Uncertainty Aversion with Second-Order Probabilities and Utilities

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May 28, 2002

Abstract

Uncertainty aversion is often conceived as a local, first-order effect that is associated with kinked indifference curves (i.e., non-smooth preferences), as in the Choquet expected utility model. This paper shows that uncertainty aversion can arise as a global, second-order effect when the decision maker has smooth preferences within the state-preference framework of choice under uncertainty, as opposed to the Savage or Anscombe-Aumann frameworks. Uncertainty aversion is defined and measured in direct behavioral terms without reference to probabilistic beliefs or consequences with state-independent utility. A simple axiomatic model of “partially separable” non-expected utility preferences is presented, in which the decision maker satisfies the independence axiom selectively within partitions of the state space whose elements have similar degrees of uncertainty. As such, she may behave like an expected-utility maximizer with respect to assets in the same uncertainty class, while exhibiting higher degrees of risk aversion toward assets that are more uncertain. An alternative interpretation of the same model is that the decision maker may be uncertain about her credal state (represented by second-order probabilities for her first-order probabilities and utilities), and she may be averse to that uncertainty (represented by a second-order utility function). The Ellsberg and Allais paradoxes are explained by way of illustration. Keywords: risk aversion, uncertainty aversion, ambiguity, non-additive probabilities, state-preference theory, Choquet expected utility, Ellsberg’s paradox.
1. Introduction

The axiomatization of expected utility by von Neumann-Morgenstern and Savage hinges on the axiom of independence, which requires preferences to be separable across mutually exclusive events and leads to representations by utility functions that are additively separable across states of the world. A strong implication of the independence axiom is that preferences among acts do not depend on qualitative properties of events but rather only on the sum of the values attached to their constituent states, which in turn depend only on the probabilities of the states and the consequences to which they lead. If the set of states of nature can be partitioned in two or more ways, the decision maker is not permitted to display uniformly different risk attitudes toward acts that are measurable with respect to different partitions, because the events in any two partitions are composed of the same states, merely grouped in different ways. There is considerable empirical evidence that individuals violate this requirement in certain kinds of choice situations. One example is provided by Ellsberg’s 2-color paradox, in which subjects consistently prefer to bet on unambiguous events rather than ambiguous events, even when they are otherwise equivalent by virtue of symmetry, a phenomenon known as uncertainty aversion. Other violations of independence are provided by Ellsberg’s 3-color paradox and Allais’ paradox, in which subjects’ preferences between two acts that agree in some events depend on how they agree there, as though there were complementarities among events.

A variety of models of non-expected utility have been proposed to accommodate violations of the independence axiom, and most of them do so by positing that the decision maker has non-probabilistic beliefs or that her preferences depend nonlinearly on probabilities. The Choquet expected utility model, in particular, assumes that the decision maker tends to overweight events leading to inferior payoffs by applying non-additive subjective probabilities
derived from a Choquet capacity. (Schmeidler 1989) The ranking of states according to the payoffs to which they lead thus plays a key role in the representation of uncertainty aversion: the decision maker violates the axioms of expected utility theory only when faced with choices among acts whose payoffs induce different rankings of states. Within sets of acts whose payoffs are comonotonic, the decision maker’s preferences have an ordinary expected-utility representation. Schmeidler derives the Choquet model within the Anscombe-Aumann framework of choice under uncertainty, which includes objective probabilities in the composition of acts, and he defines uncertainty aversion in this framework as convexity with respect to mixtures of probabilities. That is, an uncertainty-averse individual who prefers \( f \) to \( g \) also prefers \( \alpha f + (1-\alpha)g \) to \( g \) for \( 0<\alpha<1 \), where the mixture operation applies to the objective probabilities with which acts \( f \) and \( g \) lead to various consequences. Another way to view the Choquet model is to note that it implies that the decision maker has indifference curves in payoff space that are kinked in particular locations, namely at the boundaries between comonotonic sets of acts. If the decision maker’s status quo wealth happens to fall on such a kink—which is a set of measure zero—her local preferences (i.e., her preferences for “neighboring” acts) will display uncertainty aversion, otherwise they will not.

Epstein (1999) takes a different approach, in which uncertainty aversion is defined as a departure from probabilistically sophisticated behavior (Machina and Schmeidler 1992). A probabilistically sophisticated decision maker may have non-expected-utility preferences but nevertheless has an additive subjective probability measure that is uniquely determined by preferences. Epstein (following Machina and Schmeidler) uses the Savage-act modeling framework, in which objects of choice are mappings from states to consequences whose utilities are a priori state-independent. There are no objectively mixed acts; rather, Epstein’s definition
of uncertainty aversion assumes the existence of a set of special acts that are a priori *unambiguous*. A preference order \( \succeq_1 \) is defined to be *more uncertainty averse* than preference order \( \succeq_2 \) if, for all acts \( w \) and all unambiguous acts \( y \), \( y \succeq_1 w \) \((y \succeq_1 w)\) whenever \( y \succeq_2 w \) \((y \succeq_2 w)\). In other words, order 1 is more uncertainty averse than order 2 if 1 never chooses an ambiguous act over an unambiguous act when 2 would not. A preference order is then defined to be *uncertainty averse* if it is *more* uncertainty averse than some *probabilistically sophisticated* preference order. Epstein and Zhang (2001) extend the same approach to define the set of unambiguous events endogenously: an unambiguous event is one that does not give rise to preference reversals when a common consequence assigned to that event is replaced by another common consequence.

Ghirardato and Marinacci (2001, 2002) give a definition of uncertainty aversion that, like Epstein’s, is based on a definition of comparative uncertainty aversion but is more general in the sense that it does not assume the existence of acts that are a priori unambiguous, aside from the constant acts that play a key role in Savage’s framework. However, Ghirardato and Marinacci’s definition imposes the additional restriction that preferences should be *biseparable*—i.e., that when choices are restricted to binary gambles, preferences should be represented by a utility function of the form \( V(xAy) = u(x)\rho(A) + u(y)(1-\rho(A)) \), where \( u \) is a cardinal utility function for money and \( \rho \) is a “willingness to bet” measure. Uncertainty aversion is exhibited when \( \rho \) is nonadditive, i.e., \( \rho(A) + \rho(\bar{A}) < 1 \).

The preceding definitions of uncertainty aversion are motivated by the Choquet expected utility model and related models in which uncertainty aversion is associated with kinked indifference curves—thus, uncertainty aversion is conceived as a *first-order* rather than second-order effect (Segal and Spivak 1990)—and they all require a priori definitions of riskless...
(constant) acts in the tradition of Savage and Anscombe-Aumann. The aim of the present paper is to consider, instead, situations in which (i) there are no a priori riskless acts or consequences whose utilities are state-independent, (ii) the decision maker has smooth preferences so that both risk aversion and uncertainty aversion are second-order effects, and (iii) the decision maker may be locally uncertainty averse at all wealth positions, not merely at “special” wealth positions. The definition of uncertainty aversion given here is somewhat similar in spirit to that of Ghirardato and Marinacci, insofar a decision maker is considered uncertainty averse if she is “more uncertainty averse” than a decision maker with separable preferences. However, unlike Schmeidler, Epstein, and Ghirardato/Marinacci, we make no attempt to uniquely separate beliefs from state-dependent cardinal utilities, on the grounds that such a separation is impossible to achieve in practice and is not necessary for purposes of economic modeling or decision analysis (Nau 2002). Roughly speaking, a decision maker will be defined here to be uncertainty averse if she is systematically more risk averse toward acts that are “more uncertain” in a manner that cannot be explained by a state-dependent Pratt-Arrow measure of risk aversion. For example, she might be risk neutral—or even locally risk seeking—toward casino gambles, moderately risk averse toward investments in the stock market, and highly risk averse when insuring against health or property risks.

The organization of the paper is as follows. Section 2 introduces the basic mathematical framework, and section 3 defines risk aversion and uncertainty aversion for a decision maker with smooth preferences. Section 4 gives a simple example of a utility function defined on a 4-element state space that rationalizes Ellsberg’s 2-color paradox. Section 5 presents a more detailed and general version of the same model and the axioms of “partially separable preferences” on which it rests. Section 6 presents an alternative interpretation of the model in
terms of second-order uncertainty about credal states, illustrated by a model of Ellsberg’s 3-color paradox as well as the 2-color paradox. Section 7 presents two different models for the Allais paradox, one that is based on second-order uncertainty about probabilities and another that is based on second-order uncertainty about utilities. Section 8 discusses how the preference model differs from Choquet expected utility, and Section 9 presents some concluding comments.

2. Preliminaries

The modeling framework used throughout this paper is that of state-preference theory (Arrow 1953/1964, Debreu 1959, Hirshleifer 1965), which encompasses both expected-utility and non-expected-utility models of choice under uncertainty. Suppose that there are \( n \) mutually exclusive, collectively exhaustive states of nature and a single divisible commodity (money) in terms of which payoffs are measured. The wealth distribution of a decision maker is represented by a vector \( \mathbf{w} \in \mathbb{R}^n \), whose \( j^{th} \) element \( w_j \) denotes the quantity of money received in state \( j \), in addition to (unobserved) status quo wealth.

Assumption 1: The decision maker’s preferences among wealth distributions satisfy the usual axioms of consumer theory (reflexivity, completeness, transitivity, continuity, and monotonicity), as well as a smoothness property, so that they are representable by a twice-differentiable ordinal utility function \( U(\mathbf{w}) \) that is a non-decreasing function of wealth in every state.

By the monotonicity assumption, the gradient of \( U \) at wealth \( \mathbf{w} \) is a non-negative vector that can be normalized to yield a probability distribution \( \mathbf{\pi}(\mathbf{w}) \) whose \( j^{th} \) element is

\[ \pi_j(\mathbf{w}) = \frac{\partial U(\mathbf{w})}{\partial w_j} / \sum_{j=1}^{n} \frac{\partial U(\mathbf{w})}{\partial w_j} \]

1 For our purposes, the appeal of the state-preference framework is that it uses concrete sums of money as outcomes without necessarily assuming that utility for money is state-independent or that background wealth is observable or that utilities and subjective probabilities are uniquely separable.
Thus, \( \pi_j(w) \) is the rate at which the decision maker would indifferently bet infinitesimal amounts of money on or against the occurrence of state \( j \). \( \pi(w) \) is invariant to monotonic transformations of \( U \) and is observable, regardless of whether the decision maker is probabilistically sophisticated. It is commonly known as a \textit{risk neutral probability distribution} because the decision maker prices very small assets in a seemingly risk-neutral manner with respect to it. A state will be defined here to be \textit{non-null} if its risk neutral probability is positive at every wealth distribution. A decision maker’s risk neutral probabilities at wealth \( w \) are implicitly a function of her beliefs, her attitudes toward risk and uncertainty, and her personal and financial stakes in events, although their effects are often confounded.

The risk-neutral distribution \( \pi(w) \) determines the first-order properties of the decision maker’s local preferences in the vicinity of wealth \( w \). Second-order properties of local preferences (risk premia, etc.) are determined by the matrix \( R(w) \) whose \( jk^{th} \) element is

\[
r_{jk}(w) = -\frac{\partial^2 U(w)/\partial w_j \partial w_k}{\partial U(w)/\partial w_j}.
\]

\( R(w) \) generalizes the familiar Pratt-Arrow measure to the present setting and will be called the \textit{risk aversion matrix} (Nau 2001). The risk aversion matrix is closely related to the matrix of

\[2\text{ Because preferences are assumed to be both complete and smooth, we rule out first-order aversion to risk or uncertainty, in which, even for infinitesimal stakes, there might be no rate at which the decision maker would indifferently bet on or against a given event.}

\[3\text{ The matrix } R(w) \text{ is not observable in the general case, just as the ordinal utility function } U \text{ is not observable. } U \text{ is determined by preferences only up to monotonic transformations, and correspondingly } R(w) \text{ is uniquely determined.} \]
derivatives of the risk neutral probabilities, and as such it measures the relative curvature of the
decision maker’s indifference curves at wealth \( w \).

If no additional restrictions are placed on preferences, the individual is rational by the
standard of no-arbitrage, but she may have non-expected utility preferences in the sense that her
valuation of a risky asset may not be decomposable into a product of probabilities for states and
utilities for consequences. For example, she may behave as if amounts of money received in
different states are substitutes or complements for each other, which is forbidden under expected
utility theory. However, if her preferences are additionally assumed to satisfy the independence
axiom (Savage’s P2), then \( U(w) \) has an additively separable representation:

\[
U(w) = v_1(w_1) + v_2(w_2) + \ldots + v_n(w_n).
\]

In this case, \( R(w) \) reduces to an observable diagonal matrix. If preferences are further assumed
to be conditionally state-independent (Savage’s P3), then \( U(w) \) has a state-independent expected-
utility representation:

\[
U(w) = p_1u(w_1) + p_2u(w_2) + \ldots + p_nu(w_n),
\]

where \( p \) is a unique probability distribution and \( u(x) \) is a state-independent utility function that is
unique up to positive affine scaling, as in Savage’s model. Although the probabilities in the
by preferences only up to the addition of constants to each column. However, \( R(w) \) can be normalized by
subtracting a constant from each column so that the risk neutral expectation of each column is zero, and the
normalized matrix is observable. Here the “unnormalized” matrix \( R(w) \) will be used for convenience, but the results
could as well be recast in terms of the normalized matrix.

\[\text{For a decision maker who is a state-independent expected-utility maximizer, risk neutral probabilities are merely the product of true subjective probabilities and relative marginal utilities for money at the current wealth position, i.e., } \pi_j(w) \propto p_j u'(w), \text{ where } p_j \text{ is the subjective probability of state } j \text{ and } u(w) \text{ is the state-independent utility of wealth } w. \text{ If an attempt is made to elicit the decision maker’s subjective probabilities by de Finetti’s method—i.e.,}\]

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4 For a decision maker who is a state-independent expected-utility maximizer, risk neutral probabilities are merely the product of true subjective probabilities and relative marginal utilities for money at the current wealth position, i.e., \( \pi_j(w) \propto p_j u'(w) \), where \( p_j \) is the subjective probability of state \( j \) and \( u(w) \) is the state-independent utility of wealth \( w \). If an attempt is made to elicit the decision maker’s subjective probabilities by de Finetti’s method—i.e.,
latter representation are unique, it does not yet follow that they are the decision maker’s “true” subjective probabilities, because there are many other equivalent representations in which different probabilities are combined with state-dependent utilities. In order for the decision maker to be probabilistically sophisticated—i.e., in order for her preferences to determine a unique ordering of events by probability—an additional qualitative probability axiom (Savage’s P4 or Machina and Schmeidler’s P4*, 1992) is needed, together with an *a priori* definition of consequences whose utility is “constant” across states of nature. However, preference measurements among feasible acts do not reveal whether the consequences are, in fact, constant, and so the unique separation of probability from cardinal utility remains problematic and controversial (Schervish et al. 1990, Karni and Mongin 2000, Karni and Grant 2001, Nau 2002, Karni 2002). For this reason, it is of interest to try to define both risk aversion and uncertainty aversion in a way that does not depend on the a priori identification of a set of constant consequences nor on the assumption of state-independent preferences.

3. Definitions of aversion to risk and uncertainty

In order to characterize aversion to uncertainty, it is necessary to begin with a characterization of aversion to risk. Following Yaari (1969) the decision maker is defined to be *risk averse* if her preferences are *convex with respect to mixtures of payoffs*, which means that by asking which gambles she is willing to accept—the probabilities that are observed will be her risk neutral probabilities rather than her true probabilities.

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5 Epstein and Zhang (2001) explicitly invoke Savage’s state-independent utility axiom (P3) and observe: “The necessity of this axiom in our approach implies, in particular, that we have nothing to say about the meaning of ambiguity when preferences are state-dependent.” [p. 275, emphasis added]
whenever \( w \) is preferred to \( w^* \), \( \alpha w + (1-\alpha)w^* \) is also preferred to \( w^* \) for 0<\( \alpha \)<1, where \( \alpha w + (1-\alpha)w^* \) denotes the wealth distribution whose monetary value in state \( i \) is \( \alpha w_i + (1-\alpha)w_i^* \).\footnote{Payoff-convexity of preferences implies that the ordinal utility function \( U \) is quasi-concave. This definition of risk aversion does not require prior definitions of expected value or absence of risk.}

The decision maker’s degree of local risk aversion can be measured by the difference between her risk-neutral valuation of an asset and the price at which she is actually willing to buy it. Let \( z \) denote the payoff vector of a risky asset. The decision maker’s \textit{buying price} for \( z \) at wealth \( w \), denoted \( B(z; w) \), is determined by

\[
U(w+z-B(z; w)) - U(w) = 0.
\]

The (buying) \textit{risk premium} associated with \( z \) at wealth \( w \), here denoted \( b(z; w) \), is the difference between the asset’s risk neutral expected value and its buying price:

\[
b(z; w) = E_{\pi(w)\{z\}} - B(z; w).
\]

It follows (as a consequence of quasi-concavity) that the decision maker is risk averse if and only if her risk premium is non-negative for every asset \( z \) at every wealth distribution \( w \). If \( z \) is a neutral asset (\( E_{\pi(w)\{z\}} = 0 \)), its risk premium has the following second-order approximation that generalizes the Pratt-Arrow formula (Nau 2001):

\[
b(z; w) \approx \frac{1}{2} z^\top \Pi(w) R(w) z,
\]

where \( R(w) \) is the risk aversion matrix and \( \Pi(w) = \text{diag}(\pi(w)) \).

\footnote{Note that there is a suggestive duality between Yaari’s definition of risk aversion as payoff-convexity and Schmeidler’s definition of uncertainty aversion as probability-convexity. However, Schmeidler’s definition is criticized by Epstein (1999) and cannot be applied in the present framework, which lacks objective probabilities.}

\footnote{See Machina (1995) and Karni (1995) for a discussion of the implications of different definitions of risk aversion under non-expected utility preferences.}
In the special case where $U$ is additively separable (i.e., the decision maker satisfies the independence axiom), $R(w)$ is an observable diagonal matrix and the risk premium formula reduces to

$$b(z; w) = \frac{1}{2} E_{\pi(w)}[r(w) z^2],$$

where $r(w)$ is a vector-valued risk aversion measure whose $j^{th}$ element is

$$r_j(w) = -U_{jj}(w)/U_j(w).$$

Hence, for a decision maker with quasi-concave additively-separable utility, local preferences are completely described (up to second-order effects) by a pair of numbers for each state: a risk neutral probability $\pi_j(w)$ and a risk aversion coefficient $r_j(w)$ that are uniquely determined by preferences. Such a decision maker is risk averse but uncertainty neutral, inasmuch as her preferences have an expected-utility representation even if her “true” probabilities cannot be uniquely separated from her (possibly state-dependent) utilities.

It remains to characterize aversion to uncertainty in a simple behavioral way. For convenience, in the spirit of Epstein (1999), we assume that there is a reference set of events that are a priori “unambiguous.”

**Assumption 2:** There is a set of unambiguous events, closed under complementation and disjoint union, at least one of which is a union of two or more non-null states and whose complement is also a union of at two or more non-null states.

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8 The unambiguous events play no essential role in the analysis except to distinguish whether a particular pattern of behavior is ambiguity-averse or ambiguity-loving. Since unambiguous events can be created if necessary by flipping coins, this entails no great loss of generality. However, we do not assume that unambiguous events necessarily have have objective or otherwise uniquely determined probabilities.
Intuitively, a decision maker is averse to uncertainty if she prefers to bet on unambiguous rather than ambiguous events, other things being equal. To make this notion precise, we introduce the notion of an $A:B$ $\Delta$-spread (“$A$ vs. $B$ delta-spread”). Let $A$ and $B$ denote two logically independent events, let $\{ \pi_{AB}, \pi_{\overline{A}B}, \pi_{\overline{A}\overline{B}}, \pi_{AB} \}$ denote the local risk neutral probabilities (at wealth $w$) of the four possible joint outcomes of $A$ and $B$, and assume they are all strictly positive. Let $\Delta$ denote a quantity of money (just) large enough in magnitude that second-order utility effects are relevant. Then an $A:B$ $\Delta$-spread and a $B:A$ $\Delta$-spread are defined as the neutral assets whose payoffs are given by the following two contingency tables:

$$
\begin{array}{ccc}
\hline
 & B & \overline{B} \\
\hline
\overline{A} & \frac{\Delta}{\pi_{AB}} & \frac{-\Delta}{\pi_{AB}} \\
A & \frac{-\Delta}{\pi_{\overline{A}B}} & \frac{-\Delta}{\pi_{\overline{A}B}} \\
\hline
\end{array}
$$

$A:B$ $\Delta$-spread

$$
\begin{array}{ccc}
\hline
 & B & \overline{B} \\
\hline
\overline{A} & \frac{-\Delta}{\pi_{AB}} & \frac{\Delta}{\pi_{AB}} \\
A & \frac{\Delta}{\pi_{\overline{A}B}} & \frac{-\Delta}{\pi_{\overline{A}B}} \\
\hline
\end{array}
$$

$B:A$ $\Delta$-spread

Table 1: $\Delta$-spreads defined

Note that an $A:B$ $\Delta$-spread is a non-simple bet on $A$: the sign of the payoff depends only on whether $A$ occurs or not, but the magnitude may also depend on $B$. Similarly, a $B:A$ $\Delta$-spread is a non-simple bet on $B$ whose payoff magnitude may depend on $A$. In both cases, the winning and losing amounts are scaled so that the contribution to the risk neutral expected payoff is $\pm \Delta$ in every cell of the table. By construction, both bets are neutral and both must have the same risk premium for a decision maker with separable preferences, because for such a person the risk premium is obtained (to a second order approximation) by squaring the payoffs, multiplying them by the statewise risk neutral probabilities and risk aversion coefficients, and summing.
**Definition 1:** The decision maker is *locally uncertainty averse at wealth* $w$ if, for every unambiguous event $B$, every event $A$ that is logically independent of $B$, and any $\Delta$ sufficiently small in magnitude (positive or negative), a $B:A \Delta$-spread is weakly preferred to (i.e., has a risk premium less than or equal to that of) an $A:B \Delta$-spread. The decision maker is *uncertainty averse* if she is locally uncertainty averse at every wealth position.

The classic demonstration of uncertainty aversion is provided by Ellsberg’s 2-color paradox, in which a subject prefers to stake a modest prize (say, $400) on the draw of a ball from an urn with equal proportions of red and black balls rather than a draw from an urn with unknown proportions of red and black, regardless of the winning color. To adapt this experiment to the framework of Definition 1, let $B$ denote the unambiguous event that a red ball is drawn from urn 1 (with known 50-50 proportions of red and black) and let $A$ denote the potentially ambiguous event that a red ball is drawn from urn 2 (with unknown proportions of red and black). By symmetry, if there are no prior stakes, it is reasonable to suppose that the decision maker’s risk neutral probabilities for the four possible joint outcomes are identically $1/4$. Then, according to Definition 1, an uncertainty averse decision maker would prefer to receive $+4\Delta$ in the event that red is drawn from urn 1 and $-4\Delta$ in the event that black is drawn from urn 1 (a $B:A \Delta$-spread) rather receive $+4\Delta$ in the event that red is drawn from urn 2 and $-4\Delta$ in the event that black is drawn from urn 2 (an $A:B \Delta$-spread), regardless of whether $\Delta$ is $+$100 or $-$100.

In Ellsberg’s three-color paradox, there is a single urn containing 30 red balls and 60 balls that are black and yellow in unknown proportions. A typical subject prefers to stake (say) a $300 prize on the draw of a red ball rather than a black ball when $0 is to be received in either case if a yellow ball is drawn, but the direction of preference is reversed when $300 is to be received if yellow is drawn. The 3-color paradox is more transparently a violation of the independence
axiom but less transparently an example of aversion to uncertainty. It is usually presented as a set of choices among prospects that involve only gains:

<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
<th>(e)</th>
<th>(f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td>300</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>300</td>
<td>300</td>
</tr>
<tr>
<td>Black</td>
<td>0</td>
<td>300</td>
<td>0</td>
<td>300</td>
<td>0</td>
<td>300</td>
</tr>
<tr>
<td>Yellow</td>
<td>0</td>
<td>0</td>
<td>300</td>
<td>300</td>
<td>0</td>
<td>300</td>
</tr>
</tbody>
</table>

Table 2: Ellsberg’s 3-color paradox in gains-only form

The typical response pattern is \(a > b \sim c\) and \(d > e \sim f\). When the prospects are converted to “fair” gambles by subtraction of appropriate constants, the choices look like this:

<table>
<thead>
<tr>
<th></th>
<th>(a')</th>
<th>(b')</th>
<th>(c')</th>
<th>(d')</th>
<th>(e')</th>
<th>(f')</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td>200</td>
<td>-100</td>
<td>-100</td>
<td>-200</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Black</td>
<td>-100</td>
<td>200</td>
<td>-100</td>
<td>-200</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Yellow</td>
<td>-100</td>
<td>-100</td>
<td>200</td>
<td>100</td>
<td>100</td>
<td>-200</td>
</tr>
</tbody>
</table>

Table 3: Ellsberg’s 3-color paradox in fair-gamble form

The last three gambles are merely the opposite sides of the first three. We would expect an uncertainty-averse subject to give the same pattern of responses, i.e., \(a' > b' \sim c'\) and \(d' > e' \sim f'\), and indeed the Choquet expected utility model predicts exactly this result. However, \(a' > b' \sim c'\) and \(d' > e' \sim f'\) does not violate the independence axiom. For example, it is consistent with a model of state-dependent expected-utility preferences of the following form:

\[
U(w) = \frac{1}{3}(-2 \exp(-\frac{1}{2} \alpha w_1)) + \frac{1}{3}(-\exp(-\alpha w_2)) + \frac{1}{3}(-\exp(-\alpha w_3)),
\]

where \(w_1, w_2, w_3\) are the amounts of wealth received under Red, Black, and Yellow, respectively, and \(\alpha\) is any positive number. Under this utility function, the risk neutral probabilities of the three events are equal when \(w_1 = w_2 = w_3 = 0\), but the Pratt-Arrow measure of risk aversion in the event Red is only half as large as it is in the other two events, hence the decision maker prefers fair gambles in which the largest change in wealth occurs under Red, regardless of whether it is
positive or negative. For example, with $\alpha = 0.001$, the risk premium is 6.667 for both $a'$ and $d'$, while it is 9.167 for the other prospects.\footnote{This example illustrates the complications that state-dependent preferences create for some definitions of ambiguity, alluded to by Epstein and Zhang (2001).}

To embed the 3-color paradox in the framework of Definition 1, which requires the state space to consist of at least four elements and to be partitioned by at least one unambiguous event, it is necessary to create a fourth state. Therefore, consider the following extension of the 3-color experiment: a fair coin is flipped, and (only) if tails is obtained, a ball is then drawn from the urn. The four possible outcomes are then \{Red, Black, Yellow, Heads\}, where Red and Heads (and their union) are unambiguous. By symmetry, if there are no prior stakes, it is reasonable to assume that the corresponding risk neutral probabilities are (1/6, 1/6, 1/6, 1/2). Consider the following two $\Delta$-spreads:

<table>
<thead>
<tr>
<th>Event</th>
<th>Red $\cup$ Heads: Black $\cup$ Heads</th>
<th>Black $\cup$ Heads: Red $\cup$ Heads</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td>$+6\Delta$</td>
<td>$-6\Delta$</td>
</tr>
<tr>
<td>Black</td>
<td>$-6\Delta$</td>
<td>$+6\Delta$</td>
</tr>
<tr>
<td>Yellow</td>
<td>$-6\Delta$</td>
<td>$-6\Delta$</td>
</tr>
<tr>
<td>Heads</td>
<td>$+2\Delta$</td>
<td>$+2\Delta$</td>
</tr>
</tbody>
</table>

\textbf{Table 4: $\Delta$-spreads for the extended 3-color paradox}

Thus, the decision maker receives $+2\Delta$ in any case if Heads occurs. If Heads does not occur, the first column is a bet on or against Red, while the second is a bet on or against Black. (The direction of the bet depends on the sign of $\Delta$.) Because Red $\cup$ Heads is unambiguous, an uncertainty-averse decision maker prefers the first bet over the second, regardless of whether $\Delta$ is (say) $+50$ or $-50$.

The preceding examples motivate a definition of \textit{comparative ambiguity} between events:
Definition 2: (a) If $A_1$ and $A_2$ are logically independent and their four joint outcomes are non-null, then an uncertainty averse decision maker regards $A_1$ as less ambiguous than $A_2$ if for any $\Delta$ sufficiently small in magnitude (positive or negative), an $A_1:A_2 \Delta$-spread is strictly preferred to an $A_2:A_1 \Delta$-spread at every wealth distribution.

(b) If $A_1$ and $A_2$ are disjoint, and the state space can be partitioned as $\{A_1, A_2, A_3, B\}$ where all four events are non-null and $B$ is unambiguous, then an uncertainty averse decision maker regards $A_1$ as less ambiguous than $A_2$ if for any $\Delta$ sufficiently small in magnitude (positive or negative), an $A_1 \cup B:A_2 \cup B \Delta$-spread is strictly preferred to an $A_2 \cup B:A_1 \cup B \Delta$-spread at every wealth distribution.

4. A simple model of smooth preferences explaining the Ellsberg paradox

The definitions of the preceding section provide simple tests for presence of second-order uncertainty aversion and for comparing the degree of ambiguity of different events. We now present an example of a utility function that exhibits second-order uncertainty aversion in Ellsberg’s 2-color paradox. Henceforth, let $A_1$ [$A_2$] denote the event that the ball drawn from the unknown urn is red [black], and let $B_1$ [$B_2$] denote the event that the ball drawn from the known urn is red [black]. The relevant state space is then $\{A_1B_1, A_1B_2, A_2B_1, A_2B_2\}$. Unless the subject has a strict color preference and/or prior stakes in the outcomes of the events (which we assume she does not), the four states are completely symmetric when considered one-at-a-time: each is the conjunction of an ambiguous event ($A_1$ or $A_2$) and an unambiguous event ($B_1$ or $B_2$) differing only in their color associations. If the subject had to choose one of the four states on which to stake a prize, she would have no basis for a strict preference. The paradox lies in the fact that the
states are not symmetric when considered two-at-a-time: the pair of states \{A_1B_1, A_2B_1\} has an objectively known probability while the pair of states \{A_1B_1, A_1B_2\} does not.

Let \( w = (w_{11}, w_{12}, w_{21}, w_{22}) \) denote the decision maker’s wealth distribution, where \( w_{ij} \) is wealth in state \( A_iB_j \), and suppose that she evaluates wealth distributions according to the following non-separable utility function:

\[
U(w) = -p_1 \exp(-\alpha(q_{11}w_{11} + q_{12}w_{12})) - p_2 \exp(-\alpha(q_{21}w_{21} + q_{22}w_{22}))
\]

where \( \alpha \) is a positive constant and \( p_1 = p_2 = q_{11} = q_{12} = q_{21} = q_{22} = \frac{1}{2} \). It is natural to interpret \( p_i \) as a marginal probability for \( A_i \) and \( q_{ij} \) as a conditional probability of \( B_j \) given \( A_i \). Suppose that the prior wealth distribution is an arbitrary constant—i.e., the subject has no prior stakes in the draws from either urn. Then the states are symmetric with respect to changes in wealth one-state-at-a-time, from which it follows that \( \pi(w) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \). Hence, for infinitely small bets, the subject does not distinguish among the states, exactly as if she were a state-independent expected utility maximizer with uniform prior probabilities and no prior stakes. However, she does distinguish among the states when considering bets in which states are grouped together and in which the stakes are large enough for risk aversion to come into play. For example, the subject would prefer to pair a finite gain in state \( A_1B_1 \) with an equal loss in state \( A_1B_2 \) (yielding no change in \( U \)) rather than with an equal loss in state \( A_2B_1 \) or state \( A_2B_2 \) (yielding a decrease in \( U \)). The corresponding risk aversion matrix is \( R(w) \equiv \frac{1}{2} \alpha C \), where \( C \) has a block structure with 1’s in its upper left and lower right 2\times2 submatrices.

Now consider the following three neutral bets. In bet #1, the subject wins \( x > 0 \) if a red ball is drawn from the known urn and loses \( x \) otherwise, so that the payoff vector is \( (x, -x, x, -x) \), i.e., a \( B_1:A_1 \) \((x/4)\)-spread. In bet #2, the subject wins \( x \) if a red ball is drawn from the unknown urn and loses \( x \) otherwise, so that the payoff vector is \( (x, x, -x, -x) \), i.e., an \( A_1:B_1 \) \((x/4)\)-spread.
In bet #3, the subject wins $x$ if the balls drawn from urns 1 and 2 are the same color, and loses $x$ otherwise, so that the payoff vector is $(x, -x, -x, x)$. Applying the formula $b(z; \mathbf{w}) = \frac{1}{2} \mathbf{z}^\top \Pi(\mathbf{w}) R(\mathbf{w}) \mathbf{z}$, the risk premium for bet #1 is zero, and the same is true if the sign of $x$ is reversed so that black is the winning color. Hence, the subject is risk neutral with respect to bets on the known urn. Whereas, the risk premium for bet #2 is $\frac{1}{2} \alpha x^2$, and the same risk premium is obtained if the winning color is changed to black. Hence, the subject is risk averse with respect to bets on the unknown urn: she behaves toward it as if she believes red and black are equally likely but her Pratt-Arrow risk aversion coefficient is equal to $\alpha$. This pattern of risk neutral behavior toward unambiguous events and risk averse behavior toward ambiguous events exemplifies uncertainty aversion as given in Definition 1. Interestingly, the risk premium for bet #3 is the same as for bet #1, namely zero, and the same is true if the sign of $x$ is reversed so that the subject wins if the balls are of different colors. Hence, the subject behaves risk neutrally with respect to the unknown urn when the winning color is determined by “objective” randomization using the known urn, which is a well-known trick for eliminating the ambiguity.

5. A general model of partially separable preferences

The example in the preceding section suggests a novel hypothesis about the character of non-expected-utility preferences, namely that the decision maker may behave like an expected-utility maximizer with respect to assets in the same “uncertainty class,” while exhibiting higher degrees of second-order risk aversion toward assets that are “more uncertain.” This section presents an axiomatic model of such preferences. Henceforth, let the state space consist of a Cartesian product $A \times B$, where $A = \{A_1, \ldots, A_m\}$ and $B = \{B_1, \ldots, B_n\}$ are finite partitions, and $A$-measurable events are potentially ambiguous while $B$-measurable events are a priori
unambiguous. To fix ideas, it might be supposed that the elements of A are “natural” states of the world, while elements of B are states of an artificial randomization device. (Another interpretation will be suggested later.)

Let \( w, w^*, z, z^* \), denote wealth distributions over states, i.e., monetary acts. For any event \( E \) and acts \( w \) and \( z \), let \( Ew + (1-E)z \) denote the act that agrees with \( w \) on \( E \) and agrees with \( z \) on \( \bar{E} \). Suppose that preferences among acts satisfy the following partition-specific independence axioms.

**Assumption 3:**

**A-independence:**

\[
Ew + (1-E)z \geq Ew^* + (1-E)z^* \iff Ew + (1-E)z^* \geq Ew^* + (1-E)z
\]

for all acts \( w, w^*, z, z^* \) and every A-measurable event \( E \), and conditional preference \( w \geq_E w^* \) is accordingly defined for such events.

**B-independence:**

\[
Fw + (1-F)z \geq_i Fw^* + (1-F)z^* \iff Fw + (1-F)z^* \geq_i Fw^* + (1-F)z
\]

for every B-measurable event \( F \), where \( \geq_i \) denotes conditional preference given element \( A_i \) of A.

In other words, the decision maker satisfies the independence axiom unconditionally with respect to A-measurable events and conditionally with respect to B-measurable acts and events within each element of A. Such a person will be said to have *partially separable* preferences. A-independence and B-independence are similar to the time-0 and time-1 substitution axioms of Kreps and Porteus (1979), adapted to a framework of choice under uncertainty rather than risk and stripped of their temporal interpretation. Our main result is the following
PROPOSITION:

(a) Under Assumptions 1–3, preferences are represented by a utility function $U$ having the composite-additive form:

$$(2) \quad U(w) = \sum_{i=1}^{m} u_i \left( \sum_{j=1}^{n} v_{ij} (w_{ij}) \right)$$

where $w_{ij}$ denotes wealth in state $A_iB_j$, and $\{u_i\}$ and $\{v_{ij}\}$ are non-decreasing twice-differentiable state-dependent utility functions.

(b) The corresponding risk neutral probabilities satisfy

$$\pi_{ij} = \frac{u_i \left( \sum_{k=1}^{n} v_{ik} (w_{ik}) \right) v_{ij} (w_{ij})}{\sum_{h=1}^{m} \sum_{l=1}^{n} u_{hl} \left( \sum_{k=1}^{n} v_{hk} (w_{hk}) \right) v_{hl} (w_{hl})}.$$  

(c) The local risk aversion matrix $R(w)$ is the sum of a diagonal matrix and a block-diagonal matrix, with $r_{ij,kk}(w) = 0$ if $i \neq k$ and

$$r_{ij,il}(w) = \frac{u_i \left( \sum_{h=1}^{n} v_{ih} (w_{ih}) \right)}{u_i \left( \sum_{h=1}^{n} v_{ih} (w_{ih}) \right)} v_{il} (w_{il}) + v_{ij} (w_{ij})/v_{ij} (w_{ij}) 1_{jl}.$$  

(d) Let $s$ and $t$ denote $m$- and $mn$-vectors defined by:

$$s_i = \frac{u_i \left( \sum_{h=1}^{n} v_{ih} (w_{ih}) \right) \sum_{h=1}^{n} v_{ih} (w_{ih})}{u_i \left( \sum_{h=1}^{n} v_{ih} (w_{ih}) \right)}$$

$$t_{ij} = v_j''(w_{ij})/v_j'(w_{ij}).$$
Let $\pi_i = \sum_{h=1}^{n} \pi_{ih}$ and $\pi_{h|i} = \pi_{ih}/\pi_i$ denote the induced marginal and conditional probabilities, and let $\bar{z}$ be defined as the $m$-vector whose $i^{th}$ element is $\bar{z}_{i} = \sum_{h=1}^{m} \pi_{h|i} z_{ih}$, i.e., the conditional risk-neutral expectation of $z$ given event $A_i$. In these terms, the risk premium for a neutral asset $z$ is

$$b(z; w) \approx \frac{1}{2} \sum_{i=1}^{m} \pi_i s_i \bar{z}_{i}^2 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ij} t_{ij} z_{ij}^2 = \frac{1}{2} E_{\pi}[s \bar{z}^2] + \frac{1}{2} E_{\pi}[t \bar{z}^2].$$

(e) The decision maker is locally uncertainty averse if $s \geq 0$ and globally uncertainty averse if $u_i$ is concave for every $i$.

**Proof:** Part (a) follows from a double application of the usual argument showing that the independence axiom leads to an additively separable utility representation. (Debreu 1960, Wakker 1989) Parts (b)-(d) are obtained by some algebra and the risk premium formula from Nau (2001). For part (e), note that the off-diagonal terms of $R(w)$ are non-negative if $s \geq 0$, and this is true at all $w$ if $u_i$ is concave for every $i$. If $A$ is any $A$-measurable (potentially ambiguous) event and $B$ is any $B$-measurable (unambiguous) event, the risk premium of an $A:B$ $\Delta$-spread must be greater than or equal to the risk premium of the corresponding $B:A$ $\Delta$-spread, because the risk premia consist of sums of matched terms with identical magnitudes, but the terms in the former summation all have positive signs as a consequence of the fact that $R(w)$ is block-diagonal with non-negative off-diagonal elements, while those in the latter summation have both positive and negative signs.
By comparison with the risk premium formula for separable preferences, it is suggestive to think of the term $\frac{1}{2} E_\pi[tz^2]$ in the risk premium formula of part (d) as a pure risk premium while $\frac{1}{2} E_\pi[sz^2]$ is an additional premium for the uncertainty surrounding $A$-measurable events, with $s$ and $t$ serving as vector-valued measures of aversion to uncertainty and risk, respectively. If $z$ is neutral and $A$-measurable, then $z_{ij} \equiv z_i$ and the total risk premium is $\frac{1}{2} E_\pi[(s + t)z^2]$. (Here $s + t$ is understood to be the vector whose $ij^{th}$ element is $s_i + t_{ij}$.) If $z$ is neutral and $B$-measurable while $A$ and $B$ are independent under $\pi$, then $z_i = E_\pi[z] = 0$ for every $i$ and the total risk premium is $\frac{1}{2} E_\pi[tz^2]$.

As a special case of (2), suppose that the component utility functions are state-independent expected utilities of the form $u_i(v) = p_i u(v)$ and $v_i(x) = q_{ij} v(x)$, where $p$ is a marginal probability distribution on $A$ and $q_i$ is a conditional probability distribution on $B$ given $A_i$, yielding:

$$U(w) = \sum_{i=1}^m p_i u \left( \sum_{j=1}^n q_{ij} v(w_{ij}) \right).$$

Then the decision maker behaves as though she assigns probability $p_i q_{ij}$ to state $A_i B_j$ and she bets on events measurable with respect to $A$ as though her utility function were $u(v(x))$. If $A$ and $B$ are also independent, i.e., if $q_i$ is the same for all $i$, she meanwhile bets on events measurable with respect to $B$ as though her utility function for money were $v(x)$. If $u$ is concave, she is uniformly more risk averse with respect to $A$-measurable bets than to $B$-measurable bets, implying that she is averse to uncertainty. Thus, concavity of $v$ encodes the decision maker’s
aversion to risk while concavity of $u$ encodes her aversion to the additional uncertainty surrounding the $A$-measurable events.

The preference model (3) will henceforth be called partially separable utility (PSU). For a decision maker with PSU preferences, a utility function for money elicited via choices among objectively-randomized lotteries cannot be used to predict or prescribe choices among natural lotteries, contrary to usual decision-analytic practice. Nevertheless, such a decision maker is perfectly rational in the sense that her behavior does not create opportunities for arbitrage, and she can still solve a decision tree by dynamic programming provided that all-but-the-last chance node on every path is $A$-measurable. Whether she is able to use dynamic programming in practice will depend on whether she frames a dynamic decision problem in such a way that the $A$-measurable events are resolved first.

If the PSU decision maker is further assumed to have constant (i.e., nonstochastic) prior wealth $x$, then her risk neutral distribution is the product $\pi_{ij} = p_i q_{ij}$ and her local attitude toward risk and uncertainty can be summarized by a scalar risk aversion measure $t(x) = -v''(x)/v'(x)$ and a scalar uncertainty aversion measure $s(x) = -u''(v(x))/u'(v(x))v'(x)$. Under these conditions, $\bar{z}$ is the vector whose $i$th element is $\bar{z}_i = \sum_{j=1}^n q_{ij} z_j$, the conditional expectation of $z$ given $A_i$ under the distribution $q_i$. The total risk premium for a neutral asset $z$ is then

$$b(z; w) \approx \frac{1}{2} s(x)E_p[\bar{z}^2] + \frac{1}{2} t(x)E_u[z^2].$$

The measures $s(x)$ and $t(x)$ are convenient hyperbolic functions if $u$ and $v$ are utilities from the HARA (generalized log/power/exponential) family, as shown in the following table:
Table 5: Risk and uncertainty aversion measures for HARA partially separable utilities

Here, (ii) and (iv) are limiting cases of (i) and (iii) in which \( \beta \to 0 \) and \( \alpha \to 0 \), respectively, and wealth is assumed to be bounded below by \(-\gamma\). The conditions in the last column imply \( t(x) \geq 0 \) and \( s(x) + t(x) \geq 0 \), ensuring convexity of preferences. Note that if \( \alpha < 0 \) in (ii) or \( \alpha > 1 \) in (iii), \( s(x) \) is negative and the decision maker is less risk averse toward A-measurable events than B-measurable events. This could represent a situation in which the decision maker is uncertainty-loving, despite having convex preferences overall.

The preference model (1) introduced earlier is the special case of HARA partially separable utility in which \( m = n = 2 \), \( u(x) = -\exp(-\alpha x) \), \( v(x) = x \), \( q_{11} = q_{21} \), and \( q_{12} = q_{22} \). The same general construction can, of course, be extended to 3-way partitions, 4-way partitions, etc., all having different degrees of uncertainty, although the 2-way partition suffices to model the basic dichotomy between risk and uncertainty.

6. Second-order probabilities and utilities

In the discussion of partially separable preferences in the preceding section, the partitions A and B were interpreted to represent sets of observable, payoff-relevant events that were, respectively, ambiguous or unambiguous. In this section, a different interpretation of the same model will be suggested, namely that the partition B represents the observable, payoff-relevant events while the partition A represents possible credal states for the decision maker in which she
may have different probabilities and/or utilities. The set of credal states could have various interpretations in practice. For example, it could be interpreted to represent uncertainty about what the decision maker’s state of mind will be after further introspection or learning, or it could be interpreted to represent model risk—i.e., uncertainty about the model which ought to be used for purposes of decision analysis.

Henceforth, let the wealth vector \( w \) be singly-subscripted, with \( w_j \) representing wealth in event \( B_j \in \mathcal{B} \), and consider the following specialization of (2):

\[
(4) \quad U(w) = \sum_{i=1}^{m} p_i u_i^{-1}(v_i^{-1}(\sum_{j=1}^{n} q_{ij} v_j(w_j))) ,
\]

where \( p \) is a probability distribution on \( A \) and, for each \( i \), \( q_i \) is a probability distribution on \( \mathcal{B} \). (The former \( u_i(.) \) has been rewritten as \( p_i u_i^{-1}(.) \), and the former \( v_j(.) \) has been assumed to have the form \( q_{ij} v_i(.) \), which is conditionally state-independent.) The argument of \( u_i \) is now the certainty equivalent of \( w \) obtained from an expected-utility calculation with probability distribution \( q_i \) and utility function \( v_i \):

\[
CE_i(w) = v_i^{-1}(\sum_{j=1}^{n} q_{ij} v_j(w_j)) ,
\]

in terms of which the utility function (4) becomes:

\[
(5) \quad U(w) = \sum_{i=1}^{m} p_i u_i(CE_i(w)) .
\]

A utility function of essentially this same composite form was used by Segal (1989) to model behavior violating the reduction of compound lotteries axiom in two-stage lotteries under risk, and a similar function was used by Grant et al. (1998) to model intrinsic preference for information. Here the “first stage” lottery is the selection of an element \( A_i \) from partition \( \mathcal{A} \),
which can be interpreted as a credal state within which the decision maker behaves like an
expected-utility maximizer with probability distribution $q_i$ and utility function $v_i$. The
implication of (5) is that, prior to the resolution of the first-stage lottery, the decision maker is
uncertain about her credal state (as represented by a second-order probability distribution $p$) and
is potentially averse to that uncertainty (as represented by a second-order utility function $u$ which
is applied to the certainty equivalents realized in different credal states). If $u_i = v_i$ for every $i$,
then the second-order uncertainty about probabilities and utilities can be integrated out and the
decision maker has (possibly state-dependent) expected-utility preferences and is uncertainty-
neutral, but otherwise she has non-expected utility preferences and may be uncertainty averse.

The specific utility function (1), which was previously used to explain the Ellsberg 2-
color paradox when the partition $A$ was interpreted as a set of ambiguous (but observable)
events, can now be re-interpreted as a special case of (4)-(5) in which the decision maker is an
expected-value maximizer (i.e., risk neutral) within each credal state, but she is uncertain about
her probability distribution and is averse to that uncertainty with a constant degree of uncertainty
aversion quantified by $\alpha$. In particular, the model of the 2-color Ellsberg paradox is a special
case of (4) in which the decision maker thinks it is equally likely that the unknown urn contains
all red balls or all black balls, while she is certain that the known urn contains equal numbers of
red and black balls. There are four payoff-relevant events: $B_1 = \{\text{both red}\}$, $B_2 = \{\text{black from urn}
1, \text{red from urn 2}\}$, $B_3 = \{\text{red from urn 1, black from urn 2}\}$, $B_4 = \{\text{both black}\}$; the decision
maker’s two possible credal states are represented by first-order probability distributions $q_1 = (\frac{1}{2},
\frac{1}{2}, 0, 0)$ and $q_2 = (0, 0, \frac{1}{2}, \frac{1}{2})$; her first-order utility is linear, $v_i(x) \equiv x$; and her second-order
probabilities and utilities are $p = (\frac{1}{2}, \frac{1}{2})$ and $u_i(x) \equiv -\exp(-\alpha x)$. With these parameter
assignments, the decision maker is risk neutral with respect to bets on the known urn and risk
averse with respect to bets on the unknown urn, exactly as before. A two-stage lottery interpretation of the Ellsberg paradox was also given by Segal (1987), although there the underlying utility model was that of anticipated (rank-dependent) utility rather than expected utility.

Similarly, the 3-color Ellsberg paradox can be modeled as a special case of (4) in which the decision maker thinks it is equally likely that the urn contains 60 yellow balls and zero black balls or vice versa, while she is certain that it also contains 30 red balls. There are now three observable events, namely \( B_1 = \text{Red}, \ B_2 = \text{Yellow}, \) and \( B_3 = \text{Black}, \) and two credal states represented by the probability distributions \( q_1 = (1/3, 2/3, 0) \) and \( q_2 = (1/3, 0, 2/3) \) over those events; and the credal states are considered equally likely, i.e., \( p = (½, ½). \) When these values are substituted into (4), together with \( v(x) \equiv x \) and \( u(x) \equiv -\exp(-\alpha x), \) the result is:

\[
U(w) = -\frac{1}{2} \exp(-\alpha \left( \frac{1}{3}w_1 + \frac{2}{3}w_2 \right)) - \frac{1}{2} \exp(-\alpha \left( \frac{1}{3}w_1 + \frac{2}{3}w_3 \right)).
\]

A decision maker with this utility function will exhibit the paradoxical preference pattern of Table 2 as well as the somewhat less paradoxical pattern of Table 3.

In the models just presented above for the 2-color and 3-color Ellsberg paradoxes, the decision maker is assumed to have linear utility for money \( (v(x) \equiv x) \) and “constant absolute aversion to uncertainty” \( (u(x) \equiv -\exp(-\alpha x)), \) where \( \alpha^{-1} \) has the interpretation of an “uncertainty tolerance” measured in units of currency. It has often been pointed out that individuals with good employment prospects and good credit ought to have roughly linear utility for money for small to moderate risks, which contradicts the empirical fact that most individuals are risk averse even for small gambles. The model of credal uncertainty developed here suggests an explanation of this puzzle, namely that individuals may indeed have linear utility for money but a low
tolerance for uncertainty, and they may skeptically regard even supposedly “objective” gambles as though the probabilities were somewhat uncertain.

7. Two models of the Allais paradox

The utility function (6) explaining the 3-color Ellsberg paradox can also be adapted to explain the Allais paradox, which exhibits a similar direct violation of the independence axiom. The pairs of prospects are as follows:

<table>
<thead>
<tr>
<th></th>
<th>$f$</th>
<th>$g$</th>
<th>$f'$</th>
<th>$g'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>$1M$</td>
<td>$0$</td>
<td>$1M$</td>
<td>$0$</td>
</tr>
<tr>
<td>$F$</td>
<td>$1M$</td>
<td>$5M$</td>
<td>$1M$</td>
<td>$5M$</td>
</tr>
<tr>
<td>$G$</td>
<td>$1M$</td>
<td>$1M$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Table 6: the Allais paradox

The typical pattern of responses is $f > g$ and $f' < g'$. Here the three observable events are $B_1 = E$, $B_2 = F$, and $B_3 = G$, for which probability estimates are given in the table. Suppose the decision maker has two credal states represented by probability distributions $q_1 = (1, 0, 0)$ and $q_2 = (0, 10/99, 89/99)$, having second-order probabilities $p = (1/100, 99/100)$, yielding

$$U(w) = -0.01 \exp(-\alpha w_1) - 0.99 \exp(-\alpha (10/99)w_2 + (89/99)w_3)).$$

With these parameters, which agree with the probability estimates in the table, the subject will prefer $f$ over $g$ if $\alpha^{-1} < \$226,040$ but will nevertheless prefer $g'$ over $f'$ as long as $\alpha^{-1} > \$21,982. Hence, for a wide range of “uncertainty tolerances,” the subject will display the typical preference pattern of the Allais paradox. This model implies the following interpretation of the paradox: the subject thinks that the game is rigged so that one alternative is dominant over the other in both pairs, she just doesn’t know which one. In particular, she thinks there is a 1%
chance that $E$ is sure to happen, in which case $f$ and $f'$ are strictly dominant, and conversely there is a 99% chance that $E$ is sure not to happen, in which case $g$ and $g'$ are weakly dominant.

Levi (1986 and elsewhere) has suggested a fundamentally different interpretation of the Allais paradox, namely that it is due to indeterminacy of utilities rather than probabilities. That interpretation can also be accommodated by the present model, although here a second-order probability distribution is assessed over the set of credal states (possible utility functions), whereas in Levi’s model alternatives are compared on the basis of admissibility criteria referring to extremal utilities. In the Allais example, suppose that the decision maker is certain that events $E$, $F$, and $G$ have the given probabilities of 0.01, 0.10, and 0.89, respectively, but meanwhile she is uncertain about her utility function. In particular, suppose that she has an exponential utility function whose risk aversion parameter is equally likely to be 1 or 10, when payoffs are measured in $M$. In other words, her risk tolerance (the reciprocal of her risk aversion coefficient) is equally likely to be $100,000 or $1,000,000. Furthermore, assume $u_i(x) \equiv x$ in (4), so that the decision maker evaluates alternatives on the basis of the second-order expectations of their first-order certainty equivalents. For such a decision maker, the certainty equivalent of $f$ is $1M$, while the certainty equivalent of $g$ is equally likely to be $0.46M or $1.08M, whose expected value is $0.77M. Hence $f$ is preferred to $g$. Meanwhile, the certainty equivalent of $f'$ is equally likely to be $0.012M or $0.072M, yielding an expected value of $0.042M, while the certainty equivalent of $g'$ is equally likely to be $0.011M or $0.105M, yielding an expected value of $0.058M, hence $g'$ is preferred to $f'$.

Seidenfeld (1986) has shown that violations of independence in sequential decisions under risk can lead to sequential incoherence. The preference model presented in this paper refers to static decisions under uncertainty and cannot, per se, lead to sequential incoherence.
When faced with a sequential decision problem, a decision maker with partially separable preferences could either solve the problem in normal form and proceed as though “risks borne but not realized” were relevant (Machina 1989), or, more interestingly, she might regard her future decisions as stochastic due to her uncertain credal state.

8. Comparison with other preference models

This section compares the partially-separable preference model against other well-known preference models. First, as already noted, composite utility functions have previously been used by Kreps and Porteus (1979), Segal (1989), and Grant et al. (1998) to model preferences for temporal or compound lotteries under conditions of risk (known probabilities). Here, the setting is that of uncertainty—i.e., states of nature with subjective beliefs—and it is not necessary to think of the decision problem as having a temporal or compound structure, though the uncertain-credal-state interpretation could perhaps be viewed in temporal terms.

Second-order probabilities are used in hierarchical Bayesian statistical models to represent imprecise prior distributions, but in those models the second-order uncertainty has no behavioral implications: it can be integrated out to yield an equivalent representation of preferences in terms of first-order expected utility. In contrast, the partially-separable-preference model admits the possibility of a second-order utility function representing aversion to uncertainty and/or it admits uncertainty in the first-order utility, thereby rationalizing behavior that is inconsistent with standard Bayesian theory. In “quasi-Bayesian” models, incomplete preferences are represented by sets of probabilities and/or utilities. Here, the preference ordering is “completed” through the use of second-order probabilities and utilities.
Epstein (1999) has defined uncertainty aversion in relative terms by reference to sets of ambiguous and unambiguous acts, with *probabilistic sophistication* (Machina and Schmeidler 1992) serving as a benchmark for uncertainty neutrality. (A decision maker is probabilistically sophisticated if there is a probability distribution on states such that her preferences among acts depend only on the probability distributions they induce on consequences, regardless of whether she is an expected-utility maximizer.) Epstein’s definition of uncertainty aversion, like that of probabilistic sophistication, applies to a Savage-act framework in which the primitive rewards are abstract consequences whose utilities are assumed to be state-independent, providing a basis for extracting personal probabilities from preferences among acts. The analysis in this paper, in contrast, applies to a state-preference framework in which the primitive rewards are quantities of money whose utilities may be state-dependent and hence inseparable from subjective probabilities, rendering it impossible to apply the Machina-Schmeidler definition of probabilistic sophistication. Nevertheless, in the state-preference framework, a decision maker with separable preferences is unquestionably “uncertainty neutral” insofar as her preferences have an expected-utility representation, even if the utilities are state-dependent and the probabilities are not unique. Using this alternative standard of uncertainty-neutrality, a decision maker whose preferences are represented by (4) is uncertainty averse by Epstein’s definition if $u_i(v_i^{-1}(\cdot))$ is concave for every $i$. To show this, suppose that there is a decision maker whose preferences are represented by (4) with $\{q_i\}$ distinct and $u_i(v_i^{-1}(\cdot))$ concave for every $i$. For such a person, an act $w$ is unambiguous if the first-order expected utility $\sum_{j=1}^{n} q_{ij} v_i(w_{ij})$ is the same in every credal state $i$, and it is ambiguous otherwise. (Note that ambiguity measured in this way is endogenous to the decision maker. For example, the contents of Ellsberg’s urn might be known to the experimenter but not to the subject.) Now consider a second decision maker whose preferences have the same
representation except that, for the second decision maker, \( u_i = v_i \) and hence \( u_i(v_i^{-1}(\cdot)) \) is linear for every \( i \). Then the second decision maker is uncertainty neutral—she has fully additively separable utility. The two decision makers assign the same first-order expected utility to every act, and the second decision maker evaluates the first-order expected utilities in a risk-neutral manner (by taking expectations with respect to the second-order distribution), while the first decision maker evaluates them in a risk-averse manner (using the same second-order probabilities together with a concave second-order utility function). Hence, the two decision makers will assign identical certainty equivalents to unambiguous acts but the second decision maker will assign lower certainty equivalents to ambiguous acts.

The characterization of risk and uncertainty aversion in this paper applies to general smooth preferences, which differ from the Choquet expected utility preferences that are currently the most popular alternative to subjective expected utility. The differences between the two types of preferences are transparent and have testable implications. Choquet expected utility preferences are the same as subjective expected utility preferences within each comonotonic set, which is a convex cone in payoff space. For example, in two dimensions, the comonotonic sets are the half-planes above and below the line \( x = y \). In three dimensions, the comonotonic sets are six wedges whose cutting edges meet along the line \( x = y = z \). Within each such cone, the decision maker’s indifference curves and risk neutral probabilities are determined by a fixed subjective probability distribution and a state-independent utility function. At the boundaries between cones, the indifference curves are kinked: the subjective probabilities jump to new (usually more pessimistic) values while the marginal utilities remain the same, so the risk neutral probabilities also change discontinuously.
Several details are important. First, the Choquet model requires knowledge of prior wealth in order to determine the comonotonic sets, and constant acts play an even more critical role than they do in the standard theory. All of the usual caveats about the difficulties of observing prior wealth and defining constant acts under naturalistic conditions therefore apply. Second, a Choquet expected utility maximizer displays true uncertainty aversion only when comparing prospects that lie in different comonotonic sets. She is “locally risk averse but uncertainty neutral” and uses a state-independent local Pratt-Arrow measure to compute risk premia for small gambles, ambiguous or otherwise, except when her prior wealth happens to lie on the boundary between two comonotonic sets (e.g., in an idealized state of constant prior wealth). Whereas, under a general smooth preference model such as the partially separable model introduced here, a decision maker may be locally uncertainty averse everywhere in payoff space. Third, when a Choquet expected utility maximizer finds herself on the boundary between two comonotonic sets, she exhibits first-order uncertainty aversion: toward bets on ambiguous events she is risk averse even for infinitesimal stakes. The Choquet model offers one way to model behavior that is first-order uncertainty averse, but not the only way. For example, a subject who has incomplete (partially ordered) preferences due to indeterminate probabilities for ambiguous events could exhibit first-order uncertainty aversion everywhere.

The empirical questions, then, are: (i) whether uncertainty aversion is a first-order or second-order phenomenon, and (ii) whether it affects all choices that involve uncertain events or only choices between alternatives that induce different rankings of states. More specifically, is uncertainty aversion revealed by valuations of small assets only when prior wealth is constant across states? The CEU model localizes uncertainty-averse behavior on the boundaries of comonotonic sets, which just happens to be where the empirical light shines the brightest. It is
easiest to demonstrate violations of Savage’s axioms in choices among simple acts that lead to only two or three distinct consequences with state-independent valuations—e.g., a status quo and one or two prizes—which do not turn on subtle issues of cardinal utility measurement. Practically the only non-trivial choices under such conditions are those in which the acts lie in different comonotonic sets. By comparison, it is rather hard to elicit violations of SEU in choices among acts in the relative interior of the same comonotonic set, because such choices depend sensitively on many cardinal utilities. Nevertheless, it is intuitively plausible that in the Ellsberg urn problem, a subject might “feel” differently toward the two urns regardless of the complexity of the acts pegged to them.

The following hypothetical experiment illustrates the possibility—as well as the difficulty—of eliciting violations of the independence axiom in choices among comonotonic acts. Consider again a two-urn problem in which urn 1 contains equal numbers of red and black balls and urn 2 contains red and black in unknown proportions. Suppose the subject’s preferences are assessed for the following two pairs of bets: (i) win $100 if the ball drawn from urn 1 is Red (“R1”) vs. win $100 if the ball drawn from urn 2 is Red, (“R2”), and (ii) win $100 if the ball drawn from urn 1 is Black (“B1”) vs. win $100 if the ball drawn from urn 2 is Black, (“B2”). Furthermore, suppose that the subject is endowed with the following distribution of prior wealth:

<table>
<thead>
<tr>
<th>Urn 1</th>
<th>Red (50%)</th>
<th>Black (50%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$0</td>
<td>$300</td>
</tr>
<tr>
<td></td>
<td>$200</td>
<td>$100</td>
</tr>
<tr>
<td>Urn 2</td>
<td>Red (unknown %)</td>
<td>Black (unknown %)</td>
</tr>
</tbody>
</table>

Thus, the decision maker’s prior expected wealth is $150 regardless of the proportions of balls in urn 2. Against this background, the four bets are comonotonic. If the subject
nevertheless prefers to bet on the ball drawn from the known urn regardless of the winning color—i.e., if $R_1 > R_2$ and $B_1 > B_2$—then she violates CEU but still could conform to the PSU model.

9. Discussion

The resurrection of cardinal utility theory by von Neumann–Morgenstern and Savage was predicated on the argument that, under conditions of risk and uncertainty, preferences should be separable across mutually exclusive events. Although separability of risk preferences does seem reasonable in many situations, at least as a simplifying assumption, it is no longer accepted as a universal normative or descriptive principle. The currently-most-popular alternative theory, Choquet expected utility, admits a special kind of inseparability by positing that indifference curves are kinked at the boundaries of comonotonic sets—the so-called “45-degree certainty line” in payoff space—while conforming to subjective expected utility theory everywhere else. In giving a central role to the 45-degree certainty line, the CEU model depends very heavily on some other assumptions of subjective expected utility theory that are equally questionable (Shafer 1986), namely that beliefs can be uniquely recovered from preferences, that cardinal utilities are state-independent, and that it is possible to identify a set of riskless acts that have constant consequences for the decision maker.

This paper has presented a simpler, alternative model of non-expected-utility preferences that does not involve kinked indifference curves, uniquely determined beliefs, state-independent utilities, or riskless acts. The decision maker is permitted to display different degrees of risk

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10 In a pilot experiment with students at Duke University, involving $10’s rather than $100’s, a slight majority of subjects exhibited this pattern.
aversion toward different partitions of states of nature, which leads to a simple characterization of aversion to uncertainty, viz., the decision maker is uncertainty averse if she is more risk averse toward ambiguous acts than unambiguous ones. Equivalently, she may behave as though her credal state is uncertain and she is averse to the credal uncertainty. A decision maker may be uncertainty averse by this definition and yet have additive hierarchical probabilities for all events and conform to subjective expected utility theory within a subalgebra of events having the same degree of ambiguity or within a given credal state. This preference model does not necessarily invalidate conventional methods of decision analysis—rather, it suggests a simple way that decision analysis could be extended to account for model risk—but it does cast doubt on the common practice of assessing utility functions for naturalistic decisions by contemplating bets on objective randomization devices.

Acknowledgements

This research was supported by the National Science Foundation under grant 98-09225 and by the Fuqua School of Business. I am grateful for the comments of two anonymous referees at ISIPTA ‘01. The opinions expressed here, and responsibility for any errors or omissions, are entirely my own.

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