategy. The players may either choose pure or mixed (i.e. probabilistic) strategies. Let $\boldsymbol{x}$ denote an m-vector representing a possibly-mixed strategy of player 1 (i.e., $\boldsymbol{x}$ is a vector of probabilities assigned to the pure strategies of player 1), and let $\boldsymbol{y}$ denote an n-vector representing a strategy of player 2. Then the expected payoff to player 1 can be expressed by the matrix product $\boldsymbol{x} \bullet A\, \boldsymbol{y}$. Now suppose that player 1 wishes to choose $\boldsymbol{x}$ to maximize the minimum expected gain to herself over all strategies that player 2 might use. Thus, player 1 chooses $\boldsymbol{x}$ to solve: $\max_{x} \; \min_{y} \; \boldsymbol{x} \bullet A\, \boldsymbol{y}$. Meanwhile, suppose player 2 wishes to choose y so as to minimize the maximum expected gain that player 1 can possibly achieve (which is the same as maximizing her own minimum gain). Thus, player 2 chooses $\boldsymbol{y}$ to solve $\min_{y} \; \max_{x} \; \boldsymbol{x} \bullet A\, \boldsymbol{y}$. The minimax theorem states that the optimal objective value of these two problems is the same number—called the *value* of the game for player 1. Indeed, these two problems are *dual* to each other in the sense of the duality theorem of linear programming—and conversely, the duality theorem of linear programming can be proved as a consequence of the minimax theorem. Thus, it is easy to solve a zero-sum game: just use linear programming to find the unique, minimax solution. For example, consider the zero-sum game with the following payoff matrix:

	Left	Right
Top	3	-1
Bottom	-4	2

The number in each cell is the gain to the row player and the loss to the column player. It may not be apparent at first glance who has the advantage in this game, but in fact the row player has the advantage: the value of this game for row is 0.2. Her maximin strategy which guarantees this expected gain is 60% top, 40% bottom, while the minimax strategy for column that prevents her expected loss from being more than this is 30% left, 70% right.

Now consider a discrete two-player game that is not necessarily zero-sum, often called a "bimatrix" game. Let A denote the matrix of payoffs to player 1 and let B denote the matrix of payoffs to player 2. If it happens that $B = -A$, then the game is zero-sum, but henceforth we will assume this need not be the case. Suppose, as before, that players use pure or mixed strategies represented by probability vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. Then the expected payoff to player 1 is $\boldsymbol{x} \bullet A\, \boldsymbol{y}$ and the expected payoff to player 2 is $\boldsymbol{x} \bullet B\, \boldsymbol{y}$. The minimax theorem is no longer applicable, because the interests of players 1 and 2 need not be diametrically opposed. This is where Nash equilibrium rears its head as a seemingly natural generalization. A strategy pair ($\boldsymbol{x}^*$, $\boldsymbol{y}^*$) is a Nash equilibrium if $\boldsymbol{x}^*$ is a *best response* to $\boldsymbol{y}^*$ and $\boldsymbol{y}^*$ is a best response to $\boldsymbol{x}^*$, in the sense that $\boldsymbol{x}^*$ solves $\max_{x} \; \boldsymbol{x} \bullet A\, \boldsymbol{y}^*$ and $\boldsymbol{y}^*$ solves $\max_{y} \; \boldsymbol{x}^* \bullet B\, \boldsymbol{y}$. Now, it might seem as though the problem of finding a Nash equilibrium of a bimatrix game is not too much harder than the problem of finding a minimax solution of a zero-sum game. Can't you just use linear programming? Not quite! You can't formulate a single linear program whose solution is guaranteed to be a Nash equilibrium, although it is possible to devise an algorithm that solves a *sequence* of linear programs that eventually terminates in a Nash equilibrium. The first such algorithm for bimatrix games was developed by Lemke and Howson in 1964, more than a decade after Nash first proposed his equilibrium concept. The Lemke-Howson algorithm is guaranteed to find *a* Nash equilibrium, but it is possible that there is more than one—indeed, in the worst case, the number of possible Nash equilibria grows exponentially with the size of the game. Interestingly, the number of Nash equilibria in a bimatrix game is always *odd*. Recently, an efficient algorithm for enumerating *all* Nash equilibria of a bimatrix game has been developed by Charles Audet: it relies on a "branch and bound" approach similar to what is used to solve integer linear programs.

Okay, so it is straightforward to find Nash equilibria of 2-player games, even if the algorithm is complicated and the equilibria are not always attractive. What if the game has more than two players? Now things begin to get really difficult. Until recently there was no known algorithm for efficiently finding a Nash equilibrium of a matrix game with three or more players. One algorithm that is *not* particularly efficient is to guess the strategies that have positive probability in a Nash equilibrium, then look at the reduced game consisting only of these strategies and solve a system of nonlinear (polynomial) equations to find the equilibrium probabilities.[4] The equilibrium probabilities have the property that they render every player indifferent among all her own strategies that have positive probability. If the system of nonlinear equations has a feasible solution, and if the "included" strategies yield expected payoffs greater than or equal to those of the "excluded" strategies for each player, you've found a Nash equilibrium. Otherwise you have to go back and make a different guess as to the strategies that have positive probability. Another complication in games with three or more players is that the probabilities used in mixed-strategy Nash equilibria may be *irrational numbers*, even if the payoffs are all integer-valued. In his original paper, Nash gave an example of a 3-player "poker" game with a unique equilibrium in irrational mixed strategies. How are poker players supposed to come up with such numbers?

Nash's poker game reduces to a 2×2×2 game following deletion of dominated strategies. It is actually fairly easy to construct a 2×2×2 game with a unique Nash equilibrium in irrational mixed strategies. For example, consider the following 3-player game in which player 1 chooses the *row* (top or bottom), player 2 chooses the *column* (left or right), and player 3 chooses the *matrix* (up or down). The three numbers in each cell are, respectively, the utility payoffs to players 1, 2, and 3.

Up:	Left	Right
Top	3, 0, 2	0, 2, 0
Bottom	0, 1, 0	1, 0, 0

Down:	Left	Right
Top	1, 0, 0	0, 1, 0
Bottom	0, 3, 0	2, 0, 3

The game has a unique Nash equilibrium in irrational, completely mixed strategies with the following marginal probabilities:

[4] This approach is used by the PolEnum algorithm in GAMBIT: it iterates through all possible "supports" of the set of Nash equilibria, and on each support it attempts to solve a set of polynomial equations to find a completely mixed strategy (http://gambit.sourceforge.net/)

$\sigma_2(L) = (-13 + \sqrt{601})/24 \approx 0.47980$
$\sigma_1(T) = (9\sigma_2(L) - 1)/(7\sigma_2(L) + 2) \approx 0.61923128$
$\sigma_3(U) = (-3\sigma_2(L) + 2)/(\sigma_2(L) + 1) \approx 0.37882534$

The expected payoffs for the Nash equilibrium are (0.843, 0.854, 0.594)

The more "obvious" solution of this game (it seems to me) is that the players should somehow try to avoid the four outcomes in which one player receives 1 (her second-worst payoff) while the others receive zero (their worst payoff). This requires coordination between row and matrix so that they play either TU or BD. This reasoning leads to a correlated equilibrium in which column plays 2/3 L and 1/3 R while row and matrix play 3/5 TU and 2/5 BD, yielding expected payoffs of (1.467, 1.200, 1.200), strictly dominating the Nash equilibrium.

The "real" rules matrix (*G*) for this game is as follows:

	TLU	TRU	BLU	BRU	TLD	TRD	BLD	BRD
1TB	3	-1			1	-2		
1BT			-3	1			-1	2
2LR	-2		1		-1		3	
2RL		2		-1		1		-3
3UD	2	0	0	-3				
3DU					-2	0	0	3

A correlated equilibrium of the game can be found by solving a linear program in which the decision variables are elements of the joint distribution σ^* and the constraints consist of the system of linear inequalities $G\sigma^* \geq \mathbf{0}$. An *efficient* correlated equilibrium can be found by maximizing any positive weighted sum of the players' expected payoffs subject to those constraints, although this requires the introduction of "dialog 2" information not contained in *G*. By comparison, finding a Nash equilibrium requires searching for a profile of individual strategies $\{\sigma_1^*, \ldots, \sigma_n^*\}$, i.e., marginal distributions on the strategy sets of individual players, satisfying $G\sigma^* \geq \mathbf{0}$ where $\sigma^* = \sigma_1^* \times \ldots \times \sigma_n^*$. However, this is not a convex optimization problem because the joint distribution σ^* is a nonlinear function (in particular, a product) of the decision variables $\{\sigma_1^*, \ldots, \sigma_n^*\}$.

Recently Audet, Belhaïza, and Hansen (*Journal of Optimization Theory and Applications*, 2006) have extended Audet's branch-and-bound method for enumerating extreme Nash equilibria for bimatrix games to the case of "polymatrix" games, which are a special class of *n*-player games in which each player is engaged in a 2-player game against every opponent, and her total payoff is the sum of the payoffs in those games. In other words, each player's payoff function is additively separable in the strategies of the other players. The set of Nash equilibria of polymatrix games has nice geometrical properties like those of bimatrix games: it is a finite union of convex polytopes. This is not true for general *n*-player games.