ROBERT F. NAU

DE FINETTI WAS RIGHT: PROBABILITY DOES NOT EXIST

ABSTRACT. De Finetti's treatise on the theory of probability begins with the provocative statement PROBABILITY DOES NOT EXIST, meaning that probability does not exist in an objective sense. Rather, probability exists only subjectively within the minds of individuals. De Finetti defined subjective probabilities in terms of the rates at which individuals are willing to bet money on events, even though, in principle, such betting rates could depend on state-dependent marginal utility for money as well as on beliefs. Most later authors, from Savage onward, have attempted to disentangle beliefs from values by introducing hypothetical bets whose payoffs are abstract consequences that are assumed to have state-independent utility. In this paper, I argue that de Finetti was right all along: PROBABILITY, considered as a numerical measure of pure belief uncontaminated by attitudes toward money, does not exist. Rather, what exist are de Finetti's 'previsions', or betting rates for money, otherwise known in the literature as 'risk neutral probabilities'. But the fact that previsions are not measures of pure belief turns out not to be problematic for statistical inference, decision analysis, or economic modeling.

KEY WORDS: Probability, Beliefs, Values, Statistical inference, Decision theory

The numerous, different, opposed attempts to put forward particular points of view which, in the opinion of their supporters, would endow Probability Theory with a 'nobler' status, or a more 'scientific' character, or 'firmer' philosophical or logical foundations, have only served to generate confusion and obscurity, and to provide well-known polemics and disagreements, even between supporters of essentially the same framework. (de Finetti 1974, p. xi)

[E]very... more or less original result in my conception of probability theory should not be considered as a discovery (in the sense of advanced research). Everything is essentially the fruit of a thorough examination of the subject matter, carried out in an unprejudiced manner, with the aim of rooting out nonsense. (de Finetti 1974, p. xii)

1. INTRODUCTION

It is strange that the summary of a lifetime of work on the theory of X should begin by declaring that X does not exist, but so begins de Finetti's *Theory of Probability* (1970/1974):

Theory and Decision **51:** 89–124, 2001.

My thesis, paradoxically, and a little provocatively, but nonetheless genuinely, is simply this:

PROBABILITY DOES NOT EXIST

The abandonment of superstitious beliefs about the existence of the Phlogiston, the Cosmic Ether, Absolute Space and Time, ... or Fairies and Witches was an essential step along the road to scientific thinking. Probability, too, if regarded as something endowed with some kind of objective existence, is no less a misleading misconception, an illusory attempt to exteriorize or materialize our true probabilistic beliefs. (p. x)

Of course, what de Finetti meant by this was that probability does not exist *objectively*, independently of the human mind. Rather:

[I]n the conception we follow and sustain here, only *subjective* probabilities exist – i.e., the *degree of belief* in the occurrence of an event attributed by a given person at a given instant and with a given set of information. (pp. 3–4)

The subjective theory of probability, which is now widely accepted as the modern view, is jointly attributed to de Finetti (1928/1937), Ramsey (1926/1931), and Savage (1954). Ramsey and de Finetti developed their theories independently and contemporaneously, and Savage later synthesized their work and also incorporated features of von Neumann and Morgenstern's (1944/1947) expected utility theory. All three authors proposed essentially the same behavioristic definition of probability, namely that it is a rate at which an individual is willing to *bet on* the occurrence of an event. Betting rates are the primitive measurements that reveal *your* probabilities or *someone else's* probabilities, which are the only probabilities that really exist. This definition neatly inverts the objectivistic theory of gambling, in which probabilities are taken to be intrinsic properties of events (e.g., propensities to happen or long-run frequencies) and personal betting rates are later derived from them. Of course, subjective probabilities may be informed by classical, logical, or frequentist reasoning in the special cases where they apply, and as such,

all the three definitions of 'objective' probability, although useless *per se*, turn out to be useful and good as valid auxiliary devices when included as such in the subjectivistic theory. (p. xii)

For example, in the case of a normal-looking die that is about to be tossed for the first time, a classicist would note that there are six possible outcomes which by symmetry must have equal chances of occurring, while a frequentist would point to empirical evidence showing that similar dice thrown in the past have landed on each side about equally often. A subjectivist would find such arguments to be suggestive but needlessly encumbered by references to superfluous events. What matters are her *beliefs* about what will happen on the single toss in question, or more concretely how she should *bet*, given her present information. If she feels that the symmetry argument applies to her beliefs, then that is sufficient reason to bet on each side at a rate of one-sixth. But a subjectivist can find other reasons for assigning betting rates in situations where symmetry arguments do not apply and repeated trials are not possible.

In de Finetti's theory, bets are for money, so your probability of an event is effectively the *price* that you are willing to pay for a lottery ticket that yields 1 unit of money if the event occurs and nothing otherwise. De Finetti used the notation 'Pr' to refer interchangeably to Probability, Price, and Prevision ('foresight'), and he treated them as alternative labels for a single concept. The appeal of his money-based definition is that it has the same beauty and simplicity as theories of modern physics: the measurements are direct and operational, they involve exchanges of a naturally conserved quantity, and their empirical laws are deducible from a single governing principle, namely the principle of *coherence* or no-arbitrage. De Finetti was strongly influenced by the work of operationalist physicists such as Mach, Bridgman, and Einstein – particularly by Einstein's theory of special relativity. Relativity theory emphasizes that physical quantities must be defined from the perspective of a given observer using a given measuring instrument. For this reason, concepts such as length, velocity, and simultaneity cannot be defined in absolute terms but only in relative terms, and dimensions of measurement that are commonly believed to be independent (such as space and time) turn out to be strongly connected, particularly when the scale of observation is extreme compared to ordinary experience. Similar lessons come from other branches of modern physics such as quantum mechanics: what may be said to exist depends upon the measurements that are possible to perform.

Ramsey and Savage (and of course, von Neumann) were also inspired by modern developments in the theory of physical measurements, and they were even more ambitious than de Finetti. Recognizing that the marginal utility of money could vary across states of the world and levels of wealth, Ramsey and Savage simultaneously introduced measurement schemes for utility and then tied their definitions of probability to bets in which the payoffs were effectively measured in utiles rather than dollars. In this way, they obtained probabilities that were interpretable as measures of *pure belief*, uncontaminated by marginal utilities for money. Ramsey sought to separate probability from utility by the device of an *ethically neutral proposition*. (He referred to money bets as the 'old-established way of measuring a person's belief,' which he regarded as 'insufficiently general' for his purposes because 'it is universally agreed that money has a diminishing marginal utility'.) Savage introduced, instead, the notion of a *consequence*, a prize whose utility would be, by definition, the same in every state of the world.

In theories and models of choice under uncertainty developed since Savage's time, it has become conventional to adopt his notion of a consequence and to strive for a clean separation between probabilities and cardinal utilities in the representation of preferences. This approach is followed in Anscombe and Aumann's (1963) simpler 'horse lottery' axiomatization of subjective expected utility and in Karni's (1985) theory of state-dependent utility, as well as in newer non-expected-utility theories such as Schmeidler's (1989) Choquet expected utility, Gilboa and Schmeidler's (1989) maxmin expected utility, Machina and Schmeidler's (1992) probabilistically sophisticated non-expected-utility preferences, Ghirardato and Marinacci's (2002) biseparable preferences, and Grant and Karni's (2000) quantifiable beliefs, to name a few. In models of information economics and financial economics that are based on those theories, the beliefs of the actors are represented by their true subjective probabilities (or non-additive generalizations thereof), which are sometimes also assumed to be mutually consistent or empirically correct. The separation of probability and utility is also fundamental to the theory of non-cooperative games, in which the payoff functions are expressed in units of pure utility, and especially to the theory of games of incomplete information, where the players' true beliefs about exogenous states of nature are subject to the common prior assumption. On a more down-to-earth level, the separation of probability from utility is central to the 'divide and conquer' strategy of applied decision analysis: wheels for assessing probabilities and computer programs for assessing utility functions for money have been used in business schools and consulting firms since the early 1960's.

De Finetti later admitted that it might have been better to adopt the seemingly more general approach of Ramsey and Savage:

The formulation... could be made watertight... by working in terms of the utility instead of with monetary value. This would undoubtedly be the best course from the theoretical point of view, because one could construct, in an integrated fashion, a theory of decision-making... whose meaning would be unexceptionable from an economic viewpoint, and which would establish simultaneously and in parallel the properties of *probability* and *utility* on which it depends. (p. 79)

Nevertheless, he found 'other reasons for preferring' the money bet approach:

The main motivation lies in being able to refer, in a natural way to combinations of bets, or any other economic transactions, understood in terms of monetary value (which is invariant). If we referred ourselves to the scale of utility, a transaction leading to a gain of amount S if the event E occurs would instead appear as a variety of different transactions, depending on the outcome of other random transactions. These, in fact, cause variations in one's fortune, and therefore in the increment of utility resulting from the possible additional gain S: conversely, suppose that in order to avoid this one tried to consider bets, or economic transactions, expressed, let us say, in 'utiles' (units of utility, definable as the increment between two fixed situations). In this case, it would be practically impossible to proceed with the transactions, because the real magnitudes in which they have to be expressed (monetary sums or quantities of goods, etc.) would have to be adjusted to the continuous and complex variations in a unit of measure that nobody would be able to observe. (p. 81)

From the perspective of Savage's theory, the problem with de Finetti's definition of probability is the following. Suppose your 'true' probability of state i is p_i and your marginal utility for money in state i is z_i. Then the probability distribution revealed by your betting rates, denoted π, will be proportional to the *product* of your true probabilities and marginal utilities for money:

$$\pi \propto \boxed{p_1 z_1 \mid p_2 z_2 \mid \dots \mid p_n z_n}$$

(See Kadane and Winkler, 1988, for a discussion.) Of course, Savage's and de Finetti's subjective probabilities will coincide if the marginal utilities z_i happen to be constant, which will be true under 'non-relativistic' conditions involving events in which the individual has no prior personal or financial interest – conditions that prevail in most scientific and engineering applications and in recreational gambling – but more generally they will differ. For the purposes of this paper, I will henceforth refer to the two kinds of subjective probabilities as 'true probabilities' and 'previsions', respectively. The conventional (although not quite universal) view is that true probabilities are the natural belief parameters that should be used in statistical and economic decision models, while previsions reflect an undesirable confounding of beliefs and values.

This brings me to the first thesis of this paper, which is that, to paraphrase de Finetti, paradoxically, and a little provocatively, but nonetheless genuinely,

TRUE PROBABILITY DOES NOT EXIST

It is impossible to measure degrees of pure belief with the psychological instruments proposed by Savage, just as it is impossible to measure absolute velocity or simultaneity with physical instruments. Belief and value are as fundamentally inseparable as space and time, contrary to the premises of most theories of choice under uncertainty that are in common use today, expected-utility or non-expected-utility. The second thesis of the paper is that

THIS IS NOT A PROBLEM

Previsions *do* exist and, as the measurable parameters of belief, they are sufficient for decision modeling, even where they diverge from the decision maker's imaginary true probabilities. Indeed, they are already central to several important branches of economic theory, and elsewhere they can help to resolve a few puzzles. This is not to say that true probabilities are not potentially useful as a mental construct. To paraphrase de Finetti again, the various definitions of 'true' probability, although useless *per se*, turn out to be useful and good as valid auxiliary devices when included as such in the theory of previsions. But they should not be asked to bear any significant weight such as requirements of external validity or inter-agent consistency or common knowledge. In most applications, knowing someone else's true probabilities is about as important as knowing her speed relative to the Cosmic Ether.

2. WHY TRUE PROBABILITY DOES NOT EXIST

The idea that probabilistic beliefs ought to be separable from values is an old one, dating back at least to Bayes, but its modern incarnation owes a great deal to von Neumann and Morgenstern. In laying down a new game-theoretic foundation for economics, they felt it was necessary to introduce a form of 'new money' that would have all the convenient properties of 'old money' except that, by construction, everyone's objective would be to maximize its expected value under all circumstances. The new money introduced by von Neumann and Morgenstern was the old concept of cardinal utility, resurrected and re-animated by interpreting it as an index of choice under risk. Ever since von Neumann and Morgenstern's rehabilitation of utility, it has been something of a challenge for economic theorists to explain why real money, the old-fashioned kind, plays such a distinguished role in everyday life. In standard microeconomic theory, money is just an arbitrarily designated numeraire commodity. The explanation of why such a numeraire is useful often appeals, somewhat circularly, to game-theoretic arguments.

Von Neumann and Morgenstern showed that it is theoretically possible to determine pure subjective utility from preferences under conditions of *risk*, where the objects of choice are objective probability distributions over an abstract set of prizes. It remained for Savage to show how pure subjective utility and pure subjective probability might be determined simultaneously from preferences under conditions of *uncertainty*, where the objects of choice are lotteries in which prizes are attached to states of the world rather than objective probabilities. The device by which Savage accomplished this feat has become the standard tool used by most theorists who model choice under uncertainty, namely the concept of a 'consequence', which is a prize having the properties that (i) it is possible to imagine receiving or experiencing it in any state of the world, and (ii) it has the same von Neumann–Morgenstern utility index in every state of the world. A consequence is a quantum of psychic income: a state of the person as distinct from a state of the world or the state of a bank account.

What's wrong with Savage's device? Others have written eloquently on this subject, but let me highlight some problems concerning the determination of personal probabilities.

The first problem is that consequences are not, in general, directly observable: they are internal to the decision maker and their definitions are not necessarily independent of other model elements. Although most decision theories that use Savage's framework take the set of consequences to be given, Savage's own arguments – his use of the very terms 'consequence' and 'state of the person', his illustrative examples, his digression on small worlds, his rationales for key axioms – emphasize that consequences implicitly depend on prior notions of states and acts, as well as the imagination of the decision maker. A consequence is 'anything that may happen to the person' as the result of a collision between a course of action she might take and a state of the world that nature might throw at her. Formally, Savage's framework refers to a set S of states of the world, a set C of consequences, and a set F of acts that are arbitrary mappings from S to C, of which some subset F_0 consists of 'concrete' acts that are behaviorally possible – or at least plausible. But from an observer's perspective, only S and F_0 are visible landmarks. We can devise procedures by which different observers could agree that a particular concrete act has been chosen or a particular state has occurred, but the state of the person is ultimately known only to herself. Of course, this is not necessarily a drawback if the decision maker is simply modeling her own affairs, but if the theory is to also provide a foundation for interactive decisions and communication of beliefs, it may matter whether one person can know another's consequences. (The picture is much clearer if the concrete acts and states of the world lead only to the receipt of different quantities of material goods, which have their own a priori definitions, but then we are in the realm of state-preference theory, a rival modeling framework to be discussed later.) Practically speaking, then, the set C of consequences must be identified with some subset of the Cartesian product of S and F_0. Care must be taken that the sets S, F_0, and C are specified with just enough detail that the consequence c resulting from concrete act f in state s is psychologically primitive and self-contained in the way the preference axioms will later require, but this potentially leads to an infinite regress, as Aumann (1971) and Shafer (1986) observe. The pair (*S*, *C*) is called a 'small world' and is interpreted as a coarse representation of some infinitely detailed 'grand world', although Savage admits that 'in the final analysis, a consequence is an idealization that can perhaps never be well approximated' (1954, p. 84).

The next problem is that the set *F* of all acts includes every possible *counterfactual* mapping of states in *S* to consequences in *C*. In other words, it should be possible to imagine that any state of the person can be experienced in any state of the world, even though the states of the person were originally conceived as natural consequences of *particular* concrete acts in *particular* states of the world. Thus, to use some of the examples that were discussed in Savage's famous 1971 exchange of letters with Aumann (reprinted in Drèze, 1987), it should be possible to envision being hung without damage to your health or reputation, or to enjoy the companionship of your wife after she has died on the operating table. And not only is it supposed to be possible to imagine such bizarrely counterfactual acts, the decision maker should also have complete preferences with respect to them, because preferences among the counterfactual acts (especially 'constant' acts that yield the same consequence in every state of the world) are the key to uniquely separating probabilities from utilities. I won't dwell on the combinatorial complexity of this preference structure, which no one takes literally anyway. The critical point is that most of the preference measurements in Savage's model have no direct or immediate implications for real behavior: they are hypothetical choices among alternatives that are unreal and impossible by definition. Such preferences have at best an indirect and tenuous connection with behavior in the sense that, taken together, they may eventually constrain choices among concrete acts via rules of composition that the decision maker has agreed to abide by. As Shafer (1986) has observed, the only plausible way in which a decision maker could satisfy Savage's axioms would be to start from an assumed probability distribution and utility function and then construct her preferences for the myriad counterfactual acts by performing expected-utility calculations.

Even if we ignore the conceptual and practical problems of defining consequences and eliciting counterfactual preferences, two deeper and more troubling theoretical problems remain. First, a given decision problem may admit many different small world representations within the same grand world, and different small worlds may yield different values for the subjective probabilities. An example is given by Shafer (1986), in which he deconstructs Savage's example of a decision about whether and how to add a sixth egg, which might or might not be rotten, to a five-egg omelet. Shafer shows that when the decision is modeled with two different small worlds, in which the degree of freshness of the non-rotten eggs is or is not distinguished, different probabilities are obtained for the event that the sixth egg is rotten, even though the preferences are the same and the models are mutually consistent. Savage was aware that different small worlds could yield different subjective probabilities for the same events under the same grand-world preferences, and he conjectured that the correct probabilities would be approached in the limit as more detailed small worlds were envisioned, but there is no proof of convergence.

Second and more importantly, even within a given small world, true probabilities are not uniquely determined by preferences: Savage's axioms guarantee that preferences among acts are represented by a unique probability distribution and a state-independent utility function that is unique up to positive affine transformations, but they do not guarantee that the von Neumann–Morgenstern utilities of the consequences are actually state-independent. That is, they do not guarantee that the decision maker's utility functions for consequences are similarly scaled as well as similarly shaped in every state of the world. In a nutshell, the problem is as follows. Let the decision maker's preferences be represented by a probability distribution $\boldsymbol{p}$:

$$\boldsymbol{p} = \boxed{\;p_1 \;|\; p_2 \;|\; \cdots \;|\; p_n\;}$$

and a state independent utility function u, so that the utilities yielded by act f are given by the vector:

$\boldsymbol{u}(f) =$

$u(f_1)$	$u(f_2)$	$\ldots$	$u(f_n)$

where f_i is the consequence yielded by act f in state i. Then the expected utility of f can be expressed as $\boldsymbol{p}\cdot\boldsymbol{u}(f)$. Now let $\boldsymbol{a}$ be an arbitrary positive vector (suitably scaled) and define a new probability distribution $\boldsymbol{q}$ by the pointwise product $q_i = p_i a_i$:

$\boldsymbol{q} =$

$p_1 a_1$	$p_2 a_2$	$\ldots$	$p_n a_n$

Correspondingly, define a *state-dependent* utility function by the pointwise quotient $v_i(f_i) = u(f_i)/a_i$:

$\boldsymbol{v}(f) =$

$u(f_1)/a_1$	$u(f_2)/a_2$	$\ldots$	$u(f_n)/a_n$

Note that $v_i(.)$ is the same generic von Neumann–Morgenstern utility function for consequences as $u(.)$ in every state i, since the utility scale factor is arbitrary. By construction, $\boldsymbol{q}\cdot\boldsymbol{v}(f) = \boldsymbol{p}\cdot\boldsymbol{u}(f)$, so ($\boldsymbol{q}$, $\boldsymbol{v}$) represents the same preferences as ($\boldsymbol{p}$, $\boldsymbol{u}$), and the conditional preferences among consequences are technically state-independent under either representation, which is all that Savage's axioms require. The choice of ($\boldsymbol{p}$, $\boldsymbol{u}$) as the 'correct' representation is based only on the completely gratuitous assumption that the utility scale factors are the same across states. Hence, Savage's probabilities are not necessarily the decision maker's 'true' subjective probabilities, even if the latter are assumed to exist (Karni et al., 1983; Schervish et al., 1990; Karni and Mongin, 2000).

Another way to look at the problem is to consider how a probability actually would be measured in Savage's framework. Assume that Savage's world includes a roulette wheel. (His axioms imply that a subjective roulette wheel exists, but Machina (2001) has recently shown how and why objective roulette wheels exist under very general conditions.) To determine the probability of an event E, fix two consequences x and y such that x is preferred to y. Then the probability of E is the number p such that a roulette wheel lottery which yields x with probability p and y with probability $1 - p$ is indifferent to a lottery that yields x if E occurs and y otherwise. This measurement scheme assumes that the consequences x and y have the same *absolute* utilities when they are received as a result of a roulette wheel spin as when they are received as a result of event E occurring or not occurring, but the latter condition is unverifiable. Consequences are *supposed* to be defined so their utilities are the same in every state, but the decision maker's preferences among acts do not establish that fact because subjective probabilities are confounded with utility scale factors.

Professor Karni has investigated this problem more thoroughly than anyone, having written a book and numerous papers on the subject (e.g., Karni and Schmeidler, 1981; Karni et al., 1983; Karni, 1985; Karni and Safra, 1995; Karni and Mongin, 2000), and he and Professor Schmeidler have proposed an ingenious device for divining the decision maker's 'true' subjective probability distribution in the face of possibly-state-dependent utility: assume that the decision maker can articulate additional preferences over acts in which, not only are states mapped arbitrarily to consequences, but the probabilities of the states are imagined to have objectively specified values. In theory, this does the trick: the additional preferences do yield a unique probability distribution and unique state-independent utility function. The correct distribution is the one for which the additional preferences agree with the original preferences when that distribution is imagined to be objectively assigned to the states. But... the additional preferences are even more counterfactual than those required by Savage. They do not even *indirectly* constrain the decision maker's material behavior. The decision maker could simply pull an arbitrary probability distribution out of the air and call it her 'true' distribution in order to generate the additional preferences required by the Karni–Schmeidler method. There is no way to determine if she is, in fact, telling the truth.

A corollary of the preceding analysis is that one person's true subjective probabilities, even if they existed, would have little meaning to anyone else. It is not clear what should be done with them, even if somehow they could be reliably extracted. If we already knew the decision maker's preferences among acts, the revelation of her true probabilities would provide no additional predictive power with respect to her future behavior. Whereas, if we did not already know the decision maker's preferences among acts, the revelation of her probabilities would be useless without independent knowledge of her utilities and other personal data. For example, consider an encounter between two strangers in which one reveals that her true probability of an event is 0.3 and the other reveals that his true probability for the same event is 0.4. What should either of them do with this information? Should they revise their beliefs? Should they bet with each other? These questions cannot be answered without a great deal more information: utility functions, prior stakes, and likelihood functions describing how they expect to learn from each other. A third party, observing the two strangers, would be in the same quandary. How, if at all, should the third party characterize the 'consensus beliefs' of the two individuals? What should he advise them to do? The question of whether and how the 'true' probabilistic beliefs of different individuals should be combined is one of the most vexing problems in Bayesian theory.

I should hasten to point out that all of the above remarks about the indeterminacy of true probabilities in Savage's model apply to every other theory of choice under uncertainty that uses 'consequences' as a primitive, including horse-lottery theory, Choquet expected utility, maxmin expected utility, probabilistic sophistication, biseparable preferences, and quantifiable beliefs. All of these theories claim to determine a unique measure of pure belief from observations of preferences alone, but in fact, the uniqueness of the belief measure depends on the choice of a particular small world and on the arbitrary convention of scaling the cardinal utilities identically in all states. In other words, in all these theories, the uniqueness of the belief measure depends on what is counted as a 'constant of utility', which is outside the scope of the preference model. And the theories which do address this issue, developed by Professor Karni and his co-authors, do so by retreating to an even higher level of counterfactualism and non-materiality.

Thus, in any theory of choice under uncertainty whose fundamental measurements consist of revealed preferences among material acts – even counterfactual Savage acts – probabilities are inherently undetermined. True probability is a concept that makes sense only when it is regarded as a primitive (e.g., DeGroot, 1970), something that is understood without definition, evaluated by direct intuition, and revealed by unsubstantiated verbal reports. A decision maker's true probabilities are whatever she says they are, for what little that information may be worth.

3. WHAT DOES EXIST: PREVISION

Having stated the case for the non-existence (or at least, the non-materiality) of true probabilities, let me briefly give the definition and basic properties of previsions. Let $\boldsymbol{E}$ denote a real-valued uncertain quantity whose value in state i is E_i, and let $\boldsymbol{H}$ be the 0–1 indicator variable for an event. (Following de Finetti's convention, the same symbol will be used as a label for an uncertain quantity or event and for the vector whose elements are its values in different states.) Then your prevision for $\boldsymbol{E}$ given $\boldsymbol{H}$, denoted $\Pr(\boldsymbol{E}|\boldsymbol{H})$, is the marginal price you would pay for (small multiples of) a lottery ticket that yields $\$\boldsymbol{E}$ if $\boldsymbol{H}$ occurs and refunds the purchase price if $\boldsymbol{H}$ does not occur. In other words, you are willing to pay the certain amount $\alpha\Pr(\boldsymbol{E}|\boldsymbol{H})$ in exchange for the uncertain quantity $\alpha\boldsymbol{E}$, with the deal to be called off if $\boldsymbol{H}$ does not occur, where α is a 'small' number chosen at the discretion of an opponent. This transaction is a bet whose payoff vector (for you) is $\alpha(\boldsymbol{E} - \Pr(\boldsymbol{E}|\boldsymbol{H}))\boldsymbol{H}$. If complete preferences are assumed, α is permitted to be positive or negative – which means you are forced to name a marginal price at which you would indifferently buy or sell $\boldsymbol{E}$ given $\boldsymbol{H}$ – although this assumption is not essential. Under the more reasonable assumption of incomplete preferences, betting is voluntary and you may have distinct buying and selling prices (i.e., lower and upper previsions) $\underline{\Pr}(\boldsymbol{E}|\boldsymbol{H})$ and $\overline{\Pr}(\boldsymbol{E}|\boldsymbol{H})$. The key rule of composition imposed on such transactions is that bets which are individually acceptable should also be jointly acceptable. Thus, if $\Pr(\boldsymbol{E}|\boldsymbol{H})$ and $\Pr(\boldsymbol{F}|\boldsymbol{H})$ are your previsions for $\boldsymbol{E}$ and $\boldsymbol{F}$, respectively, then you should be willing to exchange $\alpha\Pr(\boldsymbol{E}|\boldsymbol{H}) + \beta \Pr(\boldsymbol{F}|\boldsymbol{H})$ for $\alpha\boldsymbol{E}+\beta\boldsymbol{F}$, conditional on $\boldsymbol{H}$, where α and β are chosen simultaneously by an opponent. This assumption is justified by the small size of the individual gambles, which renders it plausible that accepting one gamble does not affect your appetite for others, and it implies that previsions are additive: $\Pr(\alpha\boldsymbol{E}+\beta\boldsymbol{F}|\boldsymbol{H}) = \alpha\Pr(\boldsymbol{E}|\boldsymbol{H}) + \beta\Pr(\boldsymbol{E}|\boldsymbol{H})$, or in the case of incompleteness, $\underline{\Pr}(\alpha\boldsymbol{E}+\beta\boldsymbol{F}|\boldsymbol{H}) \geq \alpha\underline{\Pr}(\boldsymbol{E}|\boldsymbol{H}) + \beta\underline{\Pr}(\boldsymbol{E}|\boldsymbol{H})$.

De Finetti's **fundamental theorem of probability** states that your previsions are *coherent* – i.e., avoid sure loss, otherwise known as a Dutch book or arbitrage opportunity, at the hands of a clever opponent – if and only if there exists a distribution π on states such that for all uncertain quantities $\boldsymbol{E}$ and events $\boldsymbol{H}$, either $\Pr(\boldsymbol{E}|\boldsymbol{H}) =$

$\mathrm{P}_{\pi}(\boldsymbol{E}|\boldsymbol{H})$ or else $\mathrm{P}_{\pi}(\boldsymbol{H}) = 0$, where $\mathrm{P}_{\pi}(.)$ is the conditional expectation or probability induced by π. (With incomplete or only partially revealed preferences, there is a convex set Π of distributions such that for every π in Π, either $\underline{\mathrm{Pr}}(\boldsymbol{E}|\boldsymbol{H}) \leq \mathrm{P}_{\pi}(\boldsymbol{E}|\boldsymbol{H}) \leq \overline{\mathrm{Pr}}(\boldsymbol{E}|\boldsymbol{H})$ or else $\mathrm{P}_{\pi}(\boldsymbol{H}) = 0$.) Thus, having coherent previsions requires you to act *as if* you first assign probabilities (or sets of probabilities) to events and then determine the marginal prices you are willing to pay for lottery tickets on the basis of their expected values. This result follows from a separating hyperplane argument or, equivalently, from the duality theorem of linear programming. The primal characterization of rational betting is that you should avoid sure loss; the dual characterization is that, on the margin, you should behave like an expected value maximizer with respect to *some* probability distribution. A useful strengthening of de Finetti's theorem is obtained by applying the coherence criterion *ex post*, so that you should avoid suffering an actual loss without having had some compensating possibility of gain. By the duality argument, your previsions are ex post coherent only if the distribution π that rationalizes them also assigns positive probability to the state that was observed (Nau, 1995b), which is to say, if an event to which you assigned zero probability happens to occur, you either should have known better or acted otherwise.

Let me deal straight off with several objections that have been raised against de Finetti's method in the past – some of them by de Finetti himself. First, doesn't it require 'rigidity in the face of risk', i.e., linear utility for money? Actually, it does not. It requires only that preferences should be locally linear, i.e., 'smooth' in the payoffs, so that for gambles large enough to be taken seriously but small enough that only first-order valuation effects need to be considered, the additivity assumption is justified. Second, if previsions are based on local preferences but utility for money is globally nonlinear, then isn't $\mathrm{Pr}(\boldsymbol{E}|\boldsymbol{H})$ dependent on your current wealth? Yes it is: for generality, we should write $\mathrm{Pr}(\boldsymbol{E}|\boldsymbol{H}; \boldsymbol{w})$ to denote your prevision for $\boldsymbol{E}|\boldsymbol{H}$ in the vicinity of wealth distribution $\boldsymbol{w}$, and we should write $\pi(\mathbf{w})$ to denote the local supporting distribution on states. However, *this property is useful*: by measuring how $\pi(\boldsymbol{w})$ varies with $\boldsymbol{w}$, it is possible to characterize risk attitudes and determine optimal solutions of decisions and games (about which more will be said later). Third, isn't there a strategic element in the interaction between you and your opponent – e.g., might you not worry that the opponent has better information? Yes, but the strategic element is actually essential to the measurement process: it is the unavoidable interaction between the instrument and the object of measurement. Your previsions are revealed to a particular audience of observers, some of whom may be 'players' in your game. (It is especially appropriate to drop the completeness assumption in situations where strategic effects are likely to be strong, so that risks of exploitation can be neutralized by quoting bid-ask spreads rather than frictionless prices.) Fourth, if you change the monetary standard from dollars to (say) yen, won't this change your previsions? (Schervish et al., 1990) Yes it might – if the states refer to different exchange rates – but so what? The differences must be explainable by the exchange rates as a further condition of coherence, assuming that your opponent may convert currencies at market rates while betting with you. The suggestion that lack of standardization on money might make it hard for individuals to converse about beliefs and values is actually a powerful argument in favor of multinational currencies such as the euro, in which direction we are already headed.

Finally, there is an objection that has been raised against strict Bayesianism more generally: if tomorrow you observe $\boldsymbol{H}$ to be true while $\boldsymbol{E}$ remains uncertain, does the conditional prevision $\Pr(\boldsymbol{E}|\boldsymbol{H})$ that you hold today, which coherence requires to be equal to the ratio $\Pr(\boldsymbol{EH})/\Pr(\boldsymbol{H})$, necessarily become your 'posterior' prevision $\Pr(\boldsymbol{E}|\boldsymbol{H})$ after observing $\boldsymbol{H}$ to occur? In other words, is learning from experience nothing more than a mechanical application of Bayes theorem? [This point has been raised by Hacking (1967), among others.] The answer here is that, to the contrary, *temporal* coherence does not require that Bayes theorem should govern your actual learning over time, but rather only your *expected* learning. Realistically, beliefs evolve somewhat stochastically with the passage of time due to unforeseen or otherwise unmodeled events – or merely due to deeper introspection. Consequently, the posterior prevision $\Pr(\boldsymbol{E}|\boldsymbol{H})$ that you will hold tomorrow, after observing $\boldsymbol{H}$, may be an uncertain quantity when contemplated from today's vantage point, and as such it is an object of prevision in its own right: you may take bets today concerning the rate at which you will take bets tomorrow. Under these conditions, temporal coherence requires that the conditional prevision Pr(***E***|***H***) you hold today must equal your *prevision* of the posterior Pr(***E***|***H***) you will hold tomorrow, given ***H***. This is Goldstein's (1983, 1985) theorem on the prevision of a prevision, and it is related to the multiperiod version of the fundamental theorem of asset pricing (discussed below).

Now let me point out some obvious advantages of de Finetti's definition. First, previsions are measurable in units that are familiar to anyone, namely money, and they don't require the introduction of a contrived small world. Second, they have immediate implications for action, without independent knowledge of utilities. Third, under 'non-relativistic' conditions, which hold in a wide range of practical applications, previsions would correspond exactly to true probabilities anyway – if true probabilities existed. Fourth, as will be shown in more detail below, they can be defined even for individuals with very general non-expected utility preferences – e.g., preferences that don't necessarily satisfy the independence axiom. Fifth, they are easily aggregated across individuals. (If several individuals quote you different prices for the same lottery ticket, just take the highest buying price or the lowest selling price. If the prices are inconsistent, they can be reconciled by pumping out arbitrage profits.) And last but not least, previsions – rather than true probabilities – are the natural parameters of belief in economic models of choice under uncertainty.

Before defending the last claim, I would like to point out that previsions are already central to several other bodies of literature on the economics of uncertainty that do not start from Savage's model, namely the literatures of **state-preference theory** and **asset pricing**. There, they go by other names such as 'normalized state prices', 'risk neutral probabilities', or 'martingale measures'.

In the 1950's, contemporaneously with Savage, Arrow (1953/1964) and Debreu (1959) introduced the s*tate-preference* approach to modeling choice under uncertainty, in which consumers and investors trade bundles of money and commodities that are time- and date-stamped. For example, instead of just purchasing ice cream on a spot market, a consumer might purchase a claim to a gallon of ice cream to be delivered at noon tomorrow if the temperature is above 37 °C. The usual axioms of consumer theory (completeness, transitivity, continuity, non-satiation) imply that the individual has an *ordinal utility function U* defined over state-contingent commodities and money. Such an individual has well-defined previsions, and they are just the normalized partial derivatives of U with respect to state-contingent wealth, because the ratios of those partial derivatives are her marginal rates of substitution of wealth between states. Letting $\boldsymbol{w} = (w_1, \dots, w_n)$ denote the vector of wealth received in states 1 through n, define a probability distribution $\pi(\boldsymbol{w})$ by

$$\pi_j(\boldsymbol{w}) = \frac{U_j(\boldsymbol{w})}{\sum_{h=1}^{n} U_h(\boldsymbol{w})},$$

where $U_j(\boldsymbol{w})$ denotes the partial derivative $\partial U/\partial w_j$ evaluated at $\boldsymbol{w}$. If the economy is timeless, $\pi_j(\boldsymbol{w})$ is the price the individual is willing to pay for an Arrow–Debreu security that pays \$1 in state j and zero otherwise, which is the same as a de Finetti lottery ticket on state j. Hence, $\pi_j(\boldsymbol{w})$ is the prevision of state j when her wealth distribution is $\boldsymbol{w}$. Geometrically, the vector of local state-prices or state-previsions is just the gradient of the utility function at wealth $\boldsymbol{w}$, normalized so that its components sum to one.

Arrow and Debreu, and later authors such as Hirshleifer (1965) and Yaari (1969), showed that many basic properties of markets under uncertainty can be deduced by applying standard consumer theory to state- and time-contingent commodity bundles, without necessarily mentioning anyone's probabilities. For example, in an optimal solution to the consumer's investment problem, the ratio of her previsions for any two assets must equal the ratio of their market prices, and in a Pareto optimal allocation, the previsions of all agents must agree, and so forth.

A special case of the general state-preference model, which obtains when preferences satisfy an axiom of coordinate independence equivalent to Savage's sure-thing principle, is the case in which the utility function is *additive across states*,

$$U(\boldsymbol{w}) = v_1(w_1) + v_2(w_2) + \dots + v_n(w_n),$$

which is essentially state-dependent expected utility without unique determination of the probabilities (Wakker, 1989). Many economists who are, for one reason or another, uncomfortable with the subjective expected utility model have been quite happy to use the general state-preference model or the more restricted additive-across-states model. However, some of the state-preference literature does assume an underlying state-independent expected utility model (e.g., Hirshleifer and Riley, 1992). Under that assumption, the 45° line through the origin in payoff space assumes great significance: it represents the set of supposedly 'riskless' wealth positions, and along this line the indifference curves are all parallel with a slope that reveals the decision maker's true subjective probability distribution. That model treats sums of money as consequences in Savage's sense, and it has the same flaws: there is no way to verify that the utility of a particular sum of money is, in fact, the same in every state.

In the 1970's – the so-called 'golden age' of asset pricing theory – there was an explosion of interest among finance theorists in models of asset pricing by arbitrage. The key discovery of this period was the **fundamental theorem of asset pricing** (Ross, 1976; see also Dybvig and Ross, 1987; Duffie, 1996). The theorem states that there are no arbitrage opportunities in a financial market if and only if there exists a probability distribution with respect to which the expected value of every asset's future payoffs, discounted at the risk free rate, lies between its current bid and ask prices; and if the market is complete, the distribution is unique. This is just de Finetti's fundamental theorem of subjective probability, with discounting thrown in, although de Finetti is not usually given credit in the finance literature for having discovered the same result 40 years earlier. In finance theory, the probability distribution supporting the arbitrage-free asset prices is known as a *risk neutral probability distribution*, and, with a slight abuse of terminology, I will use that term interchangeably with *prevision* in the remainder of the paper. It is an abuse of terminology to the extent that, in the finance literature, risk neutral probabilities are usually attributed to markets rather than individuals, but every individual who posts her own prices is a micro-market, so we can speak of the risk neutral probabilities of the individual or the risk neutral probabilities of the market as long as the ownership is clear. In a multiperiod market which is complete and arbitrage-free, the discounted gain process of every asset is a *martingale* under the supporting risk neutral distribution (Cox and Ross, 1976; Harrison and Kreps, 1979). In other words, the price of any asset at any date in any state of the world must equal the conditional expectation of its discounted future value, according to the risk-neutral stochastic process. (This is a more general version of the theorem proved later by Goldstein concerning the prevision of a prevision.)

Risk neutral probabilities have an almost magical quality of adjusting for the relative amount of risk in an investment situation. Intuitively, the price of an asset depends on the parameters of its joint distribution with other assets – means, variances, covariances, etc. – as well as on the market price of risk. But once you know the risk neutral probabilities of the states – assuming market completeness – you can price any asset with a simple expected value calculation, ignoring all those unpleasant details. Of course, the market's risk neutral distribution is determined by the aggregated opinions of a large number of well-motivated investors who in their own ways have tried to take account of means, variances, covariances, etc., according to their own risk preferences. Risk neutral valuation methods are not really 'preference free', despite appearances. But in the end the collective beliefs and tastes of all the investors can be boiled down to a single distribution on the state space that prices all assets: that is the power of the no-arbitrage condition in a world where transactions are additive. Much of the literature of asset pricing, especially the pricing of derivative securities, is based entirely on arbitrage arguments, and as such has no need of anyone's true probabilities. True probabilities might be expected to serve some purpose in general equilibrium models of asset pricing, where the preference-aggregation problem is solved explicitly, and I will return to that subject below.

4. WHY TRUE PROBABILITIES ARE UNNECESSARY

I now come to the second, and perhaps more provocative, thesis of the paper, namely that true probabilities have no *essential* role to play in statistical and economic models of choice under uncertainty. By this I mean that in any theory which realistically allows for heterogeneity in beliefs and tastes among different individuals, it is neither *possible* to observe their true probabilities nor is it *necessary* to know their true probabilities for purposes of predicting or prescribing their behavior. True probabilities are not only non-material, they are immaterial. It suffices instead to know the mathematical properties of the individuals' previsions (risk neutral probabilities) or, equivalently, their marginal rates of substitution for wealth and commodities between different dates and states of the world. On the surface, this claim appears to contradict the fundamental tenets of a number of bodies of literature where subjective probabilities (or more general measures of pure belief such as Choquet capacities) have traditionally played starring roles, namely:

(1) Bayesian statistical inference;
(2) Bayesian decision analysis;
(3) Models of risk and uncertainty aversion;
(4) Noncooperative game theory;
(5) Models of markets under uncertainty.

In the remainder of the paper I will show that the need to know anyone's true probabilities in these bodies of literature is illusory.

It is appropriate to begin with **statistical inference**, because this is the area in which de Finetti's and Savage's work was grounded. Here the case is rather easy to make: in a typical Bayesian statistical inference problem, experimental data is summarized by a likelihood function that is used to update a prior distribution over exogenous states of the world. In most cases the experimental result is of no intrinsic interest to the statistician: it is merely an informational event, not a payoff-relevant event. To the extent that the statistician's previsions may be distorted by state-dependent marginal utilities for money, this distortion should affect only her prior distribution, not her likelihood function, because marginal utilities do not vary across outcomes of the experiment given the state of the world. Hence, to worry about the difference between previsions and true probabilities in this setting is tantamount to worrying about whether the statistician has the 'correct' prior distribution. From a strict subjectivistic viewpoint, that is an unreasonable and unnecessary invasion of her privacy, although not everyone defends the right to an idiosyncratic prior distribution. Bayesian statisticians have long been uncomfortable (and subject to criticism by non-Bayesians) over the role that the prior is supposed to play in their analyses, so in recent decades attention has shifted to 'robust' Bayesian methods that use 'diffuse' or 'objective' or set-valued priors, which do not dwell on the details of anyone's personal probabilities. However, the ultimate role of the prior distribution in Bayesian analysis is to provide a foundation for decision-making, and whether a true prior distribution or a risk neutral prior distribution is more useful will depend on the type of utility information that is available, about which more will be said below.

In Bayesian **decision analysis**, the decision maker (DM) is, of course, free to assess a probability distribution and a utility function for purposes of clarifying her own thinking, if she finds it helpful to do so. The relevant questions are (i) is it *necessary* for her to do so, and (ii) if she does so, how can she credibly reveal the parameters of her probability distribution and utility function to others? To address these questions, suppose that the decision maker does indeed have a subjective probability distribution and (possibly state-dependent) utility function. Let $\boldsymbol{p}$ denote the DM's true probability distribution:

$\boldsymbol{p}$ =	p_1	p_2	...	p_n

To compare two alternatives, say A and B, the minimal necessary utility information is the vector of *statewise utility differences*:

$\boldsymbol{u}_{AB}$ =	$u_{A1} - u_{B1}$	$u_{A2} - u_{B2}$	...	$u_{An} - u_{Bn}$

where $\boldsymbol{u}_{Ai}$ and $\boldsymbol{u}_{Bi}$ are the utilities of the consequences yielded by A and B in state i. It follows that A is preferred to B iff $\boldsymbol{p} \cdot \boldsymbol{u}_{AB} > 0$. Thus, $\boldsymbol{u}_{AB}$ is the coefficient vector of a linear constraint that the DM's true probabilities have to satisfy in order for her to prefer A to B. The articulation of such a constraint is a completely general way to encode utility information. In principle, the vector $\boldsymbol{u}_{AB}$ could be directly elicited – *if* true probabilities and utilities could be separated. The difficulty, as noted earlier, is that the DM's true probability distribution $\boldsymbol{p}$ and true utility difference vector $\boldsymbol{u}_{AB}$ are not uniquely determined by her preferences among material acts. If she satisfies all of Savage's axioms, there is *a* probability distribution that represents her preferences, in conjunction with a state-independent utility function, but it need not be her true distribution. There is, however, a way to decompose the DM's preferences in terms that are unique relative to a given monetary currency. Suppose that her marginal utilities for money are state- and decision-dependent, and let z_A and z_B denote her vectors of marginal utilities for money when alternatives A and B are chosen, respectively. The DM's risk neutral probabilities will then depend on her choice, being proportional to the products of her probabilities and marginal utilities. For example, if she imagined herself in possession of alternative A, the decision maker would bet on events at rates determined by a risk neutral distribution π_A in which her true probabilities are distorted by $\mathbf{z}_A$:

$$\pi_A \quad \propto \quad \boxed{p_1 z_{A1} \mid p_2 z_{A2} \mid \dots \mid p_n z_{An}}$$

Now consider $\hat{\boldsymbol{u}}_{AB}$ defined as the DM's vector of true utility differences *reciprocally* distorted by $\mathbf{z}_A$:

$$\hat{\boldsymbol{u}}_{AB} \quad = \quad \boxed{u_{AB1}/z_{A1} \mid u_{AB2}/z_{A2} \mid \dots \mid u_{ABn}/z_{An}}$$

where $u_{AB1} = u_{A1} - u_{B1}$, etc., as before. Notice that the elements of $\hat{\boldsymbol{u}}_{AB}$ are measured in dollars (or other currency): the utility units cancel out because u_{ABi} is measured in utiles while z_{Ai} is measured in utiles per dollar. By construction, the DM prefers A to B iff $\pi_A \cdot \hat{\boldsymbol{u}}_{AB} > 0$. Thus, $\hat{\boldsymbol{u}}_{AB}$ is the coefficient vector of a linear constraint that her *risk neutral probabilities given A* would have to satisfy in order for her to prefer A to B. (By focusing on B instead, the DM could assess corresponding vectors π_B and $\hat{\boldsymbol{u}}_{BA}$, although with only two alternatives this is unnecessary, since it would follow that $\pi_A \cdot \hat{\boldsymbol{u}}_{AB} = -\pi_B \cdot \hat{\boldsymbol{u}}_{BA}$.) Unlike $\boldsymbol{p}$ and $\boldsymbol{u}_{AB}$, both π_A and $\hat{\boldsymbol{u}}_{AB}$ are uniquely determined by preferences and are observable by other individuals. The representation of the DM's preferences in the latter terms embodies a certain kind of separation between subjective sources of value, but not a strict separation between probabilities of events and utilities for consequences. Rather, it is a separation between the effects of *information* about events and the effects of *control* over events. π_A is the risk neutral probability distribution that the DM would use for betting on events if she were forced to choose alternative A, possibly against her will. In principle, she could assess π_A by imagining that A is the only alternative available when contemplating bets on events. $\hat{\boldsymbol{u}}_{AB}$, on the other hand, is a monetary gamble that replicates the relative differences in utility the DM perceives between the natural consequences of A and B, when she is in possession of A. In principle, she could assess $\hat{\boldsymbol{u}}_{AB}$

by asking herself what gamble she would accept under any conditions where A would be chosen over B, regardless of any additional information that might be received in the meantime. These kinds of hypothetical questions, involving perturbations of information or control, may sound fanciful, but they are actually more concrete than hypothetical questions involving counterfactual Savage acts. Moreover, if the DM really knew her own preferences and didn't mind revealing them, it would be in her own financial interest to accept bets determined in this fashion.

The preceding method of 'decision analysis without true probabilities' has several other characteristic features. First, as might be expected, it is valid even under the additive-across-states utility representation without unique determination of probabilities. Second, it does not rely on the notion of consequences: value is attached directly to realizations of events and decisions. Third, the arbitrary and troublesome unit of utility never rears its head: measurements are expressed in dollars rather than utiles. Fourth, when the DM actually accepts bets consistent with π_A, $\hat{\boldsymbol{u}}_{AB}$, etc., her optimal decision is distinguished by the fact that it does not expose her to ex post arbitrage. Hence, this method alternatively can be described as 'decision analysis by arbitrage', and as such it is the natural extension of de Finetti's theory of coherent subjective probabilities – the ex post version – to a world in which choices as well as inferences are made. (More details are given in Nau, 1995b.)

When decision analysis takes place against the background of a complete market for contingent claims, the situation is even simpler: the optimal strategy for the decision maker is to choose the alternative that maximizes expected net present value – when discounting is performed at the risk free rate and expectations are taken with respect to the market's risk neutral distribution – and simultaneously to hedge her risks by trading securities in the market so as to equilibrate her own risk neutral probabilities with those of the market, in the context of the chosen alternative (Nau and McCardle, 1991). Thus, the optimal alternative can be determined by options-pricing methods that do not involve subjective probabilities, while the optimal risk-hedging trades can be determined by buying or selling assets according to whether the DM's own marginal prices are greater or less than current market prices after she has chosen that alternative. If the discrepancies are small, the latter trades can be determined from a simple formula involving the derivatives of the DM's risk neutral probabilities with respect to wealth (Nau 2001a). If the market is only 'partially complete', the decision tree rollback procedure can be modified to incorporate conventional certainty-equivalent calculations at 'private' event nodes, using the decision maker's true probabilities under a restricted-preference assumption of time-additive exponential utility (Smith and Nau, 1995). Under the same restriction, however, the calculations at private event nodes can be performed equally well in terms of risk neutral probabilities. At any event node in a decision tree, the expected utility is the weighted arithmetic mean of the branch utilities, using true probabilities as weights. Under exponential utility, curiously enough, the expected utility is equal to the weighted *harmonic* mean of the branch utilities when risk neutral probabilities are used as weights. Consequently, the respective formulas for the certainty equivalent at an event node with payoffs $\boldsymbol{x}$ and risk tolerance t differ only by a pair of minus signs:

$$CE = -t \ln\left(\sum_{i=1}^{n} p_i \exp(-x_i/t)\right) = t \ln\left(\sum_{i=1}^{n} \pi_i \exp(x_i/t)\right).$$

The characterization of **aversion to risk and uncertainty** appears, on the surface, to require knowledge of true probabilities. A decision maker is often defined to be risk averse if the amount she is willing to pay for a risky asset is less than its expected value or if she dislikes mean-preserving spreads of payoff distributions. Such definitions assume that the true probability distribution of the asset is given, either objectively or subjectively. Pratt's (1964) risk premium formula also appears to depend on knowledge of true probabilities: the risk premium of an asset is proportional to the product of the local risk aversion measure and the asset's true variance. Some definitions of aversion to uncertainty (e.g., Epstein, 1999) use probabilistically sophisticated behavior as a benchmark, in which the decision maker is permitted to violate the independence axiom but is nevertheless assumed to act as if she assigns definite probabilities to events (Machina and Schmeidler, 1992). However, risk and uncertainty aversion can be defined and measured equally well in terms of risk neutral probabilities, because the decision maker's attitudes toward risk and uncertainty are revealed by the functional dependence of her risk neutral probabilities on her state-contingent wealth. Yaari (1969) gives a simple definition of risk aversion in the context of state-preference theory, namely that *a decision maker is risk averse if her preferences for state-contingent wealth are convex*, which is to say, her ordinal utility function is quasi-concave. Under this definition, the decision maker is risk averse if and only if the amount she is willing to pay for a risky asset is always less than its *risk neutral* expected value, using the risk neutral probability distribution determined by her current wealth position. It also turns out that the 'correct' variance to use in Pratt's risk premium formula, when prior wealth is stochastic or utility is state-dependent, is the local risk neutral variance rather than the true variance. (It just so happens that the two variances coincide under Pratt's assumptions of state-independent utility and non-stochastic prior wealth.) If the decision maker has smooth non-expected-utility preferences, her local aversion to risk *and* uncertainty is measured more generally by the matrix of derivatives of her risk neutral probabilities with respect to changes in wealth (Nau, 2001a,b).

The theory of **non-cooperative games** is the branch of economic theory in which the strict separation of probability from utility is asked to bear the most weight. Following von Neumann and Morgenstern, the rules of the game (i.e., the payoff functions of the players) are conventionally expressed in units of pure utility and are assumed to be common knowledge, nothwithstanding the theoretical difficulties of measuring pure utility even under non-strategic uncertainty. In games of incomplete information, where there is also uncertainty about exogeneous events, the true probabilities of the players are assumed to be consistent with a common prior distribution. A solution of a game, insofar as it may involve mixed strategies, is expressed in terms of the true probabilities that the players assign to each other's moves. Not coincidentally, the common knowledge assumptions, common prior assumption, and solution concepts have been subject to a good deal of criticism by decision theorists (e.g., Kadane and Larkey, 1982, 1983; Sugden, 1991), who question whether Savage's axioms are at all applicable in a strategic environment. Even leaders in the field of game theory are circumspect about its normative or predictive power, preferring to describe the theory as a 'language' in which to discuss the possible varieties of strategic interaction. Models of games that are played for money by risk-neutral players are generally considered more believable than games in which the reciprocal measurement of utility is an issue. Solutions that involve pure strategies – or even better, dominant strategies – are more believable than those that depend on delicate equilibrium reasoning or calculations of mixture probabilities. The common prior assumption is openly rejected by some theorists (e.g., Kreps, 1990) and reduced to tautological status by others (e.g., Myerson, 1991) on the very grounds that subjective probabilities are arbitrary when utilities are state-dependent.

There is, however, a way to recast non-cooperative game theory in terms that do not require the separation of true probability from utility and, indeed, do not require the introduction of any additional rationality postulates beyond those that suffice to model non-strategic behavior. Suppose that the rules of the game are revealed through public gambling – in particular, through the acceptance of gambles such as $\hat{\boldsymbol{u}}_{AB}$ that reveal utility differences in monetary terms. Then, as a standard of strategic rationality, require only that the play of the game against the background of those gambles should not yield an ex post arbitrage profit to an outside observer (Nau and McCardle, 1990; Nau, 1992, 1995c). This approach to defining common knowledge and rational play is merely a multi-player implementation of decision–analysis-by-arbitrage, and it leads to a fundamental duality theorem which states that an outcome of the game is arbitrage-free (ex post coherent) if and only if it occurs with positive probability in a *correlated equilibrium*, a possibly-correlated generalization of Nash equilibrium originally proposed by Aumann (1974, 1987). In the general case where players may have non-linear utility for money or otherwise state-dependent utility, the common prior and the correlated equilibrium distribution are risk neutral probabilities rather than any player's true probabilities, harmonizing the fundamental equilibrium concept of games with the 'arbitrage intuition' of financial markets (Ross, 1987). The reinterpretation of the common prior distribution as a risk neutral distribution also resolves the mystery of 'no-trade' theorems (Nau 1995a).

Notwithstanding the views of Keynes (1936) concerning ‘animal spirits’, the theory of **markets under uncertainty** might be expected to be the arena in which true probabilities would be indispensable to economic analysis: the choices are well-structured, the financial stakes are high, and empirical frequency data are abundant. But even here, the role of true probabilities remains elusive. As has already been noted, models of asset pricing by arbitrage do not depend in any way on knowledge of investors’ true probabilities, although in a sense they are not completely general: they merely determine the consistency conditions that must be satisfied by prices of primary and derivative securities. More ambitious general equilibrium models attempt to explain how the prices of all securities (i.e., the risk neutral probabilities of the market) are determined by the beliefs and tastes of all investors. The most basic general equilibrium models – Arrow–Debreu equilibrium and Radner equilibrium – invoke the state-preference framework. Investors are endowed with utility functions over state-contingent consumption plans, and they form expectations expressed in terms of prices that will prevail in different states of the world at different times. Probabilities are still nowhere to be found. (See, for example, the presentation of these models in Mas-Colell et al. 1995.) In some equilibrium models, an expected-utility representation of preferences is assumed for analytic convenience, but if the representation allows state-dependent utilities, as it often does, the probabilities remain arbitrary. To be sure, there are models in which strong homogeneity restrictions are imposed on expected-utility preferences – for example, the investors may be assumed to agree on probability distributions of returns and to have similar utility functions, or the ‘fundamental’ variables of the economy may be assumed to follow an objective stochastic process – but such models assume away the core issues of subjective expected utility along with most of the incentives for trade that exist in real markets.

The heterogeneous-expectations capital asset pricing model (CAPM) illustrates the theoretical difficulties of distinguishing the effects of true probabilities in a market under uncertainty, even in a very simple setting where investors are assumed to be state-independent expected-utility maximizers. Consider a two-date economy in which there are K primary risky securities and one riskless security, as well as a complete market for contingent claims. Investors trade primary and derivative securities at time 0 and receive payoffs at time 1, and all consumption occurs at time 1. Suppose that investors have heterogeneous multivariate normal subjective probability distributions for the values of the primary securities at time 1, as well as heterogeneous exponential utility functions for consumption (i.e., constant absolute risk aversion) and arbitrary initial wealth. It is straightforward to determine the unique equilibrium allocation of risky assets that will prevail at time 0, and the solution has the following properties (Nau and McCardle, 1991). First, each investor holds a portfolio consisting of primary securities and *quadratic options*, the latter being derivative securities whose payoffs are proportional to the squares or pairwise products of the payoffs of the primary securities. The role of the quadratic options in the investors' portfolios is to equalize the covariance matrices of their risk neutral probability distributions, while the role of the primary securities in their portfolios is to equalize the means of their risk neutral distributions. (In equilibrium, every investor must have the same risk neutral distribution, and the mean return on every asset under the common risk neutral distribution must equal the risk free rate.) Second, every investor perceives that the CAPM equation applies to *her* portfolio of primary securities according to *her* true probabilities. That is, every investor perceives that her true excess return on a primary security is equal to the excess return on her primary portfolio multiplied by the 'beta' of the security, i.e., its covariance with her primary portfolio divided by the variance of her primary portfolio. Thus, everyone agrees that the CAPM applies, but they all disagree on its parameters. Third, the CAPM equation also applies to the market portfolio – i.e., the sum of the investors' portfolios of primary securities – when excess returns and betas are calculated from an aggregate probability distribution in which the investors are weighted according to their risk tolerances. Fourth, by observing the portfolio held by an investor, it is impossible to separate the two most important parameters of her beliefs and tastes, namely her true expected return on her primary portfolio and her risk tolerance. Since the excess returns on individual securities are proportional to the excess return on the primary portfolio under the CAPM formula, the true expected returns are jointly indeterminate.

Hence, it is impossible to determine the true excess returns that an investor expects she will earn on any investments without independently knowing her risk tolerance, and vice versa. The same is true at the level of the market: by observing the market portfolio and market prices (which are summarized by the observable risk neutral distribution), it is impossible to determine the true aggregate excess returns on securities without an independent estimate of the total risk tolerance of all investors. Even under such highly idealized and simplified conditions, it is impossible to recover the true probabilities of investors, either individually or in the aggregate, from a snapshot of portfolios and prices taken on any given date: their beliefs and values are inseparable. In particular, it is impossible to determine the *first moment* (mean) of anyone's true distribution of returns on any portfolio, although the second moment (covariance matrix) of the true aggregate distribution is identical to that of the market risk neutral distribution because the quadratic options exist in zero net supply. (It is theoretically possible to recover both investor beliefs and tastes when consumption is observed continuously over time, rather than at a single terminal date, but only under very restrictive additional assumptions: a single risky asset and a single representative investor with time-additive utility and a constant felicity function. See Cuoco and Zapatero, 2000.)

It might appear as though time series data could be used to obtain estimates of true probabilities of security returns, on the (optimistic) hypothesis that investors' aggregate beliefs are calibrated with historical frequencies. Alas, this too turns out to be hard, and the results obtained to date are rather anomalous. For one thing, only the covariances of returns can be estimated with any precision. The estimation of mean returns on individual securities is practically impossible because the signal-to-noise ratio is so low and because mean returns might be expected to change over intervals of time that are sufficiently long to yield precise estimates: the so-called 'blur of history' problem (Leuenberger, 1998). Hence, the most elusive parameters of the true probability distribution – the first moments – are still undetermined, which is unfortunate for empirically minded investors because their primary portfolio weights under the CAPM are linearly dependent on estimated mean returns. (Nature may be trying to tell us something by this.) Moreover, when securities are aggregated and observed over decades-long horizons, their excess returns are too great to be explained by plausible degrees of risk aversion under an expected-utility model of investor preferences, a result known as the equity premium puzzle. Efforts have been made to estimate investor risk aversion from risk neutral probabilities combined with time series data – with empirical mean returns replaced by estimates keyed to the risk free rate – and here too, the results are puzzling: in the post-1987-crash period, the inferred investor risk aversion is negative in the vicinity of status quo wealth, i.e., the center of the risk-neutral distribution is right-shifted rather than left-shifted relative to the assumed true distribution (Jackwerth, 2000). The market's risk neutral distribution seems to have a character of its own that cannot easily be factored into probabilities and marginal utilities of expected-utility-maximizing investors.

5. CONCLUDING COMMENTS

For the last 50 years, researchers following in the footsteps of von Neumann–Morgenstern and Savage have attempted to banish money from the foundations of decision theory in order to isolate the true probabilities and utilities of the decision maker. This paper has questioned both the possibility and necessity of carrying out that project. Why has so much effort and ingenuity been expended on it? Several arguments can be adduced, but none appears compelling.

First, it might be argued that we should separate true probabilities from utilities because it is *normative* to distinguish beliefs from values. Beliefs and values are commonly regarded as having different subjective sources – namely, information and tastes – and by not distinguishing them we could fall prey to errors of judgment or failures of communication. But this argument begs the question: what kinds of quantitative or qualitative judgments are really primitive in the human mind? If preferences are primitive, then perhaps we shouldn't try so hard to decompose them. If, on the contrary, preferences are constructed from more primitive beliefs and values, then perhaps we should devise separate theories and measurement schemes for beliefs and values, rather than trying to distill them from preferences. But the fact that we have *linguistic* concepts of belief and value does not necessarily entail that mathematical decision models should have real-valued parameters that correspond exactly to them. We also have a linguistic concept of risk, but it has turned out not to have a unidimensional scale of measurement. In everyday language we may speak of the 'amount of risk' in a situation, but the expression has diverse meanings, and in risk theory there is no universal measure of risk.

A second possible argument is that, in order to *predict* behavior under uncertainty, it is necessary to impose external validity assumptions on the true probabilities of different individuals – e.g., that they should hold the correct probabilities given their information, or that they should hold mutually consistent prior probabilities, or that their probabilities should be calibrated with historical frequencies. But assumptions of this character are so dubious in their realism that it is hard to see how they could improve predictions. Situations in which 'correct' probabilities can be identified are usually not economically interesting. The same can be said of situations in which it would be reasonable to assume mutually consistent true probabilities, i.e., a common prior distribution. If true probabilities are not revealed by material behavior, it is difficult to see how different individuals would ever come to agree on them except in trivial cases. The assumption that subjective probabilities should be calibrated with empirical frequencies places unreasonable demands on boundedly-rational individuals to have correct mental models of the complex economic system in which they are small cogs.

Third, it might be argued that probabilities and utilities should be separated because it is *analytically convenient*, permitting a fairly rich (if not necessarily behaviorally realistic) class of preferences to be modeled with a minimum of parameters while allowing the full toolkit of probability theory to be employed. This is a fine argument, but it does not require Savage's counterfactual house of cards. Under non-relativistic conditions – i.e., no prior financial stakes or intrinsic state-dependence of utility – we can simply use de Finetti's method and define subjective probabilities as betting rates for money. Under more general conditions, we can adopt the state-preference framework of Arrow–Debreu and then parameterize the utility function in whatever form is appropriate for the application at hand, merely taking care to not attach too much significance to the true probabilities if an expected-utility representation is chosen.

It is worth returning to the original question: why *shouldn't* money, the old-fashioned kind, play a distinguished, primitive role in quantitative decision theory? The evolutionary record suggests that money is indispensable as a yardstick when subjective judgments of belief and value must be expressed in terms that are credible, numerically precise, and commonly understood. For better or worse, the importance of money as a lingua franca in human affairs only seems to be increasing as national currencies coalesce into multinational currencies, as the last bastions of central planning surrender to market forces, as new market mechanisms are enabled by electronic media, and as individuals peg their personal fortunes to stock index funds and tune in to financial news throughout the day. On a deep level, it could well be easier for an individual to subjectively assign a fair price than to assign a true probability, perhaps because comparing prices is a more familiar and concrete activity than comparing likelihoods or abstract acts, or perhaps because it invokes different cognitive processes better suited to fine quantitative discrimination.

6. ACKNOWLEDGEMENTS

This paper is based on research supported by the National Science Foundation under grant 98-09225 and by the Fuqua School of Business. I am grateful to participants at FUR X for helpful comments.

REFERENCES

Anscombe, F.J. and Aumann, R.J. (1963), A definition of subjective probability, *Annals of Mathematical Statistics* 34, 199–205.

Arrow, K.J. (1964), The role of securities in the optimal allocation of risk-bearing, *Quarterly Journal of Economics* 31, 91–96.

Aumann, R.J. (1971), Letter to L.J. Savage. Reprinted in J. Drèze (1987).

Aumann, R.J. (1974), Subjectivity and correlation in randomized games, *Journal of Mathematical Economics*, 1, 67–96.

Aumann, R.J. (1987), Correlated equilibrium as an expression of Bayesian rationality, *Econometrica* 55, 1–18.

Cox, J. and Ross, S.A. (1976), The valuation of options for alternative stochastic processes, *Journal of Financial Economics* 3, 145–166.

Cuoco, D. and Zapatero, F. (2000), On the recoverability of preferences and beliefs, *Review of Financial Studies* 13(2), 417–432.

de Finetti, B. (1937), La prévision: ses lois logiques, ses sources subjectives, *Ann. Inst. Henri Poincaré* 7, 1–68. Translation reprinted in H.E. Kyburg and H.E. Smokler (eds.) (1980), *Studies in Subjective Probability*, 2nd edn (pp. 53–118). New York: Robert Krieger.

de Finetti, B. (1974), *Theory of Probability*, Vol. 1. New York: John Wiley and Sons.

Debreu, G. (1959), *Theory of Value*. New Haven, CT: Cowles Foundation.

DeGroot, M. (1970), *Optimal Statistical Decisions*. New York: McGraw-Hill.

Drèze, J. (1987), *Essays on Economic Decision under Uncertainty*. London: Cambridge University Press.

Duffie, D. (1996), *Dynamic Asset Pricing Theory*. Princeton, NJ: Princeton University Press.

Dybvig, P. and Ross, S.A. (1987), Arbitrage. In J. Eatwell, M. Milgate & P. Newman (eds.), *The New Palgrave*. New York: Norton.

Epstein, L. (1999), A definition of uncertainty aversion, *Review of Economic Studies* 66, 579–608.

Ghirardato, P. & Marinacci, M. (2002), Ambiguity made precise: A comparative foundation, *Journal of Economic Theory* 102(2), 251–289.

Gilboa, I. and Schmeidler, D. (1989), Maxmin expected utility with nonunique prior, *Journal of Mathematical Economics* 18(2), 141–153.

Goldstein, M. (1983), The prevision of a prevision, *Journal of the American Statistical Association* 78, 817–819.

Goldstein, M. (1985), Temporal coherence (with discussion). In J. Bernardo, M. DeGroot, D. Lindley and A. Smith (eds.), *Bayesian Statistics 2: Proceedings of the Second Valencia International Meeting* (pp. 231–248). Amsterdam: North-Holland, Amsterdam.

Grant, S. and Karni, E. (2000), A theory of quantifiable beliefs. Johns Hopkins University, Economics Department, Working paper

Hacking, I. (1967), Slightly more realistic personal probability, *Philosophy of Science* 34, 311–325.

Harrison, J. and Kreps, D. (1979), Martingales and Arbitrage in Multiperiod Securities Markets. *Journal of Economic Theory* 2(3), 381–408.

Hirshleifer, J. (1965), Investment decision under uncertainty: Choice-theoretic approaches, *Quarterly Journal of Economics* 74, 509–536.

Hirshleifer, J. & Riley, J.G. (1992), *The Analytics of Uncertainty and Information*. Cambridge: Cambridge University Press.

Jackwerth, J. (2000), Recovering risk aversion from option prices and realized returns, *Review of Financial Studies* 13(2), 433–452.

Kadane, J.B. and Larkey, P.D. (1982), Subjective probability and the theory of games, and reply, *Management Science* 28, 113–120, 124.

Kadane, J.B. and Larkey, P.D. (1983), The confusion of Is and Ought in game theoretic contexts, *Management Science* 29, 1365–1379.

Kadane, J.B. and Winkler, R.L. (1988), Separating probability elicitation from utilities, *Journal of the American Statistical Association* 83, 357–363.

Karni, E. (1985), *Decision Making under Uncertainty: The Case of State-Dependent Preferences*. Cambridge, MA: Harvard University Press.

Karni, E. and Mongin, P. (2000), On the determination of subjective probability by choices, *Management Science* 46, 233–248.

Karni, E. and Safra, Z. (1995), The impossibility of experimental elicitation of subjective probabilities, *Theory and Decision* 38, 313–320.

Karni, E. and Schmeidler, D. (1981), An expected utility theory for state-dependent preferences. Tel-Aviv University, Foerder Institute for Economic Research, Working Paper 48–80.

Karni, E., Schmeidler, D. and Vind, K. (1983), On state dependent preferences and subjective probabilities, *Econometrica* 51, 1021–1031.

Keynes, J.M. (1936), *The General Theory of Employment, Interest, and Money*. London: McMillan.

Kreps, D. (1990), *Microeconomic Theory*. Princeton, NJ: Princeton University Press.

Luenberger, D. (1998), *Investment Science*. New York: Oxford University Press.

Machina, M.J. and Schmeidler, D. (1992), A more robust definition of subjective probability, *Econometrica* 60, 745–780.

Machina, M.J. (2001), Almost-objective uncertainty. University of California at San Diego, Economics Department, Working Paper.

Mas-Colell, A., Whinston, M. and Green, J. (1995), *Microeconomic Theory*. New York: Oxford University Press.

Myerson, R. (1991), *Game Theory: Analysis of Conflict*. Cambridge, MA: Harvard University Press.

Nau, R.F. (1992), Joint coherence in games of incomplete information, *Management Science* 38(3), 374–387.

Nau, R.F. (1995a), The incoherence of agreeing to disagree, *Theory and Decision* 39, 219–239.

Nau, R.F. (1995b), Coherent decision analysis with inseparable probabilities and utilities, *Journal of Risk and Uncertainty* 10, 71–91.

Nau, R.F. (1995c), Arbitrage-free correlated equilibria. Fuqua School of Business, Working paper.

Nau, R.F. (2001a), A generalization of Pratt-arrow measure to non-expected-utility preferences and inseparable probability and utility. Fuqua School of Business, Working paper.

Nau, R.F. (2001b), Uncertainty aversion with second-order probabilities and utilities. Fuqua School of Business, Working paper.

Nau, R.F. and McCardle, K.F. (1990), Coherent behavior in noncooperative games, *Journal of Economic Theory* 50(2), 424–444.

Nau, R.F. and McCardle, K.F. (1991), Arbitrage, rationality, and equilibrium, *Theory and Decision* 31, 199–240.

Pratt, J.W. (1964), Risk aversion in the small and in the large, *Econometrica* 32, 122–136.

Ramsey, F. (1931), Truth and probability. In *Foundations of Mathematics and other Logical Essays*. London: K. Paul, Trench, Trubner and Co. Reprinted in H.E. Kyburg and H.E. Smokler (eds.) (1980), *Studies in Subjective Probability*, 2nd edn (pp. 25–52) New York: Robert Krieger.

Ross, S. (1976), Return, risk and arbitrage. In: I. Friend and J. Bicksler (eds.), *Studies in Risk and Return.* Cambridge, MA: Ballinger.

Ross, S. (1987), The interrelations of finance and economics: Theoretical perspectives, *American Economic Review* 77(2), 29–34.

Savage, L.J. (1954), *The Foundations of Statistics*. New York: John Wiley and Sons (second edition, 1972, New York: Dover).

Schervish, M.J., Seidenfeld, T. and Kadane, J.B. (1990), State-dependent utilities, *Journal of the American Statistical Association* 85, 840–847.

Schmeidler, D. (1989), Subjective probability and expected utility without additivity, *Econometrica* 57, 571–587.

Shafer, G. (1986), Savage revisited (including comments), *Statistical Science* 1, 463–501.

Smith, J.E. and Nau, R.F. (1995), Valuing risky projects: Option pricing theory and decision analysis, *Management Science* 41(5), 795–816.

Sugden, R. (1991), Rational choice: A survey of contributions from economics and philosophy, *Economic Journal* 101, 751–785.

von Neumann, J. and Morgenstern, O. (1947), *Theory of Games and Economic Behavior*. Princeton, NJ: Princeton University Press.

Wakker, P.P. (1989), *Additive Representations of Preferences: A New Foundation of Decision Analysis*. Dordrecht/Boston/London: Kluwer Academic Publishers.

Yaari, M. (1969), Some remarks on measures of risk aversion and their uses, *Journal of Economical Theory* 1, 315–329.

Addresses for correspondence: Robert F. Nau, Fuqua School of Business, Duke University, Durham, NC 27708-0120, USA Phone: +1-919-660-7763 (office); E-mail: robert.nau@duke.edu