Screening When Some Agents are Non-Strategic: Does a Monopoly Need to Exclude?

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First Version: July 20, 2002
This Version: August 24, 2004

Abstract

We characterize the optimal screening mechanism for a monopolist facing consumers who have privately known demands, some of whom have limited abilities to misrepresent their preferences. In particular, we show that communication with consumers plays an important role in the process of screening. Consumers who have better abilities to misrepresent information benefit from the presence of consumers who lack such abilities. Whenever the fraction of the latter group of consumers is positive, there is no exclusion: it is optimal for the firm to supply a positive quantity of the good to all consumers whose valuations exceeds the marginal cost of production. Our analysis reflects the view that environments in which all individuals can costlessly and effortlessly manipulate and misrepresent their private information, although standard in economics, represent an extreme point in the spectrum of various possibilities. In fact, available evidence suggests that at least some individuals have limited ability to misrepresent their true types and imitate the behavior of others.

JEL Nos: C72, D82
Keywords: mechanism design, screening, honesty, bounded rationality.

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1 Introduction.

The nature and qualitative properties of the optimal selling strategies for a profit-maximizing monopolist have been thoroughly explored by many authors. The relevant literature contains detailed analyses of a broad range of selling mechanisms, marketing techniques and pricing schemes, such as different forms of price discrimination, bundling and tying (see, for example, Tirole (1988) and Wilson (1993)), and encompasses a variety of environments. The most ubiquitous situation is one where the monopolist faces a population of heterogeneous consumers who have private information about their preferences. It is well-known that the optimal mechanism in this case can be implemented via a simple non-linear pricing schedule (e.g. Maskin and Riley (1987)). This is the essence of the Taxation Principle.

In practice, however, firms possessing significant market power do not only employ non-linear pricing, but also rely on a significant amount of communication and interaction with customers. Firms in many industries, such as car dealerships, insurance companies and airlines, try to elicit information regarding income, occupation, demographic status, as well as the tastes and habits of their customers before making a sale to them.\(^1\) This information is clearly related to customers’ willingness to pay for the good. It is often collected through voluntary questionnaires, although sometimes the customers may be requested to provide evidence supporting their claims.

The evidence shows that firms use such information in order to offer the same goods or services to different customers at different prices. For example, car dealers use various techniques and tricks to induce customers to reveal their true willingness to pay for the car and then quote prices on the basis of information obtained in this way.\(^2\) Eskeldson (2000) describes the following “Four-Square Negotiating” technique which car salespeople use to try to elicit information about the level of monthly lease payment that a customer can bear: “A car salesperson will sit you down in front of a blank piece of paper divided into four quadrants. In each quadrant (s)he will fill in values for the price, the trade-in value, the down payment and the monthly lease rate. The salesperson will then negotiate the four factors separately, crossing out numbers and writing in new ones until the customer is hopelessly confused. The problem is, each of these factors is used to build the monthly payment. By definition, therefore, they cannot be negotiated separately from it. In the end, you think you have cut a great deal on the price and trade-in, when in fact, all you’ve done is told the dealer what monthly payment you’ll put up with.”

In internet commerce, it is becoming more common for the prices quoted by internet stores to depend on the path which the person has used to access the site. The path itself, i.e. the history of the customer’s visits to other sites and her responses to questionnaires along the way, contain information about the customer’s preferences.\(^3\)

\(^1\)For example, insurance companies price car insurance on the basis of self-reported consumer characteristics, such as the percentage of travel on a vehicle that is attributable to driving to and from work. These characteristics are largely unverifiable, and indeed appear to remain unverified.

\(^2\)The leading marketing textbook on pricing (Nagle (1987), p. 158) says: “The retail price of an automobile is typically set by the salesperson who evaluates the buyer’s willingness to pay. Notice how the salesperson takes a personal interest in the customer, asking what the customer does for a living (ability to pay), how long he has lived in the area (knowledge of the market), what kinds of cars he has bought before (loyalty to a particular brand), and whether he has looked at, or is planning to look at, other cars (awareness of alternatives). By the time a deal has been put together, the experienced salesperson has a fairly good idea how sensitive the buyer’s decision will be to the product’s price.”

\(^3\)According to McKinsey & Co. (2000), companies with an on-line presence can use a multitude of sources to help determine a customer’s price sensitivity, such as “clickstream” information about the customer’s current on-
Obviously, selling mechanisms that offer identical products or services at different prices depending on the information provided by the consumer would not be feasible if all consumers could easily imitate one another and perfectly understood the precise nature of the selling mechanism: consumers would infer how their responses affect the price, and therefore provide answers signalling that their willingness to pay for the good is low.

Although environments in which individuals can costlessly and effortlessly manipulate and misrepresent their private information in a way that maximizes their payoffs are standard in the economic literature, they clearly represent an extreme point in the spectrum of various possibilities. In fact, it is likely that at least some individuals have limited skills, knowledge and abilities to misrepresent their true types and imitate the behavior of others. There are several reasons for this.

At the most basic level, some consumers may not understand whether or how their behavior affects their subsequent terms of trade. One may think about such consumers as naive or possessing bounded rationality.

The “Four Square Negotiating” technique described above provides an example of a strategy that car dealers use to elicit information from less witting customers regarding the level of monthly lease payments that they can sustain. In internet commerce, many consumers are unaware that “cookies” allow manufacturers and retailers to monitor customers’ behavior not only at their own sites, but also at the competitors’ sites. Savvy consumers employ different strategies to avoid being charged higher prices on the basis of their browsing history and/or past purchases, such as periodically erasing the cookies on their computers or getting price quotes from different computers.

Secondly, an individual may be unable or unwilling to misrepresent her information if she is naturally averse to lying. For some individuals, the act of lying may be associated with stress or discomfort (“blushing,” “feeling wrong”) causing a disutility. This may be due to psychological or ethical reasons.

The report mentions that Ford is tracking individual customer history and behavior on the Internet, enabling the company to tailor promotions for specific customers. As a result, Ford expects to significantly improve the yield on the nearly $10 billion it spends annually on promotional pricing.

Behavioral psychologists have extensively studied the physical symptoms associated with the emotional discomfort people experience when lying (see, e.g., Ekman (1973)). The idea that emotions act as predictable and powerful motivation devices guiding economic behavior has been pursued deeply by Robert H. Frank in his book "Passions within Reason: The Strategic Role of Emotions."
to be inherently honest, willing to bear their full tax burden even when faced with financial incentives to underreport their income. Evidence for such inherently honest taxpayers derives not just from casual introspection; it is also supported by econometric evidence and survey findings...” Alger and Ma (1998) advocate a similar view. They maintain that some physicians have stronger ethics and are not able to exaggerate the medical problems of a patient when requesting coverage from an HMO, while other physicians are willing to do so. Experimental evidence confirms that a nonnegligible portion of the population chooses not to lie regarding private information, even though lying increases their monetary payoff. Chen (2000) argues that individuals have a tendency to keep promises, even if this it not always in their self-interest, and shows that this may cause optimal contracts to be incomplete.

Thirdly, some individuals may be unable to conceal their personal characteristics when these characteristics are correlated with observable consumption levels or other personal attributes which are costly to conceal. For example, an individual’s wealth and income level, demographic status and even preferences can be inferred from observation of that individual’s profession, residence, or automobile. Environments where misrepresenting the truth may require costly concealment actions have been studied by Lacker and Weinberg (1989), Maggi and Rodriguez-Clare (1995) and Crocker and Morgan (1998). Lacker and Weinberg (1989) argue that ‘there are many instances in which lying about the state of nature requires more than simply sending a false signal regarding one’s private information. Often, costly actions must be taken to lend credence to the signals being sent.’ For example, a consumer may have to hide her assets or move to a less affluent neighborhood if her report of a low wealth and/or income is to be credible. Our approach differs from the one used by these authors, because we allow individuals to differ in their aversion or ability to undertake such actions.

Finally, in some situations messages may be supported by submission of credible or verifiable claims. Indeed, individuals are often asked to support their statements with some form of evidence. For example, telecommunication firms provide discounts to households who can credibly document that their incomes fall below a certain threshold. The government requires taxpayers to justify deductions on their tax returns with receipts or other forms of evidence. Clearly, failure of an individual to produce evidence known to be available to certain types can serve as proof that the given individual is of a different type. Then only the economic actors who have the skills and technology to manufacture evidence will be able to mimic others, while those who lack such technologies will not be able to conceal their private information. 8

8For example, Gneezy (2002) reports experiments with deception games in which responders were known to largely follow the sender’s recommendation. Yet the proportion of informed senders who chose not to mislead opponents even though misleading was in the senders’ best interests varied from 48% to 83% across experiments. Survey evidence paints a similar picture, with a core group of people having no qualms at all about inflating claims to insurance companies, but an even greater fraction considering it unacceptable to do so (Tennyson 1997).

9Anecdotal evidence suggests that this factor is important in the pricing of home remodelling and repair projects. See, for example Robert Strauss “For a Precious TV, Mr. Fix-It Is Still There,” (The New York Times, June 12, 2003.)

10Consider a class of environments where the ‘binding’ direction of imitation is from ‘lower’ (less able, accident-prone, not creditworthy) to ‘higher’ (more able, low-risk, creditworthy) types. For example, in order to obtain a loan or be chosen as a supplier one may need to be perceived as successful, wealthy, or creditworthy. In such environments, some individuals will not be able to exaggerate their prior performance or risk of default, while others could rely on a network of friends or associates to provide them with references, loan money and ‘status’ goods to support their exaggerated claims. Bhidé and Stevenson (1999) describe how the founders of software firm Borland International were able to get a deeply discounted rate on a crucial advertising in BYTE magazine by crafting and
Lippman and Seppi (1995) study an environment where messages can be supported by ‘credible claims.’ Green and Laffont (1986) consider a situation in which the set of types that an agent can imitate depends on the agent’s own type. This situation also admits an interpretation in the spirit of the ‘credible claims’ model. Che and Gale (2000) study an optimal mechanism for selling a good to a buyer who may be budget constrained. In their case, the ability of a buyer to misrepresent her willingness to pay is limited if the seller can ask her to post a bond and thus credibly disclose information about her budget. A situation where players have the ability to certify truthfulness of certain messages is explored by Okuno-Fujiwara, Postlewaite and Suzumura (1990).

The main goal of this paper is to examine how the presence of consumers, who have limited ability to misrepresent their private information and imitate others, affects the optimal selling mechanism of a monopolist and, in particular, whether or how the monopolist can extract private information from these consumers at little or low cost. The presence of such consumers is incorporated into a standard screening model in the simplest possible way: we assume that a certain fraction of consumers always provide true information about their willingness to pay for the good when asked to report it.

In the context of our model, the reporting of valuations need not be understood literally. It is natural to view it as a reduced form representing the ultimate result of the firm’s actions directed at discovering a consumer’s willingness to pay (such as interviewing, requesting evidence, tracking a customer’s behavior over the internet), as well as her ability or inability to conceal her type. For brevity, consumers who are unable to misrepresent or conceal their private information will be referred to as ‘honest.’ However, this term does not pertain exclusively to consumer’s ethics. Alternatively, such consumers can be called boundedly rational or naive. All other consumers can misrepresent their valuations costlessly and will do so to increase their payoffs. Such consumers will be referred to as ‘strategic.’ Since a ‘strategic’ consumer can easily imitate an ‘honest’ one, honesty or bounded rationality, or naivete is not an observable characteristic. So, the firm cannot simply segment the market into two parts, i.e. third-degree price discrimination is not feasible.

We derive the optimal selling mechanism for such an environment, and characterize its properties. Our analysis consists of two parts. First, we derive an optimal game form for the selling mechanism. Second, we characterize the unique optimal allocation profile implementable via the optimal game form.

In the standard environment where all consumers are ‘strategic’ and choose a report maximizing their expected payoffs, the choice of a game form has no real significance. This is one of the implications of the Revelation Principle (or the Taxation Principle). However, the Revelation Principle does not hold in our setup, because the mechanism designer can typically take advantage of the fact that different consumers have different sets of feasible messages and elicit more information about types by constructing a game form where some types submit non-truthful reports in equilibrium.\footnote{We establish that the following game form, which we call a ‘password mechanism,’ is optimal in our case. First, a consumer is asked to report her valuation. Then, depending on her report, executing a strategy to create an appearance that Borland was a highly successful and well-funded firm likely to generate a significant amount of repeat business.}

\footnote{The failure of the Revelation Principle in a general class of environments that includes the one studied here has been demonstrated by Green and Laffont (1986). In Deneckere and Severinov (2001), we argue that an extended version of the Revelation Principle holds in such environments.}
she is either offered a specific quantity/transfer pair, or is given a menu of quantity/transfer pairs to choose from. The optimality of the ‘password’ mechanism stems from the fact that it allows the principal to implement allocation profiles satisfying a minimal set of incentive constraints. In particular, no incentive constraints of ‘honest’ consumers have to be satisfied. Using this mechanism, we derive the optimal allocation profile for an arbitrary fraction of ‘honest’ consumers in the population and characterize its properties. Overall, the presence of ‘honest’ consumers has the following qualitative effects:

1. Less distortion for the ‘strategic’ consumers: the quantities assigned to a subset of ‘strategic’ consumers, including the ones with the lowest valuations, are strictly higher than in the standard ‘second-best’ case with no ‘honest’ consumers but still below the first-best, while the quantities assigned to the rest of ‘strategic’ consumer types (including the ones with the highest valuations) are the same as in the standard case.

2. The quantities assigned to a subset of ‘honest’ consumer types, including the lowest valuation types, are below the first-best level, but above the quantities assigned to the ‘strategic’ types with the same valuations, while the quantities assigned to the rest of ‘honest’ consumers, including the ones with the highest valuations, are at the first-best level.

3. No exclusion: all consumers whose valuations exceed the marginal cost of production consume a positive quantity, no matter how small the fraction of ‘honest’ consumers may be.

4. ‘Strategic’ consumers (as well as the firm) benefit from the presence of the ‘honest’ ones: The surplus earned by every strategic ‘consumer’ type (and the firm’s profits) is higher than in the absence of ‘honest’ consumers. All ‘honest’ consumers are held at their reservation utility level.

The most surprising qualitative property of the optimal allocation profile is the absence of exclusion. Exclusion is a robust qualitative feature of the optimal pricing mechanism in a market with no ‘honest’ types. Except for the non-generic case of perfectly inelastic demand at price equal to marginal cost (which requires either that there are no consumers with valuations near marginal cost, or that the density of valuations is infinite at this level), a profit-maximizing monopolist will choose not to sell to consumers whose willingness to pay for the good is not sufficiently higher than marginal cost, under both uniform and non-linear pricing (see Maskin and Riley (1987) or the discussion in the next section for details). This result also holds in settings with multidimensional private information (see Armstrong (1996) and Rochet and Choné (1998)).

On the other hand, if the population consisted exclusively of ‘honest’ types, then absence of exclusion would be natural. So, intuition based on continuity would suggest that the threshold level of valuation below which consumers are not served continuously decreases to zero as the fraction of ‘honest’ consumers grows in size. The surprising thing that emerges from our analysis is that this is not so: as soon as the fraction of honest consumers becomes positive, the threshold level immediately jumps to zero.

The prospect of exclusion is troubling, and may be unacceptable from a social point of view, especially when it concerns such vitally important areas as telecommunication, energy or transportation. It provides a strong argument in support of government regulation of monopolistic industries. Indeed, it is easy to construct simple examples where the monopolist optimally
excludes a significant proportion of consumers.\textsuperscript{12}

In contrast, our no-exclusion result suggests that such concerns may not be well-founded. Even unregulated monopolists will find it optimal to serve all consumers whose willingness to pay for the good is above marginal cost, as long as some consumers in the population are not able to hide their valuations. Furthermore, if the proportion of ‘honest’ consumers is nonnegligible (as the econometric and experimental evidence suggests) then the consumption of most types who would be excluded in a market without ‘honest’ consumers is substantial (see Table 1 and Figures 2 and 3 for an illustration).

In summary, we see the contribution of our paper as follows. We contribute to the theory of screening, first, by developing a method that can handle a population including both strategic and non-strategic agents and, second, by deriving an optimal allocation profile for this complex environment. Technically, our contribution lies in introducing new techniques to solve a particular class of multidimensional screening problems which are known to be quite complex. A more general lesson learned from our results is that the predictions derived for environments that include only strategic agents may differ qualitatively from the predictions for environments in which some non-strategic agents are present. This concerns both the game form in the optimal mechanism and the optimality of exclusion.

The remainder of the paper is organized as follows. Section 2 presents the model and some preliminary results. Section 3 introduces the main results, and provides intuition. Section 4 includes the proofs of the main results. Technical results are relegated to the appendix.

2 Model and Preliminaries.

A monopoly supplier faces a population of consumers with privately known preferences for the good. Specifically, a consumer of type $\theta$ gets utility $u(q, \theta) - t$ from consuming quantity $q$ of the good, acquired at cost $t$. The distribution function $F(\theta)$ of consumer types in the population is common knowledge. We assume that $F(\cdot)$ is twice continuously differentiable function, whose density function $f(\cdot)$ is strictly positive and whose support consisting of a bounded interval. Without loss of generality, we take this interval to be $[0, 1]$. The consumer’s reservation utility level is 0. The firm’s cost is additively separable across consumers.\textsuperscript{13} We let $c(q)$ denote the cost that the firm incurs when it supplies quantity $q$ to any given consumer.\textsuperscript{14}

Equivalently, we can interpret the model as one of quality provision by a monopolist when each consumer has inelastic unit demand. A consumer of type $\theta$ gets utility $u(q, \theta)$ from a good of quality $q$. The firm’s marginal cost of producing any unit of quality $q$ is equal to $c(q)$.

We maintain the following assumptions on preferences and technology throughout the paper:

\textsuperscript{12}For example, if a firm with production costs $\frac{q^2}{2}$ faces a population of consumers whose valuation for the good is equal to $\theta q$ where $\theta$ is consumer’s private information and is distributed uniformly on $[0, 1]$, then it is optimal for the firm to exclude all consumers with valuations below $1/2$. Furthermore, exclusion is optimal even in cases where the good is ‘essential’, i.e. the marginal utility at zero consumption is infinite. In particular, it is still optimal to exclude all types with valuations below $1/2$ if we change the utility function in this example to $\theta \log q$.

\textsuperscript{13}The model can also be interpreted as one of a monopolist supplying a single consumer randomly drawn from a population whose preferences are distributed according to $F(\cdot)$.

\textsuperscript{14}One additional step allows to handle the case in which the monopolist’s aggregate cost $C(Q)$ is an increasing function of aggregate output $Q = \int_0^1 q(\theta) f(\theta) d\theta$. For any given constant marginal cost level $c$, our model predicts the corresponding aggregate output level $Q$ selected by the firm. Equilibrium then obtains whenever $C'(Q) = c$. 6
Assumption 1 (i) \( u(q, \theta) \) is a \( C^3 \) function, with \( u(q, 0) = 0, u(0, \theta) = 0, u_q(q, \theta) > 0 \) and \( u_{\theta \theta}(q, \theta) > 0 \), for all \( \theta \in (0, 1) \) and \( q \geq 0 \).

(ii) \( c(q) \) is a \( C^2 \) function, with \( c(0) = 0 \) and \( c'(0) = 0 \).

(iii) \( \exists Q > 0 \) s.t. \( u(q, \theta) - c(q) < 0 \) for all \( q > Q \) and \( \theta \in [0, 1] \).

(iv) \( u(q, \theta) - c(q) \) and \( u(q, \theta) - c(q) - \frac{1-F(\theta)}{f(\theta)} u(\theta, \theta) \) are concave in \( q \) with strictly negative second derivatives \( \forall \theta \in [0, 1] \).

Parts (i) and (iv) of Assumption 1 imply that the first-best quantity \( q^*(\theta) = \arg \max u(q, \theta) - c(q) \) is unique and increasing in \( \theta \).

To this standard model we add an additional assumption that a fraction \( \gamma \in (0, 1) \) of consumers are ‘honest’. An ‘honest’ consumer is not able or not willing to misrepresent (or conceal) her valuation, and truthfully reveals it when asked to report it.\(^{15}\)

The rest of the consumers behave in a standard fashion: they can and will always misrepresent their type if this allows them to obtain a larger surplus. We refer to such consumers as ‘strategic.’ Whether a consumer is ‘honest’ or ‘strategic’ is not observable, since a ‘strategic’ consumer can imitate an ‘honest’ one.

We assume that whether a consumer is ‘honest’ or ‘strategic’ is independent of her valuation. This assumption does not qualitatively affect our results and is adopted to simplify the exposition. In the Conclusion, we show how to characterize an optimal allocation profile when the fraction of honest consumers is an arbitrary positive function on the valuation \( \theta \).

Our goal is to understand how the presence of ‘honest’ consumers affects the optimal selling mechanism. The characterization of the optimal mechanism will involve two steps. First, we will design an optimal game form for the mechanism. Then we will derive an allocation profile that maximizes the firm’s expected profits among all allocation profiles that are implementable via the chosen game form.

To define an allocation profile, let \( (q(\theta), g(\theta)) \) respectively denote the quantities obtained by a ‘strategic’ and ‘honest’ consumer with valuation \( \theta \), and let \( (t^s(\theta), t^r(\theta)) \) denote the corresponding transfers paid to the firm. Then an allocation profile is a collection of functions \( \{q(\theta), t^s(\theta), g(\theta), t^r(\theta)\} \) from \([0, 1]\) into \( R_+ \). An allocation profile is implementable via a game form if it is optimal for each consumer type to choose a strategy giving her an allocation corresponding to her true type, i.e. such that a ‘strategic’ (‘honest’) consumer with valuation \( \theta \) chooses a strategy that gives her the allocation \( q(\theta), t^s(\theta), (g(\theta), t^r(\theta)) \).

Consider now the choice of a game form. As a preliminary step, note that a mechanism that uses only a non-linear pricing schedule is not optimal in this environment, because a consumer’s choice from such schedule will only depend on her valuation. Hence, the firm would not be able to differentiate ‘honest’ consumers from ‘strategic’ ones, and exploit the presence of the former.

Next, consider a class of mechanisms which are called ‘direct’ in a standard environment: a consumer is asked to report her valuation and is assigned an allocation based on her report. Since ‘honest’ consumers always report their valuations truthfully, while ‘strategic’ consumers choose reports maximizing their payoffs, the firm would face a choice between two alternatives. First, it could offer an allocation profile which keeps the consumers reporting truthfully at their reservation utility levels. This strategy allows to extract full surplus from ‘honest’ consumers. However,

\(^{15}\)As mentioned above, one can interpret the reporting of valuations as an outcome of a pre-sale interaction between the firm and a customer in which the firm uses different methods and technique to evaluate the customer’s willingness to pay for the good and the customer may take steps to conceal it.
the ‘strategic’ consumers faced with such an allocation profile will choose to underreport their valuations, reducing the efficiency of the mechanism and the firm’s expected profits. Alternatively, the firm can extract a larger surplus from ‘strategic’ consumers by offering an allocation profile which makes reporting the true valuation incentive compatible. However, incentive compatibility comes at a cost: the firm has to leave some surplus to all consumers who report their valuations truthfully, including the ‘honest’ ones.16

In fact, we will show that the firm can do better than in any such quasi-direct mechanism.17 It can choose a game form which allows to distinguish ‘honest’ consumers from ‘strategic’ ones without leaving any surplus to the former, while also inducing the latter to make self-selecting choices in the mechanism.

Generally, a game form is optimal if it allows to implement the largest set of allocation profiles or, equivalently, if in any game form the set of incentive constraints that have to be imposed on an implementable allocation profile is (weakly) larger. Since a ‘strategic’ consumer can always imitate any other type, an allocation profile cannot be implemented if it does not satisfy all incentive constraints of ‘strategic’ consumers. Hence, a game form is optimal if an allocation profile implementable via this game form has to satisfy only the incentive constraints of ‘strategic’ consumers.

Consider the following “Password” Mechanism:

- Stage 1. A consumer reports her valuation.
- Stage 2. (a) If the reported valuation $\hat{\theta}$ is strictly greater than 0 (the lowest valuation), then the consumer is assigned the allocation $g(\hat{\theta}), t^r(\hat{\theta})$.
- (b) If the reported valuation is 0 (the lowest valuation), then the consumer is given a choice from a menu that includes $\{q(\theta), t^s(\theta)\}_{\theta \in [0,1]}$ and $g(0), t^r(0)$.

We will say that a mechanism implements an allocation profile a.e. if the set of types who in this mechanism obtain the allocations corresponding to their true types has full measure. Allocation profiles which differ only on a set of types of measure zero are associated with the same expected profits for the firm. Thus, there is no loss in considering game forms that guarantee implementation only almost everywhere.

**Theorem 1** The ‘password’ mechanism is optimal, i.e. any allocation profile implementable via another game form is also a.e. implementable via the ‘password’ mechanism

**Proof:** Consider an allocation profile $\{q(\theta), g(\theta), t^s(\theta), t^r(\theta)\}$ implementable via some game form $\Gamma$. Let us show that it is also a.e. implementable via the ‘password’ mechanism.

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16 Alger and Ma (1998) study how this tradeoff is resolved in a related model with two possible valuations.

17 A direct mechanism in our environment is one where the consumer is asked to report both her valuation and whether she is ‘honest’ or ‘strategic.’ Our analysis assumes that ‘honest’ consumers can claim to be ‘strategic.’ This assumption is plausible in many contexts. Recall that we use the term ‘honest’ to describe consumers who are not able to misrepresent their valuation for a variety of reasons, including bounded rationality, naiveté, etc. Yet, such consumers may certainly be willing and able to claim that they can act strategically. In this context, direct mechanisms perform no better than the quasi-direct mechanisms, in which consumers are only asked to report their valuations. Alternatively, if ‘honest’ consumers are unable to claim that they are capable of misrepresentation, the optimal allocation profile in the direct mechanisms would be identical to the one derived in the next section of this paper. Thus, our solution is robust to different ways of modelling restrictions on manipulating information by ‘honest’ types.
Since a ‘strategic’ consumer can imitate any other type in any game form including Γ, this allocation profile must satisfy all incentive constraints of all ‘strategic’ consumers (or, at least, a set of full measure of ‘strategic’ consumers), i.e. the allocation \((q(\theta), t^*(\theta))\) has to provide more surplus to a consumer with valuation \(\theta\) than any allocation from \((q(\bar{\theta}), t^*(\bar{\theta}))_{\bar{\theta} \neq \theta}\), or any allocation from \((g(\bar{\theta}), t^*(\bar{\theta}))_{\bar{\theta} \in [0,1]}\). So, a ‘strategic’ consumer will not imitate another type if the principal offers \((q(\theta), g(\theta), t^*(\theta), t^r(\theta))\) in the ‘password’ mechanism.

Further, in stage 1 of the ‘password’ mechanism the only feasible reporting strategy for an ‘honest’ consumer with valuation \(\theta > 0\) is to report her true valuation. So, an ‘honest’ consumer with valuation \(\theta\) cannot deviate from the allocation \((g(\theta), t^r(\theta))\).

Thus, at most one consumer type - an ‘honest’ consumer with valuation 0 who gets access to the menu \(\{(q(\theta), t^*(\theta))| \theta \in [0,1]\}\) may choose to deviate from an allocation designed for her. So an allocation profile satisfying all incentive constraints of the ‘strategic’ consumers is a.e. implementable via the ‘password mechanism.’

The ‘password’ mechanism is optimal because it allows to ignore all the incentive constraints of ‘honest’ consumers. Plainly, the report of the lowest valuation \(\theta = 0\) in the first stage can be viewed as a ‘password’ necessary to access the menu which is designed to be sufficiently attractive (and incentive compatible) for ‘strategic’ consumers. Since an ‘honest’ consumer with valuation \(\theta > 0\) cannot misrepresent it, she cannot access the menu. Thus, in the first stage the ‘honest’ types are effectively separated from the ‘strategic’ ones, and the valuations of the ‘honest’ are identified.

In the first stage of the ‘password’ mechanism almost all ‘strategic’ consumers misrepresent their valuations. This raises the question whether there exists an optimal incentive compatible mechanism in which the valuations are reported truthfully. The answer to this question is negative. Green and Laffont (1986) have demonstrated that the Revelation Principle fails, i.e. incentive compatible direct mechanism may be suboptimal in environments where some agents are not able to send certain message and, in particular, to misrepresent themselves as certain other types. Intuitively, inducing some types to lie can help to eliminate incentive constraints of other types, thereby increasing the set of implementable allocation profiles.

In a companion paper (Deneckere and Severinov 2001), we extend the analysis of Green and Laffont (1986) by developing a general approach to implementation, considering all possible game forms (rather than just quasi-direct ones) and characterizing the set of implementable social choice functions in a broad class of environments where sending certain messages can be costly for some types. This class includes the environments considered by Green and Laffont (1986), as well as the one studied in this paper. In particular, we show that the ‘password’ mechanism is just one amongst a whole class of game forms optimal for this class of environments. However, it minimizes the amount of communication among all game forms in this class.

Theorem 1 implies that an allocation profile is implementable iff it satisfies the incentive constraints of all ‘strategic’ consumers. We show below that the optimal allocation is such that the ‘honest’ consumer with valuation 0 does not wish to deviate by choosing an allocation from \(\{(q(\bar{\theta}), t^*(\bar{\theta}))| \bar{\theta} \in [0,1]\}\) instead of \((g(0), t^r(0))\). So, in fact, the optimal allocation profile is implemented everywhere. It is formally correct to consider a.e. implementation here, because no incentive constraints are imposed on \((g(0), t^r(0))\).

If the seller deals with ‘boundedly rational’ or ‘naive’ consumers, it is natural to interpret the ‘password’ mechanism as follows. The seller offers a mechanism that is complicated and difficult to understand, so that figuring out a method to access the menu would require comprehension and analytical ability which the ‘boundedly rational’ consumers lack.
constraints of all ‘strategic’ types, the individual rationality constraints of all ‘strategic’ and ‘honest’ types and the feasibility constraints $q(\theta) \geq 0, g(\theta) \geq 0 \ \forall \theta \in [0,1]$. Let $\alpha$ be a relative proportion of ‘honest’ to ‘strategic’ agents in the population, i.e.

$$\alpha = \frac{\gamma}{1 - \gamma}.$$ 

Then we can state the firm’s profit maximization problem as follows:

$$\max_{(q(\theta) \geq 0, t^s(\theta)), (g(\theta) \geq 0, t^r(\theta))} \int_0^1 (t^s(\theta) - c(q(\theta))) f(\theta) d\theta + \alpha \int_0^1 (t^r(\theta) - c(g(\theta))) f(\theta) d\theta$$

subject to:

1. $u(q(\theta), \theta) - t^s(\theta) \geq u(q(\theta'), \theta) - t^s(\theta') \ \forall \theta, \theta' \in [0,1]$ (2)
2. $u(q(\theta), \theta) - t^s(\theta) \geq u(g(\theta'), \theta) - t^r(\theta') \ \forall \theta, \theta' \in [0,1]$ (3)
3. $u(q(\theta), \theta) - t^s(\theta) \geq 0 \ \forall \theta \in [0,1]$ (4)
4. $u(g(\theta), \theta) - t^r(\theta) \geq 0 \ \forall \theta \in [0,1]$ (5)

The presence of the second set of incentive constraints for ‘strategic’ consumers, (3), illustrates the multidimensional nature of our problem, and explains why the standard approach based on replacing the whole set of incentive constraints with a single differential equation cannot be applied here. First, let us establish the following.

**Theorem 2 Existence.** There exist bounded measurable functions $q(\theta), t^s(\theta), g(\theta), t^r(\theta)$ solving Problem 1 subject to (2)-(5).

**Uniqueness.** If $u_{\theta q}(q, \theta) \geq 0 \ \forall \theta \in [0,1]$, then the solution is unique.

**Proof:** See the appendix.

### 3 Main Results.

The literature on non-linear pricing with multidimensional private information (e.g. Wilson (1993), Armstrong (1996), Rochet and Choné (1998)) points out that identifying the set of binding incentive constraints is the key step towards characterizing the optimal mechanism.

Following this approach in our analysis, we show that either one or two incentive constraints are binding for a ‘strategic’ consumer, depending on her valuation. First, as in the standard framework without ‘honest’ consumers, downward incentive constraints between ‘adjacent’ ‘strategic’ types are binding in the optimal mechanism. Consequently, the quantity allocation $q(\theta)$ must be monotonically increasing, and the net payoff (informational rent) $U(\theta)$ of a ‘strategic’ consumer with valuation $\theta$ in the optimal mechanism is equal to $U(0) + \int_0^\theta u_\theta(q(s), s)ds$. (We also establish that $U(0) = 0$.)

Second, for a strategic type with valuation $\theta$ there is exactly one other incentive constraint that may be binding: the incentive constraint between this type and an ‘honest’ consumer with valuation $r(\theta)$ satisfying $U(\theta) = u(q(\theta), \theta) - u(q(\theta), r(\theta))$. This constraint holds if and only if an ‘honest’ consumer with valuation $\bar{\theta} = r(\theta)$ is assigned a lower quantity than a ‘strategic’ consumer with valuation $\theta$.
These results (see Lemma 6 in Section 4) allow us to significantly reduce the set of incentive constraints that have to be imposed on problem (1). In particular, we conclude that \( g(\tilde{\theta}) = \min\{q^*(\theta), q(r^{-1}(\tilde{\theta}))\} \), and so the optimal schedule \( g(.) \) is uniquely determined by the quantity schedule \( q(.) \). This step reduces the dimensionality of the problem and makes it amenable to optimal control methods.

The set of ‘strategic’ consumer types can be divided into two subsets: the first including the types \( \theta \) for whom the constraint \( q(r(\theta)) \leq q(\theta) \) (Case 1) is binding, and the second including the types for whom this constraint is non-binding (Case 2). As we argue in Lemma 9, both sets are non-empty, and the nature of the solution and the optimal quantity allocation for a particular type \( \theta \) depend critically on whether Case 1 or Case 2 applies.

Intuitively, the nature of the solution is determined by the well-known tradeoff between efficiency and informational rents. For the set of valuations where Case 1 applies, this tradeoff is qualitatively different from the standard case without ‘honest’ consumers, and for the set of valuations where Case 2 applies the tradeoff is similar to the standard case.

So, let us focus attention on Case 1. It is obvious that an ‘honest’ consumer should get zero surplus irrespective of her valuation, because information regarding her valuation is elicited for free. At the same time, if the firm attempts to reduce the informational rent \( U(\theta) \) earned by a strategic consumer with valuation \( \theta \) belonging to Case 1, it will need to reduce not only the quantities assigned to ‘strategic’ consumers with valuations less than \( \theta \) (as in the standard case). It also needs to reduce the quantity \( g(r(\theta)) \) assigned to the ‘honest’ consumer with valuation \( r(\theta) \).

This modified tradeoff between the efficiency of an allocation profile and informational rents implies that in the optimal mechanism the quantities \( q(\theta) \) for \( \theta \in [0, 1) \) and \( g(\theta) \) for \( \theta \in [0, r(\tilde{\theta})] \) will be distorted below the efficient level. However, because the efficiency losses are exacerbated by the fact that the firm has to reduce the quantities assigned to the ‘honest,’ the balance of the tradeoff between higher efficiency and lower informational rents has to shift towards higher efficiency. Lemma 5 demonstrates that \( q(\theta) \) is uniformly closer to the efficient (first-best) quantity level than the optimal quantity in the standard case without ‘honest’ consumers, and \( g(.) \) is even closer to the first-best than \( q(.) \).

An important manifestation of the shift in the tradeoff between efficiency and informational rents is the following no-exclusion result.

**Theorem 3** For any \( \alpha > 0 \), the optimal allocation profile \( q(\theta), g(\theta), t^e(\theta), t^r(\theta) \) is such that \( q(\theta) > 0 \) and \( g(\theta) > 0 \) for all \( \theta \in (0, 1] \).

**Proof**: see the Appendix.

This result stands in contrast with the standard case where the optimality of exclusion is a very robust property. In fact, under our maintained assumptions, the optimal quantity schedule exhibits exclusion when the fraction of honest consumers in the population is zero.\(^{20}\) As shown by Armstrong (1996), exclusion is also generic in the multidimensional non-linear pricing model.

The intuition behind the optimality of exclusion is well-understood. Starting from a tariff under which all customers participate in the market, if the monopolist introduces a small fixed charge \( \varepsilon \), she will gain \( \varepsilon \) from every customer that remains in the market, but lose any

---

\(^{20}\) Consider virtual surplus \( u(q, \theta) - \frac{1}{\int f(\theta)} u_d(q, \theta) - c(q) \). Since \( u_d(q, \theta) \) is bounded away from zero, and since \( u(q, \theta) \) converges to zero as \( \theta \to 0 \), virtual surplus will be maximized at \( q = 0 \) for a nondegenerate interval of types containing \( \theta = 0 \). More generally, exclusion will be present provided \( u(q, \theta) / u_d(q, \theta) \to 0 \) as \( \theta \to 0 \), and \( f(0) < \infty \).
amount she collected on customers that leave the market. In the regular case, both the number of lost customers and the revenue per lost customer are of order \( \varepsilon \), so total losses are an order of magnitude smaller than total gains, and exclusion pays. Only if there are no low valuation customers (so that the tariff can be set sufficiently high without exclusion) or if the density of low valuation customers is infinite (so that the revenue lost by excluding them is large), may exclusion fail to occur.

Absence of exclusion in our mechanism is explained by two factors. First, in Lemma 4 we establish the following ‘common cutoff’ property which follows from a simple analysis of incentive compatibility: if ‘strategic’ consumers with valuations below some threshold level \( \vartheta > 0 \) are assigned zero quantity, then so are the ‘honest’ consumers with valuations below \( \vartheta \). Second, since the firm does not leave any surplus to the ‘honest’ consumers, rationing them is costly. Particularly, if the firm raises \( q(\vartheta) \) on the interval \([0, \vartheta]\) to some small \( \varepsilon > 0 \), then the profits that it collects from ‘strategic’ consumers decrease. But the firm can now assign positive quantities to ‘honest’ consumers with valuations in \([0, \vartheta]\), and collect profits from them. Theorem 3 is proven by showing that the extra profits collected from the ‘honest’ consumers is of higher order of magnitude than the loss of profits from the ‘strategic’ consumers.

The intuition for the difference in the order of magnitudes is as follows. When the firm raises \( q(\vartheta) \) on the interval \([0, \vartheta]\) from 0 to \( \varepsilon \) and all ‘strategic’ consumers obtain more informational rent, the magnitude of the associated decrease in the firm’s expected profits is of order \( \varepsilon \). Yet, the firm can now raise the quantities \( g(\cdot) \) assigned to ‘honest’ consumers with valuations in \([0, \vartheta]\) by some positive amount, up to the level where incentive constraints from ‘strategic’ consumers become binding again. In the limit as \( \varepsilon \) converges to zero, only ‘strategic’ types with valuations close to \( \vartheta \) are willing to imitate ‘honest’ types with valuations close to \( \vartheta \). This follows from the fact that ‘strategic’ consumers with valuations strictly above \( \vartheta \) have earned strictly positive utility prior to this modification. But since the monopolist extracts full surplus from ‘honest’ consumers, type \( \vartheta \)’s incentive to mimic ‘honest type’ \( \theta < \vartheta \) vanishes as \( \theta \) approaches \( \vartheta \). Consequently, \( g(\theta) \) can be an order of magnitude larger than \( \varepsilon \) for \( \theta \) near \( \vartheta \).\(^{21}\) We conclude that near \( \varepsilon = 0 \) the profits gained from ‘honest’ consumers with valuations in \([0, \vartheta]\) is much larger then the loss incurred by leaving higher informational rents to ‘strategic’ consumers.

The no-exclusion result implies that on the lower part of the type space Case 1 applies and \( q(\vartheta) > q^{sb}(\vartheta) \), where \( q^{sb}(\vartheta) \) is the optimal quantity in the standard case without ‘honest’ consumers. Yet the downwards distortions do not disappear. Specifically, we have \( q^{sb}(\vartheta) \leq q(\vartheta) < g(\vartheta) \leq q^*(\theta) \forall \theta \in (0, 1) \).

Furthermore, in Section 4 we show that \( q^{sb}(\vartheta) = q(\vartheta) \) on the upper part of the type space where Case 2 applies. The intuition for the latter property is as follows. When \( \theta \) is high, the informational rent \( U(\theta) \) of a ‘strategic’ consumer is sufficiently large that it exceeds the payoff which she could get by imitating an ‘honest’ consumer with valuation \( r(\vartheta) \), even if the latter was assigned the first-best quantity. Therefore, when \( \theta \) is high, \( q(\vartheta) \) is determined by the same efficiency vs. informational rent tradeoff as in the standard case, and coincides with the quantity schedule that is optimal there.

Nevertheless, there is a sort of a ‘domino effect’ in the informational rents of ‘strategic’ consumers: because ‘strategic’ consumers with lower valuations are assigned larger quantities and

\(^{21}\)Since the firm gives quantity \( \varepsilon \) for free to ‘strategic’ consumers with valuation in \([0, \vartheta]\), type \( \vartheta \)’s utility equals \( u(\varepsilon, \vartheta) \). A Taylor expansion on each side of the incentive constraint \( u(\varepsilon, \theta) = u(g(\theta), \theta) - w(g(\theta), \theta) \) yields \( \varepsilon u_g(0, \theta) \approx g(\theta)(\vartheta - \theta) u_{\vartheta\theta}(0, \theta) \), so \( \frac{u(\theta)}{\varepsilon} \approx \frac{1}{(\vartheta - \theta)} \frac{u(0, \theta)}{u_g(0, \theta)} \to \infty \) as \( \theta \to \vartheta \).
are paid more not to imitate ‘honest’ types, all ‘strategic’ types obtain higher surpluses than in the standard case. Thus, ‘strategic’ consumers benefit from the presence of ‘honest’ ones. The former are paid more to prevent them from imitating the latter.

In the next theorem, we characterize the optimal quantity schedule \(q(\theta)\) and \(r(\theta)\) under an assumption guaranteeing that the monotonicity constraint \(q'(\theta) \geq 0\) is non-binding.

**Theorem 4** If \(F(\theta) + f(\theta) \frac{u_q(q, \theta) - c'(q)}{u_\theta(q, \theta)}\) is increasing in \(\theta\) for all \(\theta \in [0, 1]\) and \(q \in [0, q^*(1)]\), then the optimal quantity schedule \(q(\theta)\) and \(r(\theta)\) solve the following system of differential equations with boundary conditions \(q(0) = r(0) = 0\) and \(q(1) = q^*(1)\):

\[
q'(\theta) = \frac{f'(\theta) u_q(q, \theta) - c'(q)}{u_\theta(q, \theta)} - f(\theta) \frac{u_q(q, \theta) - c'(q)}{u_\theta(q, \theta)}^2
\]

\[
r'(\theta) = q' \frac{u_q(q, \theta) - u_q(q, r)}{u_\theta(q, r)}
\]

Furthermore, if \(f(\theta) \frac{u_q(q, \theta) - c'(q)}{u_\theta(q, \theta)}\) is increasing in \(\theta\), for all \(\theta \in [0, 1]\) and \(q \in [0, q^*(1)]\), then there is exactly one such solution.

The theorem is proven in the next section. Several comments are in order here. First, the term \(\max \left\{ f(r) \frac{u_q(q, r) - c'(q)}{u_\theta(q, r)}, 0 \right\}\) on the left-hand side of (6) highlights the distinction between Cases 1 and 2. In Case 1 the constraint \(g(r(\theta)) \leq q^*(\theta)\) is binding, and so \(u_q(q, r) - c'(q) > 0\). In contrast, in Case 2 this constraint is not binding, and so \(u_q(q, r) - c'(q) \leq 0\). Thus, the term \(\max \left\{ f(r) \frac{u_q(q, r) - c'(q)}{u_\theta(q, r)}, 0 \right\}\) is positive in Case 1, and is zero in Case 2. So, one can say that the slope of \(q(\theta)\) is flatter in Case 1 than in Case 2. Lemma 9 in the next section establishes that the solution is in Case 1 when \(\theta\) is low, and is in Case 2 when \(\theta\) is high. Lemma 10 provides additional conditions under which the solution switches between Cases 1 and 2 only once. The solution in this case is depicted in Figure 4.

The system of differential equations (6)-(7) is singular at the origin. Hence it may not be easy to solve this system starting at the origin, even numerically. It could be easier then to solve the system in reverse, by starting at \(\theta = 1\). However, the boundary value \(r(1)\) is not a priori given. One recipe would be to derive solutions to (6) and (7) starting from different values of \(r(1)\) and then choose the one(s) that also satisfy \(q(0) = r(0) = 0\).

Although according to Lemma 2 the optimal allocation profile is unique when \(u_\theta(q, \theta) \geq 0\), the singularity of (6) at 0 implies that the system (6) and (7) can potentially have several solutions satisfying the boundary conditions \(q(0) = r(0) = 0\), \(q(1) = q^*(1)\). The condition that \(f(\theta) \frac{u_q(q, \theta) - c'(q)}{u_\theta(q, \theta)}\) is increasing in \(\theta\) rules out this possibility. If this condition is not met and there are, indeed, multiple solutions to (6)-(7), one would have to choose the optimal one amongst them by comparing the values of the firm’s expected profits.

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\(^{22}\)Note that this assumption is ‘standard;’ it is implied by the ones that are normally used to ensure that the monotonicity constraint is not binding in the model with \(\alpha = 0\). See Fudenberg and Tirole (1991), Chapter 7.

\(^{23}\)In the Conclusion we exhibit the counterpart of (6) for the case when \(\alpha\) (or, equivalently, \(\gamma\)) varies with \(\theta\).
When the conditions of the Theorem are not satisfied, $q(\theta)$ solving (6) may not be monotonically increasing in $\theta$, and so will not be the solution to our problem. Then to derive the correct solution one would have to use an ‘ironing’ procedure, which is described in Subsection 4.3. On intervals where $q(\theta)$ is strictly increasing, it will still be characterized by (6). At the same time, the solution will contain intervals on which $q(\theta)$ is constant (‘ironed’).

Next, we characterize the solution explicitly in a special but common case.

**Corollary 1** Suppose that $u(q, \theta) = \theta q$, $c(q) = \frac{q^2}{2}$ and $F(.)$ is uniform. Then there is a unique optimal allocation profile $(q(\theta), g(\theta), t^s(\theta), t^r(\theta))$ maximizing the monopolist’s expected profits. Both $q(\theta)$ and $g(\theta)$ are strictly increasing and continuous, and $q(\theta)$ is convex on $[0, 1]$.

The transfers satisfy $t^s(\theta) = \theta q(\theta) - \int_0^1 q(s) ds$, $t^r(\theta) = \theta g(\theta)$.

If $\alpha \neq 4$, then for all $\theta \in \left[0, \frac{2}{3} + \frac{1}{3(\sqrt{1} + 2\alpha + 1)}\right]$, $q(\theta)$ is the unique solution in the range $\left[0, \frac{1}{3} + \frac{2}{3(\sqrt{1} + 2\alpha + 1)}\right]$ satisfying the following equation:

$$
\frac{1 - \alpha}{2 - \alpha/2} q(\theta) + \frac{(1 + \frac{\alpha}{2(\sqrt{1} + 2\alpha + 1)})}{(2 - \alpha/2) \left(\frac{1}{3} + \frac{2}{3(\sqrt{1} + 2\alpha + 1)}\right)} q(\theta)^{\frac{1 + 2\alpha - 1}{2}} = \theta. \quad (8)
$$

For all $\theta \in \left[\frac{2}{3} + \frac{1}{3(\sqrt{1} + 2\alpha + 1)}, 1\right]$, we have $q(\theta) = 2\theta - 1$.

For all $\theta \in \left[0, \frac{1}{3} + \frac{2}{3(\sqrt{1} + 2\alpha + 1)}\right]$, $g(\theta)$ is the unique solution in the range $\left[0, \frac{1}{3} + \frac{2}{3(\sqrt{1} + 2\alpha + 1)}\right]$ satisfying the following equation:

$$
\frac{1 - \alpha}{4 - \alpha} g(\theta) + \frac{3}{4 - \alpha} \left(\frac{\sqrt{1 + 2\alpha + 1}}{\sqrt{1 + 2\alpha + 3}}\right) g(\theta)^{\frac{4 - 3}{2}} = \theta. \quad (9)
$$

For all $\theta \in \left[\frac{1}{3} + \frac{2}{3(\sqrt{1} + 2\alpha + 1)}, 1\right]$, we have $g(\theta) = \theta$.\(^{24}\)

For each $\alpha > 0$, $q(\theta)$ and $g(\theta)$ are characterized indirectly on the lower part of the range of valuations, i.e. $\theta$ is expressed as a function of $q$ and $g$ respectively. In the proof, we demonstrate that (8) and (9) are invertible over their respective ranges, and so $q(\theta)$ and $g(\theta)$ are well-defined.

The two-part nature of the optimal schedules $q(\theta)$ and $g(\theta)$ derives from the fact that the incentive constraint between a ‘strategic’ consumer with valuation $\theta$ and an ‘honest’ consumer with valuation $\tau(\theta)$ is binding when $\theta \in [0, \theta]$ (Case 1 applies), and is not binding if $\theta$ belongs to the interval $(\theta, 1]$ (Case 2 applies) where $\theta = \frac{2}{3} + \frac{1}{3(\sqrt{1} + 2\alpha + 1)}$. The structure of binding incentive constraints is depicted in Figure 1. Consistently with Lemma 5, $q_{\theta}(\theta) < q(\theta) < q^*(\theta)$ for all $\theta \in (0, \theta)$, $q(\theta) = q_{\theta}(\theta)$ for all $\theta \in [\theta, 1]$, while $g(\theta) = q^*(\theta) = \theta$ for all $\theta \in [\tau(\theta), 1]$. Figures 2 and 3 illustrate the solution for three different values of $\alpha$: 10, 1 and 0.02. When $\alpha = 1$, i.e. half of the population is ‘honest’ and the other half is ‘strategic,’ the optimal quantity

\(^{24}\)If $\alpha = 4$, then the equation characterizing $q(\theta)$ on the lower part of the type space $[0, 3/4]$ is $(3/2 + \log(1/2))q - q \log(q) = \theta$, while the equation characterizing $g(\theta)$ on $[0, 1/2]$ is $(1 + \log(1/2)/2)g - g \log(g)/2 = \theta$. They can be derived either directly or by taking the appropriate limits of the equations characterizing the solution for $\alpha \neq 4$.  

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schedules have the following closed form:

\[
q(\theta) = \begin{cases} 
\sqrt{3} \left( \sqrt{3} - 1 \right) \sqrt{3} + 1 \theta^{\sqrt{3}+1} & \text{if } \theta \in [0, \frac{1}{\sqrt{3}-1}] \\
2\theta - 1 & \text{if } \theta \in \left[ \frac{1}{\sqrt{3}}, 1 \right]
\end{cases}
\]

\[
g(\theta) = \begin{cases} 
\sqrt{3} \left( \sqrt{3} - 1 \right) \theta^{\sqrt{3}+1} & \text{if } \theta \in [0, \frac{1}{\sqrt{3}-1}] \\
\theta & \text{if } \theta \in \left[ \frac{1}{\sqrt{3}}, 1 \right]
\end{cases}
\]

By varying \( \alpha \), the ratio of ‘honest’ consumers to ‘strategic’ ones in the population, we establish a number of interesting comparative statics results described in the following corollary. Let us explicitly incorporate the dependence of the solution on \( \alpha \) by using the notation \( q(\cdot, \alpha) \), \( g(\cdot, \alpha), U(\cdot, \alpha) \) and \( \theta(\alpha) \) for the quantity schedules, the informational rent and the threshold value between Cases 1 and 2, respectively.

**Corollary 2** 25 Let \( \alpha_1 \) and \( \alpha_2 \) be such that \( \alpha_1 > \alpha_2 > 0 \). Then,
(i) There exists a unique \( \theta_c \in (0, \frac{2}{3} + \frac{1}{3\sqrt{1+2\alpha_1+1}}) \) s.t. \( q(\theta_c, \alpha_1) = q(\theta_c, \alpha_2) \). Furthermore, \( q(\theta, \alpha_1) > q(\theta, \alpha_2) \) for \( \theta \in (0, \theta_c) \), \( q(\theta, \alpha_1) < q(\theta, \alpha_2) \) for \( \theta \in (\theta_c, \frac{2}{3} + \frac{1}{3\sqrt{1+2\alpha_2+1}}) \).
(ii) \( U(\theta, \alpha_1) > U(\theta, \alpha_2) \) for \( \theta \in (0, 1] \).
(iii) \( g(\theta, \alpha_1) > g(\theta, \alpha_2) \) for \( \theta \in (0, \frac{1}{3} + \frac{2}{3\sqrt{1+2\alpha_2+1}}) \).

Parts (ii) and (iii) are easy to understand. Clearly, as the proportion of ‘honest’ consumers increases, the benefit to the firm from increasing \( g(\cdot) \) towards an efficient level and extracting more surplus from the ‘honest’ consumers becomes larger than the cost of an associated increase in informational rents \( U(\cdot) \) paid to ‘strategic’ consumers whose fraction has now decreased.

---

The intuition behind part (i) is similar, but more complex. Since \( g(. , \alpha) \) and hence \( U(. , \alpha) \)
increase in \( \alpha \), and \( U(\theta, \alpha) = \int_0^\theta q(s, \alpha) ds \), \( q(\theta, \alpha) \) must also be increasing in \( \alpha \), at least for small \( \theta \). Intuitively, an increase in the proportion of strategic consumers causes a ‘ripple’ effect in the form of higher \( q(.) \). However, this effect disappears when \( \theta \) is high. This happens because the informational rent \( U(\theta, \alpha) \) of a ‘strategic’ consumer with valuation \( \theta > \theta(\alpha) \) is sufficiently large that this consumer strictly prefers not to imitate an ‘honest’ consumer even when all ‘honest’ consumers with valuations that exceed \( r(\theta(\alpha), \alpha) \) are assigned an efficient quantity. Therefore, if \( \theta > \theta(\alpha) \), it is no longer optimal for the firm to raise \( q(\theta, \alpha) \) above the quantity optimal in the standard case. So, the downward distortions in the quantity schedule for ‘strategic’ consumers persist even as the fraction of ‘honest’ consumers converges to 1.

Since the firm can set \( g(. , \alpha) \) efficiently on the interval \([r(\theta(\alpha), \alpha), 1]\), it is optimal to set a lower threshold \( \theta(\alpha) \) when \( \alpha \) is large, so that \( q(\theta(\alpha), \alpha_1) < q(\theta(\alpha), \alpha_2) \) for \( \alpha_1 > \alpha_2 \). Thus, \( q(\theta, \alpha) \) must be decreasing in \( \alpha \) when \( \theta \) is sufficiently high.

Essentially, when \( \alpha \) is large, then \( q(.) \) is front-loaded: it is high when \( \theta \) is small, and relatively low when \( \alpha \) is large. The opposite is true when \( \alpha \) is small.

Table 1 below describes the aggregate welfare effects from the presence of ‘honest’ types for the uniform-quadratic case of Corollary 1. The aggregate welfare gain \( WG \) is measured as a percentage of the maximal possible welfare gain relative to the standard second-best case in which only strategic agents are present.\(^{26}\) In interpreting the Table, one should keep in mind that both the quantity schedule of the honest agents and that of the strategic agents are distorted relative to the first-best, but that the former is less distorted than the latter. When the fraction of ‘honest’ agents gets larger, welfare increases for two reasons. First, ‘strategic’ agents are replaced by the ‘honest’ ones. Second, both quantity schedules become less distorted.

### 4 Solving the Monopolist’s Maximization Problem.

#### 4.1 Preliminary Results.

In this subsection we establish several key steps that will allow us to solve Problem (1)-(5). In particular, we determine the set of binding incentive constraints in this Problem. Then, we use this result to establish a link between the quantity schedules for ‘honest’ and ‘strategic’ consumers, allowing us to express the objective as a function of the latter schedule only.

Let us first establish several restrictions that can without loss of generality be imposed on the set of implementable allocation profiles. First, note that the individual rationality constraint (5) of an ‘honest’ consumer must be binding for all \( \theta \in [0,1] \). Otherwise, the value of (1) can be increased by raising the corresponding transfer \( t^* (\theta) \) without violating any incentive or individual rationality constraints. Hence, we can substitute \( u(g(\theta), \theta) \) for \( t^* (\theta) \) in (1) and eliminate (5).

We will say that quantity schedules \( q(.) \) and \( g(.) \) are admissible if there exist transfer functions \( t^*(.) \) and \( t^* (.) \) such that the allocation profile \( (q(.), t^*(.), g(.), t^*(.) ) \) satisfies (2)-(5), i.e.

\[ \int_0^1 \theta q(\theta) - \frac{q(\theta)^2}{2} + \alpha(\theta g(\theta) - \frac{g(\theta)^2}{2}) d\theta \] / \( (1 + \alpha) \).

---

\(^{26}\) Honest \( 100 \times \alpha / (1 + \alpha) \), \%WG \( \alpha / (1 + \alpha) \) \( \equiv \) \( W(\alpha, W_{sb}) / W_{max} - W_{sb} \times 100 \), where \( W_{max} \equiv \int_0^1 \theta q(\theta) - \frac{q(\theta)^2}{2} d\theta = 1/6 \), \( W_{sb} \equiv \int_0^1 \theta \max \{ 0, 2\theta - 1 \} - \frac{\max \{ 0, 2\theta - 1 \}^2}{2} d\theta = 1/8 \), and \( W(\alpha) \equiv \left( \int_0^1 \theta q(\theta) - \frac{q(\theta)^2}{2} + \alpha(\theta g(\theta) - \frac{g(\theta)^2}{2}) d\theta \right) / (1 + \alpha) \).
is implementable. Then (2) implies that a feasible quantity schedule \( q(\cdot) \) must be nondecreasing. The following lemma allows to restrict the set of admissible schedules further:

**Lemma 1** Without loss of generality, we can impose the following restrictions on the set of admissible quantity schedules in Problem (1)-(5):

(i) \( g(\theta) \) is nondecreasing and satisfies \( g(\theta) \leq q^*(\theta) \) for all \( \theta \in [0,1] \);

(ii) \( q(0) = 0 \) and \( q(1) = q^*(1) \).

Lemma 1 has several implications. Since \( q(\theta) \) is non-decreasing and bounded, it is Riemann integrable (Rudin (1976), Theorem 6.9, p.126) and a.e. differentiable (Royden (1987), Theorem 3, p.100). Hence, \( u(q(\theta), \theta) \) is also bounded, Riemann integrable and a.e. differentiable. Let \( U(\theta) \equiv u(q(\theta), \theta) - t^s(\theta) \). Incentive constraints (2) imply that \( U(\theta) \) is increasing and satisfies the following inequalities \( \forall \theta, \theta' \in [0,1] \):

\[
    u(q(\theta'), \theta) - u(q(\theta'), \theta') \leq U(\theta) - U(\theta') \leq u(q(\theta), \theta) - u(q(\theta), \theta')
\]

By the intermediate value theorem, \( \exists \lambda_1, \lambda_2 \in [0,1] \) s.t. \( u_\theta(q(\theta'), \lambda_1 \theta + (1 - \lambda_1) \theta') - u_\theta(q(\theta), \lambda_2 \theta + (1 - \lambda_2) \theta') \leq U(\theta) - U(\theta') \leq u_\theta(q(\theta), \lambda_2 \theta + (1 - \lambda_2) \theta') - u_\theta(q(\theta), \theta') \). Since, \( u_\theta(q(\theta'), \theta) \leq \max_{\theta \in [0,1]} u_\theta(q^*(1), \theta) < \infty \), \( U(\theta) \) is absolutely continuous. Therefore by Theorem 14, p.110 in Royden (1987), we have \( U'(\theta) = u_\theta(q(\theta), \theta) \) and \( U(\cdot) \) is equal to the integral of its derivative, i.e. \( U(\theta) - U(\theta') = \int_{\theta'}^\theta u_\theta(q(s), s)ds \). This equation implies that, as in the standard case, only downwards incentive constraints between ‘adjacent’ types are binding among the ‘strategic’ consumers whenever \( q(\cdot) \) is strictly increasing.

It is easy to show that in the optimal mechanism \( U(0) \leq u(q^*(1), 1) \). Therefore, \( t^s(\theta) = u(q(\theta), \theta) - \int_0^\theta u_\theta(q(s), s)ds - U(0) \) is bounded on [0,1]. Let us use this expression to substitute \( t^s(\theta) \) out of (1) and eliminate (2). Then integrating by parts, we obtain that Problem 1 is equivalent to the following one:

\[
    \max_{q(\theta) \geq 0, g(\theta) \geq 0, U(0) \geq 0} \quad -U(0) + \int_0^1 \left( u(q(\theta), \theta) - c(q(\theta)) - u_\theta(q(\theta), \theta) \frac{1 - F(\theta)}{f(\theta)} \right) f(\theta)d\theta + \\
    \alpha \int_0^1 (u(q(\theta), \theta) - c(g(\theta))) f(\theta)d\theta
\]

subject to: (i) \( q(\theta) \) is nondecreasing

\[
    (ii) ICT(\theta, \theta') : \quad U(\theta) \equiv U(0) + \int_0^\theta u_\theta(q(s), s)ds \geq u(q(\theta'), \theta) - u(g(\theta'), \theta') \quad \forall \theta, \theta' \in [0,1]
\]

We can now establish several additional properties of a solution.

**Lemma 2** If \( q(\theta) \) is an optimal quantity schedule, then it is continuous.

**Lemma 3** In an optimal mechanism, we have \( U(0) = 0 \).

**Lemma 4** Common cutoff. Optimal quantity schedules are such that \( q(\theta) = 0 \) if and only if \( g(\theta) = 0 \).

Let \( q^{ab}(\theta) \) be an optimal quantity schedule in the standard case without ‘honest’ consumers, i.e. when \( \alpha = 0 \).\(^{27}\) The following lemma compares an optimal quantity schedule \( q(.) \) to the benchmarks \( q^*(\theta) \) and \( q^{ab}(\theta) \).

\(^{27}\)Under the assumptions of Theorem 4, \( q^{ab}(\theta) = \arg \max_q \{ u(q, \theta) - c(q) f(\theta) - u_\theta(q, \theta) (1 - F(\theta)) \} \).
Lemma 5 For any $\alpha > 0$, an optimal quantity schedule $q(.)$ satisfies $q^{ab}(\theta) \leq q(\theta) \leq q^*(\theta)$ for all $\theta \in (0,1]$.

By Lemmas (1)-(3), we can without loss of generality impose the following additional restrictions on the domain in Problem (10): (iii) $q(.)$ is continuous, (iv) $q(0) = 0$, (v) $g(\theta) \leq q(1) = q^*(1)$, (vi) $U(0) = 0$.

The next lemma is a key step in solving our two-dimensional screening problem. It shows that the family of incentive constraints $ICT(\theta, \theta')$ in (12) can be replaced with a simpler one.

Lemma 6 Assume $U(0) = 0$. For any nondecreasing function $q(\theta)$ define $r(\theta)$ as follows. When $\theta$ is such that $q(\theta) > 0$, let $r(\theta)$ be the unique solution to $U(\theta) = u(q(\theta), \theta) - u(q(\theta), r(\theta))$. When $\theta$ is such that $q(\theta) = 0$, let $r(\theta) = \theta$. Then:

(i) $r(\theta)$ is a nondecreasing function satisfying $r(0) = 0$. Furthermore, it is strictly increasing (continuous) at all $\theta$ where $q(.)$ is strictly increasing (continuous).

(ii) Suppose that $g(\theta) \leq q(1) = q^*(1)$. Then $U(\theta) \geq u(g(\theta'), \theta) - u(g(\theta'), \theta')$ for all $\theta, \theta' \in [0,1]$ if and only if $q(\theta) \geq g(r(\theta))$ for all $\theta \in [0,1]$.

Since $r(\theta)$ is continuous and nondecreasing and $r(0) = 0$, $r(.)$ maps $[0,1]$ onto $[0, r(1)]$. The inverse image $r^{-1}(\theta)$ from $[0, r(1)]$ is unique if $q(.)$ is strictly increasing at $\theta$ s.t. $r(\theta) = \theta$. However, even if $r^{-1}(\theta)$ is not unique, $q(r^{-1}(\theta))$ is unique, because $r(\theta_1) = r(\theta_2)$ only if $\theta_1 = \theta_2$. Lemma 6 allows to establish the following important result:

Lemma 7 Fix a nondecreasing continuous quantity schedule $q(\theta)$ s.t. $q(1) = q^*(1)$, and set $U(0) = 0$. Then the optimal quantity schedule $g(\theta)$ that maximizes (10) subject to (12) is given by:

$$g(\theta) = \begin{cases} \min\{q^*(\theta), q(r^{-1}(\theta))\} & \text{if } \theta \leq r(1) \\ q^*(\theta) & \text{if } \theta > r(1) \end{cases}$$

Proof: By Lemma 1, we can without loss of generality impose the restriction $g(\theta) \leq q^*(1)$. So, by (ii) of Lemma 6, the family $ICT(\theta, \theta')$ of incentive constraints in (12) can be replaced with the following family: $g(\theta) \geq g(r(\theta))$ for all $\theta \in [0,1]$ which can be rewritten as $q(r^{-1}(\theta)) \geq g(\theta)$ for all $\theta \in [0, r(1)]$. Then the result follows because the integrand $u(q(\theta), \theta) - c(q(\theta))$ of the second integral in (10) is strictly concave in $g(\theta)$ and is strictly increasing (decreasing) in $g(\theta)$ if $g(\theta) < q^*(\theta)$ (if $g(\theta) > q^*(\theta)$).

According to Lemma 7, $g(.)$ is completely determined by $q(.)$. So $q(.)$ remains the only choice variable, which reduces the dimensionality of our problem. Imposing the additional constraints (iii) - (vi), and using Lemma 7, we conclude that Problem (10) is equivalent to the following one:

$$\max_{q(0)} \int_{0}^{1} \left( u(q(\theta), \theta) - c(q(\theta)) - u_q(q(\theta), \theta) \frac{1 - F(\theta)}{f(\theta)} \right) f(\theta) d\theta + \alpha \int_{r(1)}^{1} \left( u(q^*(\theta), \tilde{\theta}) - c(q^*(\tilde{\theta})) \right) f(\tilde{\theta}) d\tilde{\theta} + \alpha \int_{0}^{r(1)} \left( u(\min\{q^*(\tilde{\theta}), q(r^{-1}(\tilde{\theta}))\}, \tilde{\theta}) - c(\min\{q^*(\tilde{\theta}), q(r^{-1}(\tilde{\theta}))\}) \right) f(\tilde{\theta}) d\tilde{\theta}$$

subject to: $q(.)$ is nondecreasing, continuous, $q(0) = 0$, and $q(1) = 1$. 

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In the Appendix, we use this characterization of the monopolist’s maximization problem to prove Theorem 3, our no exclusion result.

### 4.2 The Optimal Control Problem. Necessary Conditions for Optimality.

Our next step is to reformulate Problem (13) to make it amenable to standard methods of optimal control. We will need to use the derivative $q'(\cdot)$ as the control variable, and so $q'(\cdot)$ has to be piecewise continuous or, equivalently, $q(\cdot)$ has to belong to the space $C^1_p([0,1])$ of piecewise smooth (continuous and piecewise continuously differentiable) functions on $[0,1]$. So far, we have only established that $q(\cdot)$ must be continuous, i.e. $q(\cdot) \in C([0,1])$, and that $q'(\cdot)$ exists almost everywhere. Nevertheless, the next lemma demonstrates that we can without loss of generality assume that $q(\cdot) \in C^1_p([0,1])$.

**Lemma 8** If $q(\theta)$ is a solution to maximization problem (13) on the domain $C^1_p([0,1])$, then $q(\theta)$ also maximizes (13) on the domain $C([0,1])$.

Next, let us make a change of variables $\tilde{\theta} = r(\theta)$ in the second term of (13). By Lemma 6, $r(\theta)$ is continuous, increasing and bounded on $[0,1]$. Therefore, it is Riemann integrable and the change of variables is legitimate. Further, Theorem 3 implies that $U(\theta) = u(q(\theta), \theta) - u(q(\theta), r(\theta)) \forall \theta \in (0,1]$. Since $q(\theta) \in C^1_p([0,1])$, we can differentiate this expression at all but (at most) a finite number of $\theta$ to yield:

$$r'(\theta) = \frac{q'(\theta)(u_q(q(\theta), \theta) - u_q(q(\theta), r(\theta)))}{u_\theta(q(\theta), r(\theta))}$$

(14)

Note that $r'(\theta)$ is piecewise continuous since $q'(\cdot)$ is piecewise continuous. So, the change of variables $\tilde{\theta} = r(\theta)$ allows us to express the second term in (13) as the following Riemann integral:

$$\alpha \int_0^{r(1)} (u(\min\{q^*(r(\theta)), q(\theta)\}, r(\theta)) - c(\min\{q^*(r(\theta)), q(\theta)\})) f(\theta) r'(\theta) d\theta$$

Using (14), we can finally restate Problem (13) as follows:

$$\max_{q(\cdot)} \int_0^{1} \left( u(q(\theta), \theta) - c(q(\theta)) - u_\theta(q(\theta), \theta) \frac{1 - F(\theta)}{f(\theta)} \right) f(\theta) d\theta + \alpha \int_0^{1} (u(q^*(\theta), \theta) - c(q^*(\theta))) f(\theta) d\theta + \alpha \int_0^{r(1)} (u(\min\{q^*(r(\theta)), q(\theta)\}, r(\theta)) - c(\min\{q^*(r(\theta)), q(\theta)\})) f(\theta) r'(\theta) \frac{q'(\theta)(u_q(q(\theta), \theta) - u_q(q(\theta), r(\theta)))}{u_\theta(q(\theta), r(\theta))} d\theta$$

(15)

subject to:

$$r'(\theta) = \frac{q'(\theta)(u_q(q(\theta), \theta) - u_q(q(\theta), r(\theta)))}{u_\theta(q(\theta), r(\theta))}, \quad q(0) = 0, \quad r(0) = 0, \quad q(1) = q^*(1), \quad and \quad q'(\cdot) \geq 0$$

(16)

Observe that (15) and (16) is an optimal control problem with control variable $q'(\cdot)$, two state variables $q(\theta)$ and $r(\theta)$, and a restriction on the control $q'(\cdot) \geq 0$. It has ‘scrap value’ $S(r_1) = \alpha \int_{r(1)}^{1} (u(q^*(\theta), \theta) - c(q^*(\theta))) f(\theta) d\theta$ at $\theta = 1$. 


The existence of a solution to this problem follows from the Filippov-Cesari theorem (see Seierstad and Sydsæter (1987)). Pontryagin’s Maximum Principle provides a standard method of solution to this problem. The corresponding Hamiltonian is given by:

\[ H(q, r, λ, δ, θ) = \left( u(q, θ) - c(q) - u_θ(q, θ) \frac{1 - F(θ)}{f(θ)} \right) f(θ) + \alpha f(r) (u(\min\{q^*(r), q\}, r) - c(\min\{q^*(r), q\})) q'(u_θ(q, θ) - u_θ(q, r)) \frac{q'(u(q, θ) - u(q, r))}{u_θ(q, r)} + λq' + δq'(u_θ(q, θ) - u_θ(q, r)) \]

\[ \text{where } λ(θ) \text{ and } δ(θ) ∈ C^1_p([0, 1]) \text{ are costate variables associated with the laws of motion of } q(θ) \text{ and } r(θ), \text{ respectively. Incorporating the constraint } q'(θ) ≥ 0, \text{ we obtain the following Lagrangian:} \]

\[ L = H(q, r, λ, δ, θ) + τq' \]

where \( τ ≥ 0 \) and \( τq' = 0 \). The transversality conditions on the costate variable \( δ \) is \( δ(1) = \frac{dS(r(1))}{dr} = -α(u(q^*(r(1)), r(1)) - c(q^*(r(1))))f(r(1)). \)

According to the Maximum Principle, the necessary conditions for an optimum are:

\[ -λ'(θ) = \frac{∂H}{∂q} = \left( u_θ(q, θ) - u_θ(q, r) \frac{1 - F(θ)}{f(θ)} \right) f(θ) + α f(r)(u_θ(q, r) - c'(q)) \frac{q'(u(q, θ) - u(q, r))}{u_θ(q, r)} + \frac{∂H}{∂q} δ \]

\[ -δ'(θ) = \frac{∂H}{∂r} = (α f'(r)(u(q, r) - c(q)) + α f(r)u_θ(q, r)) q'(u_θ(q, θ) - u_θ(q, r)) \frac{1}{u_θ(q, r)} + \frac{δ}{q'}(u_θ(q, θ) - u_θ(q, r))u_θ(q, r) \]

\[ -∂L/∂q' = (α f(r)(u(q, r) - c(q)) + δ) \frac{(u_θ(q, θ) - u_θ(q, r))}{u_θ(q, r)} + λ + τ \]

To solve (19)-(21), first use (19) and (20) to solve for \( -\frac{d(λ+δ(u_θ(q, θ) - u_θ(q, r))}{dq'} δ(q^*(r(r))) \). Second, differentiate (21) to obtain another expression for \( -\frac{d(λ+δ(u_θ(q, θ) - u_θ(q, r))}{dq'} δ(q^*(r(r))) \). Equating the two expressions yields:

\[ [δ(θ) + α f(r)(u(q, r) - c(q))] \frac{u_θ(q, θ)}{u_θ(q, r)} + τ' = (u(q, θ) - c'(q)) f(θ) - u_θ(q, θ)(1 - F(θ)) \]

Consider any interval \((θ_1, θ_2)\) where \( q'(θ) > 0. \) In that case, \( τ(θ) = 0 \) and so \( τ'(θ) = 0. \) Then by (22):

\[ δ(θ) = (u(q_θ(q, θ) - c'(q)) u_θ(q, r) f(θ) - u_θ(q, r)(1 - F(θ)) - α f(r)(u(q, r) - c(q)) \]

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Differentiate this expression to get the first expression for \( \delta'(\theta) \). The second expression for \( \delta'(\theta) \) it obtained by using (22) to substitute \( \delta \) out from the right-hand side of (21). Equating the two expressions for \( \delta'(\theta) \) we get the following differential equation:

\[
q' \left( \alpha \frac{f(r)(u_q(q, r) - c(q))}{u_\theta(q, r)} + f(\theta) \frac{(u_q(q, \theta) - c(q))u_\theta q q(q, \theta)}{u_\theta(q, \theta)^2} - f(\theta) \frac{u_q q q(q, \theta) - c''(q)}{u_\theta(q, \theta)} \right) = 2f(\theta) + f'(\theta) \frac{u_q(q, \theta) - c'(q)}{u_\theta(q, \theta)} - f(\theta) \frac{(u_q(q, \theta) - c'(q))u_\theta q q(q, \theta)}{u_\theta(q, \theta)^2}
\]

(24)

So, when \( q' > 0 \), the solution in Case 1 is characterized by (24) and the ‘law of motion’ (14).

Next, suppose there is an interval \([\theta_1, \theta_2]\) where the monotonicity constraint \( q' = 0 \) is binding. Such an interval will occur in Case 1 if the solution to (24) is non-monotone. To derive the correct \( q(\cdot) \), we need to apply the so-called ‘ironing’ technique (see Guesnerie and Laffont (1979)). Specifically, if \( q' = 0 \), then \( r' = 0 \), so (20) implies that \( \delta'' = 0 \). If \( \theta_1 \) and \( \theta_2 \) are end-points of an interval on which \( q(\cdot) \) is constant, i.e. \( q(\theta) \) is strictly increasing on \((\theta_1 - \epsilon_1, \theta_1)(\theta_2, \theta_2 + \epsilon_2)\) for some \( \epsilon_1, \epsilon_2 > 0 \), then by continuity we have \( \tau(\theta_2) = \tau(\theta_1) = 0 \). Integrating (22) on \([\theta_1, \theta_2]\) yields:

\[
(\alpha f(r)(u(q, r) - c(q)) + \delta) \frac{u_q(q, \theta_1) - u_q(q, \theta_2)}{u_\theta(q, r)} = (u_q(q, \theta_2) - c'(q))(1 - F(\theta_2) - (u_q(q, \theta_1) - c'(q))(1 - F(\theta_1))
\]

(25)

Equation (25) can be used to determine the interval(s) on which \( q \) is constant. We address this issue in more detail in the next subsection.

**Case 2:** \( q \geq q^*(r) \). The necessary first-order conditions are:

\[
-\lambda'(\theta) = \frac{\partial H}{\partial q} = \left( u_q(q, \theta) - c'(q) - u_\theta q(q, \theta) \frac{1 - F(\theta)}{f(\theta)} \right) f(\theta) + (\alpha f(r)(u(q^*(r), r) - c(q^*(r)) + \delta q' \left( u_q q q(q, \theta) - u_q q(q, r) \right) \frac{u_\theta(q, r)}{u_\theta(q, r)^2} - (u_q(q, \theta) - u_q(q, r)) u_\theta q q(q, r) \right)
\]

(26)

\[
-\delta'(\theta) = \frac{\partial H}{\partial r} = (\alpha f'(r)(u(q^*(r), r) - c(q^*(r)) + \alpha f(r) u_\theta q(q^*(r), r)) \frac{q'(u_q(q, \theta) - u_q(q, r))}{u_\theta(q, r)} - (\alpha f(r)(u(q^*(r), r) - c(q^*(r)) + \delta q' \left( u_q q q(q, \theta) - u_q q(q, r) \right) \frac{u_\theta(q, r)}{u_\theta(q, r)^2} + (u_q(q, \theta) - u_q(q, r)) u_\theta q q(q, r) \right)
\]

(27)

\[
0 = \frac{\partial C}{\partial q} = (\alpha f(r)(u(q^*(r), r) - c(q^*(r)) + \delta) \frac{u_q(q, \theta) - u_q(q, r)}{u_\theta(q, r)} + \lambda + \tau
\]

(28)

Rearranging (27) we obtain:

\[
\frac{d}{d\theta} \left( \delta + f(r)(u(q^*(r), r) - c(q^*(r))) \right) u_\theta(q, r) = \left( \delta + f(r)(u(q^*(r), r) - c(q^*(r))) \right) \frac{du_\theta(q, r)}{d\theta}
\]

Hence,

\[
\delta + f(r)(u(q^*(r), r) - c(q^*(r))) = u_\theta(q, r)k_2
\]

(29)

where \( k_2 \) is a constant of integration. Substituting this expression into (28), we obtain that

\[-(\lambda + \tau) = k_2(u_q(q, \theta) - u_q(q, r)) \]. Differentiate this expression, substitute the result into (26)
and use (29) on the right-hand side of (26) to obtain:

$$\tau'(\theta) + k_2 u_{q\theta}(q, \theta) = (u_q(q, \theta) - c'(q))f(\theta) - u_{q\theta}(q, \theta)(1 - F(\theta))$$  \hspace{1cm} (30)$$

Equations (29) and (30) imply:

$$[\delta + f(r)(u(q^*(r), r) - c(q^*(r))) \frac{u_{q\theta}(q, \theta)}{u_q(q, r)} + \tau'(\theta) = (u_q(q, \theta) - c'(q))f(\theta) - u_{q\theta}(q, \theta)(1 - F(\theta))$$

which is identical to (22).

On any interval where \( q' > 0 \) we have \( \tau = \tau' = 0 \), and so the solution is characterized by

$$(u_q(q, \theta) - c'(q))f(\theta) = u_{q\theta}(q, \theta)(1 + k_2 - F(\theta))$$  \hspace{1cm} (32)$$

Lemma 5 implies that the constant of integration \( k_2 \) is non-positive. Totally differentiating (32) we obtain a differential equation that is identical to (24), but without the term \( \alpha \frac{f(\theta)(u_q(q, r) - c'(q)))}{u_q(q, r)} \) on the left-hand side. In Case 2, \( u_q(q, r) - c'(q) \) ≤ 0, so (6) follows.

If the solution to (32) is not monotonically increasing, then there will exist an interval \([\theta_1, \theta_2]\) on which \( q' = 0 \), and hence \( r'(\theta) = \delta'(\theta) = 0 \). Integrating (31) between the end-points \( \theta_1 \) and \( \theta_2 \) and using the fact that \( \tau(\theta_1) = \tau(\theta_2) = 0 \), we see that in Case 2 the intervals on which \( q(.) \) is constant are also characterized by (25).

### 4.3 Solution Properties.

The previous results allow to establish several properties of the solution. First, if any pair \( \{q(.), r(.)\} \) satisfying (24),(32) and (14) is such that \( q(.) \) is non-decreasing, then the constraint \( q' ≥ 0 \) will be non-binding and the optimal quantity schedule for the ‘strategic’ consumers will not require ‘ironing.’ To see this, simply relax the monotonicity constraint \( q'(\theta) ≥ 0 \) in Problem (15). The solution \( q^*(\theta) \) to the relaxed program is characterized by (24), (32) and (14), and hence it will be non-decreasing. Since the value of the relaxed program is at least weakly greater than the value of the non-relaxed program, \( q^*(\theta) \) will also be the solution to the non-relaxed program.

In Theorem 4, we use the differential form of (32) to characterize the optimum by (6) and (7), and also provide conditions ensuring that the constraint \( q' ≥ 0 \) is not binding and that a solution to (6) and (7) is unique. The uniqueness condition relies on Lemma 11 below.

To see why the constraint \( q' ≥ 0 \) is non-binding if \( F(\theta) + f(\theta) \frac{u_q(q, \theta) - c'(q)}{u_{q\theta}(q, \theta)} \) is nondecreasing in \( \theta \), note that the right-hand side of (6) is nonnegative in this case. Meanwhile, Assumption 1(iv) ensures that the left-hand side is positive.\(^{28}\)

If this condition does not hold, then a solution to (6) may be decreasing for some values of \( \theta \). In this case, the constraint \( q' ≥ 0 \) will be binding, and \( q(\theta) \) will need to be constant.

---

\(^{28}\)If \( u_{q\theta}(q, \theta) ≥ 0 \) then the concavity of \( u(q, \theta) - c(q) \) in \( q \), and the fact that \( q(\theta) ≤ q^*(\theta) \) (see Lemma 5), imply that \( \frac{u_q(q, \theta) - c'(q)}{u_{q\theta}(q, \theta)} - \frac{u_{q\theta}(q, \theta) - c''(q)}{u_{q\theta}(q, \theta)} > 0 \). Suppose, on the other hand, that \( u_{q\theta}(q, \theta) < 0 \). By Lemma (5), \( q^b(\theta) ≤ q(\theta) \), so \( u_q(q, \theta) - c'(q) - \frac{F'(\theta) u_{q\theta}(q, \theta)}{f(\theta)} u_{q\theta}(q, \theta) ≤ 0 \). It follows that \( \frac{u_q(q, \theta) - c'(q)}{u_{q\theta}(q, \theta)} u_{q\theta}(q, \theta) - \frac{u_{q\theta}(q, \theta) - c''(q)}{u_{q\theta}(q, \theta)} > 0 \), where the last inequality holds because the function \( u(q, \theta) - c(q) - \frac{F'(\theta)}{f(\theta)} u_q(q, \theta) \) is concave in \( q \).
on some intervals. However, \( q(.) \) will still be characterized by (6) on the intervals where it is strictly increasing. One can determine the intervals on which \( q(.) \) is constant by using an ‘ironing’ procedure. The end-points of such intervals satisfy (25). Guesnerie and Laffont (1979) demonstrate how to find the end-points of intervals where \( q \) is constant in a standard case without ‘honest’ consumers. In that case, the counterpart of equation (25) has the same right-hand side, but has 0 on the left-hand side. The ‘ironing’ algorithm designed by Guesnerie and Laffont (1979) is general and works in our case also, provided that we can compute the value of the multiplier \( \delta \) which is constant on any interval where \( q'(.) = 0 \). We refer the reader to their work for the rest of the details, and point out that in Case 1 \( \delta(\theta_1) \) is given by (23), and in Case 2 it can be computed using (29).

Note that the solution cannot switch from one Case to another on any interval where \( q(.) \) is constant, because \( r(.) \) is also constant on such interval.

The next result helps to understand the overall properties of the solution:

**Lemma 9** There exist \( \theta_l \) and \( \theta_h \) satisfying \( 0 < \theta_l < \theta_h < 1 \) such that Case 1 applies on \((0, \theta_l)\), and such that Case 2 applies and \( q(\theta) = q^{sb}(\theta) \) on \((\theta_h, 1] \).

Thus, for low values of \( \theta \) the solution is in Case 1 where the incentive constraints between ‘strategic’ and ‘honest’ consumers are binding, while for high values of \( \theta \) the solution is in Case 2 where such constraints are non-binding and ‘strategic’ consumers obtain the same quantities as in the standard model with \( \alpha = 0 \).

In the intermediate region, the solution may in general switch between Cases 1 and 2. Under some additional regularity conditions, however, this will not happen. More precisely, suppose that the hypothesis of Theorem 4 is satisfied, so that the optimal \( q(.) \) does not need ironing. Our next Lemma then asserts that we have \( \theta_l = \theta_h \), provided \( u_{qq}(q, \theta) \) is not ‘too large’ and marginal social surplus is concave. Define:

\[
M = \max_{\theta \in [0, 1], q \in \left[0, q^*(1)\right]} \frac{u_{qq}(q, \theta)}{u_q(q, \theta)} \quad N = \max_{\theta \in [0, 1], q \in \left[0, q^*(1)\right]} \left( \frac{u_{qq}(q, \theta)}{u_q(q, \theta)} - \frac{u_{qqq}(q, \theta)}{u_{qq}(q, \theta)} \right) \left( \frac{c''(q) - u_{qq}(q, \theta)}{u_q(q, \theta)} \right)
\]

**Lemma 10** Suppose that \( f(\theta) = \frac{u_{qq}(q, \theta) - c'(q)}{u_{qq}(q, \theta)} \) is strictly increasing in \( \theta \). Suppose furthermore that \( \max_{\theta \in [0, 1], q \in \left[0, q^*(1)\right]} \frac{u_{qq}(q, \theta)}{u_q(q, \theta)} \leq \min\{M, N\} \), and that \( u_{qq}(q, \theta) - c''(q) \leq 0 \). Then \( \theta_l = \theta_h \equiv \theta_0 \), i.e. there exists a unique switchpoint between Cases 1 and 2.

The system consisting of (6)-(7) with the boundary conditions \( q(0) = r(0) = 0 \), \( q(1) = q^*(1) \) may have multiple solutions, i.e. there may exist several pairs \((q(.), r(.))\) satisfying the necessary conditions for optimality. Since our problem does not satisfy the standard sufficiency conditions (Arrow’s or Mangasarian’s), the optimal pair will have to be chosen by comparing of the value of the objective. Our final lemma provides a condition sufficient to ensure that this complication does not arise.

**Lemma 11** Suppose that \( f(\theta) = \frac{u_{qq}(q, \theta) - c'(q)}{u_{qq}(q, \theta)} \) is increasing in \( \theta \) for all \( \theta \in (0, 1] \) and \( q \in \left(q^{sb}(\theta), q^*(\theta)\right) \).

Then there exists a unique solution to (6) and (7) satisfying the boundary conditions \( q(0) = r(0) = 0 \) and \( q(1) = 1 \).

Lemmas 9-11 have an important implication. Suppose the joint conditions of Lemma 10 and 11 hold. Then, to identify the solution to Problem (15) it is sufficient to find the solution

---

29The proof of the Lemma is available at http://faculty.fuqua.duke.edu/%7Esseverin/mnopextraproofs.pdf.
(\hat{q}(\theta), \hat{r}(\theta)) to the system (24) and (7) such that \(q(0) = r(0) = 0\) and such that \(\hat{q}(\overline{\theta}) = q^*(\hat{r}(\overline{\theta})) = q^{ab}(\overline{\theta})\) for some \(\overline{\theta} \in (0, 1)\). In combination, Theorem 4 and Lemma 9 imply that \(\hat{q}(\theta) < q^*(\hat{r}(\theta))\) \(\forall \theta \in (0, \overline{\theta})\). So, the optimal quantity schedule \(q(\theta)\) is given by: \(q(\theta) = \hat{q}(\theta)\) (Case 1) for \(\theta \in [0, \overline{\theta}]\) and \(q(\theta) = q^{ab}(\theta)\) (Case 2) for \(\theta \in [\overline{\theta}, 1]\). All the important qualitative features of the solution are thus identical to those of the linear quadratic example, whose solution we turn to next.

4.4 Proof of Corollary 1: Linear-Quadratic Example.

Consider now the linear quadratic example of Corollary 1: \(u(q, \theta) = \theta q, c(q) = \frac{a^2}{2}, \) and \(F(\theta) = \theta\). In this case, the first best and second-best allocation are given by \(q^*(\theta) = \theta\) and \(q^{ab}(\theta) = \max\{2\theta - 1, 0\}\), respectively. Also, we can explicitly solve the defining equation for \(r\), to yield \(r(\theta) = \theta - \frac{\theta}{q(\theta)}\).

This example satisfies the conditions of Theorem 4 and Lemmas 10 and 11. Hence, there exists a unique switchpoint \(\overline{\theta}\) such that on \([\overline{\theta}, 1]\) the solution is in Case 2 and satisfies \(q(\theta) = q^{ab}(\theta) = 2\theta - 1\). On the interval \([0, \overline{\theta}]\) the solution is in Case 1, and is characterized by a pair of differential equations (6) and (7) which in this case simplify to:

\[
\begin{align*}
    r' &= \frac{q'(\theta - r)}{q} \quad (33) \\
    q'((\alpha r + 1 - \alpha)q) &= 2q \quad (34)
\end{align*}
\]

By Lemma 11 there is a unique solution to the system (33) and (34)\(^{30}\) satisfying the correct boundary conditions \(q(\overline{\theta}) = q^{ab}(\overline{\theta}) = q^*(r(\overline{\theta}))\). So, our goal is identify this solution and determine the boundary point \(\overline{\theta}\).\(^{31}\)

Our strategy is to guess the structure of the solution. Inspection of the system (33) and (34) leads to the conjecture that \(r(\theta) = a\theta + bq(\theta)\) on the interval \([0, \overline{\theta}]\), for some constants \(a\) and \(b\). Applying this conjecture to (33) and (34) and rearranging we obtain:

\[
\theta(\alpha a^2 + 2a - 2) = -q(\theta)(4b + a(1 - \alpha + \alpha b)) \quad (35)
\]

Suppose that (35) holds as an identity,\(^{32}\) i.e. \(\alpha a^2 + 2a - 2 = 0\) and \((1 - \alpha)a + 4b + abc = 0\). Solving for the coefficients \(a\) and \(b\) yields: \(a = \frac{-1 + \sqrt{1 + 2\alpha}}{\alpha}, \quad b = \frac{-1 - \alpha}{4 + \alpha a}\).

Choose the positive root for \(a\), so that \(b = \frac{-1 - \alpha}{4 + \alpha a} = \frac{1 - \alpha}{\alpha + \frac{\alpha}{3\sqrt{1 + 2\alpha}} - 1}\). By computation we can show that \(a < 1\) and \(a + b < 1 \forall \alpha > 0\), so \(r(\theta) < \theta\). Let \(y(\theta) = \ln \frac{\theta}{q(\theta)}\), so that \(dq = \frac{d\theta}{\theta} - \frac{dy}{q}\). It follows

\(^{30}\) (34) provides another way to ascertain the no-exclusion result in the linear-quadratic case. If \(\tilde{\theta} \equiv \inf\{\theta | q(\theta) > 0\} > 0\), then by definition \(r(\tilde{\theta}) = \tilde{\theta}\). So, \(\exists \theta_h \in (\tilde{\theta}, 1)\) s.t. the solution is in Case 1 on \((\tilde{\theta}, \theta_h)\) and has to satisfy (34). Since \(q(\cdot)\) and \(r(\cdot)\) are nonnegative and nondecreasing in \(\theta\), on this interval \(q'(\theta) = \frac{2q(\theta)}{\alpha r(\theta) + (1 - \alpha)q(\theta)} < \frac{2q(\theta)}{\alpha q(\theta)}\).

Pick \(\tilde{\theta} \in (\tilde{\theta}, \min\{\theta_h, \theta_h + \frac{a}{3\alpha}\})\). Integrating, we get \(q(\theta) \leq \frac{a}{3\alpha} F(\theta) q(s)ds\). Since \(\frac{2(\theta - \tilde{\theta})}{\alpha \theta} < 1\) and \(q(\cdot)\) is non-decreasing, this inequality can only hold if \(q(\theta) = 0\), which contradicts the definition of \(\tilde{\theta} = \inf\{\theta | q(\theta) > 0\}\).

\(^{31}\) It is possible to show directly that the solution switches between Cases 1 and 2 only once. Let \(\overline{\theta}\) be the smallest switching point i.e. \(q(\theta) < r(\theta) \forall \theta \in (0, \overline{\theta})\) and \(q(\theta) \geq r(\theta) \forall \theta \in [\overline{\theta}, \theta_h]\) for some \(\theta_h \in (0, 1)\). By continuity of the optimal quantity schedule \(q(\cdot), \overline{\theta}(\cdot) = \overline{\theta}(\overline{\theta})\), and \(q'(\overline{\theta}) = r'(\overline{\theta}) = q'(\overline{\theta}) = q'(\overline{\theta})\). Thus, \(\overline{\theta} = r(\overline{\theta}) + q(\overline{\theta})\). But \(q'(\overline{\theta}) = 2\forall \theta \in (\overline{\theta}, \overline{\theta})\). Hence, \(\theta < r(\overline{\theta}) + q(\overline{\theta})\) and \(q'(\overline{\theta}) > r'(\overline{\theta}) \forall \theta \in (\overline{\theta}, \overline{\theta})\). So, \(q(\overline{\theta}) > r(\overline{\theta})\), i.e. the solution cannot switch to Case 2 at \(\overline{\theta}\). This implies that \(\theta = 1\).

\(^{32}\) Otherwise, \(q(\overline{\theta})\) must be a linear function of \(\theta\), in which case (33) and (34) can be solved to yield \(q(\theta) = \theta \frac{\alpha + \alpha}{2\alpha}\) and \(r(\theta) = \theta/2\). But \(q(\overline{\theta}) > r(\theta)\), and so we can rule out this possibility.
from (34) that \( \frac{dq}{q} = \frac{2}{\alpha a + (1 - \alpha + \alpha b) e^y} \frac{dy}{\theta} \). Hence we obtain:

\[
\frac{d\theta}{\theta} = dy \left( \frac{c_0 + c_1 e^y}{c_2 - c_1 e^y} \right)
\]

where \( c_0 = \alpha a = \sqrt{1 + 2\alpha} - 1, c_1 = 1 - \alpha + \alpha b = \frac{4(1 - \alpha)}{5 + \sqrt{1 + 2\alpha}}, \) and \( c_2 = 2 - \alpha a = 3 - \sqrt{1 + 2\alpha} \). When \( \alpha \neq 4 \) so that \( c_2 \neq 0 \), we can integrate both sides of this equation to obtain:

\[
\ln \theta = k + \frac{c_0}{c_2} y - \frac{c_0 + c_2}{c_2} \ln |c_2 - c_1 e^y|,
\]

where \( k \) is a constant of integration. Exponentiating both sides, substituting \( y(\theta) = \ln \frac{q(\theta)}{\theta} \), and simplifying finally produces an implicit equation for \( q(\theta) \):

\[
[(2 - \alpha/2)\theta - (1 - \alpha)q]^2 = h(\alpha)q^{\sqrt{1 + 2\alpha} - 1}
\]

where \( h(\alpha) = \frac{(3 + \sqrt{1 + 2\alpha})^2}{16} e^{k(3 - \sqrt{1 + 2\alpha})} \). When \( \alpha = 4 \), (36) can be rewritten as \( \frac{d\theta}{\theta} = \frac{dy}{y} \left( \frac{1 - e^y}{e^y} \right) \), which can be solved directly to yield:

\[
\theta = h(4)q - q \ln(q)
\]

Note that \( h(\alpha) \) defines a family of solutions to the system (34)-(33) reflecting the singularity of this system at the origin. To determine \( h(\alpha) \), we will exploit the fact that only one member of this family satisfies the boundary condition \( q(\overline{\theta}) = q^*(r(\overline{\theta})) = q^{sb}(\overline{\theta}) \). Since \( r(\theta) = a\theta + bq(\theta) \), the condition \( q(\overline{\theta}) = q^*(r(\overline{\theta})) \) implies \( q(\overline{\theta}) = \frac{a}{1 - b} \overline{\theta} \). Combining this with the condition \( q(\overline{\theta}) = q^{sb}(\overline{\theta}) = 2\overline{\theta} - 1 \) yields \( \overline{\theta} = \frac{1 - b}{2 - 2\alpha} = \frac{2}{3} + \frac{1}{3(\sqrt{1 + 2\alpha} + 1)} \), so that \( q(\overline{\theta}) = r(\overline{\theta}) = \frac{1}{3} + \frac{2}{3(\sqrt{1 + 2\alpha} + 1)} \).

For \( \alpha = 4 \), we substitute \( \overline{\theta}(4) = 3/4 \) and \( q(\overline{\theta}(4)) = 2\overline{\theta}(4) - 1 = 1/2 \) into \( \theta = h(4)q - q \ln(q) \) to obtain \( h(4) = 3/2 + \ln(1/2) \). For \( \alpha \neq 4 \), substituting \( \overline{\theta} \) and \( q(\overline{\theta}) \) into (37) yields:

\[
h(\alpha) = \frac{\left( 1 + \frac{\alpha}{2(\sqrt{1 + 2\alpha} + 1)} \right)^2}{\left( \frac{1}{3} + \frac{2}{3(\sqrt{1 + 2\alpha} + 1)} \right)^{\sqrt{1 + 2\alpha} - 1}}
\]

Note that only the positive root of equation (37) holds as an equality at \( \overline{\theta} \). So, if \( \exists \theta_1 \in (0, \overline{\theta}) \) s.t. \( (2 - \alpha/2)\theta_1 - (1 - \alpha)q(\theta_1) < 0 \), then by continuity \( \exists \theta_2 \in (\theta_1, \overline{\theta}) \) satisfying \( (2 - \alpha/2)\theta_2 - (1 - \alpha)q(\theta_2) = 0 \). But since \( q(\theta_2) > 0 \), (37) cannot hold at \( \theta_2 \). Thus, \( q(\theta) \) is a solution to:

\[
\theta = \frac{1}{2 - \alpha/2} q + \frac{h(\alpha)^{1/2}}{(2 - \alpha/2) q^{\sqrt{1 + 2\alpha} - 1}}
\]

Both (38) and (39) characterize \( q(.) \) as an implicit function of \( \theta \) for given \( \alpha \). Since these equations may have multiple roots, we need to establish that \( q(\theta) \) is well-defined. Start with \( \alpha \neq 4 \). Consider \( \theta(q) \) as a function of \( q \) defined by (39) on \([0, q(\overline{\theta})]\). Substitution yields:

\[
r(\theta(q)) - q = a\theta(q) + bq - q = \frac{\sqrt{1 + 2\alpha} - 1}{\alpha} \left( \frac{1}{2 - \alpha/2} q + \frac{h(\alpha)^{1/2}}{2 - \alpha/2 q^{\sqrt{1 + 2\alpha} - 1}} \right) - \left( 1 + \frac{1 - \alpha}{\alpha} \frac{\sqrt{1 + 2\alpha} - 1}{3 + \sqrt{1 + 2\alpha}} \right) q
\]

Note that \( \overline{\theta} \) is decreasing in \( \alpha \). It converges to 2/3 as \( \alpha \) increases to infinity (almost all consumers are 'honest'), and converges to 5/6 as \( \alpha \) decreases to 0 (almost all consumers are strategic).
Observe that \( r(\theta(q)) - q \) is strictly concave in \( q \), and \( r(\theta(0)) = 0 \), while our choice of \( h(\alpha) \) guarantees that \( r(\theta(q)\theta)) - q(\theta)) = 0 \). Then, by strict concavity, \( r(\theta(q)) - q > 0 \) \( \forall q \in (0, q(\theta)) \). Since \( \theta(q) \) must also satisfy (34), \( r(\theta(q)) - q > 0 \) implies that \( \theta \) is strictly increasing in \( q \) on \( [0, \theta] \). Therefore, on \( [0, \theta] \) (39) admits a unique increasing continuous solution \( q(\theta) \) s.t. \( q(\theta) = 2\theta - 1 \) and \( r(\theta) > q(\theta) \) \( \forall \theta \in (0, \theta) \). Since \( \theta(q) \) is strictly concave in \( q \), \( q(\theta) \) is convex. The case \( \alpha = 4 \) can be handled in a similar way.

We have thus found the unique solution to (34) and (33) that satisfies the condition
\[
q(\theta) = q^b(\theta) = q^*(r(\theta)) \text{ for some } \theta, \text{ and hence the solution to Problem (15).}
\]

It remains to determine the optimal schedule for the ‘honest’ types. On the interval \( [\theta, 1] \) the solution is in Case 2, so \( g(\theta) = q^*(\theta) = \theta \) for \( \theta \in [r(\theta), 1] \), where \( r(\theta) = \frac{1}{3} + \frac{2}{3(\sqrt{1+2a_1}+1)} \).

Meanwhile, on the interval \( [0, \theta] \) the solution is in Case 1, so \( g(\theta) = q(r^{-1}(\theta)) \) on \( [0, r(\theta)] \). To determine \( g(.) \) on this interval, note that \( r(.) \) is strictly increasing on \( [0, \theta] \), so its inverse \( r^{-1}(\theta) \) is well-defined \( \forall \theta \in [0, r(\theta)] \). For \( \alpha \neq 4 \), we can use (39) to obtain:
\[
r^{-1}(\theta) = \frac{(1 - \alpha)}{2 - \alpha/2} g(\theta) + \frac{h(\alpha)^{1/2}}{(2 - \alpha/2)} g(\theta)^{\frac{\sqrt{1+2\theta} - 1}{2}}
\]
(40)

From \( r(\theta) = a\theta + bq(\theta) \) it follows that \( \theta = a r^{-1}(\theta) + bq(\theta) \). Substituting this into (40) and solving for \( \theta \) produces (9). To show that \( g(\alpha) \) is well-defined by (9) and is continuous and increasing on \( [0, \frac{1}{3} + \frac{2}{3(\sqrt{1+2a_1}+1)}] \), use an argument similar to the one establishing these properties for \( q(\theta) \). In a similar way, we can compute \( g(\theta) \) for \( \alpha = 4 \).

5 Conclusion.

In this paper, we offered an explanation for a frequently observed pricing practice of using information obtained directly from the customers to sell the same quality or quantity to different consumers at different prices. Such pricing strategies would be infeasible if, as the traditional screening literature maintains, all consumers can freely and costlessly manipulate their private information.

We offered several explanations for why some consumers may not be able to misrepresent or conceal their preferences. Consumers may be boundedly rational and not fully understand the implications of their responses for the terms of trade they will subsequently face. For ethical or moral reasons, consumers may be averse to lying. Finally, consumers may differ in their ability or cost of presenting evidence supporting their claims.

Econometric and experimental evidence suggests that the fraction of the population that “act honestly” is non-negligible, and may in fact be surprisingly large. We have shown that such behavior has important consequences for the optimal pricing policy of profit maximizing firms. Specifically, the presence of honest consumers in the population reduces the allocational distortions associated with monopoly power. Furthermore, the traditional telltale sign of monopoly power, firms’ refusal to serve customers whose value is close to the marginal cost of production, is simply absent in our model. Our model also offers an explanation why the firms may wish to offer complicated or non-transparent mechanisms. Namely, when the seller faces some ‘boundedly’ rational consumers, it is optimal for it to construct a mechanism in which finding discounts is difficult and requires analytical abilities lacked by the ‘boundedly rational’ customers.
Our theory has implications that go beyond the problem of optimal screening by profit maximizing forms, or the closely related issue of optimal regulatory policy. Indeed, the traditional exclusion motive also appears forcefully in bilateral bargaining. With one-sided asymmetric information, exclusion manifests itself in the form of absence of intertemporal price-discrimination or ‘haggling’ (see Stokey (1995) and Riley and Zeckhauser (1983)).

With two-sided asymmetric information, it appears in the form of no trade when the difference between seller’s cost and buyer’s valuation is small (see Myerson and Satterthwaite (1983) and Williams (1987)). The exclusion motive is also present in auctions where the seller selects an optimal reserve price above her true cost. Exploring the implication of our model for these environments is a topic we leave for future research.

One extension that is more immediate involves relaxing the assumption that a consumer’s likelihood of telling the truth is independent of her valuation for the object. As a first approximation, this assumption is certainly reasonable. However, it would be comforting to know that our results are robust to perturbations in which the likelihood of being honest is allowed to depend upon the underlying valuation \( \theta \). Assuming that \( \gamma(\theta) \) is smooth and satisfies \( \gamma(\theta) > 0 \) for all \( \theta \in [0,1] \), a careful perusal of our proofs reveals that analogous results to Theorems 3 and 4 hold.

In particular, no consumer is ever excluded from the market. Furthermore, if the solution to the unconstrained problem (ignoring the monotonicity constraint on the quantity schedule for ‘strategic’ types) is monotone, the optimal quantity schedule is again characterized by two differential equations, equation (7), and the following counterpart to equation (6):

\[
q' \left( \max \left\{ \gamma(r)f(r)\frac{u_q(q,r)-c'(q)}{u_\theta(q,r)}, 0 \right\} + (1-\gamma(\theta))f(\theta) \left[ \frac{(u_q(q,\theta)-c'(q))u_{\theta q}(q,\theta)}{u_{\theta q}(q,\theta)} - \frac{u_{qq}(q,\theta)-c''(q)}{u_{\theta q}(q,\theta)} \right] \right) = [(1-\gamma(\theta))f'(\theta) - f(\theta)\gamma'(\theta)] \frac{u_q(q,\theta)-c'(q)}{u_\theta(q,\theta)} + (1-\gamma(\theta))f(\theta) \left[ 2 - \frac{(u_q(q,\theta)-c'(q))u_{\theta \theta q}(q,\theta)}{u_{\theta q}^2(q,\theta)} \right]
\]

The key insight to why this should be true is that the perturbed model is equivalent to the original one, but where there is now a density \( (1-\gamma(\theta))f(\theta) \) of ‘strategic’ consumers and a density \( \gamma(\theta)f(\theta) \) of ‘honest’ consumers.

### 6 Appendix

**Proof of Theorem 2.**

**Existence.** We wish to show that there exists a 4-tuple of measurable bounded functions \( \{q(\theta), t^*(\theta), g(\theta), t^*(\theta)\} \) solving the maximization problem (1)-(5).

Note that since the monopolist will never select a sell quantity larger than \( Q = \max\{q|u(q,1)-c(q) \geq 0\} \), consumers will never pay more than \( M = u(Q,1) \). Also, without loss of generality, \( t^*(\theta) \geq 0 \) and \( t^*(\theta) \geq 0 \). Thus we may w.l.g. restrict the domain of maximization to functions whose range is contained in \([0,K]\) where \( K = \max\{Q,M\} \).

The set of measurable functions with range \([0,K]\) coincides with \( L^2(\mu) \), where \( \mu \) is the measure associated with the distribution function \( F(.) \). Let us endow this space of functions with the weak* topology. To be more precise, a sequence \( x_n(\theta) \) converges to \( x(\theta) \) in the weak* topology iff \( \int_0^1 x_n(\theta)y(\theta)f(\theta)d\theta \to \int_0^1 x(\theta)y(\theta)f(\theta)d\theta \ \forall y \in L^2(\mu) \). Setting \( y(\theta) \equiv 1 \) then shows that \( x_n(\theta) - x(\theta) \to 0 \), a.e.-\( \theta \).
Since \( c(\cdot) \) is continuous, and since the components of \( \{q(\theta), t^*(\theta), g(\theta), t^*(\theta)\} \) are bounded by \( K \), it follows from the Lebesgue dominated convergence Theorem that the objective function is a continuous functional under the weak* topology.

By Alaoglu’s Theorem (Royden (1987), Theorem 6.17) a \( K \)-ball is compact in the weak* topology. Furthermore, by Tychonoff’s Theorem the product of \( 4 \) \( K \)-balls is compact in the product topology generated by the weak* topology. Since the set \( S \) of all 4-tuples \( \{q(\theta), t^*(\theta), g(\theta), t^*(\theta)\} \) whose components lie in \([0, K]\) and satisfy the constraints (2)-(5) is a closed subset of the \( K \)-ball, we conclude that \( S \) is compact in the product topology generated by the weak* topology. It follows from the Weierstrass Theorem that there exists a 4-tuple of \( L^2 \) functions \( \{q(\theta), t^*(\theta), g(\theta), t^*(\theta)\} \) in \( S \) that attains the maximum in (1).

**Uniqueness.** Since Problem (1)-(5) is equivalent to Problem (10)-(12), we will prove the uniqueness of a solution to the latter problem.

Suppose to the contrary that there existed two distinct pairs \( (q_1(\theta), g_1(\theta)) \) with \( \theta \in (0, 1) \), and let \( g_3(\theta) = \rho q_1(\theta) + (1 - \rho)g_2(\theta) \), \( q_3(\theta) = \rho q_1(\theta) + (1 - \rho)g_2(\theta) \).

Now define \( U_\tau(\theta) = \max_{\rho \in [0, 1]} u(g_3(\theta^*(\theta)), \theta) - u(g_3(\theta'), \theta') \), and let \( \theta^*(\theta) \) be the largest corresponding maximizer. Since \( u_{\theta q}(q, \theta) > 0 \) and \( g_3(\theta) \) is increasing, the maximand is supermodular in the choice variable, and so \( \theta^*(\theta) \) is an increasing function. Furthermore, at any point \( \theta \) where \( \theta^* \) is continuous (which excludes at most a countable number of points), the function \( U_\tau \) is differentiable with derivative \( U_{\tau}^*(\theta) = u_\theta(g_3(\theta^*(\theta)), \theta) \). At any discontinuity point of \( \theta^* \) we nevertheless have

\[
\limsup_{\tau' \uparrow \tau} \frac{U_{\tau}(\theta) - U_{\tau'}(\theta)}{\theta' - \theta} \leq u_\theta(g_3(\theta^*(\theta)), \theta) \quad \text{and} \quad \liminf_{\tau' \downarrow \tau} \frac{U_{\tau}(\theta) - U_{\tau'}(\theta)}{\theta' - \theta} \geq u_\theta(g_3(\theta^*(\theta)), \theta),
\]

where \( \theta^* \) is the left limit of \( \theta^* \).

We will show that the following modification of the quantity schedules \( (\bar{q}(\theta), \bar{g}(\theta)) \) improves the firm’s expected profits. Set \( \bar{g}(\theta) = g_3(\theta) \) and let \( \bar{q}(\theta) \) satisfy (in a recursive fashion):

\[
\bar{q}(\theta) = \begin{cases} 
q_3(\theta) & \text{if } \int_0^\theta u_\theta(\bar{q}(s), s) \, ds > U_\tau(\theta) \\
\max\{q_3(\theta), g_3(\theta^*(\theta))\} & \text{if } \int_0^\theta u_\theta(\bar{q}(s), s) \, ds = U_\tau(\theta)
\end{cases}
\]

Let us show that the tuple \( (\bar{q}(\theta), \bar{g}(\theta)) \) is feasible, i.e. satisfies the constraints (11) and (12). (12) holds because \( \bar{q}(\theta) \) is constructed so that whenever \( U(\theta) = \int_0^\theta u_\theta(\bar{q}(s), s) \, ds = U_\tau(\theta) \) we have \( U'(\theta) = u_\theta(\bar{q}(\theta), \theta) \geq u_\theta(g_3(\theta^*(\theta)), \theta) \leq \liminf_{\tau' \downarrow \tau} \frac{U_{\tau}(\theta) - U_{\tau'}(\theta)}{\theta' - \theta} \), where the inequality holds because \( u_{\theta q}(q, \theta) > 0 \) and \( \bar{q}(\theta) \geq g_3(\theta^*(\theta)) \). Thus, whenever \( U(\theta) \) equals \( U_\tau(\theta) \), it cannot decrease below it.

To see that (11) holds note that both \( g_3(\theta) \) and \( \max\{q_3(\theta), g_3(\theta^*(\theta))\} \) are increasing functions. \( \bar{q}(\theta) \) can fail to be increasing only if \( \exists \theta_d \) s.t. \( U(\theta_d) = U_\tau(\theta_d) \), \( U(\theta) > U_\tau(\theta) \) in a right neighborhood of \( \theta_d \), and \( g_3(\theta^*(\theta_d)) > g_3(\theta_d) \). The latter inequality, the monotonicity of \( g_3(\theta^*(\cdot)) \), and the continuity of \( g_3(\cdot) \) would then imply that there exists a right neighborhood of \( \theta_d \) over which \( g_3(\theta^*(\theta)) > g_3(\theta) \), and hence \( U'(\theta) = u_\theta(g_3(\theta), \theta) < u_\theta(g_3(\theta^*(\theta)), \theta) \leq \liminf_{\tau' \downarrow \tau} \frac{U_{\tau}(\theta) - U_{\tau'}(\theta)}{\theta' - \theta} \). This contradicts that \( U(\theta) > U_\tau(\theta) \) in a right neighborhood of \( \theta_d \).

Finally, let us show that the objective (10) attains a strictly higher value under \( (\bar{q}(\theta), \bar{g}(\theta)) \) than under either \( (q_1(\theta), g_2(\theta)) \) or \( (g_2(\theta), g_2(\theta)) \). First, note that strict concavity of \( u(q, \theta) - c(q) \) implies that \( u(g_3(\theta), \theta) - c(g_3(\theta)) > \rho(u(q_1(\theta), \theta) - c(q_1(\theta))) + (1 - \rho)(u(q_1(\theta), \theta) - c(q_1(\theta))) \). Thus, the second term in (10) increases.

Second, the first term in (10) can be rewritten as

\[
\int_0^1 \left( u(q(\theta), \theta) - c(q(\theta)) - \int_0^\theta u_\theta(q(s), s) \, ds \right) \, dF(\theta).
\]

Lemmas 5 and 7 imply that \( pq_1(\theta) + (1 - \rho)q_2(\theta) \leq \bar{q}(\theta) \leq q^*(\theta) \). Since \( u(q, \theta) - c(q) \) is strictly concave in \( q \), we have \( u(q(\theta), \theta) - c(q(\theta)) > \rho(u(q_1(\theta), \theta) - c(q_1(\theta))) + (1 - \rho)(u(q_1(\theta), \theta) - c(q_1(\theta))). \)
Hence the first term under the integrand of the above expression strictly increases. To complete the proof, let us now show that the second term decreases, i.e. that for all \( \theta \in [0, 1] \) we have:

\[
\int_0^\theta u_\theta(\tilde{q}(s), s) ds \leq \rho \int_0^\theta u_\theta(q_1(s), s) ds + (1 - \rho) \int_0^\theta u_\theta(q_2(s), s) ds
\]

First, suppose \( \theta \) is such that \( \int_0^\theta u_\theta(\tilde{q}(s), s) ds = U_\tau(\theta) \equiv \int_0^\theta u_\theta(g_3(\theta^*(\theta)), s) ds \). Now (3) implies that for both \( i = 1, 2 \) we have \( \int_0^\theta u_\theta(g_i(\theta^*(\theta)), s) ds \leq \int_0^\theta u_\theta(q(s), s) ds \). Since \( u_{\theta q}(q, \theta) \geq 0 \), we also have \( u_\theta(g_3(\theta), \theta) \leq \rho u_\theta(g_1(\theta), \theta) + (1 - \rho) u_\theta(g_2(\theta), \theta) \), and the desired inequality holds. Second, if \( \theta \) is such that \( U(\theta) > U_\tau(\theta) \), then

\[
\frac{d}{d\theta} \int_0^\theta u_\theta(\tilde{q}(s), s) ds = u_\theta(q_3(\theta), \theta) \leq \rho u_\theta(q_1(\theta), \theta) + (1 - \rho) u_\theta(q_2(\theta), \theta)
\]

where the inequality follows because \( q_3(\theta) = \rho q_1(\theta) + (1 - \rho) q_2(\theta) \) and \( u_{\theta q}(q, \theta) \geq 0 \). So, (41) holds.

**Proof of lemma 1:**

We will demonstrate that, if an admissible quantity schedule does not satisfy the condition in part (i) (part (ii)), then there exists an alternative admissible quantity schedule that satisfies this condition and also guarantees a higher value of the objective (1).

Part (i). Consider an admissible schedule s.t. \( g(\theta) > q^*(\theta) \) for some \( \theta \in [0, 1] \). Then the value of the integrand in (1) can be increased by the following modification: set \( g(\theta) = q^*(\theta) \) and reduce the transfer \( \tau^*(\theta) \) appropriately to make (5) binding. It is easy to see that this modification does not violate any incentive constraints in (3). So, the modified schedule is also admissible.

Now, suppose that \( g(\theta_2) > g(\theta_1) \) for some \( \theta_2 < \theta_1 \). Then \( u(g(\theta_2), \theta) - u(g(\theta_2), \theta_2) > u(g(\theta_1), \theta) - u(g(\theta_1), \theta_2) \) \( \forall \theta > \theta_2 \). Hence, \( U(\theta) > u(g(\theta_1), \theta) - u(g(\theta_1), \theta_1) \) \( \forall \theta \in [0, 1] \). So, if \( g(\theta) < q^*(\theta) \), let us increase \( g(\theta) \) by some \( \epsilon \) s.t. \( q^*(\theta) - g(\theta) \geq \epsilon > 0 \). This modification does not violate any incentive constraints, and causes an increase in the value of (1). If \( g(\theta_1) = q^*(\theta_1) \), then \( g(\theta_2) > q^*(\theta_2) \) which contradicts the result that \( g(\theta) \leq q^*(\theta) \).

Part (ii). Fix an admissible schedule \( q(.) \) s.t. \( q(\theta) > q^*(1) \) for some \( \theta \in [0, 1] \). Let \( \hat{\theta} = \inf\{\theta | q(\theta) > q^*(1)\} \). Since \( q(.) \) is non-decreasing, \( q(\theta) > q^*(1) \) \( \forall \theta \in (\hat{\theta}, 1] \). Consider a modified quantity/transfer schedule \( (\tilde{q}(\theta), \tilde{\tau}(\theta)) \) s.t. \( \tilde{q}(\theta) = q(\theta) \), \( \tilde{\tau}(\theta) = \tau^*(\theta) \) if \( \theta \in [0, \hat{\theta}] \), and \( \tilde{q}(\hat{\theta}) = q^*(1), \tilde{\tau}(\hat{\theta}) = \tau^*(\hat{\theta}) + u(q^*(1), \theta) - u(q(\theta), \theta) \) if \( \theta \in [\hat{\theta}, 1] \). It is easy to check that \( (\tilde{q}(\theta), \tilde{\tau}(\theta)) \) satisfies all incentive constraints. In particular, all constraints in (3) hold because \( g(\theta) \leq q^*(1) \) \( \forall \theta \in [0, 1] \).

The new quantity schedule \( \tilde{q}(\theta) \) generates a larger total surplus. Furthermore, any ‘strategic’ consumer with valuation \( \theta \geq \hat{\theta} \) gets a lower payoff than prior to this modification, while the payoffs earned by the other consumer types do not change. So, the firm’s expected profit increases.

Now suppose that \( q(1) = \mu < q^*(1) \). Since \( q(\theta) \) is nondecreasing, \( q(\theta) \leq \mu \ \forall \theta \in [0, 1] \). Let \( \theta_m \) be well-defined by the following equality: \( u(q^*(\theta_m), \theta_m) - c(q^*(\theta_m)) = u(\mu, 1 - c(\mu)) \). Since \( \mu < q^*(1), \theta_m < 1 \) and \( q^*(\theta_m) > \mu \). So, \( \theta_m > \theta_\mu \) where \( \theta_\mu \) satisfies \( q^*(\theta_\mu) = \mu \). Therefore, \( u(q^*(\theta_m), \theta_m) - c(q^*(\theta_m)) > u(\mu, \theta) - c(\mu) > u(q^*(\theta), \theta) - c(q(\theta)) \ \forall \theta > \theta_m \).

Let us show that the firm’s expected profit goes up when it offers a modified schedule \( (\tilde{q}(\theta), \tilde{\tau}(\theta)) \) which differs from \( (q(\theta), \tau^*(\theta)) \) on the interval \( [\theta_m, 1] \) as follows: \( \tilde{q}(\theta) = q^*(\theta_m) \),
\( \hat{t}^*(\theta) = t^*(\theta_m) + u(q^*(\theta_m), \theta_m) - u(q(\theta_m), \theta_m) \). Note that all incentive constraints remain satisfied because all ‘strategic’ consumers’ with valuations in \([\theta_m, 1]\) earn a higher surplus than before this modification.

To see that the firm’s expected profits increases, note that it earns the same profit from selling to a consumer with valuation \( \theta \leq \theta_m \). Also, simple computation demonstrates that the firm’s expected profit from selling to a consumer with valuation \( \theta > \theta_m \) changes by:

\[
u(q^*(\theta_m), \theta_m) - c(q^*(\theta_m)) - u(q(\theta), \theta) - c(q(\theta)) - \int_{\theta_m}^{\theta} u_\theta(q(s), s)ds > 0\]

The proof that it is optimal to set \( q(0) = 0 \) can be provided along similar lines. Q.E.D.

**Proof of lemma 2:**
Since the schedule \( q(\cdot) \) is non-decreasing, by Theorems 4.29 and 4.30, p.96 in Rudin (1976) it has at most countably many points of discontinuity on \([0, 1]\), and both the left-hand and the right-hand limits exist at all discontinuity points of \( q(\cdot) \).

Suppose that the optimal quantity schedule \( q(\cdot) \) is discontinuous at \( x \in (0, 1) \). Let \( q(x-) \) and \( q(x+) \) be, respectively, the left-hand and the right-hand limits of \( q(\cdot) \) at \( x \). Consequently, we have \( q_-(x) = q_+(x) - 2\delta \) for some \( \delta > 0 \).

Let \( G(q, \theta) = u(q, \theta) - c(q) - u_\theta(q, \theta)\frac{1-F(\theta)}{f(\theta)} \) and \( \Delta(x) = q_+(x) - q_-(x) \). We will consider two different cases.

**Case 1:** \( G(q_-(x), x) < G(q_+(x), x) \). By continuity of \( G(q, \theta) \) and \( f(\theta) \), \( \exists \epsilon > 0 \) s.t. \( \forall \theta \in (x - \epsilon, x) \), \( G(q(\theta) + \Delta(x), \theta) > G(q(\theta), \theta) \). Then let us replace the schedule \( q(\theta) \) with modified quantity schedule \( \tilde{q}(\theta) \) s.t. \( \tilde{q}(\theta) = q(\theta) \forall \theta \in [0, x - \epsilon] \cup [x, 1] \), \( \tilde{q}(\theta) = q(\theta) + \Delta(x) \forall \theta \in (x - \epsilon, x) \) and \( \tilde{q}(x) = q(x) \). Note that \( \tilde{q}(\theta) \) is increasing in \( \theta \), and all incentive constrains in (12) still hold because \( \tilde{U}(\theta) \geq U(\theta) \forall \theta \in [0, 1] \). At the same time, the value of the objective (10) increases.

**Case 2:** \( G(q_-(x), x) \geq G(q_+(x), x) \). By concavity of \( G(q, \theta) \) in \( q \), \( G((q_-(x) + q_+(x))/2, x) > G(q_-(x), x)/2 + G(q_+(x), x)/2 \). Furthermore, by continuity of \( G(q, \theta) \) and \( f(\theta) \), \( \exists \epsilon > 0 \) s.t. \( \forall \theta \in (x - \epsilon, x), G((q_-(x) + q_+(x))/2, \theta)f(\theta) + G((q_-(x) + q_+(x))/2, \theta + \epsilon)f(\theta + \epsilon) > G(q(\theta), \theta)f(\theta) + G(q(\theta + \epsilon), \theta + \epsilon)f(\theta + \epsilon) \).

So, let \( \tilde{q}(\theta) = q(\theta) \forall \theta \in [0, x - \epsilon] \cup [x + \epsilon, 1] \) and \( \tilde{q}(\theta) = (q_-(x) + q_+(x))/2 \forall \theta \in (x - \epsilon, x + \epsilon) \). Note that \( \tilde{q}(\theta) \) is increasing in \( \theta \). If \( u_\theta((q_-(x) + q_+(x))/2, x) > u_\theta(q_-(x), x)/2 + u_\theta(q_+(x), x)/2 \), then \( \epsilon \) can be chosen small enough that \( \forall \theta \in (x - \epsilon, x), u_\theta(q_-(x) + q_+(x))/2, \theta)f(\theta) + u_\theta(q_-(x) + q_+(x))/2, \theta + \epsilon)f(\theta + \epsilon) > u_\theta(q(\theta), \theta)f(\theta) + u_\theta(q(\theta + \epsilon), \theta + \epsilon)f(\theta + \epsilon) \). So under the quantity schedule \( \tilde{q}(\theta) \), \( U(\theta) \equiv \int_0^0 u_\theta(q(s), s)ds \geq U(\theta) \equiv \int_0^0 u_\theta(q(s), s)ds \forall \theta \in [0, 1] \). The value of (10) changes by:

\[
\int_{x - \epsilon}^{x + \epsilon} (G((q_-(x) + q_+(x))/2, \theta) - G(q(\theta), \theta))f(\theta)d\theta > 0
\]

If \( u_\theta((q_-(x) + q_+(x))/2, x) \leq u_\theta(q_-(x), x)/2 + u_\theta(q_+(x), x)/2 \), then it is possible that \( \Delta U(x + \epsilon) = \int_{x - \epsilon}^{x + \epsilon} u_\theta(q(s), s) - u_\theta(q(\theta), s)ds < 0 \). In this case, \( \forall \theta \in (x - \epsilon, x + \epsilon) \) set \( \tilde{q}(\theta) = \tilde{q}(\theta) \) s.t. \( \int_{x - \epsilon}^{x + \epsilon} u_\theta(q(s), s) - u_\theta(\varphi(\theta), s)ds = 0 \). Note that \( \tilde{q} > (q_-(x) + q_+(x))/2 \). If \( \epsilon \) is sufficiently small, then the value of the Problem (10) changes approximately by:

\[
\int_{x - \epsilon}^{x + \epsilon} (u(\varphi(\theta) - c(\varphi(\theta)) - u(q(\theta), \theta) + c(q(\theta))])f(\theta)d\theta > 0
\]

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The inequality holds by concavity of $u(\theta, q) - c(q)$ and the fact that $\tilde{q} > (q_- + q_+)/2$. \textit{Q.E.D.}

\textbf{Proof of Lemma 3:} Suppose that in the optimal mechanism $U(0) = u > 0$. Consider set $Z \subset \Theta$ s.t. $\theta \in Z$ iff
\begin{equation}
 u + \int_0^\theta u_\theta(q(x), x)dx = \sup_{\theta' \in [0,1]} u(g(\theta'), \theta) - u(g(\theta'), \theta') \tag{42}
\end{equation}
The set $Z$ is non-empty, because otherwise the firm could reduce $U(0)$ and hence increase its expected profits without violating any of the incentive constraints in (12). Let $\hat{\theta}$ be the minimal element of $Z$. $\hat{\theta}$ exists because both the left-hand side and the right-hand side of (42) are continuous in $\theta$.

Define $U_\tau(\theta) \equiv \sup_{\theta' \in [0,1]} u(g(\theta'), \theta) - u(g(\theta'), \theta')$. Note that $U_\tau(\theta)$ is continuous and strictly increasing in $\theta$. Since $g(\theta) \leq q^*(1) \forall \theta$, $|U_\tau(\theta_1) - U_\tau(\theta_2)| \leq |\theta_1 - \theta_2| \max_{\theta \in [0,1]} u(\theta, q^*(1))$, $U_\tau(\theta)$ is absolutely continuous. Hence, it is almost everywhere differentiable and has finite left-hand and right-hand derivatives for all $\theta \in [0,1]$.

The following lemma will be useful below:

\textbf{Lemma 12} Suppose that $U_\tau(\theta_2) - U_\tau(\theta_1) > u(g, \theta_2) - u(g, \theta_1)$ for some $\theta_1, \theta_2$ s.t. $\theta_2 > \theta_1$ and $g$, then $U_\tau(\theta_4) - U_\tau(\theta_3) > u(g, \theta_4) - u(g, \theta_3)$ s.t. $\theta_4 > \theta_3 \geq \theta_2$.

\textbf{Proof:} Suppose that sequence $\{\theta_n\}_{n=1}^{\infty}$ is s.t. $\lim_{n \to \infty} u(g(\theta_n), \theta_2) - u(g(\theta_n), \theta_1) = U_\tau(\theta_2)$ and $\lim_{n \to \infty} g(\theta_n) = \tilde{g}_2$. (Such a sequence exists because $g(\theta) \in [0, q^*(1)] \forall \theta \in [0,1]$, and any sequence in a compact set has a converging subsequence.) Then $\tilde{g}_2 > g$. For suppose not, i.e. $\tilde{g}_2 \leq g$. Then
\begin{equation*}
 U_\tau(\theta_2) - U_\tau(\theta_1) \leq \lim_{n \to \infty} u(g(\theta_n), \theta_2) - u(g(\theta_n), \theta_1) = u(\tilde{g}_2, \theta_2) - u(\tilde{g}_2, \theta_1) < u(g, \theta_2) - u(g, \theta_1)
\end{equation*}
Contradiction.

Next, consider a sequence $\{\theta_m\}_{m=1}^{\infty}$ s.t. $\lim_{m \to \infty} u(g(\theta_m), \theta_3) - u(g(\theta_m), \theta_m) = U_\tau(\theta_3)$ and $\lim_{m \to \infty} g(\theta_m) = \tilde{g}_3$. Then $\tilde{g}_3 > g$. Again, suppose otherwise i.e. $\tilde{g}_3 \leq g$. We have:
\begin{equation*}
 U_\tau(\theta_3) = \lim_{m \to \infty} u(g(\theta_m), \theta_3) - u(g(\theta_m), \theta_m) \geq \lim_{n \to \infty} u(g(\theta_n), \theta_3) - u(g(\theta_n), \theta_n)
\end{equation*}
Since $\tilde{g}_3 < \tilde{g}_2$ by assumption, it follows that $\exists N, M$ s.t. $\forall n \geq N$ and $m \geq M, g(\theta_m) < g(\theta_n)$, and so $u(g(\theta_m), \theta_2) - u(g(\theta_n), \theta_2) > u(g(\theta_m), \theta_3) - u(g(\theta_n), \theta_2)$.

But then $\lim_{m \to \infty} u(g(\theta_m), \theta_2) - u(g(\theta_m), \theta_m) > \lim_{m \to \infty} u(g(\theta_n), \theta_2) - u(g(\theta_n), \theta_n) = \sup_{\theta' \in [0,1]} u(g(\theta_2), \theta') - u(g(\theta'), \theta')$. Contradiction. Finally note that
\begin{equation*}
 U_\tau(\theta_4) - U_\tau(\theta_3) \geq \lim_{m \to \infty} u(g(\theta_m), \theta_4) - u(g(\theta_m), \theta_3) = u(\tilde{g}_3, \theta_4) - u(\tilde{g}_3, \theta_1) > u(g, \theta_4) - u(g, \theta_3)
\end{equation*}
\textit{Q.E.D.}

Now, let us demonstrate that the firm can strictly increase its expected profits by offering a modified quantity schedule $\tilde{q}(\theta)$ and setting $U(0) = 0$. To define $\tilde{q}(\theta)$, let $g_-(\theta)$ denote the left-hand limit of $g(\cdot)$ at $\theta$ (such limit exists $\forall \theta \in [0,1]$ since $g(\cdot)$ is increasing and bounded), and let $\theta^m = \min\{\theta, \sup\{\theta | g_-(\theta) \leq q(\theta)\}\}$. Then for $\theta \in [0, \theta^m]$ define:
\begin{equation*}
 V(\theta) = \int_0^\theta u_\theta(\max\{g(s), q(s)\}, s)ds + \int_0^\theta u_\theta(\max\{g_-(\theta), q(s)\}, s)ds
\end{equation*}
We will show that \( \exists \theta_0 \in [0, \theta^m] \) s.t. \( V(\theta_0) = u + \int_0^{\theta_0} u_\theta(q(s), s)ds \). Note that \( V(\theta) \) is continuous in \( \theta_0 \) and \( V(0) = \int_0^{\theta_0} u_\theta(q(s), s)ds < u + \int_0^{\theta_0} u_\theta(q(s), s)ds \).

Next, let us establish that \( V(\theta^m) \geq U_\tau(\hat{\theta}) \). Note that \( V(\theta^m) = \int_0^{\theta^m} u_\theta(\max\{g(s), q(s)\}, s)ds + u(q(\hat{\theta}), \hat{\theta}) - u(q(\hat{\theta}), \theta^m) \). Since \( g(.) \) is nondecreasing, \( \int_0^{\theta^m} u_\theta(\max\{g(s), q(s)\}, s)ds \geq U_\tau(\theta) \forall \theta \in [0, \theta^m] \).

Since \( U'(\hat{\theta}) = u_\theta(q(\hat{\theta}), \hat{\theta}) \) and \( \hat{\theta} \) satisfies (42), there exists \( \hat{\delta} > 0 \) s.t. \( \forall \theta \leq \hat{\theta} \), \( U(\hat{\theta} - \hat{\delta}) \leq u(q(\hat{\theta}), \hat{\theta}) - \hat{\delta} \leq U(\hat{\theta}) \leq U(\theta) \) \( \forall \theta \leq \hat{\theta} \). Otherwise, \( \exists \delta \) small enough that \( U(\tau + \delta) > U(\theta) \). By Lemma 12, this implies that \( U_\tau(\theta) - U_\tau(\theta^m) \geq u(q(\hat{\theta}), \hat{\theta}) - u(q(\hat{\theta}), \theta^m) \), and so \( V(\theta^m) \geq U_\tau(\theta) \). Hence, by continuity of \( V(\theta) \), \( \exists \theta_0 \in [0, \theta^m] \) s.t. \( V(\theta_0) = U_\tau(\theta) \).

Define \( \bar{q}(\theta) = \max\{g(\theta), q(\theta)\} \) \( \forall \theta \in [0, \theta_0] \). Then we have \( \bar{q}(\theta) = \max\{g(\theta), q(\theta)\} \) \( \forall \theta \in [0, \theta_0] \). So, by Lemma 12 \( \bar{U}(\theta) > U(\theta) \) \( \forall \theta \in [0, \theta_0] \). Contradiction.

When the firm implements quantity schedule \( q(\theta) \) rather than \( \bar{q}(\theta) \), sets \( U_0 = 0 \), and does not modify \( g(.) \), the change in the firm’s expected profits is equal to:

\[
\int_0^{\theta^m} (u(\bar{q}(\theta), \theta) - u(q(\theta), \theta))f(\theta)d\theta - \int_0^{\theta^m} (u_\theta(\bar{q}(\theta), \theta) - u_\theta(q(\theta), \theta))(1 - F(\theta))d\theta = \int_0^1 (u(\bar{q}(\theta), \theta) - u(q(\theta), \theta))f(\theta)d\theta + \int_0^1 (u_\theta(\bar{q}(\theta), \theta) - u_\theta(q(\theta), \theta))F(\theta)d\theta > 0
\]

The equality follows from the fact that \( \bar{U}(1) = U(1) \). The inequality follows because both terms in the expression on the second line are positive. The second term is positive because \( \bar{q}(\theta) \geq q(\theta) \) \( \forall \theta \in [0, 1] \). The first term is positive, because in addition, whenever \( \bar{q}(\theta) > q(\theta) \), \( \bar{q}(\theta) \leq g(\theta) \leq q^*_s(1) \), and so, since \( u(q, \theta) \) is quasiconcave in \( q \), \( u(\bar{q}(\theta), \theta) - u(q(\theta), \theta) > 0 \).

**Proof of Lemma 4:**

Suppose that \( \exists \theta \) s.t. \( q(\theta) = 0 \) and \( g(\theta) > 0 \). By continuity of \( q(\theta) \), \( \exists \theta' > \theta \) s.t. \( q(\theta') < g(\theta) \). Since \( q(\theta) \) is nondecreasing, \( U(\theta') = \int_0^{\theta'} u_\theta(q(s), s)ds < u(g(\theta), \theta') - u(g(\theta), \theta) \) i.e., ICT(\( \theta', \theta \)) in (12) fails.

Next, suppose that \( \exists \hat{\theta} > 0 \) s.t. \( g(\hat{\theta}) = 0 \) and \( \hat{\theta} > 0 \). By Lemma 1 \( \hat{\theta} > 0 \). Since \( g(.) \) is nondecreasing and \( q(.) \) is continuous, \( \exists \epsilon > 0 \) s.t. \( q(\theta) > 0 = g(\theta) \forall \theta \in [\hat{\theta} - \epsilon, \hat{\theta}] \).

But then without violating any incentive constraints in (12), the firm can increase its profits by setting \( g(\theta) = \min\{q^*(\theta), q(\theta)\} > 0 \) and \( t^*(\theta) = u(g(\theta), \theta) \forall \theta \in [\hat{\theta} - \epsilon, \hat{\theta}] \). Q.E.D.

**Proof of Lemma 5:** (i) Suppose that \( \exists \theta_a \in (0, 1] \) s.t. \( q(\theta_a) < q^{ab}(\theta_a) \). Let \( \theta_b = \sup\{\theta|q(\theta) < q^{ab}(\theta)\} \). By continuity of \( q(.) \), \( \theta_b > \theta_a \). Then the firm can strictly increase its profits by offering a modified quantity schedule \( q_a(.) \) s.t. \( q_a(\theta) = q(\theta) \forall \theta \in [0, \theta_a \cup (\theta_b, 1] \) and \( q_a(\theta) = q^{ab}(\theta) \forall \theta \in [\theta_a, \theta_b] \) and adjusting the transfers to preserve the incentive compatibility. Inspecting (10) and (12), one can see that this modification does not violate any incentive constraints and leads to an increase in the value of the objective function (10).

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∀U because it holds under constraints in (12) continue to hold. Inspecting (10) one can also see that as a result of this modification the value of the first integral goes up, while the second remains unchanged.

It remains to consider the following case: ∃∀1, θ2 s.t. q(θ) ≤ q*(θ) ∀θ ∈ [θ2, 1], q(θ) > q*(θ) ∀θ ∈ (θ1, θ2), and at least one is true: q(θ1) = q*(θ1) or θ1 = 0. Note that by continuity q(θ2) = q*(θ2).

Let us construct a modified quantity schedule qa(.) in the following way: qa(θ) = q(θ) ∀θ ∈ [0, θ1], qa(θ) = q*(θ) ∀θ ∈ (θ1, θ2). Further, if ∃∀ ∈ (θ2, 1) s.t.

\[ \int_{θ_1}^{θ} u_θ(q^*(s), s)ds + \int_0^{θ_1} u_θ(\max\{q^*(θ), q(s)\}, s)ds = \int_{θ_1}^{θ_1} u_θ(q(s), s)ds \]  

then set qa(θ) = q*(θ) ∀θ ∈ (θ2, θ1] and qa(θ) = max{q*(θ), q(θ)} ∀θ ∈ (θ1, 1]. Otherwise, if the left-hand side of (43) is strictly smaller than its right-hand side ∀θ ∈ [θ2, 1], then set qa(θ) = q*(θ) ∀θ ∈ (θ2, 1].

Leaving the quantity schedule g(.) unmodified, we need to establish two claims. **Claim 1:** all incentive constraints (12) still hold after this modification leads to an increase in the firm’s expected profits given by (10).

To establish **Claim 1**, consider θ3 = sup{θ | θ ≥ θ, q(θ) < q*(θ)}. Then ∀θ ∈ [θ1, θ1] ∪ [θ3, 1], Ua(θ) ≡ \[ \int_{θ_1}^{θ} u_θ(q_a(s), s)ds = U(θ) \equiv \int_{θ_1}^{θ} u_θ(q(s), s)ds \]. So, (12) holds ∀θ ∈ [θ1, θ1] ∪ [θ3, 1] under qa(.) because it holds under q(.)

Next suppose that θ ∈ [θ1, θ1]. Then qa(θ) = min{q*(θ), q*(θ)}. Then, since g(θ′) ≤ q*(θ′) and qa(θ) ≥ q*(θ1), the inequality U(θ1) ≥ u(g(θ′), θ1) − u(g(θ′), θ′) ∀θ′ ∈ [0, θ] implies that U(θ) ≥ u(g(θ′), θ) − u(g(θ′), θ′) ∀θ ∈ [θ1, θ1] and θ′ ∈ [0, θ1].

Similarly, ∀θ′ ∈ [θ1, θ1], we have U(θ) ≥ \[ \int_{θ_1}^{θ} u_θ(q_a(s), s)ds ≥ u(g(θ′), θ) − u(g(θ′), θ′) \]. It also follows immediately that U(θ) ≥ u(g(θ′), θ) − u(g(θ′), θ′) ∀θ′ ∈ (θ1, θ1) if θ′ ∈ (θ1, θ1) and g(θ′) ≤ q*(θ).

Finally, if θ ∈ (θ1, θ1) and θ′ ∈ (θ1, θ1) and g(θ′) > q*(θ), we can use U(θ1) ≥ u(g(θ′), θ1) − u(g(θ′), θ′) and qa(θ) = q*(θ) ∀θ ∈ (θ1, θ1) to show that U(θ) ≥ u(g(θ′), θ) − u(g(θ′), θ′). So, all the incentive constraints (12) hold when we replace q(.) with qa(.)

To prove **Claim 2**, focus on the first integral in (10). Note that \[ \int_{θ_1}^{θ} (u(q_a(θ), θ) − c(q(θ)))f(θ)dθ < \int_{θ_1}^{θ} (u(q_a(θ), θ) − c(q_a(θ)))f(θ)dθ \], because pointwise u(q(θ), θ) − c(q(θ)) ≤ u(q_a(θ), θ) − c(q_a(θ)), ∀θ ∈ [0, 1] and the inequality is strict ∀θ ∈ (θ1, θ2). If ∃∀ satisfying (43), then we have:

\[ \int_{θ_1}^{θ} (u_θ(q_a(θ), θ) − u_θ(q(θ), θ))(1 − F(θ))dθ = \int_{θ_1}^{θ} (u_θ(\min\{q^*(θ), q^*(θ)\}, θ) − u_θ(q(θ), θ))(1 − F(θ))dθ \]

\[ \leq (1 − F(θ_2)) \int_{θ_1}^{θ_3} (u_θ(\min\{q^*(θ), q^*(θ)\}, θ) − u_θ(q(θ), θ))dθ = 0 \]

where the first equality holds by definition of qa(.), the inequality holds because \( \min\{q^*(θ), q^*(θ)\} = q^*(θ) > q(θ) ∀θ ∈ (θ1, θ2) \) and \( \min\{q^*(θ), q^*(θ)\} = q^*(θ) < q(θ) ∀θ ∈ (θ2, θ3) \), and the last equality holds by (43). The same result obtains if ∃∀ satisfying (43). To see this simply replace θ by
1. Thus, after this modification the value of the first integral in (10) increases, i.e. the firm’s expected profits go up. Q.E.D

Proof of Lemma 6:
(i) Since \( q(\theta) \) is nondecreasing and \( U(0) = 0 \), \( 0 \leq U(\theta) = \int_0^\theta u_\theta(q(s),s)ds \leq u(q(\theta),\theta) \). It is then immediate from the definition of \( r(\theta) \) that \( 0 \leq r(\theta) \leq \theta \) and so \( r(0) = 0 \). Let \( \theta_2 > \theta_1 \) and \( q(\theta_1) > 0 \). Then, using the definition of \( r(\theta) \) and the fact that \( q(\cdot) \) is nondecreasing we have:

\[
u(q(\theta_2),r(\theta_2)) - u(q(\theta_1),r(\theta_1)) = u(q(\theta_2),\theta_2) - u(q(\theta_1),\theta_1) - \int_{\theta_1}^{\theta_2} u_\theta(q(s),s)ds \geq u(q(\theta_2),\theta_1) - u(q(\theta_1),\theta_1)
\]

(44)

The first equality implies that \( r(\theta_2) = r(\theta_1) \) if \( q(\theta_2) = q(\theta_1) \), while the inequality in combination with the fact that \( u_\theta(q,\theta) > 0 \) implies that \( r(\theta_2) > r(\theta_1) \) if \( q(\theta_2) > q(\theta_1) \).

Next, define \( \theta \equiv \sup\{\theta'\mid q(\theta') = 0\} \). The continuity of \( r(\theta) \) at \( \theta = \theta \) follows from the continuity of \( U(\theta) \), \( q(\theta) \) and continuity of \( u(q,s) \) in both arguments. Since \( q(\cdot) \) is nondecreasing, \( q(\theta) = 0 \) and hence by definition \( r(\theta) = \theta \) for all \( \theta < \theta \). Thus \( r(\theta) \) is continuous on \([0,\theta]\). To establish the continuity of \( r(\theta) \) at \( \theta \), note that \( \forall \theta > \theta \) we have:

\[
\int_{r(\theta)}^\theta u_\theta(q(s),s)ds = u(q(\theta),\theta) - u(q(\theta),r(\theta)) = \int_{\theta}^{\theta} u_\theta(q(s),s)ds \leq \int_{\theta}^\theta u_\theta(q(s),s)ds
\]

The last inequality can hold only if \( r(\theta) \geq \theta \). Since \( r(\theta) \leq \theta \forall \theta \), we conclude that the right-hand limit of \( r(\theta) \) at \( \theta \) is equal to \( \theta \).

(ii) Suppose that \( ICT(\theta,\theta') \) in (12) holds \( \forall \theta,\theta' \in [0,1] \). If \( q(\theta) = 0 \), then \( r(\theta) = \theta \). By Lemma 4, we have \( g(\theta) = 0 \), so \( g(r(\theta)) = g(\theta) = q(\theta) = 0 \).

If \( q(\theta) > 0 \), then using the definition of \( r(\theta) \) and the fact that \( ICT(\theta,r(\theta)) \) holds, we have:

\[
u(q(\theta),r(\theta)) = U(\theta) \geq u(q(r(\theta)),\theta) - u(q(r(\theta)),r(\theta))
\]

Then the single-crossing implies that \( q(\theta) \geq g(r(\theta)) \).

Now suppose that \( q(\theta) \geq g(r(\theta)) \forall \theta \in [0,1] \). Consider any pair \((\theta,\hat{\theta})\). Since \( r(\theta) \) is continuous, nondecreasing and \( r(0) = 0 \), either \( \hat{\theta} = r(\theta) \) for some \( \theta \in [0,1] \) or \( \theta > r(1) \). If \( \hat{\theta} = r(\theta) \) for some \( \theta \in [0,1] \), then by assumption \( g(\theta) \leq q(\theta) \). Take any \( \theta \geq \theta \). Then:

\[
U(\theta) = \int_\theta^\theta u_\theta(q(s),s)ds + U(\hat{\theta}) \geq u(q(\theta),\theta) - u(q(\theta),\hat{\theta}) + u(q(\theta),\hat{\theta}) - u(q(\theta),\hat{\theta}) \geq u(g(\theta),\theta) - u(g(\hat{\theta}),\theta)
\]

(45)

The first inequality follows from the fact that \( q(\cdot) \) is nondecreasing and the definition of \( r(\hat{\theta}) \), while the second inequality follows from single-crossing and \( q(\hat{\theta}) \geq g(\hat{\theta}) \). So, \( ICT(\theta,\hat{\theta}) \) holds.

If \( \hat{\theta} < \theta < \theta \), then \( q(\theta) \leq q(\hat{\theta}) \). So, \( ICT(\theta,\hat{\theta}) \) holds because

\[
U(\theta) = U(\hat{\theta}) - \int_\theta^{\hat{\theta}} u_\theta(q(s),s)ds \geq u(q(\hat{\theta}),\hat{\theta}) - u(q(\hat{\theta}),\hat{\theta}) - \left( u(q(\hat{\theta}),\hat{\theta}) - u(q(\theta),\theta) \right) \geq u(g(\hat{\theta}),\theta) - u(g(\hat{\theta}),\theta)
\]

\[\text{34} \hat{\theta} \text{ need not be unique. However, if } \hat{\theta} = r(\hat{\theta}_1) = r(\hat{\theta}_2), \text{ then as established above } q(\hat{\theta}_1) = q(\hat{\theta}_2).\]
Finally, suppose that \( \hat{\theta} > r(1) \). Since \( g(\hat{\theta}) \leq g^*(1) = q(1) \), for \( \theta > \hat{\theta} \) we have:

\[
U(\theta) = U(1) - \int_{\theta}^{1} u_\theta(q(s), s)ds \leq u(q(1), 1) - u(q(1), r(1)) = (u(q(1), 1) - u(q(1), \theta)) = u(q(1), \theta) - u(q(1), r(1)) \geq u(g(\hat{\theta}), \theta) - u(g(\hat{\theta}), r(1)) > u(g(\hat{\theta}), \theta) - u(g(\hat{\theta}), \hat{\theta})
\]

Thus, \( ICT(\theta, \hat{\theta}) \) also holds in this case.

**Proof of Theorem 3:** Suppose that \( \widehat{\theta} \equiv \inf \{ \theta | q(\theta) > 0 \} > 0 \). We will show that there exists \( \epsilon \in (0, \overline{\theta}/4) \), s.t. the firm can strictly increase its profits by replacing the quantity schedule \( q(\theta) \) with \( \tilde{q}(\theta) = \max \{ \epsilon, q(\theta) \} \forall \theta \in [0, 1] \).

Since \( q(\theta) \) is continuous and nondecreasing, for any sufficiently small \( \epsilon > 0 \) there exists a unique \( \tilde{\theta}(\epsilon) \) s.t. \( q(\tilde{\theta}(\epsilon)) = \epsilon \) and \( q(\tilde{\theta}(\epsilon)) > \epsilon \forall \theta \in [\tilde{\theta}(\epsilon), 1] \). Under the modified schedule \( \tilde{g}(\theta) \), 'strategic' consumer obtains surplus \( \tilde{U}(\theta) = u(\epsilon, \theta) \forall \theta \in [0, \tilde{\theta}(\epsilon)] \), and \( \tilde{U}(\theta) = u(\epsilon, \tilde{\theta}(\epsilon)) + \int_{\theta}^{\tilde{\theta}(\epsilon)} u_\theta(q(s), s)ds \forall \theta \in [\tilde{\theta}(\epsilon), 1] \).

Define \( \tilde{r}(\theta) \) as the solution to \( \tilde{U}(\theta) = u(\tilde{q}(\theta), \theta) - u(\tilde{q}^*(\theta), \tilde{r}(\theta)) \). By lemma 7 \( g(\theta) = \min \{ q^*(\theta), q(r^{-1}*(\theta)) \} \), and \( \tilde{g}(\theta) = \min \{ q^*(\theta), \tilde{q}(\tilde{r}^{-1}(\theta)) \} \). Also, define \( G(q, \theta) = u(q, \theta) - c(q) - u_\theta(q(s), s)ds \forall \theta \in [0, \tilde{\theta}(\epsilon)] \).

Using (10), we conclude that after this modification the firm’s expected profit changes by:

\[
\Delta(\epsilon) = \int_{0}^{1} (G(\tilde{g}(\theta), \theta) - G(q(\theta), \theta)) f(\theta)d\theta + \alpha \int_{0}^{1} (u(\tilde{g}(\theta), \theta) - c(\tilde{g}(\theta)) - u(g(\theta), \theta) - c(g(\theta))) f(\theta)d\theta
\]

We will show that \( \Delta(\epsilon) > 0 \) if \( \epsilon \) is sufficiently small, because the second term is positive and is of higher order than \( \epsilon \), while the first term may be negative, but is at most of order \( \epsilon \).

To establish the claim regarding the first term, pick some \( \epsilon > 0 \) and define \( \omega = \max_{\theta \in [0, \tilde{\theta}(\epsilon)]} \frac{\partial G(q, \theta)}{\partial \theta} \). Note that \( \omega < \infty \), and by Weierstrass Theorem, \( \forall \epsilon \leq \epsilon G(\tilde{q}(\theta), \theta) - G(q(\theta), \theta) \leq \omega(\tilde{q}(\theta) - q(\theta)) \leq \omega \epsilon \), and so \( \int_{0}^{1} (G(\tilde{g}(\theta), \theta) - G(q(\theta), \theta)) f(\theta)d\theta \leq \omega \epsilon \).

Now, let us focus on the second term. If \( \theta \geq \tilde{\theta}(\epsilon) \), then \( q(\theta) = q(\theta) \) and \( U(\theta) = u(\epsilon, \theta) \), so \( \tilde{r}(\theta) < r(\theta) \). Also, \( \forall \theta \in [0, \tilde{\theta}(\epsilon)] \), \( \tilde{r}(\theta) = 0 \), while \( r(\theta) = \theta \forall \theta \in [0, \tilde{\theta}(\epsilon)] \). Therefore, \( \tilde{r}(\theta) < r(\theta) \forall \theta \in (0, 1) \). Lemma 7 implies that \( q^*(\theta) \geq \tilde{g}(\theta) \geq g(\theta) \forall \theta \in [0, 1] \). Therefore, since \( u(q(\theta) - c(g(\theta)) \geq u(g(\theta), \theta) - c(g(\theta)) \forall \theta \in [0, 1] \).

By Lemma 4, \( g(\theta) = 0 \forall \theta \in [0, \tilde{\theta}] \). So,

\[
\int_{0}^{1} (u(\tilde{g}(\theta), \theta) - c(\tilde{g}(\theta)) - u(g(\theta), \theta) - c(g(\theta))) f(\theta)d\theta \geq \int_{0}^{\tilde{\theta}/2} (u(\tilde{g}(\theta), \theta) - c(\tilde{g}(\theta))) f(\theta)d\theta
\]

Let us show that \( \tilde{g}(\theta) = q(\tilde{r}^{-1}(\theta)) < q^*(\theta) \forall \theta \in [\theta/2, \tilde{\theta}] \). First, let us establish that \( \lim_{\epsilon \to 0} \tilde{r}^{-1}(\theta) = \theta \). For suppose, to the contrary, that there exists a sequence \( \epsilon_n, \lim_{n \to \infty} \epsilon_n = 0 \), and \( \eta > 0 \) s.t.

\(^{35}\)If such \( \hat{\theta}(\epsilon) \) fails to exist for all \( \epsilon > 0 \), then \( q(\theta) = 0 \) everywhere which is suboptimal.

\(^{36}\)Note that, as shown in Lemma 6, \( q(r^{-1}(\theta)) \) is well-defined, but the preimage \( \tilde{r}^{-1}(\theta) \) may be an interval. However, all the arguments in this paper apply to every element in the preimage \( \tilde{r}^{-1}(\theta) \) i.e. each \( \theta' \) s.t. \( \tilde{r}(\theta') = \theta \). Formally, set \( \tilde{r}^{-1}(\theta) = \max \{ \theta' | \tilde{r}(\theta') = \theta \} \).

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\[ \theta_n = \tilde{r}^{-1}(\theta, \epsilon_n) \] (we explicitly incorporate the dependence of \( \tilde{r}^{-1}(\theta) \) on \( \epsilon \)) and \( \lim_{n \to \infty} \theta_n \geq \theta + \eta \).

Let \( \theta_1 \) denote the limit of a converging subsequence of \( \theta_n \). Obviously, \( \theta_1 \geq \theta + \eta \).

Note that \( \tilde{U}(\theta_n, \epsilon_n) = u(q(\theta_n), \theta_n) - u(q(\theta_n), \theta) \). Since \( \tilde{U}(\theta, \epsilon_n) \) converges to \( U(\theta) \) uniformly as \( \epsilon \) converges to zero, and \( U(\theta) \) is continuous, we have \( \tilde{U}(\theta_n) = u(q(\theta_n), \theta_n) - u(q(\theta_n), \theta) \). On the other hand, \( U(\theta_1) = \int_0^{\theta_1} u_{\theta}(q(s), s)ds < u(q(\theta_1), \theta_1) - u(q(\theta_1), \theta) \) where the inequality follows from continuity of \( q(\theta) \). This contradiction implies that \( \lim_{\epsilon \to 0} \tilde{r}^{-1}(\theta) \leq \theta \). But since \( \tilde{r}^{-1}(\theta) > \theta \) \( \forall \epsilon > 0 \), we conclude that \( \lim_{\epsilon \to 0} \tilde{r}^{-1}(\theta) = \theta \).

Let us now fix some \( \psi \in (0, q^*(\theta/2)) \). Combining \( \lim_{\epsilon \to 0} \tilde{r}^{-1}(\theta) = \theta \) with the continuity of \( q(\theta) \) we obtain that \( \exists \tilde{\epsilon} > 0 \) s.t. \( \forall \epsilon < \tilde{\epsilon} \) \( q(\tilde{r}^{-1}(\theta)) = q^*(\theta/2) - \psi \), and so \( \tilde{g}(\theta) = q(\tilde{r}^{-1}(\theta)) \leq q^*(\theta/2) - \psi \forall \theta \in (0, 2) \).

Let \( \zeta = \min_{\theta \in [0, 2]} q^*(\theta/2) - \psi \) \( u_q(\theta, \psi) - c' \) and \( \hat{f} = \min_{\theta \in [0, 2]} f(\theta) \). Note that \( \zeta > 0 \) and \( \hat{f} > 0 \). Then \( \int_{\beta/2}^{\beta} (u(q, \theta) - c(q)) f(\theta) d\theta = \zeta \hat{f} \int_{\beta/2}^{\beta} \tilde{g}(\theta) d\theta \).

Next we establish a lower bound on \( \tilde{g}(\theta) \) for \( \theta \in [\theta, \beta] \). Let \( m = \min_{\theta \in [0, 1], q \in [0, q^*(\beta/2)]} u_q \) and \( M = \max_{\theta \in [0, 1], q \in [0, q^*(\beta/2)]} u_q(\theta, q) \). Our assumptions on \( u(q, \theta) \) imply that \( 0 < m \leq M < \infty \). Then

\[ \tilde{U}(\tilde{r}^{-1}(\theta)) = u(q, \tilde{r}^{-1}(\theta)) - u(q, \theta) = \int_{\tilde{r}^{-1}(\theta)}^{\tilde{r}(\theta)} u_{\theta}(q, s)ds = \int_{\tilde{r}^{-1}(\theta)}^{\tilde{r}(\theta)} \tilde{g}(\theta) u_{\theta}(q, s)ds \]

\[ \leq M \tilde{g}(\theta)(\tilde{r}^{-1}(\theta) - \theta) \]

(47)

On the other hand, since \( \tilde{r}^{-1}(\theta) > \theta \),

\[ \tilde{U}(\tilde{r}^{-1}(\theta)) = \tilde{U}(\theta) = \int_{\theta}^{\theta} u_{\theta}(\epsilon, s)ds = \int_{\theta}^{\theta} \int_{\theta}^{\epsilon} u_{\theta q}(q, s)dsd\epsilon \geq m\epsilon \theta \]

(48)

Combining (47) and (48), we obtain that \( \tilde{g}(\theta) \geq \frac{m\epsilon \theta}{M(\tilde{r}^{-1}(\theta) - \theta)} \). Collecting these results together, we have:

\[ \int_{\beta/2}^{\beta} (u(q, \theta) - c(q)) f(\theta) d\theta \geq \frac{\zeta \hat{f} m\epsilon \theta}{m \tilde{g}(\theta) M(\tilde{r}^{-1}(\theta) - \theta)} \]

To complete the proof, we will show that \( \int_{\beta/2}^{\beta} \frac{d\theta}{\tilde{r}^{-1}(\theta) - \theta} \) increases to \( \infty \) as \( \epsilon \) converges to \( 0 \). Fix some \( \rho \in (0, \beta/2) \). Since \( \lim_{\epsilon \to 0} \tilde{r}^{-1}(\theta) = \theta \) \( \forall \theta \in [\beta/2, \beta] \), by Lebesgue’s dominated convergence theorem,

\[ \lim_{\epsilon \to 0} \int_{\beta/2}^{\beta} \frac{d\theta}{\tilde{r}^{-1}(\theta) - \theta} = \int_{\beta/2}^{\beta} \frac{d\theta}{\theta - \theta} = \log(\beta/2) - \log(\rho) \]

Note that \( \lim_{\rho \to 0} \log(\rho) = -\infty \), which proves the desired result.

\[ Q.E.D. \]

**Proof of Lemma 8:**

Suppose that \( q(\theta) \) is a solution to Problem (13) on the domain \( C_p^1([0, 1]) \), but there exists an admissible schedule \( \hat{q}(\theta) \in C([0, 1]) \setminus C_p^1([0, 1]) \) s.t. the objective function in (13) takes a strictly higher value under \( \hat{q}(\theta) \) than under \( q(\theta) \).

By the Stone-Weierstrass theorem, the space of continuously differentiable functions \( C^1([0, 1]) \), which is a subspace of \( C_p^1([0, 1]) \) is dense in \( C([0, 1]) \). Therefore, \( C_p^1([0, 1]) \) is dense in \( C([0, 1]) \). So, there exists a sequence \( \hat{q}_n(\theta) \in C_p^1([0, 1]) \) converging to \( \hat{q}(\theta) \) in the sup-norm. The objective
function (13) is continuous in the \( sup \)-norm. Therefore, \( \exists N > 0 \) s.t. \( \forall n \geq N \) (13) takes a strictly higher value under \( \tilde{q}_n(\theta) \) than under \( q(\theta) \). This contradicts the hypothesis that \( q(\theta) \) is a solution on \( C_p^1([0, 1]) \).

Proof of Lemma 9: The Lemma will be established in a sequence of steps.

Step 1. \( \exists \theta_1, \theta_2 \in [0, 1] \), \( \theta_1 < \theta_2 \), s.t. Case 1 applies i.e. \( q(\theta) < q^*(r(\theta)) \) on \( (\theta_1, \theta_2) \).

For suppose not. Then by continuity, we must have \( q(\theta) \geq q^*(r(\theta)) \) \( \forall \theta \in [0, 1] \). This implies that for any \( \theta \) s.t. \( q'(\theta) > 0 \), \( q(\theta) \) is given by the solution to (32) with the same constant of integration \( k_2 \) \( \forall \theta \in [0, 1] \).

Since \( q(1) = q^*(1) \), it follow from (32) that \( k_2 = 0 \). Then, however, (32) cannot hold as an equality near \( \theta = 0 \) if \( q(\theta) > 0 \) \( \forall \theta > 0 \). To see this, recall that \( u_q(q, 0) = 0 \) \( \forall q \geq 0 \), and so \( u_q(q, \theta) = \int_0^\theta u_q(q, s)ds \). Therefore, \( u_q(0, \theta) < u_q(0, \theta) \frac{1-F(\theta)}{f(\theta)} \) when \( \theta \) is sufficiently small. Consequently, \( \theta_0 \) solving \( u_q(0, \theta_0) = u_q(0, \theta_0) \frac{1-F(\theta_0)}{f(\theta_0)} \) is strictly positive, and we must have \( q(\theta) = 0 \) \( \forall \theta \leq \theta_0 \).

Step 2. \( \exists \theta_1 \in (0, 1) \) s.t. \( q(\theta) < q^*(r(\theta)) \), i.e. Case 1 applies, on \( (0, \theta_1) \).

Suppose not. Then \( \exists \theta_a, \theta_b \in (0, 1) \), s.t. Case 2 applies on \( (0, \theta_a) \) and Case 1 applies on \( (\theta_a, \theta_b) \). So, \( q(\theta_a) = q^*(r(\theta_a)) \). Since \( r(\theta_a) < \theta_a \), we have \( u_q(q(\theta_a)) < u_q(q(\theta_a), r(\theta_a)) = c^r(q(\theta_a)) \). Hence, since (32) holds on \( (0, \theta_a) \), the constant of integration \( k_2 \) on this interval needs to satisfy \( k_2 > -1 \). Therefore, by the same argument as in Step 2, \( \hat{\theta} \) solving \( u_q(0, \hat{\theta}) = u_q(0, \hat{\theta}) \frac{1+k_2-F(\hat{\theta})}{f(\hat{\theta})} \) is strictly positive, and we must have \( q(\theta) = 0 \) \( \forall \theta \leq \hat{\theta} \). Contradiction.

Step 3. \( \exists \theta_n \in (\theta_1, 1) \) s.t. \( q(\theta) > q^*(r(\theta)) \), i.e. Case 2 applies, on \( (\theta_1, 1) \).

Since \( \forall \theta < \theta_1 \), we have \( q^*(r(\theta)) \leq q^*(r(1)) < q^*(1) = q(1) \). By continuity of \( q(.) \) it follows that \( q^*(r(\theta)) < q(\theta) \) for all \( \theta \) sufficiently close to 1. This establishes the result.

Step 4. \( \exists \theta_h \in [\theta_n, 1) \) s.t. \( q(.) \) is strictly increasing on \( [\theta_n, 1) \).

Suppose otherwise. Since by Lemma (1) \( q(1) = q^*(1) \), \( \exists \theta^f \in (0, 1] \) s.t. \( q(\theta) = q^*(1) \) \( \forall \theta \in [\theta^f, 1] \). Also, since \( q(.) \) is continuous and \( q(0) = 0 \), \( \exists \theta^l \) s.t. \( q(\theta^l) = q^*(\theta^l) \) and \( q(\theta) > q^*(\theta) \) \( \forall \theta \in (\theta^l, 1) \).

Let us show that the value of the objective can be strictly increased by replacing the quantity schedule \( q(.) \) with modified quantity schedule \( q_m(.) \) s.t. \( q_m(\theta) = q(\theta) \) \( \forall \theta \in [0, \theta^l] \), and \( q^m(\theta) = q^*(\theta) \) \( \forall \theta \in [\theta^l, 1] \).

First, from the definition of \( r(\theta) \) it follows that \( r(\theta) \leq \theta \) \( \forall \theta \in [0, 1] \).\footnote{This also follows from the differential equation (14) and the initial condition \( r(0) = 0 \), because by (14) \( r'(\theta) < 0 \) if \( r(\theta) > \theta \).} Therefore, \( \forall \theta \in (\theta^l, 1) \), \( q(\theta) > q_m(\theta) \geq q^*(r(\theta)) \), i.e. the solution is in Case 2 and \( g(\theta) = q^*(\theta) \) \( \forall \theta \in [r(\theta), 1] \). Thus, under both the original and the modified quantity schedule \( g(\theta) \) is the same \( \forall \theta \in [0, 1] \).

Inspecting the objective in (10) it is easy to see that its value increases as a result of this modification, because the value of the first integral goes up, while the second integral remains unchanged.

Step 5 \( q(\theta) = q^{ab}(\theta) \) on \( [\theta_h, 1] \).

Since \( q(.) \) is strictly increasing and is in Case 2 on \( [\theta_h, 1] \), it is given by the solution to (32). From \( q(1) = q^*(1) \), it follows that \( k_2 = 0 \) on this interval, and hence \( q(\theta) = q^{ab}(\theta) \). Q.E.D.
The existence of at least one solution to (6) and (7) with the boundary conditions \(q(0) = r(0) = 0\) and \(q(1) = 1\) follows because the optimal solution, which does exist, must possess these properties. Now, suppose that there are two pairs of functions \((q_1(\theta), r_1(\theta))\) and \((q_2(\theta), r_2(\theta))\) with these properties.

Lemma 9 implies that for \(i = 1, 2\) \(\exists \theta^i > 0\) s.t. \(q_i(\theta^i) = q^*(r_i(\theta^i))\), and \(\forall \theta \in (\theta^i, 1] \ q_i(\theta) \geq q^*(r_i(\theta))\), so \(q_i(\theta)\) satisfies (32) with \(k_2 = 0\). Suppose without loss of generality that \(\theta^1_1 > \theta^2_2\). (We can rule out \(\theta^1_1 = \theta^2_2\), because in this case \((q_1(\theta), r_1(\theta))\) would be identical to \((q_2(\theta), r_2(\theta))\).)

It follows that, \(q_1(\theta^1_1) = q_2(\theta^1_1)\) and \(U_1(\theta^1_1) \equiv \int_0^{\theta^1_1} u_\theta(q_1(s), s)ds < U_2(\theta^1_1) \equiv \int_0^{\theta^1_1} u_\theta(q_2(s), s)ds\). Then there must exist \(\bar{\theta} \in (0, \theta^1_1]\) and \(\epsilon_1 > 0\) s.t. \(q_1(\theta) \geq q_2(\theta) \forall \theta \in [\bar{\theta}, \theta^1_1]\) and \(q_1(\theta) < q_2(\theta) \forall \theta \in (\bar{\theta} - \epsilon_1, \bar{\theta}]\).

Therefore, \(q'_1(\tilde{\theta}) \geq q'_2(\tilde{\theta})\) and there exists \(\epsilon_2 > 0\) s.t. \(q'_1(\theta) > q'_2(\theta) \forall \theta \in (\bar{\theta} - \epsilon_2, \bar{\theta})\). By inspection of (6), this implies that \(\forall \theta \in (\bar{\theta} - \epsilon_2, \bar{\theta})\) \(q^*(r_2(\theta)) > q_1(\theta)\), and so by continuity \(q^*(r_2(\theta)) \geq q_1(\tilde{\theta})\).

Since \(q_1(\theta) \geq q_2(\theta) \forall \theta \in [\bar{\theta}, \theta^1_1]\) and \(U_1(\theta^1_1) < U_2(\theta^1_1)\), it follows that \(U_1(\tilde{\theta}) < U_2(\tilde{\theta})\), and so \(r_1(\tilde{\theta}) > r_2(\tilde{\theta})\). But since \(q^*(r_2(\theta)) \geq q_2(\tilde{\theta})\), \(q_1(\tilde{\theta}) = q_2(\tilde{\theta})\) and \(\frac{f'(r)(u_\theta(q,r) - c'(q))}{u_\theta(q,r)}\) is increasing in \(r\), it follows from (6) that \(q'_1(\tilde{\theta}) < q'_2(\tilde{\theta})\). A contradiction. \(Q.E.D.\)

References


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Figure 2: Optimal quantity schedule $q(.)$ of the ‘strategic’ consumers.
Figure 3: Optimal quantity schedule $g(.)$ of the ‘honest’ consumers.
Figure 4: Optimal quantity schedules with a unique switchpoint $\bar{\theta}$. 

Legend:
- First-best quantity $q^*(\bar{\theta})$
- Second-best quantity $q^\beta(\bar{\theta})$
- Optimal quantity for 'honest' types $g(\theta)$
- Optimal quantity for 'strategic' types $q(\theta)$