UNCERTAINTY AVERSION WITH SECOND-ORDER PROBABILITIES AND UTILITIES

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Uncertainty Aversion

- The tendency of a decision maker to prefer “unambiguous” acts or gambles over ambiguous ones, as illustrated in Ellsberg’s paradoxes

- Has a variety of (conflicting!) formal definitions that are motivated by features of different analytical frameworks and preference models

- Most definitions of ambiguity are framed as generalizations of Savage’s or Anscombe-Aumann’s axiomations of subjective probability

  Schmeidler 1989  
  Epstein 1999, Epstein & Zhang 2001  
  Ghiradato/Marinacci 2001  
  Ghirardacho/ Maccheroni/Marinacci 2002  
  Casadesus et al. 2000  
  Nehring 2001

- Their goal is to generalize the representation of beliefs so as to embrace some notion of “multiple priors”
In most of those models:

• *Unique* separation of a *belief measure* from a *state-independent cardinal utility function* is central to the characterization of uncertainty aversion, in the tradition of Savage

• Uncertainty aversion is conceived as a form of *first order* risk aversion, i.e., it is associated with kinked indifference curves (*unlike* the Pratt-Arrow characterization of risk aversion)

• Uncertainty aversion is displayed only for choices among prospects that straddle a kink, i.e., the decision maker may not be locally uncertainty averse at all (or even most) prior wealth positions.

• Prior wealth is assumed to be observable, because it is important to know the decision maker’s position relative to the kink (i.e., to know how states are ranked by different acts in terms of utility of final consequences)
Objective of this paper: frame the characterization of ambiguity aversion as an extension of the Pratt-Arrow characterization of risk aversion, as generalized to the state-preference framework by Yaari (1969) and Nau (2001)

- Preferences are assumed to be smooth, so that both risk aversion and uncertainty aversion are second-order effects

- An uncertainty averse decision maker is one who is locally uncertainty averse everywhere

Intuition:

- If the decision maker is permitted to be risk averse, why can’t she simply be more risk averse toward Ellsberg’s second urn?

- Even when she has significant prior stakes, the DM may still “feel” differently toward the two urns
Advantages of this approach:

- There is no need to uniquely separate beliefs from cardinal utilities—*the definition and measurement of risk aversion and uncertainty aversion do not require it*
- Preferences may be state-dependent
- Prior wealth may be stochastic and unobservable
- “Riskless” acts play no distinguished role

*IMHO, very little of decision theory or economics really requires the unique separation of beliefs from cardinal utilities.*
Key results:

- Uncertainty aversion has a simple behavioral definition, generalizing the two-urn paradox

- Extremely simple axioms of “partially separable” preferences lead to a representation of uncertainty-averse preferences of the form

$$U(w) = \sum_{i=1}^{m} u_i \left( \sum_{j=1}^{n} v_{ij}(w_{ij}) \right)$$

where risk aversion is measured by $$-v''_i / v'_i$$ and uncertainty aversion is measured by $$-u''_i / u'_i$$

- Similar to composite utility functions previously used by Kreps/Porteus, Segal, and others, in models of decision under risk

- A state-independent version of the essentially the same representation has been axiomatized by Klibanoff et al. (2002, this meeting)
Preliminaries: how to generalize the definition of risk aversion?

- There are several different approaches to defining risk aversion, which are all equivalent for state-independent EU preferences.
- They are not equivalent for state-dependent and/or non-EU preferences.

**Definition I (Epstein 1999 & others):** a preference order is risk averse if it is *more risk averse than some risk neutral preference order*, i.e., if there is some probability distribution $p$ such that “constant” wealth equal to $E_p[w]$ is always preferred to risky wealth $w$.

  ➢ This definition generalizes the notion that a risk averse DM prefers a riskless wealth position to a risky position with the same expected value.

**Definition II (Yaari 1969):** a preference order is risk averse if it is *payoff-convex*, i.e., if $w$ is preferred to $y$, then $\alpha w + (1-\alpha)y$ is preferred to $y$.*

  ➢ This definition generalizes (and strengthens) the notion that a risk averse DM has diminishing marginal utility for money in every state of the world.

*${\alpha w + (1-\alpha)y}$ means the wealth distribution whose value in state $s$ is $\alpha w(s) + (1-\alpha)y(s)$, i.e., a pointwise deterministic mixture of payoffs, not a probabilistic mixture.*
Risk-averse non-EU preferences: Definition I

Risk-neutral indifference curves

“45-degree certainty line”

DM is locally risk-averse at "riskless" wealth positions...

Risk-averse indifference curves enclose not-necessarily-convex sets of more-preferred wealth-distributions

...but may be locally risk-seeking elsewhere

DM is locally risk-averse at all wealth positions

Risk-averse indifference curves enclose convex sets of more-preferred wealth-distributions

Risk-averse non-EU preferences: Definition II
Features of Definition I:

- Riskless wealth distributions (i.e., “constant” acts) play a key role.

- A risk averse decision maker is not necessarily locally risk averse except when prior wealth is riskless.

- At some stochastic wealth positions, the DM may be locally risk seeking in that she would pay a positive amount for any fair gamble.

- State-dependence of utility leads to additional complications in the definition of a riskless act (e.g., Karni 1985).

Features of Definition II:

- Riskless acts do not necessarily play a distinguished role.

- A risk averse decision maker is locally risk averse everywhere.

- State-dependence of utility and stochasticity of prior wealth do not lead to complications.

Definition II will be adopted here.
DM is first-order risk- or uncertainty-averse at her current wealth position if the risk premium of $z$ is proportional to $|z|$, i.e., if her indifference curve is “kinked” there.

DM is second-order risk- or uncertainty averse if the risk premium of $z$ is proportional to $|z|^2$, i.e., if the indifference curve is “smooth” at the current wealth position.

- Uncertainty aversion in the Choquet expected utility model & related multiple-prior models is a form of 1st-order risk aversion.

- Pratt-Arrow measure is a measure of 2nd-order risk aversion.

(Only) second-order risk and uncertainty aversion will be considered here.
The Model (State-PREFERENCE Framework)

- There are \( n \) states of the world

- Wealth distributions are vectors in [some convex subset of] \( \mathbb{R}^n \)

**Assumption 1:** The decision maker’s preferences among wealth distributions satisfy the usual axioms of consumer theory (reflexivity, completeness, transitivity, continuity, and monotonicity), as well as a smoothness property, so that they are representable by a twice-differentiable ordinal utility function \( U(w) \) that is a non-decreasing function of wealth in every state.

- The DM is defined to be *risk averse* if her preferences are payoff-convex, which is equivalent to quasi-concavity of \( U \).
First-order properties of local preferences are determined by the risk neutral probability distribution

- $U$ is unique only up to monotonic transformations, and hence is not observable.

- The gradient of $U$ at wealth $w$, normalized so that its components sum to 1, is observable, and is called the DM’s risk neutral probability distribution $\pi(w)$ whose $j^{th}$ element is

$$
\pi_j(w) = \frac{\partial U(w)/\partial w_j}{\sum_{i=1}^n \partial U(w)/\partial w_i}.
$$

- The risk neutral distribution determines the decision maker’s marginal prices for risky assets and betting rates for very small (infinitesimal) stakes

- A state will be defined to be non-null if it has a strictly positive risk neutral probability at every wealth position
Indifference curves in payoff space:

Risk neutral probabilities are the observable parameters of belief in decisions, games and markets.
**Prices and risk premia**

Let $P(z; w)$ denote the *marginal price* that the decision maker is willing to pay for $z$, in the sense that she is willing to pay $\alpha P(z; w)$ to receive $\alpha z$ in the limit as $\alpha$ goes to zero. Then $P(z; w)$ satisfies:

$$P(z; w) = z \cdot \pi(w) \equiv E_{\pi(w)}[z] \quad \text{("risk neutral valuation")}$$

The buying price for a finite asset $z$, denoted $B(z; w)$, is determined by

$$U(w + z - B(z; w)) - U(w) = 0$$

The buying ("compensating") risk premium of $z$ at wealth $w$ is

$$b(z; w) = E_{\pi(w)}[z] - B(z; w).$$

(Proposition) The decision maker is risk averse if and only if her buying risk premium is non-negative for every asset at every wealth distribution.
Second-order properties of local preferences are determined by the risk aversion matrix

The local risk aversion matrix is the matrix $R(w)$ whose $jk^{th}$ element is the following ratio of second to first derivatives:

$$r_{jk}(w) = -\left(\frac{\partial^2 U(w)}{\partial w_j w_k} / \frac{\partial U(w)}{\partial w_j}\right)$$

…a multivariate generalization of $r(x) = -u''(x)/u'(x)$.

Risk premium formula: The risk premium of a small neutral asset $z$ satisfies

$$b(z; w) \approx \frac{1}{2} z \cdot \Pi(w) R(w) z$$

where $\Pi(w) = \text{diag}(\pi(w))$. Hence is it appropriate to consider $R(w)$ as a matrix-valued generalization of the Pratt-Arrow measure. (Nau 2001)

Claim: the structure of the risk aversion matrix $R(w)$ encodes both attitude toward risk and attitude toward uncertainty, in terms that do not require the unique separation of beliefs from cardinal utilities. The local risk premium may be decomposed into a “risk” component and an “uncertainty” component.
**Uncertainty Neutrality**

*(Separable Preferences)*

If the DM’s preferences satisfy the coordinate independence axiom (Savage’s P2), the utility function is additively separable:

\[
U(w) = v_1(w_1) + \ldots + v_n(w_n).
\]

- Its cross-derivatives are zero and \(R(w) = \text{diag}(r(w))\), where \(r(w)\) is a vector-valued Pratt-Arrow measure of risk aversion whose \(j^{\text{th}}\) element is

\[
r_j(w) = - \left( \frac{\partial^2 U(w) / \partial w_j^2}{\partial U(w) / \partial w_j} \right) = - \frac{v_j''(w_j)}{v_j'(w_j)}.
\]

- Hence the DM’s local preferences are described (up to second order) by a pair of numbers for each state: a risk neutral probability and a risk aversion coefficient.

**Risk premium formula revisited:** For a decision maker with separable preferences, the risk premium of a small neutral asset \(z\) is:

\[
b(z; w) \approx \frac{1}{2} E_{\pi(w)}[r(w)z^2]
\]
Remarks:

• A decision maker with separable preferences is “uncertainty neutral” in the sense that her preferences have a state-dependent expected-utility representation.

• Under general conditions of state-dependent preferences and stochastic prior wealth, the decision maker’s probabilities cannot be uniquely separated from her probabilities—nor do they need to be!

• Local preferences are completely described by the risk neutral probability distribution and the risk aversion matrix, “finessing away” the complications of state-dependent preferences and stochastic prior wealth.
UNCERTAINTY AVERSION

A decision maker is *uncertainty averse* if she is more averse toward bets on ambiguous events than bets on unambiguous events.

- In the present framework, “more averse” means having a higher second-order risk premium.
- Because no attempt is made to isolate “true” probabilities of events, and since uncertainty aversion is revealed (only) by patterns of variation in risk premia, it is necessary to either (i) assume the direction of uncertainty attitude (averse/seeking), or (ii) assume the existence of events that are a priori unambiguous.

**Assumption 2:** There is a set of unambiguous events, closed under complementation and disjoint union, at least one of which is a union of two or more non-null states and whose complement is also a union of at two or more non-null states.
**Δ-spreads:** assets that reveal departures from separability

Let $A$ and $B$ denote two logically independent events, let $\{\pi_{AB}, \pi_{A\bar{B}}, \pi_{\bar{A}B}, \pi_{\bar{A}\bar{B}}\}$ denote the local risk neutral probabilities (at wealth $w$) of the four possible joint outcomes of $A$ and $B$.

Let $\Delta$ denote a quantity of money (just) large enough in magnitude that second-order utility effects are relevant.

Then an $A:B \Delta$-spread and a $B:A \Delta$-spread are defined as the neutral assets whose payoffs are given by the following tables:

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<tbody>
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<td>$A$</td>
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<td>$\bar{A}$</td>
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<td>$-\Delta/\pi_{\bar{A}\bar{B}}$</td>
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**Observations:**

- Each cell contributes $\pm \Delta$ to the total risk neutral expected value.
- A DM with separable preferences must be indifferent between the two assets (i.e., assign the same risk premia)
A behavioral definition of uncertainty aversion

The decision maker is *locally uncertainty averse at wealth* $w$ if, for every unambiguous event $B$, every event $A$ that is logically independent of $B$, and any $\Delta$ sufficiently small in magnitude (positive or negative), a $B:A$ $\Delta$-spread is weakly preferred to (i.e., has a risk premium less than or equal to that of) an $A:B$ $\Delta$-spread. The decision maker is *uncertainty averse* if she is locally uncertainty averse at every wealth position.

Comparative ambiguity aversion

If $A_1$ and $A_2$ are logically independent and their four joint outcomes are non-null, then an uncertainty averse decision maker regards $A_1$ as less ambiguous than $A_2$ if for any $\Delta$ sufficiently small in magnitude (positive or negative), an $A_1:A_2$ $\Delta$-spread is strictly preferred to an $A_2:A_1$ $\Delta$-spread at every wealth distribution.
Additional structure: Let the state space consist of a Cartesian product \( A \times B \), where \( A = \{ A_1, \ldots, A_m \} \) and \( B = \{ B_1, \ldots, B_n \} \) are finite partitions.

Possible interpretations:
I. \( A \)-measurable events are *potentially ambiguous* while \( B \)-measurable events are a priori unambiguous

II. \( A \)-measurable events are internal *credal states* of the decision maker while \( B \)-measurable events are external, payoff relevant events.

Assumption 3 (“partial separability” of preferences)

**A-independence:**  
\[ Ew + (1-E)z \geq Ew^* + (1-E)z \iff Ew + (1-E)z^* \geq Ew^* + (1-E)z^* \]  
for all acts \( w, w^*, z, z^* \) and every \( A \)-measurable event \( E \), and conditional preference \( w \geq_E w^* \) is accordingly defined for such events.

**B-independence:**  
\[ Fw + (1-F)z \geq_i Fw^* + (1-F)z \iff Fw + (1-F)z^* \geq_i Fw^* + (1-F)z^* \]  
for every \( B \)-measurable event \( F \), where \( \geq_i \) denotes conditional preference given element \( A_i \) of \( A \).
PROPOSITION:
(i) Under the preceding assumptions, preferences are represented by a utility function $U$ having the composite-additive* form:

$$U(w) = \sum_{i=1}^{m}u_{i}\left(\sum_{j=1}^{n}v_{ij}(w_{ij})\right)$$

where $w_{ij}$ denotes wealth in state $A_{i}B_{j}$, and \{u_{i}\} and \{v_{ij}\} are non-decreasing twice-differentiable state-dependent utility functions.

(ii) The local risk aversion matrix $R(w)$ is the sum of a diagonal matrix and a block-diagonal matrix, with $r_{ij,kl}(w) = 0$ if $i \neq k$ and

$$r_{ij,ii}(w) = \frac{u_{i}''\left(\sum_{h=1}^{n}v_{ih}(w_{ih})\right)}{u_{i}'\left(\sum_{h=1}^{n}v_{ih}(w_{ih})\right)}v_{il}'(w_{il}) + (v_{ij}''(w_{ij})/v_{ij}'(w_{ij}))\delta_{jl}$$

(iii) The decision maker is uncertainty averse if $u_{i}$ is concave for every $i$.

* Similar composite utility functions have been used by Kreps/Porteus (1979), Segal (1989), and Grant et al. (1998) in models of decision under risk involving temporal resolution of uncertainty or 2-stage lotteries.
Risk premium of an $A:B$ $\Delta$-spread

\[
\begin{bmatrix}
+ & + & 0 & 0 \\
+ & + & 0 & 0 \\
0 & 0 & + & + \\
0 & 0 & + & +
\end{bmatrix}
\]

Risk premium of a $B:A$ $\Delta$-spread

\[
\begin{bmatrix}
+ & + & 0 & 0 \\
+ & + & 0 & 0 \\
0 & 0 & + & + \\
0 & 0 & + & +
\end{bmatrix}
\]

- The diagonal part of $R(w)$ is composed of terms proportional to $-v_{ij}''/v_i'$ that measure aversion to risk.
- The block-diagonal part of $R(w)$ is composed of terms proportional to $-u_i''/u_i'$ that measure aversion to uncertainty.
- Because the off-diagonal elements are positive (if $u$ is concave) and fall into a block-diagonal pattern, the $B:A$ $\Delta$-spread has a lower risk premium—i.e., the DM is uncertainty averse.
Special case of interpretation I: “partially separable utility”

Suppose the component utility functions are state-independent expected utilities of the form $u_i(v) = p_i u(v)$ and $v_{ij}(x) = q_{ij}v(x)$, where $p$ is a marginal probability distribution on $A$ and $q_i$ is a conditional probability distribution on $B$ given $A_i$, yielding:

$$U(w) = \sum_{i=1}^{m} p_i u\left(\sum_{j=1}^{n} q_{ij}v(w_{ij})\right)$$

- Then the decision maker behaves as though she assigns probability $p_i q_{ij}$ to state $A_iB_j$ and she bets on events measurable with respect to $A$ as though her utility function were $u(v(x))$.
- If $A$ and $B$ are also independent, i.e., if $q_i$ is the same for all $i$, she meanwhile bets on events measurable with respect to $B$ as though her utility function for money were $v(x)$.
- If $u$ is concave, she is uniformly more risk averse with respect to $A$-measurable bets than to $B$-measurable bets, implying she is averse to uncertainty. Thus, concavity of $v$ encodes aversion to risk while concavity of $u$ encodes aversion to uncertainty of $A$-measurable events.
Example: Ellsberg’s 2-color paradox

Let $A_1 [A_2]$ denote the event that the ball drawn from the *unknown* urn is red [black], and let $B_1 [B_2]$ denote the event that the ball drawn from the *known* urn is red [black]. The relevant state space is then $\{A_1B_1, A_1B_2, A_2B_1, A_2B_2\}$.

Let $w = (w_{11}, w_{12}, w_{21}, w_{22})$ denote the decision maker’s wealth distribution, where $w_{ij}$ is wealth in state $A_iB_j$, and suppose that she evaluates wealth distributions according to the following non-separable utility function:

$$U(w) = -\frac{1}{2} \exp(-\alpha(\frac{1}{2}w_{11} + \frac{1}{2}w_{12})) - \frac{1}{2} \exp(-\alpha(\frac{1}{2}w_{21} + \frac{1}{2}w_{22}))$$

For bets on the known urn, the DM is risk neutral. For bets on the unknown urn, she is risk averse with a Pratt-Arrow risk aversion coefficient equal to $\alpha$. 
Special case of interpretation II: second-order utilities and probabilities ("SOUP?")

Let the partition \( A \) consist of credal states while \( B \) consists of payoff relevant events, and henceforth let \( w_j \) denote wealth in event \( B_j \in B \). Let the composite-additive utility function be written as

\[
U(w) = \sum_{i=1}^{m} p_i u_i(CE_i(w)),
\]

where

\[
CE_i(w) = v_i^{-1}\left(\sum_{j=1}^{n} q_{ij} v_i(w_j)\right)
\]

is the first-order certainty equivalent of wealth \( w \) in credal state \( i \), based on utility function \( v_i \) and first-order distribution \( q_i \).
Model for Ellsberg’s 2-color paradox, revisited:

\[ U(w) = -\frac{1}{2} \exp(-\alpha(\frac{1}{2}w_{11} + \frac{1}{2}w_{12})) - \frac{1}{2} \exp(-\alpha(\frac{1}{2}w_{21} + \frac{1}{2}w_{22})) \]

**SOUP interpretation:** \( u(x) = -\exp(-\alpha x) \) and \( v(x) = x \). The decision maker is risk neutral, but she is uncertain about the number of red balls in the unknown urn, which she feels is equally likely to be 100 or 0. She is averse to uncertainty with an uncertainty-aversion coefficient equal to \( \alpha \).

Model for Ellsberg’s 3-color paradox:

A single urn contains 30 red balls and 60 balls that are black and yellow in unknown proportions. Let \( (w_1, w_2, w_3) \) denote the decision maker’s wealth in outcomes \( R, B, \) and \( Y \), respectively, and suppose that her utility function is the following:

\[ U(w) = -\frac{1}{2} \exp(-\alpha(\frac{1}{3}w_1 + \frac{2}{3}w_2)) - \frac{1}{2} \exp(-\alpha(\frac{1}{3}w_1 + \frac{2}{3}w_3)). \]

**SOUP interpretation:** same as preceding 2-color model, except that here the decision maker thinks it is equally likely that the number of black balls is 60 or 0.
SOUP versus MEU:

• As \( u \) becomes more risk averse (e.g., \( u(x) = -\exp(-\alpha x) \) as \( \alpha \to \infty \)), the SOUP model converges to MEU because it pays attention to only the worst first-order certainty equivalent, just as a pathologically risk averse SEU decision maker pays attention to only the worst-case outcome.

SOUP versus hierarchical Bayes:

• If \( u \) is linear, the SOUP model is equivalent to a hierarchical Bayes model.
• As \( u \) becomes nonlinear (e.g., \( u(x) = -\exp(-\alpha x) \) as \( \alpha \) increases from 0), valuations produced by the SOUP model move smoothly away from those produced by the hierarchical Bayes model.
• The SOUP decision maker violates independence, hence does not have well-defined conditional beliefs at downstream events, but nevertheless is “close” to being Bayesian.
SOUP vs. SEU as an explanation of risk aversion in small-stakes gambles

- The decision maker could have almost linear utility for money, but a low tolerance for uncertainty. Even supposedly “objective” gambles could be regarded skeptically as having somewhat uncertain probabilities.
- Example: in the stock market, mean returns are highly uncertain while variances are can be measured and predicted with considerable precision. Investors may have a high tolerance for risk but a low tolerance for uncertainty in the mean returns. (Possible explanation of equity premium puzzle??)

SOUP model versus Klibanoff et al. model of smooth ambiguity-averse preferences

- Essentially the same representation of preferences, although derived from every different axioms.
- State-independence of utility and unique separation of beliefs and utilities are not important in the SOUP model.
CONCLUSIONS

• Using the state-preference framework and an assumption of smooth preferences, the Pratt-Arrow measure can be generalized and decomposed into risk-aversion and uncertainty-aversion components.

• Simple axioms of partially separable preferences lead to representations of uncertainty-averse preferences in terms of “partially separable utility” and “second-order utilities and probabilities”.

• Risk neutral probabilities, rather than “true” probabilities, play a central role as local measures of belief: ambiguity can be characterized without unique measures of pure belief.