Market Microstructure With Uncertain Information Precision: A New Framework

Simon Gervais[1]
Finance Department
Wharton School
University of Pennsylvania
Steinberg Hall – Dietrich Hall
Suite 2300
Philadelphia, PA 19104-6367
(215) 898-2370
e-mail: gervais@wharton.upenn.edu

[1]I would like to thank Gail Belonsky, Jeff Bohn, Raul Espejel, Robert Goldstein, Steve Hillegeist, Hayne Leland, Richard Lyons, Jocelyn Martel, Stewart Mayhew, David Romer, Mark Rubinstein, and seminar participants at l'Université Laval, UC Berkeley, and the 1995 meetings of the Northern Finance Association and the Financial Management Association for their advice and comments. I would also like to especially thank Matthew Spiegel and Brett Trueman for their numerous suggestions and helpful comments. Financial Support by the Conseil de recherches en sciences humaines du Canada (Award No. 752-92-0225) is gratefully acknowledged.

Abstract

This paper provides the financial market microstructure literature with a relatively simple framework to accomodate the possibility for the precision of information to be random. Indeed, many researchers have discovered that the usual normal/exponential framework used in asymmetric information models is intractable when the precision of information is random. This is why we introduce a setting similar to that of Glosten and Milgrom (1985) in which shares of a risky asset can only be traded one at a time.

We find that the bid-ask spread and insider profits both increase with a first-order stochastic shift of the information precision, but decrease with a second order stochastic shift. On the other hand, the effect of these stochastic shifts on trading volume is in general ambiguous, since such shifts produce two counter-balancing effects. However, conditions are derived in which the net effect is known. Also, the model's conclusions are shown to be robust to settings that allow for multiple trade sizes, insider risk aversion, elacticity of liquidity trader demand, and market power of the market-maker.

The problem solved in this paper is in fact part of a more general problem of uncertain trader characteristics, which has been largely neglected by the financial economics literature. Indeed, financial economists always assume that the characteristics of traders (risk aversion, wealth, information distribution) are common knowledge. Our study is therefore a first step towards understanding the effects of uncertain trader characteristics on financial markets.

1 Introduction

Most of the models of financial market microstructure make the seemingly inconsequential assumption that all the trader characteristics are common knowledge. In particular, it is always assumed that the traders' risk aversion, their wealth, and the distribution of their information is known by all the market participants. Casual observation of financial markets reveals that this is unlikely to be the case.

In this paper, we take a first step in taking uncertain trader characteristics into account by introducing uncertain information precision into an asymmetric information model of a financial market. This is done by assuming that, in addition to observing a private signal about the future value of a risky asset, the insider is also the only market participant to know the precision of that information. To avoid the difficult problem of updating normal distributions with random variances encountered by many financial economists, we adopt a framework inspired by that of Glosten and Milgrom (1985) in which shares of a risky asset are traded one at a time.[1] In addition to making the introduction of random precision easier, this framework also allows us to use very general distributions for the risky asset's payoff, the informed trader's information, and its precision.

Before describing the model, it is important to clarify what we mean by information precision, and the role it plays in existing market microstructure models. In particular, it is essential to understand the differences between information, as financial models usually describe it, and information precision, which only a few authors have considered.

The first and main difference between information and information precision comes from how they are interpreted by their owner: the information tells its owner whether to buy (good information) or sell (bad information), whereas the precision level of a piece of information dictates the aggressiveness with which to do so. Since good precision translates into more aggressive trading (buying *or* selling), we can already conjecture that there will be a relationship between the properties of precision and the properties of trading volume in a given economy.

Another difference between information and information precision comes from their respective link with the assets that they are related to. In particular, information precision does not need to be correlated with the underlying assets' returns itself to have an impact on the market equilibrium. Indeed, as long as information precision describes the properties of the relationship between the information and the underlying assets, it will have effects on the price, transaction volume, and volatility of these underlying assets. This second difference between information and information

[1] For models which avoid the normality assumption, see Ausubel (1990), and Rochet and Vila (1994). For models which use the normality assumption along with a random variance, see Romer (1993), Blume, Easley, and O'Hara (1994), and Spiegel and Subrahmanyam (1994).

precision has important conceptual implications. For one, the knowledge of an informed trader's information precision may well be completely useless without the knowledge of his information. The opposite is not true, since the possession of information, even without its precision, tells its owner about the direction of the market.

In this paper, we introduce a market microstructure framework for analyzing issues brought about by the possibility of uncertain information precision. In particular, we use the above observations to construct a model which agrees with our intuitive understanding of random precision, and we then look at the impact that this relatively new concept has on a number of variables in the economy. Our model has the advantages of being fairly general while reasonably tractable.

We find that a first order stochastic shift in the distribution of information precision has a positive impact on the bid-ask spread and insider profits, but that a second order stochastic shift reduces these quantities. The effects of these stochastic shifts on trading volume are in general ambiguous, since such shifts produce two counter-balancing effects: an insider aggressiveness effect, and a bid-ask spread effect. However, conditions are derived in which the sign of the net effect is known. In particular, a positive shift in the whole distribution of precision induces an increase in volume, since the insider aggressiveness effect then always dominates the bid-ask spread effect.

These results suggest that the adverse selection component of the bid-ask spread, as described by Copeland and Galai (1983) and Glosten (1987) among others, can itself be decomposed into two subcomponents: the asymmetric information component, and the random precision component. Also, since insiders realize profits only when their precision is high enough to allow them to trade, their profits can be viewed as the payoff of a call option on their precision. This is why we also find that insiders prefer the distribution of their information precision to be disperse.

Again, we insist on the fact that our analysis of uncertain information precision is not unlike that which would result if we assumed uncertain trader risk aversion or wealth. Indeed, like information precision, both of these trader characteristics dictate the intensity with which a trader will buy or sell risky securities,[2] and neither is directly correlated with the securities' payoffs. So it is conjectured that the results derived in this paper could very well apply to a market with these uncertain trader characteristics.

The paper is organized as follows. We start by introducing in section 2 the economy and the assumptions of our one-period model. We show in section 3 that the model possesses a unique equilibrium, whose properties are derived in section 4. To show the robustness of our framework, an extension of the model is considered in section 5. Finally, section 6 concludes and discusses

[2]As a trader's risk aversion decreases (increases), his trading gets more (less) intense. Similarly, when absolute risk aversion is decreasing, a trader trades more (less) aggressively as his wealth goes up (down).

some topics for future research.

Throughout the paper, the symbol "$\subset$" is taken to mean strict inclusion, i.e. "$A \subset B$" is equivalent to "$A \subseteq B$ and $A \neq B$." Also, for any random variable $\hat{X}$ and information set $\mathcal{I}$, the expression $\mathit{Supp}(\hat{X}|\mathcal{I})$ will be used to denote the support of the variable $\hat{X}$ given information $\mathcal{I}$.

2 The Economy

We consider a two-date (zero and one) one-period economy consisting of two securities, one trader, and one market-maker. The first security is a riskless asset with a sure payoff of one dollar at date one. For convenience, its return is normalized to be zero, so that its price remains at 1 at both date zero and date one. The other security is a risky asset whose payoff at date one is a (non-degenerate) random variable $\hat{v}$, which takes values in $\mathbb{R}$. For tractability, we assume that the risky asset can only be traded one share at a time, that is the trader can only buy one share, sell one share, or decide not to trade.[3]

At the beginning of the period, the trader privately observes his own type $\hat{\tau}$, which takes one of two possible values: 1 or 0. A priori, $\Pr\{\hat{\tau} = 1\} = 1 - \Pr\{\hat{\tau} = 0\} = q$, where $q \in (0, 1)$. If $\hat{\tau} = 1$, the trader is a risk-neutral *informed trader* (also called *insider*) who seeks to maximize his expected profits for the period. Before observing the bid and ask prices b and a, and optimally choosing his demand $\hat{x}_{a,b}$, this insider receives two pieces of information: a signal $\hat{\theta}$ about $\hat{v}$, and the precision $\hat{s}$ of that signal. Both $\hat{\theta}$ and $\hat{s}$ take values in $\mathbb{R}$. For reasons to be made clear later, we assume that $\mathit{Supp}(\hat{v}, \hat{\theta}, \hat{s})$ is closed and bounded. For notational convenience, we let $\bar{v} \equiv \max \mathit{Supp}(\hat{v})$ and $\underline{v} \equiv \min \mathit{Supp}(\hat{v})$, and we define $\bar{\theta}$, $\underline{\theta}$, $\bar{s}$ and $\underline{s}$ similarly. Also, for now, we only make two assumptions about the distribution of $\hat{v}$ conditional on $\hat{\theta}$ and $\hat{s}$. The first assumption is meant to capture the intuition that a higher signal translates into higher expectations of $\hat{v}$.

Assumption 1 *For any $s \in \mathit{Supp}(\hat{s})$, the conditional mean of $\hat{v}$ given that $(\hat{\theta}, \hat{s}) = (\theta, s)$, $\mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)]$, is increasing in $\theta \in \mathit{Supp}(\hat{\theta})$.*

The second assumption concerns the weight that the insider should place on that signal. More precisely, we would expect an insider with a very high signal precision $\hat{s}$ to put a larger weight on his signal $\hat{\theta}$ than an insider with a low signal precision. In other words, if a given signal $\hat{\theta}$ tells the insider that the risky asset should appreciate (depreciate), the insider will expect the final payoff of the risky asset to be higher (lower) when $\hat{s}$ is large. The following assumption captures this tendency.

[3]Such an assumption was first introduced by Glosten and Milgrom (1985). Its use here is only meant to simplify our analysis; section 5 shows that it is not crucial for our results to obtain.

Assumption 2 *There is a $\theta_0 \in \mathbb{R}$ such that for any $\theta \in Supp(\hat{\theta})$ with $\theta > \theta_0$ ($\theta < \theta_0$), the conditional mean of $\hat{v}$ given that $(\hat{\theta}, \hat{s}) = (\theta, s)$, $\mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)]$, is increasing (decreasing) in $\hat{s} \in Supp(\hat{s})$, and is greater (smaller) than $\mathrm{E}[\hat{v}]$.*

Intuitively, θ_0 represents the point at which the insider type trader shifts his interest between buying and selling the risky asset; that is, for $\hat{\theta} < \theta_0$ ($\hat{\theta} > \theta_0$), the insider type trader will be interested in selling (buying) the risky asset, and will be more so the higher his information precision $\hat{s}$ is. Also, if $\theta_0 \in Supp(\hat{\theta})$, then $\mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] = \mathrm{E}[\hat{v}]$ for all $s \in Supp(\hat{s})$; that is $\hat{\theta} = \hat{\theta}_0$ is an uninformative signal for the insider type trader, since it does not tell him whether the risky asset's payoff is expected to be high or low, whatever his information precision may be.

An interesting analogy can be made between Assumption 2 and the parameter uncertainty literature, as described in Coles, Lowenstein and Suay (1995), and Barry and Brown (1985), among others. Indeed, Assumption 2 says that an informed investor will put more weight on his posteriors the higher his information precision is. Similarly, the parameter uncertainty literature concludes that investors will put more portfolio weight on the securities for which they have less parameter uncertainty.

Note also that these two assumptions and the compact supports for $\hat{v}$, $\hat{\theta}$ and $\hat{s}$, are all that we require until section 4.3, where we put a little more structure on the distributions of these three variables. In particular, we do not require that the information precision $\hat{s}$ be independent of the asset's future payoff $\hat{v}$, even though such an assumption would make economic sense.

If $\hat{\tau} = 0$, the trader is a *liquidity trader* who trades non-strategically for liquidity or hedging purposes. We assume for tractability that his demand for the risky asset is perfectly inelastic. More precisely, we assume that his demand for the risky asset is a random variable $\hat{z}$ which is taken to be independent from all the other random variables in the economy. In particular, we assume that $\hat{z} = 1$ (purchase) with probability t_1, $\hat{z} = -1$ (sale) with probability t_{-1}, and $\hat{z} = 0$ with probability $1 - t_1 - t_{-1}$, where both t_1 and t_{-1} are in $(0, 1)$ with $t_1 + t_{-1} \leq 1$.

Finally, there is a risk-neutral market-maker who quotes bid and ask prices b and a at the beginning of the period. The market-maker is required to clear the market at his quoted prices upon receiving the (anonymous) order $\hat{\omega}_{a,b}$, which can be written as

$$\hat{\omega}_{a,b} = \hat{\tau}\hat{x}_{a,b} + (1 - \hat{\tau})\hat{z}. \tag{2.1}$$

Since the market-maker does not observe the insider's type, the presence of the liquidity trader type in the economy serves as camouflage for the insider type, just as it did in Glosten and Milgrom (1985) and Easley and O'Hara (1987) for example.[4] Until section 5 where we consider variations of the

[4]Note that the added precision uncertainty cannot serve as camouflage, since it always results in the insider type

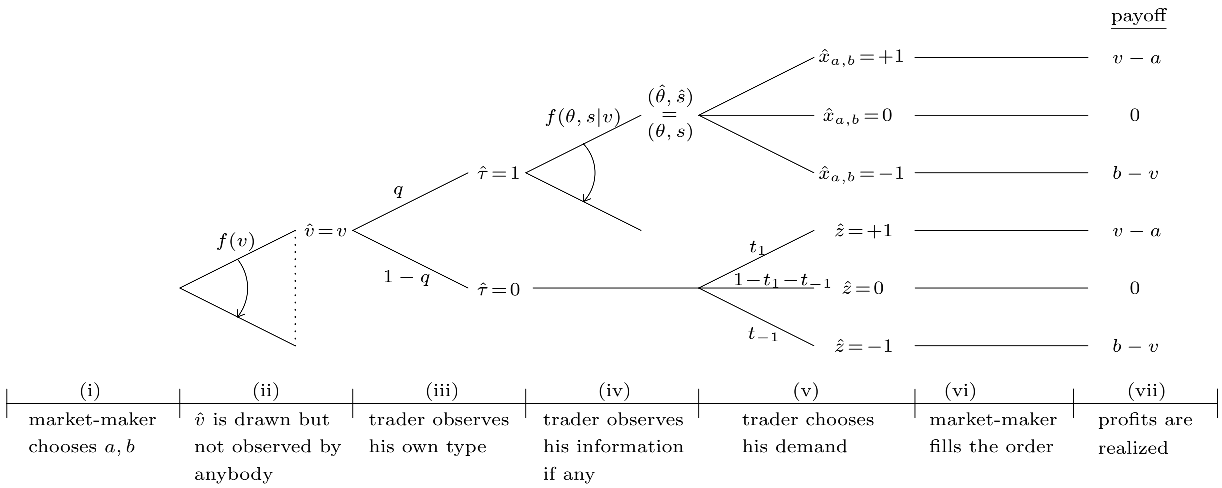


Figure 1: Sequence of events. In the figure, the dotted line indicates that no one knows the realized value of the risky asset's payoff $\hat{v} = v$. Also, $f(v)$ and $f(\theta, s|v)$ indicate the unconditional density function of $\hat{v}$ and the joint density function of $(\hat{\theta}, \hat{s})$ conditional on $\hat{v} = v$ respectively.

model, we assume that the market-maker is competitive, and thus sets the bid and ask prices so as to break even on average. Following Glosten and Milgrom (1985), the competitive market-maker will therefore set the bid and ask prices b and a so they satisfy[5]

$$b = \mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = -1], \text{ and} \tag{2.2a}$$

$$a = \mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = 1]. \tag{2.2b}$$

The sequence of events is as follows: (i) at date zero, the market-maker announces his bid and ask prices b and a, and (ii) the value of $\hat{v}$ is realized but not observed by anybody in the economy. (iii) Then, the trader observes his own type; (iv) if the trader is an insider ($\hat{\tau} = 1$), he immediately observes $\hat{\theta}$, $\hat{s}$, b and a; (v) the trader chooses his demand for the risky asset ($\hat{x}_{a,b}$ if he is an insider, and $\hat{z}$ if he is a liquidity trader). (vi) The market-maker fills the order. (vii) At date one, the liquidating value of the risky asset is announced, and each market participant realizes his profits. This sequence is illustrated in Figure 1.

trader being more informed than the market-maker. In the absence of liquidity trading, Milgrom and Stokey (1982) have shown that no trading would ever occur in equilibrium.

[5]Note that this setup can also be interpreted as in Kyle (1985), in which the strategic insider does not observe the price before sending his market order. However, knowing how the market-maker will set the price upon receiving his order, he takes his impact on the price into consideration when choosing his demand for the risky asset. Here, since the risky asset can only be traded one share at a time, and since the market-maker does not need to set a price for an order of zero share, we have a nice analogy to the bid and ask prices observed in financial markets.

3 The Equilibrium

The market-maker knows how the insider type trader will react to his bid and ask prices. In particular, he knows that, if he sets the bid and ask prices at $b \in [\underline{v}, \bar{v}]$ and $a \in [\underline{v}, \bar{v}]$ respectively, the insider type trader who observes a signal $\hat{\theta} = \theta$ with precision $\hat{s} = s$ will have expected profits of $\mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] - a$ if he buys a share of the risky asset, $b - \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)]$ if he sells a share, and 0 otherwise. Therefore, since the market-maker does not observe either $\hat{\theta}$ or $\hat{s}$, he knows that the insider type trader will buy a share of the risky asset if $(\hat{\theta}, \hat{s}) \in \mathcal{B}_{a,b}$, where[6]

$$\mathcal{B}_{a,b} \equiv \Big\{(\theta, s) \in \mathit{Supp}(\hat{\theta}, \hat{s}) : \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] - a > \max\big\{0, b - \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)]\big\}\Big\}, \tag{3.1a}$$

and will sell a share of the risky asset if $(\hat{\theta}, \hat{s}) \in \mathcal{S}_{b,a}$, where

$$\mathcal{S}_{b,a} \equiv \Big\{(\theta, s) \in \mathit{Supp}(\hat{\theta}, \hat{s}) : b - \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] > \max\big\{0, \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] - a\big\}\Big\}. \tag{3.1b}$$

In view of Assumptions 1 and 2, the definition of $\mathcal{B}_{a,b}$ $(\mathcal{S}_{b,a})$ captures the intuitive roles of information and information precision discussed in the introduction: the higher (lower) the signal $\hat{\theta}$ is and the higher its precision $\hat{s}$ is, the more likely that the insider will buy (sell) the risky asset.[7] Typical buy and sell sets $\mathcal{B}_{a,b}$ and $\mathcal{S}_{b,a}$ are illustrated in Figure 2 for distributions of $\hat{v}$, $\hat{\theta}$, and $\hat{s}$ which satisfy assumptions 1 and 2. Note that, in that figure, the buy and sell sets $\mathcal{B}_{a,b}$ and $\mathcal{S}_{b,a}$ are convex; this need not be the case. Assumptions 1 and 2 only imply that the line that delineates the buy (sell) set $\mathcal{B}_{a,b}$ $(\mathcal{S}_{b,a})$ be decreasing (increasing) in θ. Before proceeding with the market-maker's problem, let us look at a few simple properties that $\mathcal{B}_{a,b}$ and $\mathcal{S}_{b,a}$ satisfy.

Lemma 3.1 *For any pair of prices $(a, b) \in [\underline{v}, \bar{v}]^2$, the sets $\mathcal{B}_{a,b}$ and $\mathcal{S}_{b,a}$ satisfy the following properties:*

(i) If $(\theta, s) \in \mathcal{B}_{a,b}$, then $\mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] > a$. If $(\theta, s) \in \mathcal{S}_{b,a}$, then $\mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] < b$.

(ii) $\mathcal{B}_{a,b}$ and $\mathcal{S}_{b,a}$ are disjoint.

(iii) If $b \leq a$, $\mathcal{B}_{a,b}$ does not depend on b, and $\mathcal{S}_{b,a}$ does not depend on a.

[6] Notice that, in the definition of $\mathcal{B}_{a,b}$ $(\mathcal{S}_{b,a})$ in equation (3.1a) (equation (3.1b)), we have to make sure that buying (selling) a share of the risky asset is better than doing nothing *and* selling (buying) a share of the risky asset. This explains why we need to use the 'max' function in both these equations. However, Lemma 3.1 below will show that, in equilibrium, we only need to compare the expected profits of buying (selling) a share with those of doing nothing.

[7] Note that, in this one-share setting, insider trading aggressiveness translates into a higher probability of trading. In section 5, when we study a multi-share setting, we will see that insider trading aggressiveness translates into both higher probability of trading *and* more shares traded on average.

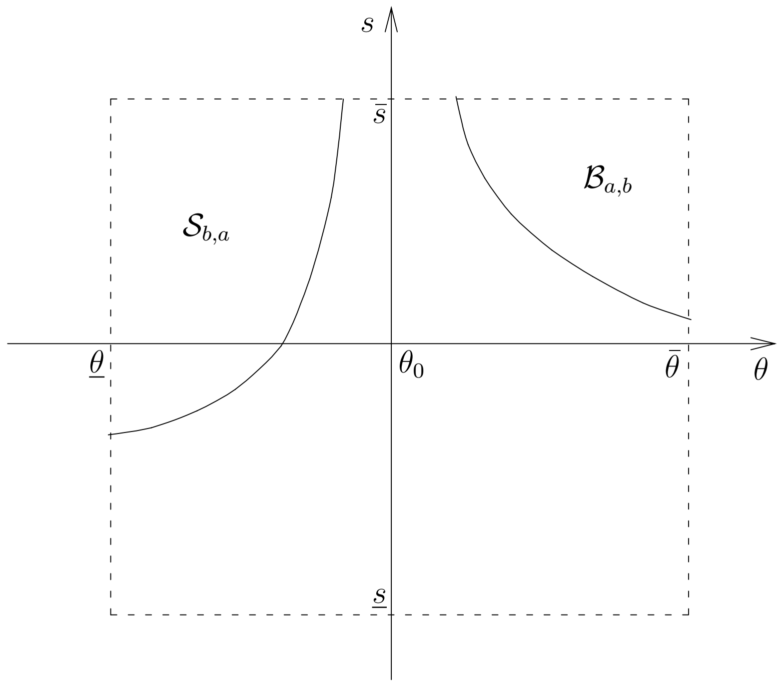


Figure 2: Typical buy ($\mathcal{B}_{a,b}$) and sell ($\mathcal{S}_{b,a}$) regions for the insider type trader. The insider type trader is more tempted to buy (sell) the higher (lower) his signal $\hat{\theta} = \theta$ is, and the higher his precision $\hat{s} = s$ is.

Proof : (i) If $(\theta, s) \in \mathcal{B}_{a,b}$, then

$$\mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] - a > \max\big\{0, b - \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)]\big\} \geq 0,$$

and similarly for $(\theta, s) \in \mathcal{S}_{b,a}$. (ii) Suppose that $(\theta, s) \in \mathcal{B}_{a,b} \cap \mathcal{S}_{b,a}$. Then, from the definition of $\mathcal{B}_{a,b}$, this means that

$$\begin{aligned} \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] - a &> \max\big\{0, b - \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)]\big\} \\ &\geq b - \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)]. \end{aligned}$$

Also, from the definition of $\mathcal{S}_{b,a}$, this means that

$$\begin{aligned} b - \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] &> \max\big\{0, \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] - a\big\} \\ &\geq \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] - a. \end{aligned}$$

These two statements are obviously contradictory. (iii) Take any $b \leq a$, and any $(\theta, s) \in \mathcal{B}_{a,b}$. By the result of part (i), $\mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] > a$, so that

$$b - \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] \leq a - \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] < 0.$$

In view of the definition of $\mathcal{B}_{a,b}$ in (3.1a), this means that $\mathcal{B}_{a,b}$ can be written as

$$\mathcal{B}_{a,b} = \left\{(\theta, s) \in \mathit{Supp}(\hat{\theta}, \hat{s}) : \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] - a > 0\right\},$$

which does not depend on b. A similar argument can be made for $\mathcal{S}_{b,a}$. ∎

As we mentioned in section 2, the bid and ask prices, as set by the risk-neutral competitive market-maker, must satisfy $b = \mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = -1]$ and $a = \mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = 1]$. Since the market-maker can only receive a purchase order if the trader is an insider ($\hat{\tau} = 1$) and $(\hat{\theta}, s) \in \mathcal{B}_{a,b}$, or the trader is a liquidity buyer ($\hat{\tau} = 0, \hat{z} = 1$), the information set $\{\hat{\omega}_{a,b} = 1\}$ is equivalent to $\left\{\{\hat{\tau} = 1, (\hat{\theta}, \hat{s}) \in \mathcal{B}_{a,b}\} \cup \{\hat{\tau} = 0, \hat{z} = 1\}\right\}$. Similarly, the information set $\{\hat{\omega}_{a,b} = -1\}$ is equivalent to $\left\{\{\hat{\tau} = 1, (\hat{\theta}, \hat{s}) \in \mathcal{S}_{b,a}\} \cup \{\hat{\tau} = 0, \hat{z} = -1\}\right\}$. Therefore, a straightforward application of Bayes' rule yields the following result.

Lemma 3.2 *The equilibrium bid and ask prices set by the risk-neutral competitive market-maker will satisfy*

$$b = \frac{q \Pr\{\mathcal{S}_{b,a}\}\mathrm{E}[\hat{v} \mid \mathcal{S}_{b,a}] + (1-q)t_{-1}\mathrm{E}[\hat{v}]}{q \Pr\{\mathcal{S}_{b,a}\} + (1-q)t_{-1}} \tag{3.2a}$$

and

$$a = \frac{q \Pr\{\mathcal{B}_{a,b}\}\mathrm{E}[\hat{v} \mid \mathcal{B}_{a,b}] + (1-q)t_{1}\mathrm{E}[\hat{v}]}{q \Pr\{\mathcal{B}_{a,b}\} + (1-q)t_{1}}, \tag{3.2b}$$

where, for a set $D \in \mathbb{R}^2$, $\Pr\{D\} \equiv \Pr\{(\hat{\theta}, \hat{s}) \in D\}$ *and* $\mathrm{E}[\hat{v} \mid D] \equiv \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) \in D]$.

Proof: See Appendix A.

Note that since the market-maker makes zero profits on average, it must be that the insider type trader's expected profits are equal to the liquidity type trader's expected losses. This can be easily seen by rewriting equations (3.2a) and (3.2b) as

$$q \Pr\{\mathcal{S}_{b,a}\}\left\{b - \mathrm{E}[\hat{v} \mid \mathcal{S}_{b,a}]\right\} = (1-q)t_{-1}\left\{\mathrm{E}[\hat{v}] - b\right\} \tag{3.3a}$$

and

$$q \Pr\{\mathcal{B}_{a,b}\}\left\{\mathrm{E}[\hat{v} \mid \mathcal{B}_{a,b}] - a\right\} = (1-q)t_{1}\left\{a - \mathrm{E}[\hat{v}]\right\}. \tag{3.3b}$$

In these two equations, the left-hand side represents the expected profits of an insider type trader, whereas the right-hand side represents the expected losses of a liquidity type trader.

In this economy, a Nash equilibrium is a set of bid and ask prices such that the market-maker breaks even on average, and such that his conjectures about the behavior of the insider type trader are always right. In view of the above analysis, such an equilibrium is simply defined as follows.

Definition 3.1 *An equilibrium for this economy is a pair of bid and ask prices* $(b, a) \in [\underline{v}, \bar{v}]^2$ *which satisfies the set of equations (3.2a)-(3.2b). The set of such pairs will be denoted by* $\mathcal{E}$.

Of course, for the model to be useful, we need to be able to better describe the set $\mathcal{E}$. This is what we do in the remainder of this section. The first characterization of $\mathcal{E}$ concerns the relative size of the bid and ask prices in equilibrium. For the model to have any economic meaning, we want the bid to be lower than the ask price. This is confirmed in the following lemma.

Lemma 3.3 *If* $(b, a) \in \mathcal{E}$, *then* $b \leq \mathrm{E}[\hat{v}] \leq a$.

Proof : By (3.2b), a is a weighted average between $\mathrm{E}[\hat{v}]$ and $\mathrm{E}[\hat{v} \mid \mathcal{B}_{a,b}]$. Since $\mathrm{E}[\hat{v} \mid \mathcal{B}_{a,b}] > a$ by Lemma 3.1, we must have $a \geq \mathrm{E}[\hat{v}]$. The proof to show that $b \leq \mathrm{E}[\hat{v}]$ is similar. ∎

This result will greatly simplify our task of finding an equilibrium to our model. Indeed, since we can restrict our search to bid and ask prices which satisfy $b \leq \mathrm{E}[\hat{v}] \leq a$, Lemma 3.1(iii) implies that we can solve the system of equations (3.2a)-(3.2b) one equation at a time. Since this is always going to be the case, in the remainder of the paper, we will denote the insider type trader's buy and sell sets simply by

$$\mathcal{B}_a = \left\{(\theta, s) \in \mathit{Supp}(\hat{\theta}, \hat{s}) : \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] > a\right\} \tag{3.4a}$$

and

$$\mathcal{S}_b = \left\{(\theta, s) \in \mathit{Supp}(\hat{\theta}, \hat{s}) : \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] < b\right\}, \tag{3.4b}$$

in order to alleviate the notation. This also means that we can rewrite (3.2a) and (3.2b) as

$$b = \frac{q \Pr\{\mathcal{S}_b\}\mathrm{E}[\hat{v} \mid \mathcal{S}_b] + (1-q)t_{-1}\mathrm{E}[\hat{v}]}{q \Pr\{\mathcal{S}_b\} + (1-q)t_{-1}} \tag{3.5a}$$

and

$$a = \frac{q \Pr\{\mathcal{B}_a\}\mathrm{E}[\hat{v} \mid \mathcal{B}_a] + (1-q)t_1\mathrm{E}[\hat{v}]}{q \Pr\{\mathcal{B}_a\} + (1-q)t_1}. \tag{3.5b}$$

Since Lemma 3.1(i) implies that $\mathrm{E}[\hat{v} \mid \mathcal{B}_a] > a$ and $\mathrm{E}[\hat{v} \mid \mathcal{S}_b] < b$, Lemma 3.3 also tells us that, for $(b, a) \in \mathcal{E}$,

$$\mathrm{E}[\hat{v} \mid \mathcal{S}_b] < \mathrm{E}[\hat{v}] < \mathrm{E}[\hat{v} \mid \mathcal{B}_a]. \tag{3.6}$$

Because the right-hand side of (3.5b) is a weighted average between $\mathrm{E}[\hat{v} \mid \mathcal{B}_a]$ and $\mathrm{E}[\hat{v}]$, this means that, at $a = \mathrm{E}[\hat{v}]$, the right-hand side of that equation is (strictly) greater than its left-hand side,

$\mathrm{E}[\hat{v}]$. Therefore, $a = \mathrm{E}[\hat{v}]$ cannot be an equilibrium. On the other hand, at $a = \bar{v}$, the right-hand side of (3.5b) is equal to $\mathrm{E}[\hat{v}]$ (since $\Pr\{\mathcal{B}_a\} \to \Pr\{\emptyset\} = 0$ as $a \to \bar{v}$), which is (strictly) smaller than $\bar{v}$, the left-hand side of (3.5b). Since the same kind of analysis can be performed for the bid price, this seems to suggest that an equilibrium exists with $b \in (\underline{v}, \mathrm{E}[\hat{v}])$ and $a \in (\mathrm{E}[\hat{v}], \bar{v})$, even if it does not prove it.[8,9]

The next two results will show that this conjecture is right, that is there exists at least one equilibrium to our model, even if the distributions of $\hat{v}$, $\hat{\theta}$, or $\hat{s}$ are discrete or mixture of discrete and continuous distributions. The idea for the proof for the ask price is as follows.[10] We know that we are looking for an ask price a that will satisfy (3.5b). The left-hand side of that equation is obviously continuous as a function of a; it is simply the 45° line. If the right-hand side of the equation is also continuous as a function of a, the last paragraph shows the existence of an equilibrium. However, if the right-hand side has jumps, it can be shown that these jumps never go from above the 45° line to below the 45° line. Since at $a = \mathrm{E}[\hat{v}]$, the left-hand side is smaller than the right-hand side, it can therefore never be the case that a jump prevents an equilibrium to exist. This is described rigorously in the next lemma and corollary, and illustrated in Figure 3.

Lemma 3.4 *Take any $(b, a) \in [\underline{v}, \bar{v}]^2$ satisfying $b \leq \mathrm{E}[\hat{v}] \leq a$. If $\lim_{\alpha \to a^-} \mathrm{E}[\hat{v} \mid \hat{\omega}_{\alpha,b} = 1] > a$, then $\mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = 1] > a$. Similarly, if $\lim_{\beta \to b^+} \mathrm{E}[\hat{v} \mid \hat{\omega}_{a,\beta} = -1] < b$, then $\mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = -1] < b$.*

Proof: See Appendix A.

We are now ready to state one of the two main results of this section which, in view of the last lemma and the paragraph preceding it, becomes almost trivial to prove.

Corollary 3.1 *The set $\mathcal{E}$ is non-empty. In other words, there exists an equilibrium to our model.*

Proof: See Appendix A.

The existence of an equilibrium is certainly reassuring.[11] However, without a uniqueness result, this existence result could prove useless for analyzing the model's implications, since different equilibria could imply different properties. For example, we have not yet ruled out the possibility that $\mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = 1]$ jumps from below the 45° line to above the 45° line. Fortunately, the next lemma shows that such jumps are impossible since $\mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = 1]$ can only decrease after it crosses a.

[8] Actually, this argument does prove the existence of an equilibrium for continuous distributions of $\hat{v}$, $\hat{\theta}$, and $\hat{s}$.

[9] Note that this is where our boundedness assumption on $\mathit{Supp}(\hat{v}, \hat{\theta}, \hat{s})$ is critical.

[10] The proof for the bid price is similar.

[11] Note however that the result of Corollary 3.1 is not surprising in view of Lindsay (1991), who show that such equilibria always exist as long as all the traders are risk-neutral.

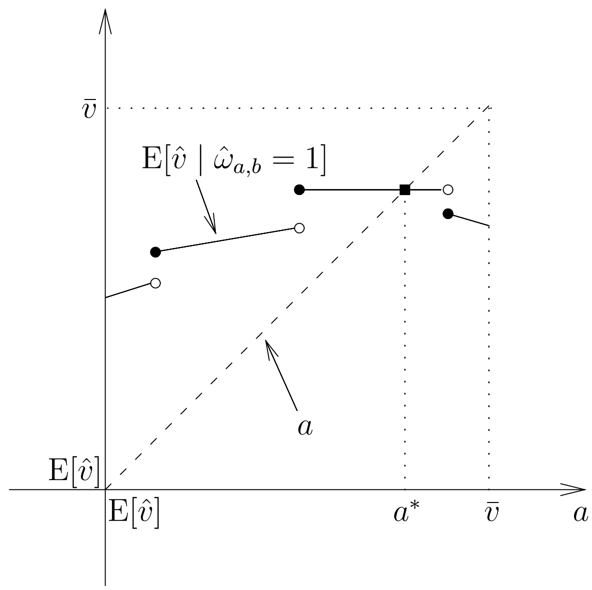


Figure 3: Typical equilibrium ask price a^* when $\mathrm{E}[\hat{v} \,|\, \mathcal{B}_a]$ contains jumps. Both the left-hand and the right-hand sides of equation (3.5b) are represented as functions of a. The dashed line represents the left-hand side of equation (3.5b) (i.e. the 45° line), whereas the continuous line with jumps represents the right-hand side of that same equation. The fact that $\mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = 1]$ can never jump from above the 45° line to below the 45° line insures that an equilibrium will always exist. In this picture, the black square represents a point where an equilibrium is reached.

Lemma 3.5 *Suppose that $(b, a) \in \mathcal{E}$. Then,*

(i) for any $b' \leq \mathrm{E}[\hat{v}]$ and $a' > a$, $\mathrm{E}[\hat{v} \mid \hat{\omega}_{a',b'} = 1] \leq a$;

(ii) for any $a' \geq \mathrm{E}[\hat{v}]$ and $b' < b$, $\mathrm{E}[\hat{v} \mid \hat{\omega}_{a',b'} = -1] \geq b$.

Proof: See Appendix A.

This result is essential in showing the main result of this section, namely that there exists exactly one equilibrium to our model.

Proposition 3.1 *The set $\mathcal{E}$ of equilibria is a singleton. In other words, there exists a unique equilibrium to our model.*

Proof: Existence has already been proved in Corollary 3.1. Suppose that both (b, a) and (b', a') are in $\mathcal{E}$. Then, by Lemma 3.3, we must have $b \leq \mathrm{E}[\hat{v}] \leq a$ and $b' \leq \mathrm{E}[\hat{v}] \leq a'$. Suppose without loss of generality that $a' > a$. From Lemma 3.5, we know that

$$\mathrm{E}[\hat{v} \mid \omega_{a',b'} = 1] \leq a < a',$$

so that (b', a') cannot satisfy (2.2b), and therefore cannot be an equilibrium of the model. This completes the proof. ∎

This result is interesting in that it only required two intuitively compelling assumptions about the joint distributions of the risky asset payoff $\hat{v}$, the insider type trader's signal $\hat{\theta}$, and its precision $\hat{s}$. This is due to the convenient assumption that shares can only be traded one at a time. In fact, this level of generality can even be increased to an economy where shares can be traded in a finite number of trade sizes, the insider type trader is risk-averse, the liquidity type trader's demand is not perfectly inelastic, and the market-maker has some market power. The critical assumption remains the fact that shares cannot be traded in unbounded trade sizes.[12]

4 Properties of the Model

In this section, we use the equilibrium derived in section 3 to analyze some of the properties of our model. We do this by looking at the properties of the different variables usually studied in the market microstructure literature, namely the bid-ask spread, expected insider profits, and trading volume. Of course, we would also like to study how these variables evolve through time. Since this requires a lot more structure on the market participants and their decision processes, we relegate this problem to Chapter 2. Also, since the primary objective of this paper is to show the effects of random precision on market microstructure theory, we concentrate on the effects that specific changes in the distribution of information precision will have on the above economic variables.

4.1 The Bid-Ask Spread

Studies of the bid-ask spread abound in the market microstructure literature. Many of these studies have been preoccupied by decomposing the different components of the bid-ask spread, either theoretically[13] or empirically.[14] In our model, the information component (also called the adverse selection component) is the only component of the bid-ask spread. Indeed, the inventory and order cost components discussed by Glosten (1987), Glosten and Harris (1988), and Stoll (1989) are driven to zero by two of our assumptions: first, the market-maker is risk-neutral and therefore does not have any inventory preoccupation; second, the market-maker is competitive and therefore cannot, in equilibrium, charge an order cost as part of the bid-ask spread. However, as the next two results will show, this information component of the bid-ask spread can itself be decomposed

[12] If the risk-neutral insider could trade any number of shares after observing the bid and ask prices, he would trade an infinite number of shares whenever he decides to trade.

[13] See for example Ho and Stoll (1983), Copeland and Galai (1983), and Glosten (1987).

[14] See for example Roll (1984), Glosten and Harris (1988), and Stoll (1989).

into two subcomponents: the asymmetric information subcomponent, and the random precision subcomponent.

The first subcomponent is the usual adverse selection component that is discussed in Copeland and Galai (1983), and Glosten (1987). It represent the market-maker's compensation for his informational disadvantage when trading with possibly informed traders. Without such compensation, the market-maker would be driven out of business by such informed traders. In both the papers mentioned above, the information always has the same precision, that is other market participants (including the market-maker) know how good the insider's information is. In contrast, in the present paper, the precision will on average be equal to a certain value but, in general, will not be equal to that average. We call the asymmetric information subcomponent of the bid-ask spread the adverse selection component which would have resulted had the information precision always been equal to that average precision. The difference between the total adverse selection component of the bid-ask spread (the total spread in our model) and the asymmetric information subcomponent then represents the random precision subcomponent.

In what follows, let $BA \equiv b - a$ denote the equilibrium bid-ask spread for $(b, a) \in \mathcal{E}$. We consider a change in the distribution of $\hat{s}$ from the original probability measure to a new probability measure, that we denote by a prime. Also, all the primed economic variables represent the new equilibrium values for these variables.[15] Let us start with the following proposition.

Proposition 4.1 *Suppose that we change the distribution of $\hat{s}$ such that the new distribution of $\hat{s}$ first-order stochastically dominates*[16] *the original distribution of $\hat{s}$. Then $BA \leq BA'$, that is the equilibrium bid-ask spread is larger with the new distribution.*

Proof : See Appendix B.

In view of the existing literature on bid-ask spreads, this result makes intuitive sense since first-order stochastic dominance implies that the new information precision variable will have a larger expected value. This result is then in essence similar to that of numerous papers which increase the (public) precision parameter to find that the equilibrium bid-ask spread goes up.[17] Intuitively, the insider with the new distribution of precision has a bigger informational advantage over the

[15] For example, the new equilibrium bid and ask prices will be denoted b' and a' respectively, and the new equilibrium bid-ask spread will be denoted BA'.

[16] Take any two real random variables X and Y with cumulative density functions F and G respectively. Then X is said to first-order stochastically dominate Y if $F(x) \leq G(x)$ for all $x \in \mathbb{R}$. See for example Shaked and Shanthikumar (1994).

[17] In papers where the market-maker does not quote just bid and ask prices but an entire price schedule [e.g. Kyle (1985)], this result is derived in terms of the slope of that price schedule.

market-maker. To protect himself, the marker-maker has to increase his bid-ask spread. The next result, however, is new to this literature, since it concerns second-order stochastic dominance.

Proposition 4.2 *Suppose that we change the distribution of $\hat{s}$ such that the new distribution of $\hat{s}$ is second-order stochastically dominated*[18] *by the original distribution of $\hat{s}$. Then $BA \leq BA'$, that is the equilibrium bid-ask spread is larger with the new distribution.*

Proof : See Appendix B.

Intuitively, a more disperse distribution for the information precision $\hat{s}$ results in a larger spread. This makes sense since the average precision faced by the market-maker only depends on the precision of traders who would choose to trade (i.e. it is equal to $\mathrm{E}[\hat{s}\,|\,\mathcal{B}_{a,b}]$ or $\mathrm{E}[\hat{s}\,|\,\mathcal{S}_{b,a}]$). So, although the fact that the information precision is more disperse does not affect the unconditional mean of $\hat{s}$, it does affect the average precision faced by the market-maker, which is given by the mean of $\hat{s}$ conditional on a trade.

Note that the asymmetric information subcomponent of the bid-ask spread does not change with a second-order shift of the precision's distribution, but that the bid-ask spread is affected nevertheless. This is because the random precision subcomponent of the bid-ask spread increases as we spread the distribution of $\hat{s}$ around its mean.

This last proposition is better explained with an example which will also be useful in later sections. Assume that the final payoff $\hat{v}$ of the risky security can only take values of $+1$ or -1 with equal probabilities of 1/2. Also, assume that, conditional on the information precision being $\hat{s} = s \in [0, 1]$, the insider type trader's signal $\hat{\theta}$ has the following distribution:

$$\begin{aligned}
\Pr\{\hat{\theta} = 1 \mid \hat{v} = 1, \hat{s} = s\} = &\ \frac{1}{2}\left(\frac{1+s}{2}\right) &= \Pr\{\hat{\theta} = -1 \mid \hat{v} = -1, \hat{s} = s\}; \\
\Pr\{\hat{\theta} = 0 \mid \hat{v} = 1, \hat{s} = s\} = &\ \frac{1}{2} &= \Pr\{\hat{\theta} = 0 \mid \hat{v} = -1, \hat{s} = s\}; \\
\Pr\{\hat{\theta} = -1 \mid \hat{v} = 1, \hat{s} = s\} = &\ \frac{1}{2}\left(\frac{1-s}{2}\right) &= \Pr\{\hat{\theta} = 1 \mid \hat{v} = -1, \hat{s} = s\}.
\end{aligned} \tag{4.1}$$

It can be easily shown that this implies that

$$\mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] = \theta s, \tag{4.2}$$

[18] Take any two real random variables X and Y with cumulative density functions F and G respectively. Then X is said to second-order stochastically dominate Y if $\int_{-\infty}^{x} F(y)dy \leq \int_{-\infty}^{x} G(y)dy$ for all $x \in \mathbb{R}$. See for example Shaked and Shanthikumar (1994).

which is increasing in s for $\theta = 1$, and decreasing in s for $\theta = -1$. The signal $\hat{\theta} = 0$ can be interpreted as the absence of signal since

$$\Pr\{\hat{v} = 1 \mid \hat{\theta} = 0, \hat{s} = s\} = \Pr\{\hat{v} = -1 \mid \hat{\theta} = 0, \hat{s} = s\} = 1/2 = \Pr\{\hat{v} = 1\} = \Pr\{\hat{v} = -1\}.$$

This says that no matter what value the precision $\hat{s}$ takes, the insider does not have any 'inside information' when receiving that signal. For that reason, we call $\hat{\theta} = 0$ the *uninformative signal*, and $\hat{\theta} = 1$ and $\hat{\theta} = -1$ the *informative signals*.[19] It can be seen from (4.1) that the insider type trader's signal is more precise the larger s is. Indeed, when $s = 0$, the insider type trader's information is useless since both informative signals are equally likely when $\hat{v} = 1$ or $\hat{v} = -1$. However, as s increases, $\hat{\theta} = 1$ ($\hat{\theta} = -1$) becomes more likely than $\hat{\theta} = -1$ ($\hat{\theta} = 1$) when $\hat{v} = 1$ ($\hat{v} = -1$). We can therefore interpret the signal $\hat{\theta} = 1$ ($\hat{\theta} = -1$) as a positive (negative) signal. Finally, when $s = 1$, the insider type trader's information is perfect in the sense that he knows that $\hat{v} = 1$ ($\hat{v} = -1$) for sure when he observes $\hat{\theta} = 1$ ($\hat{\theta} = -1$).

We also assume that the information precision $\hat{s}$ is independent from $\hat{v}$, and that its unconditional distribution is given by

$$\hat{s} = \begin{cases} s_H & \text{prob. } 1/2 \\ s_L & \text{prob. } 1/2, \end{cases}$$

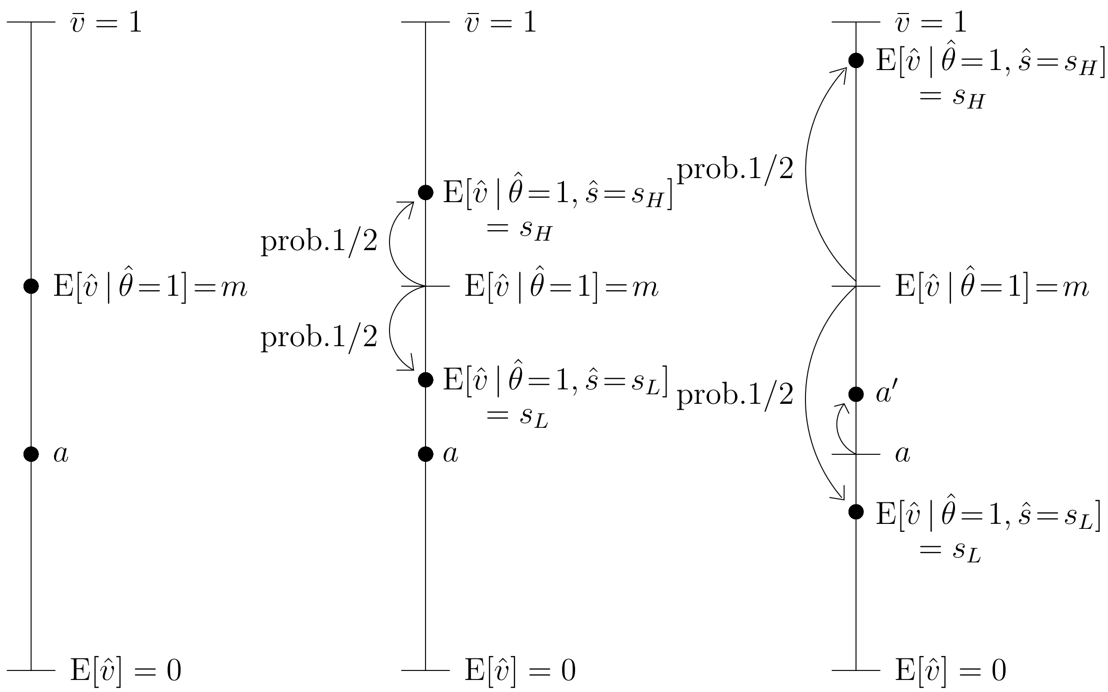
where $s_H \geq s_L$. Finally, we assume that $t_1 = t_{-1} = 1/4$, that is the liquidity type trader buys or sell a share of the risky asset with equal probabilities of 1/4, and does nothing otherwise.

In what follows, we consider a second-order stochastic shift of the information precision. We do this by spreading s_H and s_L symmetrically around the constant precision mean $m = \frac{1}{2}(s_H + s_L)$. When $s_H = s_L \equiv m$, information precision is not random. Since the market-maker cannot always prevent the insider type trader from trading,[20] the market-maker will then set an ask price a somewhere between $\mathrm{E}[\hat{v}] = 0$ and $\mathrm{E}[\hat{v} \mid \hat{\theta} = 1] = m$. This is illustrated in Panel (a) of Figure 4. In fact, this ask price is the one that would be quoted by the market-maker in the models of Glosten and Milgrom (1985), and Easley and O'Hara (1992). Since the insider's precision is not random, the information component of the bid-ask spread consists only of the asymmetric information subcomponent.

Suppose now that we start spreading s_H and s_L symmetrically around their mean m, so that the new distribution of $\hat{s}$ is stochastically dominated by the original degenerate (one-point) distribution. If we do so by only a small amount, the market-maker will still quote the same ask price a as before,

[19] In fact, the parameter θ_0 which, as discussed in section 2, represents the point at which the insider type trader shifts his interest between buying and selling the risky asset, is equal to zero in this model.

[20] Otherwise, the insider would never lose to insider type traders and would make profits out of the liquidity traders; this is clearly not a competitive outcome for the market-maker.



(a) Constant precision. (b) Small second order shift. (c) Large second order shift.

Figure 4: Effect of a second-order stochastic shift in the distribution of the information precision $\hat{s}$. The distribution of $\hat{s}$ in panel (a) second-order stochastically dominates the distribution of $\hat{s}$ in panel (b), which in turn second-order stochastically dominates the distribution of $\hat{s}$ in panel (c).

since he knows that on average he is facing the same information precision. This is illustrated in Panel (b) of Figure 4.

However, if we spread s_H and s_L by a large enough amount, the original ask price quoted by the market-maker is now too big for a low precision insider type trader ($\hat{s} = s_L$) to trade on a positive signal ($\hat{\theta} = 1$). Indeed, this ask price exceeds the value that this trader puts on the risky asset conditional on his information, $\mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (1, s_L)] = s_L < a$. This means that a low precision insider would then decide to drop out of the market, so that the market-maker could now only be facing a high precision insider type trader. This in turn implies that the average precision that the market-maker is facing is now $s_H > \frac{1}{2}(s_H + s_L)$, so that the market-maker needs to increase his ask price to $a' > a$. As seen in Panel (c) of Figure 4, the average insider precision remains at m, so that the asymmetric information subcomponent of the bid-ask spread remains constant. The random precision subcomponent, on the other hand, is increased from 0 in Panel (a) to $a' - a$ in Panel (c).[21]

[21] Although not illustrated here, it is possible to change the distribution of insider precision so that the two

4.2 Insider Profits

Recall from equations (3.3a) and (3.3b) that the expected insider profits are exactly equal to the expected liquidity trader's losses. It is also obvious from these two equations that the liquidity trader's losses increases as the bid price decreases and as the ask price increases. Therefore, the next result follows immediately from section 4.1.

Corollary 4.1 *Suppose that we change the distribution of the information precision $\hat{s}$ as in Proposition 4.1 or Proposition 4.2. Then the expected insider profits are larger with the new distribution in either case.*

Again, the first part of this result mimics results obtained by a number of authors who used constant precision and found that insider profits increase with their information precision. A similar but more general result is obtained here, since a first-order stochastic dominance shift implies an increase in the average precision. So, we therefore reach the following intuitively obvious conclusion: insiders are better off as their information gets 'more precise' on average. What is less obvious is the second part of Corollary 4.1: a second-order stochastic dominance shift in insider random precision makes insiders better off when the new distribution is *dominated*. In other words, insiders prefer their information to be as disperse as possible, since they do not care how imprecise their information is when they do not trade, but benefit from a higher precision when they do trade. In that respect, insider trading profits are much like a call option on information precision, and the value of that option is larger the more disperse (volatile) the distribution of precision is.[22]

4.3 Volume

In this one-share setting, trading volume can only take two values: 1 or 0. So trading volume can equivalently be interpreted as the probability of a trade during the period. In what follows, let $\hat{\psi}$ denote the trading volume in the period. It can easily be shown, for $(b, a) \in \mathcal{E}$, that the equilibrium expected volume for the period is

$$\mathrm{E}[\hat{\psi}] = q\big[\Pr\{\mathcal{B}_a\} + \Pr\{\mathcal{S}_b\}\big] + (1-q)\,(t_1 + t_{-1}). \tag{4.3}$$

Since volume does not distinguish between a purchase or a sale, we assume for the remainder of this section that $t_1 = t_{-1} \equiv t$ and that the distribution of $\hat{\theta}$ is symmetrical around θ_0. This symmetry

subcomponents of the information component of the bid-ask spread move in opposite directions. The effect of such a distribution change on the bid-ask spread can then be close to zero, but cannot be interpreted as a null effect on the asymmetric information subcomponent of the spread, as existing models would suggest.

[22] Note however that the strike price of that option is implicitly set by the market-maker when he sets his bid and ask prices. A similar observation was also made by Copeland and Galai (1983).

trivially implies that both $\mathcal{B}_a$ and $\mathcal{S}_b$ will have the same probability in equilibrium, so we can write

$$\mathrm{E}[\hat{\psi}] = 2\big[q\,\Pr\{\mathcal{B}_a\} + (1-q)\,t\big]. \tag{4.4}$$

To simplify the analysis and the intuition, we also assume that $\hat{v}$ and $\hat{\theta}$ are distributed as in equation (4.1) of the example in section 4.1, so that

$$\mathrm{E}[\hat{\psi}] = 2\big[q\,\Pr\{\hat{s} > a\} + (1-q)\,t\big]. \tag{4.5}$$

The change in volume due to a change in the distribution of precision is harder to study than that of the bid-ask spread, since such a distribution change has two often counter-balancing effects:

(i) The *insider aggressiveness effect*: the insider's propensity to trade shares of the risky asset are directly related to his information precision. Changing the distribution of that precision will have an effect on his willingness to trade.

(ii) The *bid-ask spread effect*: the market-maker, knowing that the insider's decisions are affected by a change in his information precision, will adjust hid bid-ask spread accordingly.

For example, a first-order shift in the insider's precision of information makes the insider more confident about his information, and therefore makes him more aggressive. However, as we showed in Proposition 4.1, such a distribution change forces the market-maker to increase his bid-ask spread, thereby reducing the chance that an insider will take advantage of him. The net effect of that distribution change is therefore ambiguous.

To see this, suppose that we have a family of precision distributions which is indexed by a parameter $\gamma \in [\underline{\gamma}, \bar{\gamma}]$. To make this dependence on γ obvious, let the cumulative density function and the density function for $\hat{s}$ be respectively written as

$$F(s;\gamma) \equiv \Pr\{\hat{s} \le s; \gamma\} \tag{4.6}$$

and

$$f(s;\gamma) \equiv \frac{\partial}{\partial s} F(s;\gamma), \tag{4.7}$$

and let $a(\gamma)$ denote the equilibrium ask price when the insider's precision comes from the distribution γ. We are now interested in the movements in expected trading volume when we change γ. In particular, equation (4.5) tells us that we can concentrate our analysis on

$$\begin{aligned} \frac{\partial}{\partial \gamma}[1 - F(a(\gamma);\gamma)] &= -F_\gamma(a(\gamma);\gamma) - F_a(a(\gamma);\gamma)\frac{da(\gamma)}{d\gamma} \\ &= -F_\gamma(a(\gamma);\gamma) - f(a(\gamma);\gamma)\frac{da(\gamma)}{d\gamma}. \end{aligned} \tag{4.8}$$

These two terms represent respectively the insider aggressiveness effect and the bid-ask spread effect.[23] Since $f(a(\gamma);\gamma)$ is always positive, the net effect of a change in γ depends on the signs and relative sizes of $F_\gamma(a(\gamma);\gamma)$ and $\frac{da(\gamma)}{d\gamma}$. In what follows, we look for precision distribution families and conditions under which the sign of the net effect is unambiguous.

Proposition 4.3 *Suppose that $F(s;\gamma) = F(s-\gamma;\underline{s})$. Then as γ increases, the insider's precision undergoes a first-order stochastic dominance shift and volume increases.*

Proof : See Appendix B.

The intuition for this special first-order stochastic shift is relatively straightforward. Suppose that the entire precision distribution undergoes a positive shift, e.g. suppose that we increase γ from γ_0 to γ_1. Since such a shift is a first-order stochastic shift, Proposition 4.1 tells us that the ask price will increase, that is $a(\gamma_1) > a(\gamma_0)$. Suppose that the ask price does increase to the point where $[1 - F(a(\gamma_1);\gamma_1)] = [1 - F(a(\gamma_0);\gamma_0)]$. At that point the average profits that the insider makes from every trade are exactly the same as before, that is

$$\mathrm{E}[\hat{v} \mid \hat{\theta} = +1, \hat{s} > a(\gamma_1);\ \gamma_1] - a(\gamma_1) = \mathrm{E}[\hat{v} \mid \hat{\theta} = +1, \hat{s} > a(\gamma_0);\ \gamma_0] - a(\gamma_0).$$

Since the insider trades with the same intensity and with the same average profits as before, the market-maker loses exactly the same to the insider as before. However, since the ask price has now increased to $a(\gamma_1) > a(\gamma_0)$, the market-maker now makes more profits from the liquidity type trader. This situation cannot prevail in a competitive market, so that $a(\gamma_1)$ must go down. This in turn implies that $[1 - F(a(\gamma_1);\gamma_1)]$ and volume must increase.

We now turn our attention to second-order stochastic shifts of information precision. Rothschild and Stiglitz (1970) show that a random variable X second-order stochastically dominates another random variable Y if the density function of Y is obtained by adding a (possibly infinite) sequence of mean preserving spreads to the density function of X. In what follows, we consider a family of precision densities which are obtained by adding an increasingly larger mean preserving spread. In other words, we assume that for $\gamma \in [0, \bar{\gamma}]$,

$$f(s;\gamma) = f(s;0) + h(s;\gamma), \tag{4.9}$$

[23]Notice that these two effects are in essence similar to the income and substitution effects traditionally studied in the microeconomics literature.

where

$$h(s;\gamma) = \begin{cases} \gamma\lambda_1 & \text{for } s_1 < s < s_1 + r \\ -\gamma\lambda_1 & \text{for } s_1 + d_1 < s < s_1 + d_1 + r \\ -\gamma\lambda_2 & \text{for } s_2 < s < s_2 + r \\ \gamma\lambda_2 & \text{for } s_2 + d_2 < s < s_2 + d_2 + r \\ 0 & \text{otherwise,} \end{cases} \tag{4.10}$$

in which

$$0 \le s_1 \le s_1 + r \le s_1 + d_1 \le s_1 + d_1 + r \le s_2 \le s_2 + r \le s_2 + d_2 \le s_2 + d_2 + r \le 1,$$

and $\lambda_1 d_1 = \lambda_2 d_2$.

The last proposition of this section shows that the net effect on expected volume from the addition of a mean preserving spread to the original distribution of information precision depends on the sizes of the original spread and the ratio $\frac{(1-q)t}{q}$. This last ratio in effect measures the intensity of liquidity trading, since such trading occurs most often when q is small and t is large.

Proposition 4.4 *Suppose that $f(s;\gamma) = f(s;0) + h(s;\gamma)$, where $h(s;\gamma)$ is given in (4.10). (i) If $a(\gamma) \le s_2$, then expected volume decreases as γ increases. (ii) If $a(\gamma) > s_2$, then a sufficient condition for expected volume to increase with γ is that $\frac{(1-q)t}{q}$ be large enough.*

Proof : See Appendix B.

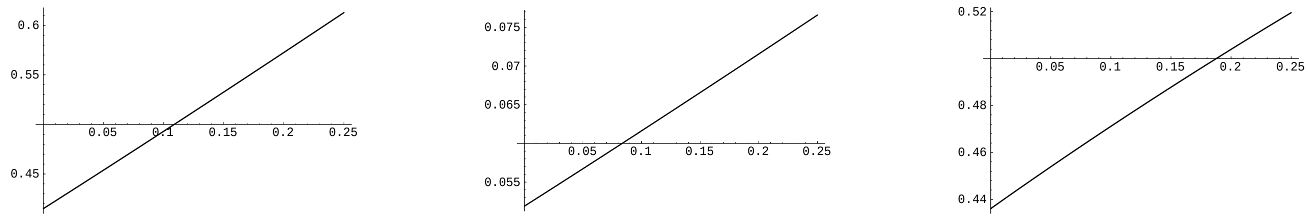
4.4 An Example

To illustrate the results derived so far, let us use an example in which $q = 0.8$ and $\hat{s}$ has a beta distribution with parameters α and β between γ and $\gamma + \delta$, that is let us assume that its probability density function is given by

$$f(s) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{s-\gamma}{\delta}\right)^{\alpha-1} \left(1 - \frac{s-\gamma}{\delta}\right)^{\beta-1}, \quad \text{for } \gamma \le s \le \gamma + \delta. \tag{4.11}$$

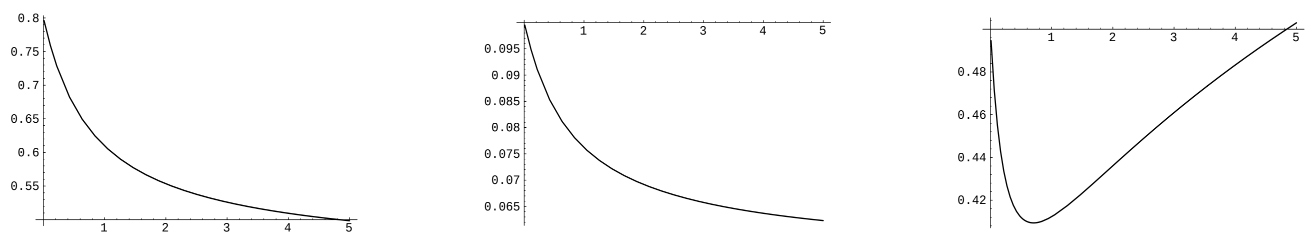
This specification allows us to capture a rich set of possible priors for the information precision. For example, consider the following first-order stochastic dominance shift in the distribution for the random precision: increase the parameter γ from 0 to 1/4 while $\alpha = \beta = 2$ and $\delta = 3/4$. Figure 5 illustrates what happens to the equilibrium bid-ask spread, the expected insider profits, and the expected trading volume.

As can be seen from that picture, the bid-ask spread, expected insider profits and expected trading volume all move in accordance with Proposition 4.1, Corollary 4.1, and Proposition 4.3.



(a) Bid-ask spread. (b) Insider profits. (c) Volume.

Figure 5: Effect of first-order stochastic dominance on (a) the bid-ask spread; (b) the expected insider profits; (c) expected volume. The horizontal axis represents the value of the parameter γ. An increase in that parameter while $\alpha = \beta = 2$ and $\delta = 3/4$ causes a first-order stochastic dominance shift of the beta distribution.



(a) Bid-ask spread. (b) Insider profits. (c) Volume.

Figure 6: Effect of second-order stochastic dominance on (a) the bid-ask spread; (b) the expected insider profits; (c) expected volume. In these graphs, $\beta = \alpha$, and the horizontal axis represents the value of the parameter α. Also, we set $\gamma = 0$ and $\delta = 1$. As α increases, the beta distribution undergoes a second-order stochastic dominance shift.

Namely, all three quantities increase as the whole precision distribution is subject to a positive shift.

Next, consider another change in the distribution of the information precision. This time however, let us keep the expected value of $\hat{s}$ constant at $1/2$ while we vary the variance of $\hat{s}$ down from $1/4$. This is done by letting $\beta = \alpha$ increase from 0 while $\gamma = 0$ and $\delta = 1$. As α and β increase, the distribution of the information precision $\hat{s}$ undergoes a second-order stochastic dominance shift. The effects of this shift on the equilibrium bid-ask spread, the expected insider profits and the expected trading volume are illustrated in Figure 6.

As can be seen in that figure and as predicted by Proposition 4.2 and Corollary 4.1, both the bid-ask spread and expected insider profits decrease as $\alpha = \beta$ decrease, i.e. as the distribution of insider precision becomes more second-order stochastically dominated. Expected trading volume on the other hand decreases as $\alpha = \beta$ decreases for large values of $\alpha = \beta$, but increase for small values of $\alpha = \beta$. This is because the ask price becomes large enough for the second part of Proposition 4.4 to kick in when $\alpha = \beta$ is small enough.

5 An Extension of the Model

The one-period one-share model analyzed so far has produced a number of interesting results about bid-ask spreads, insider profits, and volume. In this section, we seek to assess the robustness of these results by extending our one-period model to account for multiple trade sizes, insider risk aversion, elasticity of liquidity trader demand, and market power of the market-maker.[24]

To insist on the effect of random information precision and to simplify the analysis, we keep using the distribution of $\hat{v}$ and $\hat{\theta}$ used in the example of section 4.4. However, we now assume that the risky security can be traded in any integer trade sizes from 1 through N. Also, we assume that the insider type trader has a stricly increasing and weakly concave utility function $U(\cdot)$ over his profits for the period; that is, if we denote the insider type trader's profits by $\hat{\pi}_I$, he will choose his demand for the risky asset so as to maximize $\mathrm{E}[U(\hat{\pi}_I)]$.

The risk-neutral market-maker is again responsible for quoting both a bid price b and an ask price a at which he is willing to trade each share of the risky asset. Note that we could have chosen to make the market-maker quote different unit prices depending on the size of the order, like Easley and O'Hara (1987). However, since market-makers in real markets only quote one bid price and one ask price at which any integer number of shares up to a quoted a maximum can be sold or bought, we decided that this assumption would be more realistic. Note also that this assumption could not have been made in all the Kyle (1985) type models with risk-neutral insiders, since such insiders would then have chosen to trade an infinite number of shares. Here, they are limited to trading a maximum of N shares at a time, just like they are on the NYSE or other stock exchanges. Furthermore, we now consider two possible market-making scenarios.

Competitive Scenario: This scenario is similar to the scenario we have used so far, in which the market-maker quotes a bid price b_c and an ask price a_c to realize zero expected profits on each trade in equilibrium.

Monopolistic Scenario: In this scenario, the market-maker quotes his bid price b_m and ask price a_m to maximize his expected profits for the periods.

[24]An extension to a multi-period framework is considered in Chapter 2.

Notice that we have used subscripts on b and a to denote which scenario we use. Finally, assume that the liquidity type trader, after observing the bid and ask prices b and a, will decide to sell n shares of the risky asset with probability $t_{-n}(b)$, $n = 1, \ldots, N$, and buy n shares of the risky asset with probability $t_n(a)$, $n = 1, \ldots, N$. The probability that he does not trade at all is $1 - \sum_{n=1}^{N} [t_{-n}(b) + t_n(a)]$. To account for the elasticity of the liquidity trader type's demand for the risky asset, we assume that $t_{-n}(b)$ increases with b, and $t_n(a)$ decreases with a. More precisely, we assume that for $n = 1, \ldots, N$,

$$t_{-n}(b) = \begin{cases} \dfrac{1}{2N+1} & \text{if } b > 0 \\ \dfrac{(1+b)^{\varepsilon}}{2N+1} & \text{if } -1 \leq b \leq 0 \\ 0 & \text{if } b < -1, \end{cases} \tag{5.1a}$$

and

$$t_n(a) = \begin{cases} \dfrac{1}{2N+1} & \text{if } a < 0 \\ \dfrac{(1-a)^{\varepsilon}}{2N+1} & \text{if } 0 \leq a \leq 1 \\ 0 & \text{if } a > 1. \end{cases} \tag{5.1b}$$

In both (5.1a) and (5.1b), the parameter ε measures the level of liquidity demand elasticity. For ε tending to zero, the demand of the liquidity trader is completely inelastic, and the probability that he buys (or sells) n shares of the risky asset, $n = 1, \ldots, N$, as well as the probability that he does not trade all tend to $\frac{1}{2N+1}$. For ε tending to infinity, the probability that he buys or sells any positive number of shares tends to zero for any non-zero ask or bid prices.

As in the one-share model of previous sections, it can be shown that the insider type trader only considers the ask price when he wants to buy, and only considers the bid price when he wants to sell. Let $\mathcal{B}_a^n$ ($\mathcal{S}_b^n$) denote the set of $(\theta, s) \in \mathit{Supp}(\hat{\theta}, \hat{s})$ for which the insider wants to buy (sell) n shares of the risky asset when the quoted ask (bid) price is a (b). It can be shown that, for $n = 1, \ldots, N$,

$$\mathcal{S}_b^n = \left\{ (\theta, s) \in \mathit{Supp}(\hat{\theta}, \hat{s}) : \theta = -1, \alpha_n(-b) < s \leq \alpha_{n+1}(-b) \right\}, \tag{5.2a}$$

and

$$\mathcal{B}_a^n = \left\{ (\theta, s) \in \mathit{Supp}(\hat{\theta}, \hat{s}) : \theta = +1, \alpha_n(a) < s \leq \alpha_{n+1}(a) \right\}, \tag{5.2b}$$

where

$$\alpha_n(a) = \frac{U[(n-1)(-1-a)] - U[n(-1-a)]}{\left\{ U[n(1-a)] - U[n(-1-a)] \right\} - \left\{ U[(n-1)(1-a)] - U[(n-1)(-1-a)] \right\}}, \tag{5.3}$$

and $\alpha_{N+1}(a) = 1$. This is simply saying that, given an ask price of a, the insider type trader will choose to buy exactly n shares of the risky asset if he observes a positive signal ($\hat{\theta} = +1$) and his information precision s is in $(\alpha_n(a), \alpha_{n+1}(a)]$, and similarly for the sell side. Since it can be shown that $\alpha_1(a) \leq \alpha_2(a) \leq \cdots \leq \alpha_N(a)$, the higher an insider type trader's precision is, the more shares he will trade. This reflects the increased aggressiveness of the insider with high information precision.

Let $\hat{\pi}_M(a, b)$ denote the market-maker's profits for the period when he quotes a bid price b and an ask price a. As before, the symmetry of our problem allows us to concentrate on the ask side, since the bid side can be derived similarly. This is why we will only write $\hat{\pi}_M(a)$ in the rest of this section, and we will only talk about the ask side of the market. Expected market-maker profits can be written as

$$\mathrm{E}[\hat{\pi}_M(a)] = \sum_{n=1}^{N} n \left\{ (1-q)\, t_n(a)\, a - q \int_{\alpha_n(a)}^{\alpha_{n+1}(a)} (s-a) f(s) ds \right\}. \tag{5.4}$$

As in the one-share setting, the competitive market-maker will set prices so as to make zero expected profits on every trade. In other words, he will set the ask price a_c so as to make the right-hand side of (5.4) identically equal to zero, that is

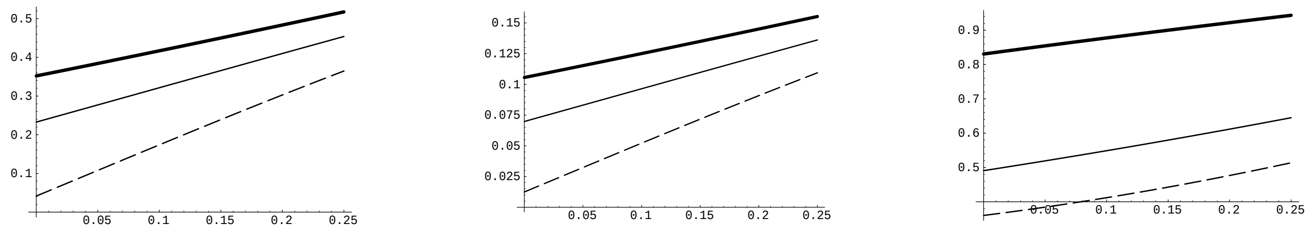
$$\sum_{n=1}^{N} n \left\{ (1-q)\, t_n(a_c)\, a_c - q \int_{\alpha_n(a_c)}^{\alpha_{n+1}(a_c)} (s-a_c) f(s) ds \right\} = 0$$

On the other hand, the monopolistic market-maker can and will set his ask price a_m to maximize (5.4).

In what follows, we set $q = 1/2$ and we plot the bid-ask spread, expected insider profits, and expected trading volume as the distribution of $\hat{s}$ undergoes a first or a second-order stochastic shift. We do this for two different scenarios. First, we consider an economy where traders can trade one or two units of the risky security ($N = 2$), the insider type trader has negative exponential utility over his period profits,

$$U(\pi_I) = \frac{1 - e^{-r\pi_I}}{r},$$

the liquidity type trader's demand is perfectly inelastic ($\varepsilon \to 0$), and the market-maker is competitive. Figure 7 illustrates the effect of a first-order stochastic shift on (a) the equilibrium bid-ask spread, (b) the expected insider profits, and (c) the expected trading volume. This first-order stochastic shift is obtained by increasing γ from 0 to 1/4 while $\alpha = \beta = 1$ and $\delta = 3/4$. In all three panels of Figure 7, we consider three values for the risk aversion parameter r: $r = 0$ (thick line), $r = 1/2$ (solid line), $r = 1$ (dashed line). As can be seen from that figure, all three variables increase with γ for all three values of r.



(a) Bid-ask spread. (b) Insider profits. (c) Volume.

Figure 7: Effect of first-order stochastic dominance on (a) the bid-ask spread; (b) the expected insider profits; (c) expected volume. The horizontal axis represents the value of the parameter γ. An increase in that parameter while $\alpha = \beta = 1$ and $\delta = 0.75$ causes a first-order stochastic dominance shift of the precision distribution. In all three panels, $N = 2$, $U(\pi_I) = \frac{1-e^{-r\pi_I}}{r}$, $\varepsilon = 0$, and the market-maker is competitive. Also, the thick line is plotted with $r = 0$, the solid line with $r = 1/2$, and the dashed line with $r = 1$.

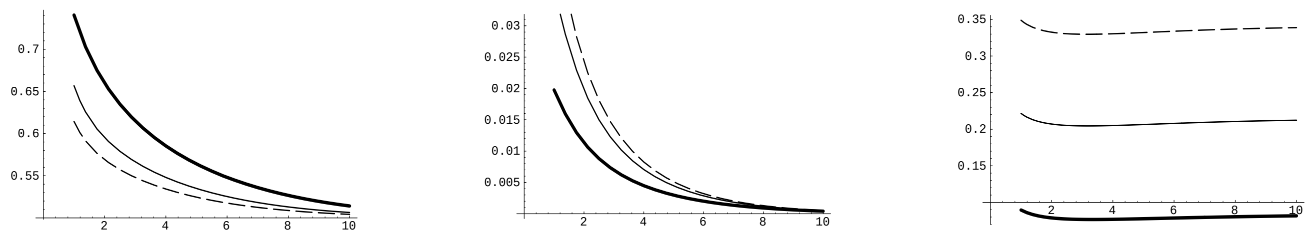
Next, we consider an economy where the insider type trader has negative exponential utility with risk aversion $r = 2$,

$$U(\pi_I) = 1 - e^{-\pi_I},$$

the elasticity of the liquidity type trader's demand is $\varepsilon = 1$, and the market-maker is monopolistic. Here, we consider the effect of a second-order stochastic shift on the same three variables. Such a stochastic shift is obtained by reducing $\alpha = \beta$ from 10 to 1 while $\gamma = 0$ and $\delta = 1$. The results are illustrated in Figure 8. In all three panels of that figure, we allow for three different trade size ranges: $N = 1$ (thick line), $N = 3$ (solid line), and $N = 5$ (dashed line). As expected, both the bid-ask spread and the expected insider profits are decreasing as $\alpha = \beta$ decreases. Also, as in Figure 6(c), expected trading volume decreases (increases) as $\alpha = \beta$ decreases when $\alpha = \beta$ is large (small).

6 Conclusion

In this paper, we introduce a new framework for analyzing the effect of random information precision in information-based models of market microstructure. This new framework avoids the traditional use of normal variables by restricting the number of shares that can be traded at one time. This assumption, which was first introduced by Glosten and Milgrom (1985), enables us to derive the unique equilibrium of a model in which the joint distribution of the risky asset's payoff, the insider's information, and its precision only have to satisfy two reasonable assumptions.



(a) Bid-ask spread. (b) Insider profits. (c) Volume.

Figure 8: Effect of second-order stochastic dominance on (a) the bid-ask spread; (b) the expected insider profits; (c) expected volume. In these graphs, $\beta = \alpha$, and the horizontal axis represents the value of the parameter α. Also, we set $\gamma = 0$ and $\delta = 1$. As α decreases, the precision distribution become more second-order stochastically dominated. In all three panels, $U(\pi_I) = 1 - e^{-2\pi_I}$, $\varepsilon = 1$, and the market-maker is monopolistic. Also, the thick line is plotted with $N = 1$, the solid line with $N = 3$, and the dashed line with $N = 5$.

The model is then used to study the effect of random information precision on the market microstructure variables usually studied in the literature. We find that a first-order stochastic shift in the distribution of the information precision has positive impact on the bid-ask spread and the insider trading profits. Trading volume also increases when the whole distribution of precision undergoes a positive shift. These results are not too surprising since a first-order stochastic shift implies an increase in the mean, and these effects have been documented in the case of known constant precision. However, completely new to the market microstructure literature is the effect of a second-order stochastic shift in the distribution of information precision. We find that such shifts imply a decrease in both the bid-ask spread and insider trading profits, but can increase or decreases trading volume according to the size of the bid-ask spread and the intensity of liquidity trading.

Since, as we mention in the introduction, information precision is not correlated with the asset's future payoffs, and since the information precision of an insider drives the intensity with which he will use his information, our model is also useful in describing the effects of uncertain risk aversion and uncertain wealth of traders in the economy. Indeed, if the market-maker does not know how risk-averse and/or wealthy traders are, he will sets his quotes knowing that the less risk-averse and the wealthiest traders will trade more aggressively, just like traders with high information precision. So our model tells us how spreads, insider profits, and trading volume are going to be affected by these additional uncertainties in the economy.

Of course, it would be interesting to see how random information precision affects the microstructure of financial markets through time, a question that our static one-period model cannot answer. However, we tackle this problem in Gervais (1995).

Appendix A

Proof of Lemma 3.2

The proof is only done for the ask price since the other part part of the proof is virtually the same. From the observation made in the paragraph preceding the lemma, we have

$$\begin{aligned}
&\mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = 1] \\
&\quad = \mathrm{E}[\hat{v} \mid \{\hat{\tau} = 1, (\hat{\theta}, \hat{s}) \in \mathcal{B}_{a,b}\} \cup \{\hat{\tau} = 0, \hat{z} = 1\}] \\
&\quad = \frac{\Pr\{\hat{\tau} = 1, (\hat{\theta}, \hat{s}) \in \mathcal{B}_{a,b}\}\mathrm{E}[\hat{v} \mid \hat{\tau} = 1, (\hat{\theta}, \hat{s}) \in \mathcal{B}_{a,b}] + \Pr\{\hat{\tau} = 0, \hat{z} = 1\}\mathrm{E}[\hat{v} \mid \hat{\tau} = 0, \hat{z} = 1]}{\Pr\{\{\hat{\tau} = 1, (\hat{\theta}, \hat{s}) \in \mathcal{B}_{a,b}\} \cup \{\hat{\tau} = 0, \hat{z} = 1\}\}},
\end{aligned}$$

where the last equality comes from Bayes' rule. Since neither $\hat{\tau}$ nor $\hat{z}$ contain information about $\hat{v}$, and since $\{\hat{\tau} = 1\}$ and $\{\hat{\tau} = 0\}$ are disjoint events, we can rewrite this as

$$\mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = 1] = \frac{\Pr\{\hat{\tau} = 1, (\hat{\theta}, \hat{s}) \in \mathcal{B}_{a,b}\}\mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) \in \mathcal{B}_{a,b}] + \Pr\{\hat{\tau} = 0, \hat{z} = 1\}\mathrm{E}[\hat{v}]}{\Pr\{\hat{\tau} = 1, (\hat{\theta}, \hat{s}) \in \mathcal{B}_{a,b}\} + \Pr\{\hat{\tau} = 0, \hat{z} = 1\}},$$

which can in turn be rewritten as (3.2b). ∎

Proof of Lemma 3.4

The proof is only done for the ask price since the one for the bid price is completely similar. Let $E_a \equiv \Big\{(\theta, s) : \mathrm{E}[\hat{v} \mid (\hat{\theta}, \hat{s}) = (\theta, s)] = a\Big\}$, so that $\mathcal{B}_{a^-} = \mathcal{B}_a \cup E_a$, where $\mathcal{B}_a$ and E_a are disjoint. By assumption, we have

$$\mathrm{E}[\hat{v} \mid \hat{\omega}_{a^-,b} = 1] = \frac{q \Pr\{\mathcal{B}_{a^-}\}\mathrm{E}[\hat{v} \mid \mathcal{B}_{a^-}] + (1-q)t_1\mathrm{E}[\hat{v}]}{q \Pr\{\mathcal{B}_{a^-}\} + (1-q)t_1} > a,$$

which is equivalent to

$$q \Pr\{\mathcal{B}_{a^-}\}\{a - \mathrm{E}[\hat{v} \mid \mathcal{B}_{a^-}]\} + (1-q)t_1\{a - \mathrm{E}[\hat{v}]\} < 0. \tag{A.1}$$

Now, suppose that

$$\mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = 1] = \frac{q \Pr\{\mathcal{B}_a\}\mathrm{E}[\hat{v} \mid \mathcal{B}_a] + (1-q)t_1\mathrm{E}[\hat{v}]}{q \Pr\{\mathcal{B}_a\} + (1-q)t_1} < a. \tag{A.2}$$

Then, since

$$PrE[\mathcal{B}_{a^-}]\mathrm{E}[\hat{v} \mid \mathcal{B}_{a^-}] = \Pr\{E_a\}\underbrace{\mathrm{E}[\hat{v} \mid E_a]}_{a} + \Pr\{\mathcal{B}_a\}\mathrm{E}[\hat{v} \mid \mathcal{B}_a]$$

and $\Pr\{\mathcal{B}_{a^-}\} = \Pr\{E_a\} + \Pr\{\mathcal{B}_a\}$, (A.2) can be shown to be equivalent to

$$q \Pr\{\mathcal{B}_{a^-}\} \{a - \mathrm{E}[\hat{v} \mid \mathcal{B}_{a^-}]\} + (1-q)t_1 \{a - \mathrm{E}[\hat{v}]\} > 0,$$

which is an obvious contradiction to (A.1). This completes the proof. ∎

Proof of Corollary 3.1

Again the proof is only done for the ask price. Take any $b \leq \mathrm{E}[\hat{v}]$. At $a = \mathrm{E}[\hat{v}]$, we have

$$a = \mathrm{E}[\hat{v}] < \mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = 1],$$

where the inequality comes from Lemma 3.1(i). Also, at $a = \bar{v}$, we have

$$a > \mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = 1] = \mathrm{E}[\hat{v}].$$

As a increases (continuously) from $\mathrm{E}[\hat{v}]$ to $\bar{v}$, Lemma 3.4 tells us that $\mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = 1]$ can never jump from above a to below a. So it must be the case that a and $\mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = 1]$ eventually cross, showing the existence of an equilibrium. ∎

Proof of Lemma 3.5

Again, since the proofs to the two parts are essentially the same, we only consider part (i). By the definition of $\mathcal{B}_a$ in equation (3.4a), $a' > a$ implies that $\mathcal{B}_{a'} \subseteq \mathcal{B}_a$. If $\mathcal{B}_{a'} = \mathcal{B}_a$, then $\mathrm{E}[\hat{v} \mid \hat{\omega}_{a',b'} = 1] = \mathrm{E}[\hat{v} \mid \hat{\omega}_{a,b} = 1] = a$ and the result holds. If, on the other hand, $\mathcal{B}_{a'} \subset \mathcal{B}_a$, then there exists a $(\theta, s) \in \Delta\mathcal{B}_{a,a'} \equiv \mathcal{B}_a - \mathcal{B}_{a'}$. Since every $(\theta, s) \in \Delta\mathcal{B}_{a,a'}$ is also in $\mathcal{B}_a$, Lemma 3.1(i) tells us that

$$\mathrm{E}[\hat{v} \mid \Delta\mathcal{B}_{a,a'}] > a = \frac{q \Pr\{\mathcal{B}_{a,b}\}\mathrm{E}[\hat{v} \mid \mathcal{B}_{a,b}] + (1-q)t_1\mathrm{E}[\hat{v}]}{q \Pr\{\mathcal{B}_{a,b}\} + (1-q)t_1}. \tag{A.3}$$

Using the fact that

$$\Pr\{\mathcal{B}_a\}\mathrm{E}[\hat{v} \mid \mathcal{B}_a] = \Pr\{\mathcal{B}_{a'}\}\mathrm{E}[\hat{v} \mid \mathcal{B}_{a'}] + \Pr\{\Delta\mathcal{B}_{a,a'}\}\mathrm{E}[\hat{v} \mid \Delta\mathcal{B}_{a,a'}], \tag{A.4}$$

manipulations of (A.3) result in

$$(1-q)t_1 \left\{\mathrm{E}[\hat{v} \mid \Delta\mathcal{B}_{a,a'}] - \mathrm{E}[\hat{v}]\right\} > q \Pr\{\mathcal{B}_{a'}\} \left\{\mathrm{E}[\hat{v} \mid \mathcal{B}_{a'}] - \mathrm{E}[\hat{v} \mid \Delta\mathcal{B}_{a,a'}]\right\}. \tag{A.5}$$

Now suppose that $\mathrm{E}[\hat{v} \mid \hat{\omega}_{a',b'} = 1] > a$ or, equivalently, that

$$\frac{q \Pr\{\mathcal{B}_{a'}\}\mathrm{E}[\hat{v} \mid \mathcal{B}_{a'}] + (1-q)t_1\mathrm{E}[\hat{v}]}{q \Pr\{\mathcal{B}_{a'}\} + (1-q)t_1} > a = \frac{q \Pr\{\mathcal{B}_{a,b}\}\mathrm{E}[\hat{v} \mid \mathcal{B}_{a,b}] + (1-q)t_1\mathrm{E}[\hat{v}]}{q \Pr\{\mathcal{B}_{a,b}\} + (1-q)t_1}.$$

Again, tedious but straightforward manipulations of this equation using (A.4) show that it is equivalent to

$$
\begin{aligned}
&q \Pr\{\Delta\mathcal{B}_{a,a'}\}\Big((1-q)t_1 \left\{\mathrm{E}[\hat{v} \mid \Delta\mathcal{B}_{a,a'}] - \mathrm{E}[\hat{v}]\right\}\Big) \\
&\qquad\qquad < q \Pr\{\Delta\mathcal{B}_{a,a'}\}\Big(q \Pr\{\mathcal{B}_{a'}\} \left\{\mathrm{E}[\hat{v} \mid \mathcal{B}_{a'}] - \mathrm{E}[\hat{v} \mid \Delta\mathcal{B}_{a,a'}]\right\}\Big),
\end{aligned}
$$

which obviously contradicts (A.5). So it must be the case that $\mathrm{E}[\hat{v} \mid \hat{\omega}_{a',b'} = 1] \leq a$, as desired. ∎

Appendix B

Proof of Proposition 4.1

As we discussed in section 3, each of the bid and ask prices can be found independently from one another. And since solving for the equilibrium bid price b is essentially the same problem as solving for the equilibrium ask price a, we only need to consider the effect of this change of distribution on the equilibrium ask price.

Notice that the first order stochastice dominance of the new ditribution of $\hat{s}$ over its original distribution along with Assumptions 1 and 2 imply the following: for any (β, α) satisfying $\beta \leq \mathrm{E}[\hat{v}] \leq \alpha$, we have $\Pr\{\mathcal{B}_\alpha\} \leq \Pr'\{\mathcal{B}_\alpha\}$ and $\Pr\{\mathcal{S}_\beta\} \leq \Pr'\{\mathcal{S}_\beta\}$.

Also observe that, for any distribution of $\hat{s}$ satisfying our assumptions,

$$\begin{aligned}\Pr\{\mathcal{B}_a\}\left\{\mathrm{E}[\hat{v} \mid \mathcal{B}_a] - a\right\} &= -\int_a^{\bar{v}} (\alpha - a) d \Pr\{\mathcal{B}_\alpha\} \\ &= -(\alpha - a)\Pr\{\mathcal{B}_\alpha\}\big|_a^{\bar{v}} + \int_a^{\bar{v}} \Pr\{\mathcal{B}_\alpha\} d\alpha \\ &= \int_a^{\bar{v}} \Pr\{\mathcal{B}_\alpha\} d\alpha, \end{aligned} \tag{B.1}$$

where we have used integration by parts to go from the first to the second line. If the new equilibrium ask price a' were strictly smaller than the original equilibrium ask price a, Lemma 3.4 tells us that we would have

$$a > \mathrm{E}'[\hat{v} \mid \hat{\omega}_{a,b'} = 1] = \frac{q\Pr'\{\mathcal{B}_a\}\mathrm{E}'[\hat{v} \mid \mathcal{B}_a] + (1-q)t_1\mathrm{E}[\hat{v}]}{q\Pr'\{\mathcal{B}_a\} + (1-q)t_1},$$

which is equivalent to

$$q\Pr'\{\mathcal{B}_a\}\left\{\mathrm{E}'[\hat{v} \mid \mathcal{B}_a] - a\right\} < (1-q)t_1\left\{a - \mathrm{E}[\hat{v}]\right\}. \tag{B.2}$$

Also, from the definition of the equilibrium ask price a in (3.5b), we have

$$a = \frac{q\Pr\{\mathcal{B}_a\}\mathrm{E}[\hat{v} \mid \mathcal{B}_a] + (1-q)t_1\mathrm{E}[\hat{v}]}{q\Pr\{\mathcal{B}_a\} + (1-q)t_1},$$

or equivalently,

$$q\Pr\{\mathcal{B}_a\}\left\{\mathrm{E}[\hat{v} \mid \mathcal{B}_a] - a\right\} = (1-q)t_1\left\{a - \mathrm{E}[\hat{v}]\right\}. \tag{B.3}$$

In view of this last equation, (B.2) implies

$$q\Pr'\{\mathcal{B}_a\}\left\{\mathrm{E}'[\hat{v} \mid \mathcal{B}_a] - a\right\} < q\Pr\{\mathcal{B}_a\}\left\{\mathrm{E}[\hat{v} \mid \mathcal{B}_a] - a\right\},$$

or, using (B.1),

$$q \int_{a}^{\bar{v}} \Pr'\{\mathcal{B}_\alpha\} d\alpha < q \int_{a}^{\bar{v}} \Pr\{\mathcal{B}_\alpha\} d\alpha. \tag{B.4}$$

which contradicts the fact that $\Pr\{\mathcal{B}_\alpha\} \leq \Pr'\{\mathcal{B}_\alpha\}$ for all $\alpha \geq \mathrm{E}[\hat{v}]$. Thus it must be the case that $a' \geq a$, and therefore that $BA' \geq BA$, as desired. ∎

Proof of Proposition 4.2

First observe that the second-order stochastic dominance of the original distribution of $\hat{s}$ over its new distribution along with Assumptions 1 and 2 imply the following: for any (β, α) satisfying $\beta \leq \mathrm{E}[\hat{v}] \leq \alpha$, we have $\int_\alpha^{\bar{v}} \Pr\{\mathcal{B}_\gamma\} d\gamma \leq \int_\alpha^{\bar{v}} \Pr'\{\mathcal{B}_\gamma\} d\gamma$ and $\int_{\underline{v}}^{\beta} \Pr\{\mathcal{S}_\gamma\} d\gamma \leq \int_{\underline{v}}^{\beta} \Pr'\{\mathcal{S}_\gamma\} d\gamma$.

As in the proof for Proposition 4.1, we only need to consider the effect on the equilibrium ask price. The proof is the same up to equation (B.4) which contradicts the fact that $\int_a^{\bar{v}} \Pr\{\mathcal{B}_\alpha\} d\alpha \geq \int_a^{\bar{v}} \Pr'\{\mathcal{B}_\alpha\} d\alpha$. Therefore, we must have $a' \geq a$, and $BA' \geq BA$. ∎

Proof of Proposition 4.3

Since $F(s;\gamma) = F(s-\gamma;\underline{\gamma})$, we have

$$F_\gamma(s;\gamma) = -\frac{\partial}{\partial s} F(s-\gamma;\underline{\gamma}) = -f(s-\gamma;\underline{\gamma}) = -f(s;\gamma). \tag{B.5}$$

So we can rewrite (4.8) as

$$\frac{\partial}{\partial \gamma}[1 - F(a(\gamma);\gamma)] = f(a(\gamma)-\gamma;\underline{\gamma}) \left[1 - \frac{da(\gamma)}{d\gamma}\right].$$

Since $f(a(\gamma)-\gamma;\underline{\gamma})$ is positive, expected volume increases if $\frac{da(\gamma)}{d\gamma} < 1$. The equilibrium condition (B.3) can be written as

$$\int_{a(\gamma)}^{1} [1 - F(\alpha;\gamma)] d\alpha = \frac{(1-q)t}{q} a(\gamma).$$

Differentiation of this expression with respect to γ leads to

$$-\frac{da(\gamma)}{d\gamma}[1 - F(a(\gamma);\gamma)] - \int_{a(\gamma)}^{1} F_\gamma(\alpha;\gamma) d\alpha = \frac{(1-q)t}{q} \frac{da(\gamma)}{d\gamma},$$

which can be rearranged to give

$$\begin{aligned} \frac{da(\gamma)}{d\gamma} &= -\frac{\int_{a(\gamma)}^{1} F_\gamma(\alpha;\gamma) d\alpha}{[1 - F(a(\gamma);\gamma)] + \frac{(1-q)t}{q}} \\ &= \frac{\int_{a(\gamma)}^{1} f(\alpha;\gamma) d\alpha}{[1 - F(a(\gamma);\gamma)] + \frac{(1-q)t}{q}} \\ &= \frac{[1 - F(a(\gamma);\gamma)]}{[1 - F(a(\gamma);\gamma)] + \frac{(1-q)t}{q}} < 1. \quad ∎ \end{aligned} \tag{B.6}$$

Proof of Proposition 4.4

(i) Since $f(s;\gamma) = f(s;0) + h(s;\gamma)$, it must be the case that $F(s;\gamma) = F(s;0) + H(s;\gamma)$, where

$$H(s;\gamma) = \begin{cases} \gamma\lambda_1(s - s_1) & \text{for } s_1 < s < s_1 + r \\ \gamma\lambda_1 r & \text{for } s_1 + r \leq s \leq s_1 + d_1 \\ \gamma\lambda_1(s_1 + d_1 + r - s) & \text{for } s_1 + d_1 < s < s_1 + d_1 + r \\ -\gamma\lambda_2(s - s_2) & \text{for } s_2 < s < s_2 + r \\ -\gamma\lambda_2 r & \text{for } s_2 + r \leq s \leq s_2 + d_2 \\ -\gamma\lambda_2(s_2 + d_2 + r - s) & \text{for } s_2 + d_2 < s < s_2 + d_2 + r \\ 0 & \text{otherwise.} \end{cases}$$

This means that

$$F_\gamma(s;\gamma) = H_\gamma(s;\gamma) = \begin{cases} \lambda_1(s - s_1) & \text{for } s_1 < s < s_1 + r \\ \lambda_1 r & \text{for } s_1 + r \leq s \leq s_1 + d_1 \\ \lambda_1(s_1 + d_1 + r - s) & \text{for } s_1 + d_1 < s < s_1 + d_1 + r \\ -\lambda_2(s - s_2) & \text{for } s_2 < s < s_2 + r \\ -\lambda_2 r & \text{for } s_2 + r \leq s \leq s_2 + d_2 \\ -\lambda_2(s_2 + d_2 + r - s) & \text{for } s_2 + d_2 < s < s_2 + d_2 + r \\ 0 & \text{otherwise,} \end{cases} \tag{B.7}$$

and we can therefore rewrite (4.8) as

$$\frac{\partial}{\partial\gamma}[1 - F(a(\gamma);\gamma)] = -H_\gamma(a(\gamma);\gamma) - f(a(\gamma);\gamma)\frac{da(\gamma)}{d\gamma}. \tag{B.8}$$

For $a(\gamma) \leq s_2$, (B.7) tells us that $H_\gamma(a(\gamma);\gamma)$ is positive, so that the first term of this last equation is negative. Also, we know from Proposition 4.2 that $\frac{da(\gamma)}{d\gamma} > 0$ for such a second-order stochastic shift, so that the second term of (B.8) is also negative. This completes the first part of the proof.

(ii) From (B.6) and (B.7), we have

$$\frac{da(\gamma)}{d\gamma} = \frac{-\int_{a(\gamma)}^{1} H_\gamma(\alpha;\gamma)d\alpha}{[1 - F(a(\gamma);\gamma)] + \frac{(1-q)t}{q}},$$

so that (4.8) can be written as

$$\frac{\partial}{\partial\gamma}[1 - F(a(\gamma);\gamma)] = -H_\gamma(a(\gamma);\gamma) + \frac{f(a(\gamma);\gamma)\int_{a(\gamma)}^{1} H_\gamma(\alpha;\gamma)d\alpha}{[1 - F(a(\gamma);\gamma)] + \frac{(1-q)t}{q}}. \tag{B.9}$$

From (B.7), $H_\gamma(\alpha;\gamma) \leq 0$ for all $a \in (s_2, 1]$. This means that the first term of (B.9) is always positive, whereas the second term is always negative. For the first term to prevail, $\frac{(1-q)t}{q}$ needs to be small. ∎

References

Ausubel, L. M., 1990, "Insider Trading in a Rational Expectations Economy," *American Economic Review*, 80, 1022-1041.

Barry, C. B., S. J. Brown, 1985, "Differential Information and Security Market Equilibrium," *Journal of Financial and Quantitative Analysis*, 20, 407-422.

Blume, L., D. Easley, and M. O'Hara, 1994, "Market Statistics and Technical Analysis: The Role of Volume," *Journal of Finance*, 49, 153-181.

Coles, J. L., U. Lowenstein, and J. Suay, 1995, "On Equilibrium Pricing under Parameter Uncertainty," *Journal of Financial and Quantitative Analysis*, 30, 347-364.

Copeland, T., and D. Galai, 1983, "Information Effects on the Bid/Ask Spread," *Journal of Finance*, 38, 1457-1469.

Easley, D., and M. O'Hara, 1987, "Price, Trade Size, and Information in Securities Markets," *Journal of Financial Economics*, 19, 69-90.

Easley, D., and M. O'Hara, 1992, "Time and the Process of Security Price Adjustment," *Journal of Finance*, 47, 577-605.

Glosten, L., 1987, "Components of the Bid-Ask Spread and the Statistical Properties of Transaction Prices," *Journal of Finance*, 42, 1293-1307.

Glosten, L., and L. Harris, 1988, "Estimating the Components of the Bid/Ask Spread," *Journal of Financial Economics*, 19, 123-142.

Glosten, L., and P. Milgrom, 1985, "Bid, Ask, and Transaction Prices in a Specialist Market with Heterogenously Informed Traders," *Journal of Financial Economics*, 14, 71-100.

Ho, T., and H. Stoll, 1983, "The Dynamics of Dealer Markets Under Competition," *Journal of Finance*, 38, 1053-1074.

Kyle, A., 1985, "Continuous Auctions and Insider Trading," *Econometrica*, 53, 1315-1335.

Lindsay, R., 1991, "Heterogeneous Beliefs, Asymmetric Information and Rational Expectations," Unpublished Dissertation, UC Berkeley.

Milgrom, P., and N. Stokey, 1982, "Information, Trade and Common Knowledge," *Journal of Economic Theory*, 26, 17-27.

Rochet, J. C., and J. L. Vila, 1994, "Insider Trading Without Normality," *Review of Economic Studies*, 61, 131-152.

Roll, R., 1984, "A Simple Implicit Measure of the Effective Bid-Ask Spread in an Efficient Market," *Journal of Finance*, 39, 1127-1139.

Romer, D., 1993, "Rational Asset-Price Movements Without News," *American Economic Review*, 83, 1112-1130.

Rothschild, M., and J. E. Stiglitz, 1970, "Increasing Risk: I. A Definition," *Journal of Economic Theory*, 2, 225-243.

Shaked, M., and J. G. Shanthikumar, 1994, *Stochastic Orders and their Applications*, Academic Press, San Diego, CA.

Spiegel, M., and A. Subrahmanyam, 1994, "Optimal Trading Halts in Financial Markets with Asymmetric Information," working paper, Haas School of Business, University of California at Berkeley.

Stoll, H. R., 1989, "Inferring the Components of the Bid-Ask Spread: Theory and Empirical Tests," *Journal of Finance*, 44, 115-134.