Supplement to "The Shape of Incomplete Preferences"
Mathematical programs for the example in section 5

The original basis vectors for $\mathcal{B}^{**}$ in this example are as follows:

$$\begin{aligned}
\mathbf{B}_1 &= \mathbf{E}_1(\mathbf{H}_2 - \mathbf{H}_0), \quad \mathbf{B}_2 = \mathbf{E}_2(\mathbf{H}_2 - \mathbf{H}_0), \quad \mathbf{B}_3 = \mathbf{E}_3(\mathbf{H}_2 - \mathbf{H}_0),\\
\mathbf{B}_4 &= \mathbf{E}_1(\mathbf{H}_1 - \mathbf{H}_2), \quad \mathbf{B}_5 = \mathbf{E}_2(\mathbf{H}_1 - \mathbf{H}_2), \quad \mathbf{B}_6 = \mathbf{E}_6(\mathbf{H}_1 - \mathbf{H}_2),\\
\mathbf{B}_7 &= \mathbf{H}_{\mathbf{E}_1} - \mathbf{H}_{0.1}, \quad \mathbf{B}_8 = \mathbf{H}_{\mathbf{E}_2} - \mathbf{H}_{0.1}, \quad \mathbf{B}_9 = \mathbf{H}_{\mathbf{E}_3} - \mathbf{H}_{0.1},\\
\mathbf{B}_{10} &= \mathbf{X} - \mathbf{H}_{0.5}\\
\mathbf{B}_{11} &= \mathbf{X} - \mathbf{H}_{0.5} + \mathbf{E}_1(\mathbf{H}_2 - \mathbf{H}_{0.9})\\
\mathbf{B}_{12} &= \mathbf{X} - \mathbf{H}_{0.5} + \mathbf{E}_2(\mathbf{H}_{0.1} - \mathbf{H}_2)
\end{aligned}$$

Note that $\mathbf{B}_1, \ldots, \mathbf{B}_6$ are the contribution of axioms A4 and A6 together; $\mathbf{B}_7$, $\mathbf{B}_8$, and $\mathbf{B}_9$ are the differences between LHS and RHS of (5.1) for $\mathbf{E} = \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$; and $\mathbf{B}_{10}$, $\mathbf{B}_{11}$, and $\mathbf{B}_{12}$ are the differences between LHS and RHS of (5.2)–(5.4), after dropping the multiplicative constants $\frac{1}{2}$ and cancelling the common $\mathbf{Z}$ terms.

Axioms A1–A4 establish that the cone $\mathcal{B}^{**}$ representing this preference relation minimally consists of the convex hull of the rays whose directions are $\mathbf{B}_1, \ldots, \mathbf{B}_{12}$. The question is whether axiom A6 enlarges this cone. To prove that it does not, we can solve a series of linear programs to show that there are no non-negative linear combinations of $\mathbf{B}_1, \ldots, \mathbf{B}_{12}$ which are of the form $\mathbf{EB}$, where $\mathbf{E}$ is an event and $\mathbf{B}$ is constant, and which are nontrivial in the sense that $[\mathbf{B}]_{\min} < 0$. (Recall that $[\mathbf{B}]_{\min}$ is defined as the minimum possible state-dependent expected utility assignable to $\mathbf{B}$. If this is not negative, then $\mathbf{B}$ is simply a linear combination of $\mathbf{B}_1, \ldots, \mathbf{B}_6$, as is $\mathbf{E}'\mathbf{B}$ for any other event $\mathbf{E}'$, so the application of A6 to this $\mathbf{B}$ does not enlarge the cone.)

If $\mathbf{B}$ is constant, then $[\mathbf{B}]_{\min}$ is the minimum of $B(s,1)$ and $B(s,1)+B(s,2)$ for any event s. To formulate this problem as a linear program, we must separately minimize each of these two quantities over all non-negative $\alpha_1, \ldots, \alpha_{12}$ such that

$$\sum_i \alpha_i \mathbf{B}_i = \mathbf{EB},$$

$$B(1,c) = B(2,c) = B(3,c), \quad c = 0, 1, 2,$$

for every possible conditioning event $\mathbf{E}$. There are seven distinct events that are subsets of S in this case, namely $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, $\{\mathbf{E}_1, \mathbf{E}_2\}$, $\{\mathbf{E}_1, \mathbf{E}_3\}$, $\{\mathbf{E}_2, \mathbf{E}_3\}$, $\{\mathbf{E}_1\}$, $\{\mathbf{E}_2\}$, and $\{\mathbf{E}_3\}$. Altogether, we must solve 14 linear programs (2 objective functions $\times$ 7 conditioning events) to see whether any has a negative optimal objective value. As it happens, none does. Hence, the cone $\mathcal{B}^{**}$ is merely the set of non-negative linear combinations of $\mathbf{B}_1, \ldots, \mathbf{B}_{12}$. To establish the lower expected utility for $\mathbf{X}$ that can be deduced from A1–A6, we can solve the primal inference problem, namely find the maximum value of u for which

$$\sum_i \alpha_i \mathbf{B}_i = \mathbf{X} - \mathbf{H}_u$$

has a solution in non-negative $\alpha_1, \ldots, \alpha_{12}$. This maximum turns out be 0.5.

Alternatively, we can find the lower expected utility for $\mathbf{X}$ by solving the dual problem, which is:

$$\min_v \; U_v(\mathbf{X})$$

subject to

$$U_v(\mathbf{B}_i) \geq 0, \quad i = 1, \ldots, 12,$$
$$v \geq 0, \quad \sum_s v(s,1) = 1$$

This yields, as before, a value of 0.5 for the lower expected utility of $\mathbf{X}$ given A1–A6.

To determine the lower expected utility of $\mathbf{X}$ among all *probability/utility pairs* agreeing with the basis, which would be the lower expected utility given A1–A7, we can solve the same problem with the additional nonlinear constraints:

$$v(1,2)/v(1,1) = v(2,2)/v(2,1) = v(3,2)/v(3,1).$$

Non-negativity of the denominators is ensured by the lower-probability constraints. This nonconvex program has a global optimum of 0.564314, which is achieved at $p(1) = 0.41,\ p(2) = 0.1,\ p(3) = 0.49,\ u(2) = 0.743137$. There is also a local minimum of 0.565306 achieved at $p(1) = 0.1,\ p(2) = 0.44444\ldots,\ p(3) = 0.45555\ldots,\ u(2) = 0.246939$. These programs were formulated and solved using Microsoft Excel; copies of the worksheet file are available from the author.