Iterative Auction Design for Graphical Valuations
Part II: General Graphs

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Abstract

In this paper we develop efficient iterative auctions for a class of value functions that admit a graphical representation and allow both for value complementarity and substitutability. It is known that for graphical valuations a Walrasian equilibrium (an equilibrium concept that relies on anonymous item prices) does not exist in general. Our first contribution is to establish the existence of a pricing equilibrium when the auctioneer uses a graphical pricing rule that involves bidder-specific prices for each item and markups/discounts for pairs of items. We provide a linear programming formulation of the efficient allocation problem, whose solutions can be used to identify such pricing equilibria. By focusing on iterative algorithms that can be used for the solution of this linear program and designing payment schemes that guarantee truthful bidding, we obtain a novel iterative auction format for graphical valuations. The auction format relies on the bidder-specific graphical pricing rule and implements the efficient outcome at an ex-post perfect equilibrium. Moreover, the linear programming formulation and the auction format generalize to settings where valuations are not graphical but admit a more general additively decomposable structure.

1 Introduction

Iterative auctions are mechanisms where the auctioneer sets prices for the items she is selling, bidders reveal their demand, and the auctioneer adjusts the prices until termination with a final allocation of items to bidders. These auctions are commonly employed in practice, for instance in the context of spectrum and electricity markets (Cramton et al. 2006). The existing iterative auction formats, on the other hand, mainly guarantee efficiency either under a restrictive gross

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substitutes assumption or when the auctioneer uses a complex pricing rule. The auctions in the former category do not allow for complementarity in valuations, which is commonly observed in practice. Those in the latter category may be impractical since the auctioneer needs to offer a price for every bundle of items (or specify exponentially many prices) at each stage of these iterative auctions.

Motivated by this observation, we study the question of iterative auction design for multi-item environments. Our main contribution is to provide novel iterative auction formats that rely on simple pricing rules and guarantee efficiency for a class of valuations that admit a graphical representation and exhibit both value complementarity and substitutability.

The value functions that belong to this class are associated with a value graph, nodes of which correspond to the items that are sold by the auctioneer. There are edges between items that can exhibit pairwise value complementarity or substitutability. We associate weights with the nodes and edges of the underlying graph. Positive weights associated with an edge capture value complementarity between the nodes (items) at the end points of this edge, and negative weights capture substitutability. The value a bidder has for a set of items is equal to the sum of the node and edge weights of the induced subgraph obtained by restricting the original value graph to this set of nodes (items). Graphical valuations allow for a compact representation of the value functions of bidders, and capture pairwise complementarity/substitutability present in practical settings. By exploiting the special structure of this class of valuations, we show that it is possible to obtain novel iterative auction formats that rely on simple pricing rules.

Often iterative auctions terminate when a market clearance condition holds (e.g., see Ausubel (2006)), i.e., bidders demand disjoint sets of items, and at the given prices the auctioneer maximizes her revenue by assigning each bidder a set of items that she demands. At such an outcome, no bidder needs to compete with the remaining bidders to acquire the set of items that she demands (since the demand sets are disjoint), thereby making this outcome a natural termination point for the auction. When the auctioneer restricts herself to a pricing rule that involves an anonymous price for each item (i.e., price $p_i$ for each item $i$), this condition holds if and only if the prices in the auction converge to a Walrasian equilibrium. This suggests that it may be possible to design iterative auction formats that rely on anonymous item pricing and the aforementioned termination condition if and only if a Walrasian equilibrium exists.

In our companion paper (Candogan et al., 2013) we study the iterative auction design question for a subclass of graphical valuations where the underlying value graph is a tree (and satisfies a technical sign-consistency condition). For this class of valuations, we establish that a Walrasian equilibrium exists and develop an iterative auction that relies on the anonymous item pricing rule. However, we also show that in settings where the underlying graph is not a tree, a Walrasian equilibrium need not exist. This implies that in these more general settings, it is not always possible to develop efficient iterative auction formats that rely on anonymous item pricing and terminate when a natural market clearance condition holds.

In this paper, we focus on iterative auction design for general graphical valuations. Since, a
Walrasian equilibrium need not exist for this class, we focus on more general pricing rules, and characterize the existence of a pricing equilibrium (a generalization of the Walrasian equilibrium to more complex pricing rules). In particular, we establish that when the auctioneer offers bidder-specific prices for each item and discounts/markups for pairs of items, a pricing equilibrium always exists. Then we develop iterative auction formats that rely on this pricing rule, terminate when a pricing equilibrium is reached (or equivalently a market clearance condition holds), and guarantee efficiency for all graphical valuations. Our paper delineates how the special structure in valuations can be exploited to design iterative auction formats that rely on simple pricing rules.

Our starting point is to provide a linear programming formulation of the efficient allocation problem that has optimal solutions that are integral if and only if a pricing equilibrium (with bidder-specific graphical pricing) exists. This formulation generalizes similar linear programming formulations that can be used to study Walrasian equilibria (Bikhchandani and Mamer, 1997) and is closely related to formulations that reveal pricing equilibria for bundle pricing rules (see Bikhchandani and Ostro2002]; Parkes 2006]. Moreover, the optimal primal-dual solutions of this LP can be used to obtain the pricing equilibria.

We then focus on developing an efficient iterative auction format that relies on bidder-specific graphical prices. The main idea behind the design of this auction is to devise an iterative process that converges to a pricing equilibrium (which is a natural termination point for iterative auctions) when bidders truthfully reveal their demand, and to guarantee that it is an equilibrium for bidders to do so by complementing this process with an appropriate payment scheme. We obtain such iterative processes by focusing on iterative algorithms (similar to primal-dual algorithms) for the solution of the linear programming formulation of the efficient allocation problem (and its dual) associated with bidder-specific graphical pricing. We guarantee truthful demand revelation is an (ex-post perfect) equilibrium, by ensuring that when her opponents bid truthfully, it is possible to charge each bidder the corresponding VCG payment (or the externality she creates on the rest of the system) at the end of the iterative process. The idea of designing iterative auctions by focusing on iterative algorithms that solve linear programming formulations of the efficient allocation problem, is also employed in the existing literature (e.g., see Vohra 2011]. However, majority of the existing iterative auction formats designed using this approach focus on general classes of valuations, and implement the efficient outcome by relying on complex pricing rules that involve exponentially many prices [Bikhchandani et al., 2002; De Vries et al., 2007; Ausubel and Milgrom 2002; Mishra and Parkes 2007; Vohra, 2011]. In contrast, in this work, we follow a similar approach, but by exploiting the properties of graphical valuations we develop an iterative auction format that relies on simple pricing rules, and guarantees efficiency for all graphical valuations (including those that exhibit complementarity). Moreover, our auction uses...
a novel descending price update structure that ensures that bidders demand progressively larger bundles of items and the auction eventually terminates when market clears.

Our auction relies on identifying a special dual optimal solution of the LP formulation of the efficient allocation problem, which can be used to compute the VCG payments of bidders. Having a price update rule that converges to such a special dual optimal solution is critical in order to implement the efficient outcome using the above described approach. Motivated by this observation, we also provide an alternative LP formulation of the efficient allocation problem, whose corresponding dual optimal solutions can always be used to compute final payments that guarantee truthful bidding. Due to this structure, solutions of this LP formulation with any primal-dual algorithm can be employed to develop iterative auction formats that implement the efficient allocation. Moreover, this result extends to settings where valuations are not necessarily graphical, but admit a more general additively decomposable structure. In particular, by appropriately generalizing the linear programming formulation associated with bidder-specific graphical pricing, we show that whenever valuations of bidders additively decompose over subsets of items, a pricing equilibrium with a pricing rule that exhibits the same structure always exists, and an efficient iterative auction format that relies on this pricing rule can be designed. Our results suggest that for iterative auction design, it suffices to restrict attention to prices that are at most as “complex” as the value functions of bidders.

Outline: In Section 2, we discuss the graphical valuation model, and introduce the concept of a pricing equilibrium. For graphical valuations, we establish the existence of a pricing equilibrium with bidder-specific graphical pricing rule in Section 3. In this section, we also provide an LP formulation of the efficient allocation problem, and present an iterative auction format that implements the efficient outcome. We provide an alternative LP formulation, and generalize our results to additively decomposable valuations in Section 4. We conclude in Section 5 by summarizing our contributions and outlining future directions. We relegate all proofs to the appendix.

Related literature: At full generality implementing the efficient outcome in a multi-item setting is a hard problem both from a computational complexity and a communication complexity point of view (Lehmann et al., 2006; Nisan and Segal, 2006; Cramton et al., 2006; Blumrosen and Nisan, 2010). This motivates considering classes of value functions with additional structure (Blumrosen and Nisan, 2010; Cramton et al., 2006). Recently, Zhou et al. (2009) and Abraham et al. (2012) studied a graphical valuation model that is similar to the one we consider in this paper. They characterized the computational complexity of efficient auction design for (hyper) graphical valuations, and developed approximately efficient sealed-bid auctions for settings where valuations do not exhibit substitutabilities. In contrast, in this work, we adopt a similar value model to the one present in these papers, but develop simple efficient iterative auctions for a class of valuations that allows for both complementarities and substitutabilities.

Due to the common use of iterative auctions in practice, and their desirable properties such
as privacy preservation and price discovery (Ausubel and Milgrom 2006), a number of papers in the recent literature focused on the design of multi-item iterative auctions. Examples include the package bidding auction (Ausubel and Milgrom 2002), clinching auction and its variants (Ausubel 2004, 2006), auctions that rely on universally competitive equilibria (UCE) (Mishra and Parkes 2007), and best response auction of Nisan et al. (2011). An important component of iterative auction design is the choice of the pricing rule used for running the auction and the termination condition. A class of auctions that are commonly employed (both in practice and in theory), involve anonymous item prices and terminate when a market clearance condition holds. However, this condition can hold if and only if a Walrasian equilibrium exists. Thus, iterative auction formats that rely on the aforementioned pricing rule and termination condition can guarantee efficiency only in settings where a Walrasian equilibrium exists (Ausubel 2006; Sun and Yang 2009; Candogan et al. 2013). Walrasian equilibrium, on the other hand, exists only under restrictive assumptions, which either disallow for complementarities (Gul and Stacchetti 1999, 2000; Ausubel 2006), or allow for only restrictive complementarity structures (Sun and Yang 2006; Candogan et al. 2013). Unlike these works, in this paper we focus on the design of iterative auctions for general graphical valuations, for which a Walrasian equilibrium need not exist. Thus, auction formats that rely on potentially more general pricing rules or termination conditions are necessary to guarantee efficiency.

Earlier literature explored generalizations of the Walrasian equilibrium concept to more general pricing rules, and studied the conditions under which these generalized equilibria (referred to as pricing equilibria or competitive equilibria) exist. In particular, (Bikhchandani and Ostroy 2002) considered settings with potentially multiple bidders and sellers, and characterized the existence of a pricing equilibrium (using a definition similar to ours), for anonymous/bidder-specific pricing rules where the sellers set a price for each item/bundle of items. They established that a pricing equilibrium always exists if (there is a single seller and) the seller offers a bidder-specific price for each bundle of items. A similar result is also established by Mishra and Parkes (2007), who also studied the relation between the VCG payments and pricing equilibria with bidder-specific bundle pricing rule. The existence of these more general pricing equilibria is also used for iterative auction design. However, since the focus has been on settings where valuations do not exhibit any special structure, the pricing equilibria and auction formats proposed by the earlier literature relied on complex bundle pricing schemes (Bikhchandani et al. 2002; De Vries et al. 2007; Ausubel and Milgrom 2002; Ausubel 2006; Mishra and Parkes 2007; Vohra 2011). In this work, on the other hand, our objective is to design iterative auction formats that rely on simple pricing rules, and guarantee efficiency for graphical valuations. In order to obtain our results, we extend the characterization of pricing equilibria to “graphical prices” introduced in this paper, and exploit the structure of this pricing rule for the design of novel iterative auction formats.

In order to guarantee truthful demand revelation in our auctions, we focus on a pricing equilibrium that remains to be a pricing equilibrium if any one of the bidders is removed from the auction. We use this pricing equilibrium to compute the VCG payments for the bidders. A
similar idea is present in (Mishra and Parkes, 2007), where the authors focus on the universal competitive equilibrium concept, which has the aforementioned structural property and allows for the computation of VCG payments. Using this idea, authors design an iterative auction that implements the efficient outcome for general valuations by relying on a complex bundle pricing scheme. Interestingly, for graphical valuations, we show that a much simpler pricing rule (bidder-specific graphical pricing) suffices to guarantee truthfulness, and leads to a novel efficient iterative auction format. Additionally, in our analysis we employ a stronger solution concept (ex-post perfect equilibrium rather than ex-post equilibrium) than the one considered by (Mishra and Parkes, 2007). This implies that in our auctions bidders have no incentive to deviate from the truthful bidding strategy after any realization of the history of the auction game, whereas this may not be the case in the existing multi-item auctions. Another difference between our auction format and the ones present in literature is the price update structure. The iterative auction we develop relies on descending prices, and at each step the auctioneer may decrease the prices of items (or pairs of items) that bidders do not demand. On the other hand, in (Mishra and Parkes, 2007), the price of every demanded bundle of items is increased for some bidder, unless the auction terminates. Note that this suggests updating potentially exponentially many (in the number of items) prices at each stage of the auction, whereas polynomially many price updates take place in our auction. Finally, in our paper we provide an alternative linear programming formulation of the efficient allocation problem whose dual solutions can always be used to compute the VCG payments. Thus, unlike the other formulations in the literature (which rely on convergence to a special dual optimal solution of the efficient allocation problem), this formulation gives us the flexibility of designing truthful iterative auctions (that rely on iterative solutions of the linear programming formulation of the efficient allocation problem) by using any primal-dual algorithm.

2 Model and Preliminaries

In this paper, we focus on settings where an auctioneer sells N (heterogeneous) items to M bidders. We denote the set of items by $\mathcal{N}$ and the set of bidders by $\mathcal{M}$. For each bidder $m \in \mathcal{M}$, the value function $v^m : 2^{\mathcal{N}} \to \mathbb{R}^+$ captures the value $v^m(S)$ this bidder has for any set $S \subset \mathcal{N}$ of items. We make two standard assumptions about the value functions:

**Assumption 2.1.** Bidders have value zero for not receiving any items, i.e., $v^m(\emptyset) = 0$. Additionally, valuations are monotone, i.e., $v^m(S_1) \leq v^m(S_2)$ if $S_1 \subset S_2$.

The value a bidder has for a set $S$ need not be additive over the items in this set, i.e., $v^m(S) \neq \sum_{i \in S} v^m(\{i\})$. If items $i$ and $j$ are such that $v^m(\{i, j\}) \geq v^m(\{i\}) + v^m(\{j\})$, then we say that these items are pairwise complementary. On the other hand, if $i$ and $j$ are such that $v^m(\{i, j\}) \leq v^m(\{i\}) + v^m(\{j\})$, we refer to them as pairwise substitutes. In this work we are mainly interested in pairwise complementarity/substitutability, and unless noted otherwise, we refer to pairwise complementarity/substitutability simply as complementarity/substitutability.
Our focus throughout this paper is on value functions that admit a compact graphical representation:

**Definition 2.1 (Graphical Valuations).** Let $G = (\mathcal{N}, E)$ be a graph such that the set of nodes corresponds to the set of items $\mathcal{N}$. An edge between two nodes represents value complementarity or substitutability and the set of edges is denoted by $E$. We refer to $G$ as a value graph for set of items $\mathcal{N}$. The value function $v : 2^\mathcal{N} \rightarrow \mathbb{R}^+$ is a graphical valuation (with respect to $G$) if:

- There exist nonnegative node weights $w_i \geq 0$ for each $i \in \mathcal{N}$,
- There exist (positive or negative) edge weights $w_{ij}$ for each $(i, j) \in E$,
- $v$ is such that $v(S) = \sum_{i \in S} w_i + \sum_{(i, j) \in E \mid i, j \in S} w_{ij}$.

**Assumption 2.2.** There exists a value graph $G = (\mathcal{N}, E)$ such that the value function of each bidder is a graphical valuation with respect to $G$. That is, for each bidder $m \in \mathcal{M}$, there exist weights $\{w^m_i\}_{m \in \mathcal{M}}$ and $\{w^m_{ij}\}_{(i, j) \in E}$ such that $v^m(S) = \sum_{i \in S} w^m_i + \sum_{(i, j) \in E \mid i, j \in S} w^m_{ij}$.

Graphical valuations are not fully general, i.e., there are value functions that cannot be represented using graphical valuations. On the other hand, they naturally capture pairwise complementarity and substitutability in valuations. In particular, a positive (negative) edge weight $w_{ij}$ associated with an edge between two items $i$ and $j$ captures pairwise complementarity (substitutability) between these items. Due to this structure, this class captures the value complementarity/substitutability in many important combinatorial auction settings, such as spectrum auctions.$^3$

In this paper our objective is to design auctions for graphical valuations that allocate items to bidders efficiently.

**Definition 2.2 (Efficient allocation).** Given bundles of items $S^m \subset \mathcal{N}$ for all $m \in \mathcal{M}$, we say that $\{S^m\}_{m \in \mathcal{M}}$ is a feasible allocation if each item is assigned to at most one bidder, i.e., $S^m \cap S^l = \emptyset$ for $m, l \in \mathcal{M}$ with $m \neq l$. An efficient allocation is a feasible allocation $\{S^m\}_{m \in \mathcal{M}}$ that maximizes the welfare or total value, i.e., $\sum_m v^m(S^m) = \max\{\{Z^m\}_{m \in \mathcal{M}} \mid Z^m \cap Z^l = \emptyset\} \sum_m v^m(Z^m)$.

We say that a feasible allocation is complete if for every item $i$, there exists a bidder $m$ such that $i \in S^m$. Observe that Assumption 2.1 guarantees that there exists an efficient allocation that is also complete. We refer to an auction that terminates with an efficient allocation for any valuations of bidders as an efficient auction.

We are interested in auctions with an iterative structure: the auctioneer sells items to bidders through a dynamic process whereby she posts prices and in response bidders report the bundles of items they demand at these prices. The auctioneer uses this information to update the prices until a final allocation of bundles to bidders is determined. We refer to such auctions as iterative auctions.

$^3$For an extended discussion of the graphical valuation model and its applications, we refer the reader to our companion paper Candogan et al. (2013).
An important component of iterative auction design is the choice of the pricing rule (i.e., the rule with which prices of bundles of items are determined) and the termination condition. A simple pricing rule that is commonly used for iterative auction design is the anonymous item pricing rule (e.g., Gul and Stacchetti (1999, 2000); Ausubel (2006)). This pricing rule involves offering a price $p_i$ for each item $i \in N$ to all bidders, and it compactly captures the price of every bundle as a summation of prices of items contained in it. Auctions that rely on anonymous item prices can be terminated when bidders demand disjoint bundles of items. Observe that this is a natural termination point for the auction, since bidders do not compete with each other for the items that they demand. This outcome coincides with the classical Walrasian equilibrium concept from microeconomic theory.

In recent work, Candogan et al. (2013) considered graphical valuations and showed that a Walrasian equilibrium exists if the underlying graph has a tree structure, and weights satisfy a “sign-consistency” condition (which requires the weights bidders associate with different edges to have the same sign). On the other hand, if the tree or the sign-consistency assumption is relaxed, then a Walrasian equilibrium need not exist. This observation suggests that in order to design iterative auctions for more general graphical valuations, it may be necessary to consider other simple pricing rules and the corresponding generalizations of the Walrasian equilibrium concept. In particular, in this paper we will focus on pricing rules that extend anonymous item pricing by allowing for bidder-specific discounts/markups for pairs of items, and use them to design simple iterative auction formats that guarantee efficiency for general graphical valuations.

For such pricing rules, we will use the notation $p^m(S)$ to denote the price offered to bidder $m$ for bundle $S$ (note that for anonymous item pricing $p^m(S) = \sum_{i \in S} p_i$). We refer to the quantity $v^m(S) - p^m(S)$ as bidder $m$’s surplus for bundle $S$. We say that a bundle $S^*$ is demanded by bidder $m$ if the maximum surplus is achieved for this bundle, i.e., $v^m(S^*) - p^m(S^*) = \max_S v^m(S) - p^m(S)$. We denote the set of bundles bidder $m$ demands by $D^m$, i.e., $D^m = \arg\max_S v^m(S) - p^m(S)$.

A generalization of the Walrasian equilibrium concept to general pricing rules and settings with multiple sellers was provided in Bikhchandani and Ostroy (2002). We adapt this definition below to the single seller setting considered in this paper and employ it subsequently for the design of our iterative auctions.

**Definition 2.3 (Pricing equilibrium).** The tuple $\{p^m(S)\}_{m,S},\{S^m\}_m$ is a pricing equilibrium if

\begin{enumerate}
  \item $\{S^m\}_m$ is a feasible allocation,
  \item For each $m$, $S^m$ is demanded by bidder $m$: $v^m(S^m) - p^m(S^m) \geq v^m(S) - p^m(S)$ for any $S \subset N$, and $m \in M$,
  \item At the given prices, the auctioneer maximizes revenue using allocation $\{S^m\}$: $\sum_m p^m(S^m) \geq \sum_m p^m(Z^m)$ for any feasible allocation $\{Z^m\}_m$.
\end{enumerate}

A Walrasian equilibrium is a pricing equilibrium with anonymous item prices $\{p_i \geq 0\}$. 
It can be seen from Definition 2.3 that each bidder \( m \) receives a demanded bundle and the auctioneer maximizes revenue at the given prices. Hence, a pricing equilibrium constitutes a natural termination point for auctions that rely on general pricing rules. Moreover, the allocation \( \{S^m\}_m \) is efficient. In this paper, we focus on pricing equilibria, where the prices admit a simple structure (such as the graphical pricing structure discussed in the next section), and develop iterative auctions that implement the efficient outcome by terminating at such pricing equilibria.

Bikhchandani and Ostrov (2002) show that in the single seller setting, a pricing equilibrium with a general nonnegative bundle pricing rule \( \{p^m(S)\}_m,S \) exists. Moreover, the equilibrium allocation and prices can be obtained as the optimal solutions of the primal/dual efficient allocation problems, which are the LP formulations given below.

\[
\begin{align*}
\text{(LP1)} & \quad \max & \sum_m x^m(S^m)v^m(S^m) \\
& \text{s.t.} & \sum_S x^m(S) \leq 1 \quad \forall m \\
& & \sum_{\mu} \delta(\mu) \leq 1 \\
& & x^m(S) \leq \sum_{\mu: \mu m = S} \delta(\mu) \quad \forall m, S \\
& & x^m(S), \delta(\mu) \geq 0 \quad \forall m, S, \mu \\
\end{align*}
\]

\[
\begin{align*}
\text{(D1)} & \quad \min & \sum_m \pi^m + \pi^s \\
& \text{s.t.} & \pi^m \geq v^m(S) - p^m(S) \quad \forall m \\
& & \pi^s \geq \sum_m p^m(\mu m) \quad \forall \mu \\
& & \pi^m, \pi^s, p^m(S) \geq 0 \quad \forall m, S \\
\end{align*}
\]

In LP1, \( \mu \) denotes a feasible allocation, and \( \mu m \) denotes the bundle this allocation associates with bidder \( m \). In the integral feasible solutions of this optimization formulation, having \( x^m(S) = 1 \) is interpreted as bidder \( m \) requesting bundle \( S \), and \( \delta(\mu) = 1 \) is interpreted as the auctioneer choosing allocation \( \mu \). The first and second constraints respectively guarantee that each bidder requests at most one bundle, and the auctioneer chooses at most one allocation. The third constraint suggests that the bundles requested by the bidders are consistent with the allocations

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3To see this note that for any feasible allocation \( \{Z^m\}_m \neq \{S^m\}_m \), we have \( \sum_m v^m(S^m) - p^m(S^m) \geq \sum_m v^m(Z^m) - p^m(Z^m) \) (Definition 2.3 (ii)), and \( \sum_m p^m(S^m) \geq \sum_m p^m(Z^m) \) (Definition 2.3 (iii)). These two inequalities suggest that \( \sum_m v^m(S^m) \geq \sum_m v^m(Z^m) \). Since, this is true for any feasible allocation, we conclude that \( \{S^m\}_m \) is efficient.

4Observe that Definition 2.3 suggests that pricing equilibria are invariant to constant changes in the prices, i.e., if at a pricing equilibrium all prices of \( m \) are increased by a constant, the new prices also support a pricing equilibrium. Bikhchandani and Ostrov (2002) focus on nonnegative prices \( p^m(S) \geq 0 \), and show that pricing equilibria with such prices can be obtained through LP1/D1. Note that the nonnegativity assumption has no impact on the optimal objectives of LP1/D1. To see this observe that at optimality (due to the nonnegativity of valuations), the third constraint of LP1 always holds with equality. Hence, replacing this constraint with equality (or dropping the nonnegativity constraint on prices) has no impact on the optimal objective value, and general pricing equilibria can be obtained through such a formulation. Thus, in this paper, we do not impose nonnegativity on prices of all bundles. Additionally, even in the absence of nonnegativity constraints the prices of bundles that constitute a pricing equilibrium are nonnegative (provided that \( p^m(\emptyset) = 0 \)) since otherwise Definition 2.3 (iii) suggests that such a bundle would not be assigned to a bidder at a pricing equilibrium.

5The efficient allocation formulation given in Bikhchandani and Ostrov (2002) allows for multiple sellers, and can be used for checking the existence of a pricing equilibrium for this more general setting. This formulation reduces to LP1 for a single seller.
chosen by the auctioneer. Hence, the presence of this constraint (and the $\delta(\mu)$) variables guarantee that bidders’ bundle assignments are consistent with a feasible allocation. At integral solutions where the first constraints hold with equality for all $m$, the objective function is the total value generated by an assignment of items according to the allocation that corresponds to $\{x^m(S)\}_m$, i.e., the allocation $\{S^m\}$ with $S^m$ such that $x^m(S^m) = 1$.

We denote the multipliers corresponding to the first three constraints of LP1 respectively by $\pi^m$, $\pi^s$, $p^m(S)$, and using these variables we provide the corresponding dual problem in D1. In D1, $p^m(S)$ can be interpreted as the price offered to bidder $m$ for bundle $S$. The constraints in the dual problem suggest that for any bundle $S$, $\pi^m$ is an upper bound on $v^m(S) - p^m(S)$, i.e., the surplus bidder $m$ associates with acquiring bundle $S$ at the given prices. Moreover, at optimality, it can be seen that $\pi^m$ is equal to the maximum surplus that can be associated with some bundle. This suggests that $\pi^m$ can be interpreted as the surplus of bidder $m$. Similarly, for any allocation $\mu$, $\sum_m p^m(\mu^m)$ captures the associated revenue of the seller, and $\pi^s$ is an upper bound on this quantity. At optimality, $\pi^s$ is equal to the maximum revenue of the seller.

Bikhchandani and Ostroy (2002) show that LP1 has optimal solutions that are integral and such solutions correspond to efficient allocations. Additionally, they show that the prices at optimal dual solutions $\{p^m(S)\}$ together with such allocations constitute a pricing equilibrium. On the other hand, Bikhchandani and Ostroy (2002) establish the existence of pricing equilibria for general valuations by focusing on the bidder-specific bundle pricing rule $\{p^m(S)\}_{m,S}$. In this paper, we restrict attention to graphical valuations and establish the existence of pricing equilibria which rely on a simpler pricing rule that involves pairwise discounts/markups for pairs of items.

### 3 Iterative Auctions for General Graphical Valuations

In this section, we first focus on graphical valuations and investigate the existence of pricing equilibria that rely on simple pricing rules. We then provide a linear programming formulation of the efficient allocation problem, and associate its primal/dual optimal solutions with such pricing equilibria (in a similar fashion to LP1/D1 of Section 2). After establishing the existence of pricing equilibria, we focus on the design of iterative auctions that rely on bidder-specific graphical pricing and terminate at a pricing equilibrium with this pricing rule.

#### 3.1 Existence of Pricing Equilibrium

In this section, we focus on pricing equilibria with the following simple pricing rule:

**Definition 3.1 (Bidder-specific Graphical Pricing).** Bidder-specific graphical pricing involves pricing parameters $\{p^m_i, p^m_{ij}\}_{m \in M, i \in N, (i,j) \in E}$ where $p^m_i$ represents the price the auctioneer offers for item $i \in N$ to bidder $m \in M$, and $p^m_{ij}$ represents the discount/markup the auctioneer offers for a pair of items $i, j$ that are connected by an edge.
Under this pricing rule, the total price of bundle $S \subset N$ for bidder $m \in M$ is given by $p^m(S) = \sum_{i \in S} p_i^m + \sum_{i,j \in S | ij \in E} p_{ij}^m$. We refer to a pricing equilibrium, where the prices $\{p^m(S)\}$ admit such an expression in terms of $\{p_i^m, p_{ij}^m\}_{m \in M, i \in N, (i,j) \in E}$, as pricing equilibrium with bidder-specific graphical pricing.

Observe that bidder-specific graphical pricing generalizes anonymous item pricing in two dimensions. First, the bidders may receive discounts/markups for acquiring larger bundles of items, due to the presence of price terms associated with the edges of the underlying graph. Second, different bidders may be offered different prices for the same bundle, i.e., the prices are bidder-specific.

We next show that a pricing equilibrium with bidder-specific graphical pricing rule exists for graphical valuations. In order to establish this result, we start by providing an LP formulation of the efficient allocation problem (in a similar fashion to LP1/D1 of Section 2) and show that this problem has integral optimal solutions if and only if a pricing equilibrium exists. We then show that integral optimal solutions exist for graphical valuations.

In the dual formulation D1, the variables $p^m(S)$ represent the price offered to bidder $m$ for bundle $S$ of items. For bidder-specific graphical prices, this variable can be expressed as $p^m(S) = \sum_{i \in S} p_i^m + \sum_{i,j \in S | ij \in E} p_{ij}^m$. Substituting this expression for $p^m(S)$ and dropping the nonnegativity constraint on prices (which as explained in Section 2 has no impact on the optimal objective values of the LP1/D1) we obtain the following dual formulation of the efficient allocation problem:

\[
\begin{align*}
& \min \pi^s + \sum_m \pi^m \\
& \text{s.t. } \pi^m \geq v^m(S) - \sum_{i \in S} p_i^m - \sum_{i,j \in S | ij \in E} p_{ij}^m \quad \forall S,m \\
& \pi^s \geq \sum_m \left( \sum_{i \in \mu^m} p_i^m + \sum_{i,j \in \mu^m | ij \in E} p_{ij}^m \right) \quad \forall \mu \\
& \pi^m, \pi^s \geq 0.
\end{align*}
\]

In this formulation, we interpret $\mu = \{\mu^m\}_m$ as a feasible allocation of items to bidders, and denote by $\mu^m$ the bundle assigned to bidder $m$ in allocation $\mu$. Consistently with the bidder-specific graphical pricing structure, we interpret variables $p_i^m$ and $p_{ij}^m$ respectively as the price term for each node (item) $i$, and edge (pair of items) $ij \in E$. The variable $\pi^m$ is an upper bound on the surplus $(v^m(S) - \sum_{i \in S} p_i^m - \sum_{i,j \in S | ij \in E} p_{ij}^m)$ a bidder can raise by acquiring the bundle $S$ of items at the given prices. The variable $\pi^s$ is an upper bound on the quantity $\sum_m \left( \sum_{i \in \mu^m} p_i^m + \sum_{i,j \in \mu^m | ij \in E} p_{ij}^m \right)$ for all allocations $\mu$. Here the quantity $\left( \sum_{i \in \mu^m} p_i^m + \sum_{i,j \in \mu^m | ij \in E} p_{ij}^m \right)$ corresponds to the total price of bundle $\mu^m$, and the summation of this quantity over all bidders captures the total revenue the auctioneer can raise by assigning items according to allocation $\mu = \{\mu^m\}$. Hence, $\pi^s$ can be interpreted as the revenue of the auctioneer.
The corresponding primal LP is given by:

\[
\begin{align*}
\text{max} & \quad \sum_{m} \sum_{S} x^m(S)v^m(S) \\
\text{s.t.} & \quad \sum_{S} x^m(S) \leq 1 \quad \forall m \in \mathcal{M} \\
& \quad \sum_{\mu} \delta(\mu) \leq 1 \\
& \quad \sum_{S|i \in S} x^m(S) = \sum_{\mu|i \in \mu^m} \delta(\mu) \quad \forall i \in \mathcal{N}, m \in \mathcal{M} \\
& \quad \sum_{S|ij \in S} x^m(S) = \sum_{\mu|ij \in \mu^m} \delta(\mu) \quad \forall ij \in \mathcal{E}, m \in \mathcal{M} \\
& \quad \delta(\mu) \geq 0, x^m(S) \geq 0.
\end{align*}
\]

(LP2)

Similar to LP1 of Section 2 in LP2 the variables \(\{x^m(S)\}\) and \(\{\delta(\mu)\}\) respectively represent the bundles requested by bidder \(m\), and the allocations chosen by the auctioneer, i.e., if \(x^m(S^m) = 1\), then bidder \(m\) requests bundle \(S^m\), and if \(\delta(\mu) = 1\), then the auctioneer chooses an allocation that assigns each bidder \(m\) a bundle \(\mu^m\). The first two constraints are identical to those in LP1. The third constraint suggests that if an item \(i\) is requested by bidder \(m\), then the allocation chosen by the auctioneer should assign it to her. Similarly, the fourth constraint guarantees that if two items are jointly requested by a bidder, then the auctioneer should choose an allocation that jointly assigns these to the same bidder. These constraints are in contrast to LP1 which imposes similar constraints for all bundles \(S\).

The objective function of this LP formulation is exactly the same as that of LP1, i.e., maximizing total value. We next establish that this LP has integral optimal solutions if and only if a pricing equilibrium with bidder-specific graphical prices exist, and this always is the case for graphical valuations.

**Theorem 3.1.**

(i) Let \(\hat{\mu}\) be a feasible allocation. Then \(x^m(S) = 0\) if \(S \neq \hat{\mu}^m\), \(x^m(\hat{\mu}^m) = 1\) and \(\delta(\mu) = 0\) if \(\mu \neq \hat{\mu}\), \(\delta(\hat{\mu}) = 1\) is a feasible solution of LP2. Moreover, the corresponding objective value of LP2 is the total value associated with this allocation.

(ii) LP2 has an optimal solution that is integral if and only if a pricing equilibrium with bidder-specific graphical pricing exists. Moreover, if a pricing equilibrium exists, then the prices at a dual optimal solution of D2, and the allocation suggested by the integral optimal solution of LP2 (i.e., \(\{S^m\}_m\), where \(S^m\) is such that \(x^m(S^m) = 1\) at the optimal solution of LP2) constitute a pricing equilibrium.

(iii) Assume that bidders have graphical valuations. Then, LP2 has an optimal solution that is integral, and a pricing equilibrium with bidder-specific graphical pricing exists.

Theorem 3.1 implies that a pricing equilibrium with bidder-specific graphical prices always exists for graphical valuations. Interestingly, the prices in D2 have the same structure as the
graphical valuations themselves, i.e., prices additively decompose over the nodes/edges of the underlying value graph. As we discuss in Section 4, this observation is more general, and it is always the case that the efficient outcome can be found by solving a linear program, whose dual uses a pricing rule with the same “structure” as the valuations.

The proof of this theorem follows from feasibility and complementary slackness for D2 and LP2. We provide the complementary slackness (CS) conditions here, and relegate the formal proof to the appendix:

\[(i) \quad \pi^m > v^m(S) - \sum_{i \in S} p^m_i - \sum_{i, j \in S | i \neq j} p^m_{ij} \rightarrow x^m(S) = 0 \]

\[(CS) \quad (ii) \quad \pi^s > \sum m \left( \sum_{i \in \mu^m} p^m_i + \sum_{i, j \in \mu^m | i \neq j} p^m_{ij} \right) \rightarrow \delta(\mu) = 0 \]

\[(iii) \quad \pi^m > 0 \rightarrow \sum S x^m(S) = 1 \]

\[(iv) \quad \pi^s > 0 \rightarrow \sum_{\mu} \delta(\mu) = 1 \]

Condition (i) (respectively (ii)) implies that unless a bundle (allocation) has maximum surplus for the bidder (revenue for the seller) it does not appear in the optimal solution (i.e., \(x^m(S) = \delta(\mu) = 0\)). Conditions (iii) and (iv) guarantee that if there is a bundle/allocation with nonnegative surplus/revenue then such a bundle/allocation appears at the optimal solution. These complementary slackness (CS) conditions for LP2/D2 will be used in the subsequent sections for characterizing the optimal solutions of LP2/D2 and the structure of the pricing equilibria.

### 3.2 Ex-post Perfect Equilibrium

We next introduce the solution concept (ex-post perfect equilibrium) that we use for analyzing the outcome of iterative auctions and provide a condition that can be used for its characterization.

In an iterative auction, bidders participate in a multi-stage game. In this game we denote the (behavior) strategy of bidder \(m\) whose valuation is \(v^m\) by \(s^m(v^m)\). This strategy maps every history \(H_t\) (i.e., the bids revealed until step \(t\) of the auction) to an action. With a slight abuse of notation, we denote by \(s^m(H_t|v^m) \in \Sigma^m(H_t)\) the action bidder \(m\), whose valuation is \(v^m\), uses at time \(t\), after observing history \(H_t\). Here, \(\Sigma^m(H_t)\) denotes the set of actions bidder \(m\) can use after history \(H_t\).

For given valuations of bidders \(\{v^k\}_k\), the payoff bidder \(m\) receives at the end of the auction game is denoted by \(u^m(s^m(v^m), s^{-m}(v^{-m})|v^m)\), where \(s^{-m}(v^{-m})\) denotes the strategies of all bidders but \(m\). Similarly, we denote by \(u^m(s^m(v^m), s^{-m}(v^{-m})|H_t, v^m)\), the payoff bidder \(m\), who is of type \(v^m\), receives by using strategy \(s^m(v^m)\), after history \(H_t\), given that her opponents use strategies \(s^{-m}(v^{-m})\). Using this notation we next introduce the solution concept we focus on:
\textbf{Definition 3.2 (Ex-post perfect equilibrium).} A strategy profile \( s = \{s^k\} \) is an ex-post perfect equilibrium if after any history \( H_t \), it satisfies
\[
    u^m(s^m(v^m), s^{-m}(v^{-m})|H_t, v^m) \geq u^m(z^m, s^{-m}(v^{-m})|H_t, v^m),
\]
for any valuations \( \{v^k\} \) of bidders, bidder \( m \), and strategy \( z^m \).

This definition suggests that a strategy profile is an ex-post perfect equilibrium if after any history, for any valuations of bidders, given strategies of her opponents, no agent has incentive to deviate from her strategy. In other words, after any realization of the history and payoffs, the given strategy profile remains to be a Nash equilibrium of the induced subgame, where types of agents are public knowledge.

We next provide a sufficient condition for a strategy profile to be an ex-post perfect equilibrium in an iterative auction game. Before we state our result, we introduce one more definition:

\textbf{Definition 3.3 (VCG Mechanism).} Consider a collection of value functions \( \{v^m\} \). A mechanism (mapping from types/valuations to allocations and payments) is called a VCG (Vickrey - Clarke - Groves) mechanism if it

- chooses an efficient allocation, i.e., \( \{S^m\} \in \arg \max_{z^m \subseteq N, Z^k \cap Z^l = \emptyset} \sum_m v^m(Z^m) \)
- assigns each bidder \( m \) a payment \( \gamma^m(\{S^k\}, \{v^k\}_{k \neq m}) = h^m(v^{-m}) - \sum_{k \neq m} v^k(S^k) \), where \( h^m \) is any real-valued function.

If \( h^m \) is such that
\[
    h^m(v^{-m}) = \max_{z^k : Z^k \cap Z^l = \emptyset} \sum_{k \neq m} v^k(Z^k),
\]
then, we say that payments of bidders \( \gamma^m(\{S^k\}, \{v^k\}_{k \neq m}) = \max_{z^k : Z^k \cap Z^l = \emptyset} \sum_{k \neq m} v^k(Z^k) - \sum_{k \neq m} v^k(S^k) \) are VCG payments with the Clarke pivot rule. Intuitively, for a given agent, the VCG payment with the Clarke pivot rule captures the opportunity cost she creates on the rest of the system, i.e., the difference between the maximum total value that can be achieved by the remaining agents and the total value those agents have at the efficient allocation. In this paper, we will only employ VCG payments with the Clarke pivot rule. For simplicity, we refer to these payments as VCG payments\(^0\).

Consider valuations \( \{v^k\} \), where bidder \( m \) receives a set of items \( S^m \) in the efficient allocation. Observe that efficiency requires allocating the remaining items to bidders \( k \neq m \) according to
\[
    \arg \max_{(z^k)_{k \neq m} : Z^k \cap Z^l = \emptyset} \sum_{k \neq m} v^k(Z^k).
\]
This suggests that for any \( \{v^k\} \) where bidder \( m \) receives a set of items \( S^m \) in the efficient allocation, her VCG payment can alternatively be expressed as
\[
    \gamma^m(\{S^k\}, \{v^k\}_{k \neq m}) = \tilde{\gamma}^m(S^m, \{v^k\}_{k \neq m}),
\]
\(^0\)In sealed bid auctions, charging bidders VCG payments guarantees that the efficient outcome can be implemented in a dominant strategy (and hence ex-post) equilibrium \cite{Nisan2007, Krishna2009}.
where
\[
\hat{\gamma}^m(S^m, \{v^k\}_{k \neq m}) \triangleq \max_{\{Z^k\} | Z^k \cap Z^l = \emptyset, Z^k \cap S^m = \emptyset} \sum_{k \neq m} v^k(Z^k) - \max_{\{Z^k\} | Z^k \cap Z^l = \emptyset} \sum_{k \neq m} v^k(Z^k) .
\] (2)

Thus, under the Clarke pivot rule the VCG payment of an agent can be expressed only as a function of her opponents’ valuations, and the set of items she acquires in the auction. We make this dependence explicit by using the function \(\hat{\gamma}^m\).

The main result of this subsection makes use of the VCG payments to obtain a sufficient condition for a strategy profile to be an ex-post perfect equilibrium in an iterative auction.

**Theorem 3.2.** Consider an auction where bidders receive zero payoffs if the auction does not terminate. Let \(\{v^k\}\) denote the valuations of bidders and \(\{s^k\}\) be the truthful bidding strategy. Assume that \(\{s^k\}\) satisfies the following conditions:

- After any history \(H_t\), if all bidders \(m\) use strategies \(\{s^m(v^m)\}\), then the auction terminates with the efficient allocation.
- After any history \(H_t\), if bidders \(k \neq m\) use strategies \(\{s^k(v^k)\}_{k \neq m}\), and \(m\) uses a strategy \(s^m(v^m)\) (possibly identical to \(s^m(v^m)\)), such that the auction terminates with bidder \(m\) receiving a set of items \(S^m\), then the corresponding payment of bidder \(m\) is equal to \(\hat{\gamma}^m(S^m, \{v^k\}_{k \neq m})\).

Then, the strategy profile \(\{s^m\}\) is an ex-post perfect equilibrium.

This theorem suggests that if a strategy profile leads to the efficient outcome, and when the opponents of bidder \(m\) follow their strategies the payment of \(m\) (i) is only a function of her opponents’ true valuations and the final bundle of items she receives, and (ii) corresponds to VCG payments specified by this bundle and the opponents’ valuations, then it is an ex-post perfect equilibrium. Intuitively, this structure of payments ensures that conditional on the final allocation of items, the actions a bidder chooses in the course of the auction do not impact her final payoffs. Thus, it is possible to obtain implementation in ex-post perfect equilibrium by ensuring that the final payments correspond to VCG payments, and disregarding the actions bidders use in the course of the auction. In the subsequent sections, we propose iterative auction formats where the conditions of the above theorem are satisfied, and the efficient outcome is obtained at an ex-post perfect equilibrium.

### 3.3 An Iterative Auction Format

In this section, we focus on iterative solutions of LP2 and D2 of Section 3.1 and use these to develop an iterative auction. Before we explain our iterative auctions in detail, we introduce some necessary definitions, and present important properties of the optimal solutions of D2. In particular, we identify particular optimal solutions of D2 that can be used to compute the VCG payments. Then, we propose an iterative auction that updates the prices to ensure convergence
to such optimal solutions. Finally, we establish that this auction can be used to implement the efficient allocation.

### 3.3.1 Special optimal solution

We define a restriction of a feasible solution \( \{(p^m_i, p^m_{ij}) \}_{m \in \mathcal{M}}, \{\pi^m_i\}_{m \in \mathcal{M}}, \pi_s \) of D2 to a set of bidders \( \mathcal{M}_0 \subset \mathcal{M} \) as the tuple \( \{(p^m_i, p^m_{ij}) \}_{m \in \mathcal{M}_0}, \{\pi^m_i\}_{m \in \mathcal{M}_0}, \pi_s \). Similarly, we refer to the tuple \( \{p^m_i, p^m_{ij}, \pi^m\}_{m \in \mathcal{M}_0} \) as the restriction of prices and bidder surpluses to a set of bidders \( \mathcal{M}_0 \subset \mathcal{M} \). Additionally, we say that D2 is formulated with subset \( \mathcal{M}_0 \) of bidders, if (i) it involves \( p^m_i, p^m_{ij}, \pi^m \) variables only for \( m \in \mathcal{M}_0 \), (ii) the first constraint is imposed only for \( m \in \mathcal{M}_0 \), and (iii) the second constraint is present only for allocations \( \mu \), where bidders \( m \notin \mathcal{M}_0 \) do not receive any items, i.e., \( \mu^m = \emptyset \).

Using these definitions, we next show that there exists an optimal solution of D2, whose restriction to any subset \( \mathcal{M}_0 \) of bidders gives prices and bidder surpluses that appear at an optimal solution of a formulation of D2 with subset of bidders \( \mathcal{M}_0 \).

**Lemma 3.1.** There exists an optimal solution of D2, such that for any \( \mathcal{M}_0 \subset \mathcal{M} \), the restriction of prices and bidder surpluses of this optimal solution to \( \mathcal{M}_0 \) agrees with prices and bidder surpluses at an optimal solution of a formulation of D2 with set of bidders \( \mathcal{M}_0 \).

The proof of Lemma 3.1 is given in the appendix. Intuitively, the solutions of D2 correspond to market clearing prices (or pricing equilibria), and when valuation are graphical, and prices have a similar structure (i.e., when they are bidder-specific and graphical), it is possible to choose prices that clear the market for different subsets of bidders jointly. One such optimal solution of D2 (but not the only one) can be obtained by setting the prices equal to valuations (for all items/edges and for every bidder). This optimal solution remains optimal when D2 is formulated with fewer bidders.

Consider an optimal solution of D2, such that for all \( m \in \mathcal{M} \), the prices and bidder surpluses of this solution agree with the prices and bidder surpluses at an optimal solution of D2, formulated with bidders \( \mathcal{M} - \{m\} \). We refer to such a solution of D2 as a special optimal solution (which exists by Lemma 3.1). A similar concept (universal competitive equilibrium, or UCE) was previously employed in Mishra and Parkes (2007, 2009). In particular, UCE prices correspond to “competitive equilibrium” prices for sets of bidders \( \mathcal{M} \) and \( \mathcal{M} - \{k\} \) (for every \( k \)). On the other hand, UCE prices associate a bidder-specific price with every bundle of items, and hence potentially consist of \( M2^N \) distinct parameters. In contrast, the special optimal solutions we focus on here, associate a bidder-specific price with each node and edge of the underlying graph (hence consist of \( O(MN^2) \) parameters). In Mishra and Parkes (2007), it was established that UCE prices can be used to obtain VCG payments for general settings. We next establish that when valuations are graphical, bidder-specific graphical pricing (and special optimal solutions of D2) suffices for the computation of VCG payments.
3.3.2 Computing VCG payments

Let \( \{p_i^m\}, \{p_{ij}^m\}, \{\pi^m\}, \pi^s \) be a special optimal solution of D2. By Theorem 3.1 (ii) and (iii) it follows that LP2 has an optimal solution that is integral, and this solution can be used to construct an efficient allocation of items to bidders (recall that pricing equilibria are efficient). We denote this efficient allocation by \( \{S_0^m\}_{m \in \mathcal{M}} \), and note that the corresponding optimal solution of LP2 is such that \( x^m(S_0^m) = 1 \) for all \( m \). Similarly, when D2 is formulated with a set of bidders \( \mathcal{M} - \{k\} \), the corresponding formulation of LP2 has an optimal solution that is integral and that identifies an efficient allocation for these bidders. For every \( k \in \mathcal{M} \), denote this allocation by \( \{S_k^m\}_{m \in \mathcal{M} - \{k\}} \), and note that the associated optimal solution of LP2 has \( x^m(S_k^m) = 1 \) for all \( m \neq k \). Since \( \{p_i^m\}, \{p_{ij}^m\}, \{\pi^m\}, \pi^s \) is a special optimal solution, using complementary slackness (CS) in LP2 and D2 (formulated with sets of bidders \( \mathcal{M} \) and \( \mathcal{M} - \{k\} \)), it follows that

\[
\pi^m = v^m(S_0^m) - p^m(S_0^m) = v^m(S_k^m) - p^m(S_k^m),
\]

where we use the shorthand notation \( p^m(S) = \sum_{i \in S} p_i^m + \sum_{ij \in E} p_{ij}^m \). This suggests that at a special optimal solution, we have

\[
v^m(S_k^m) - v^m(S_0^m) = p^m(S_k^m) - p^m(S_0^m).
\]  \( (3) \)

Since by definition \( \{S_0^m\} \) and \( \{S_k^m\} \) are the efficient allocations for sets of bidders \( \mathcal{M} \) and \( \mathcal{M} - \{k\} \) respectively, it follows that VCG payment (see Definition 3.3) for bidder \( k \) is given by \( \sum_{m \neq k} v^m(S_k^m) - v^m(S_0^m) = \sum_{m \neq k} p^m(S_k^m) - p^m(S_0^m) \). This expression suggests that if a special optimal solution of D2 can be found, then this solution can be used to compute the VCG payments.

3.3.3 An iterative auction

We next propose an iterative auction format that implements the efficient allocation for all graphical valuations. The prices in this auction are updated so that when bidders reveal their demand truthfully, they converge to prices in a special optimal solution of D2. This allows for charging bidders final payments that are equal to VCG payments, and guarantees that truthful demand revelation is an ex-post perfect equilibrium (following Theorem 3.2).

We make two assumptions before we state our auction format: (i) the node and edge weights are integer-valued, (ii) there exists some integer \( \bar{w} \) such that \( \bar{w} \geq w^m_{ij}, w_i^m \) for all \( m, i \), and \( (i, j) \in E \), i.e., an upper bound on node/edge weights is known. We will use these assumptions in our auction to obtain a price update rule that is descending and has integral price updates.

Our auction is presented in Table 2. When stating the auction, we use the notation \( D^m = \arg \max_S v^m(S) - p^m(S) \) to denote the set of bundles bidder \( m \) demands (where as before \( p^m(S) \) is a short hand notation for \( \sum_{i \in S} p_i^m + \sum_{ij \in E} p_{ij}^m \), and \( \Psi^k \) to denote the set of items that are demanded in isolation (i.e., demanded bundles are singletons) at some (prior) stage of the auction.
Table 2: An iterative auction for general graphical valuations.

S1 (Initialize): Set \( p^m_i = p^m_{ij} = \bar{w} \) for all \( m, i \) and \( (i, j) \in E \). Set \( \Psi^m = \emptyset \) for all \( m \).

S2 (Termination Cond.?): Ask every bidder \( m \) the sets she demands at the current prices, i.e., \( D^m = \arg \max_S v^m(S) - p^m(S) \).

If there exists an allocation \( \{S^m_0\} \) such that
- \( S^m_0 \in D^m \), for \( m \in M \)
- \( \sum_m p^m(S^m_0) \geq \sum_m p^m(S^m) \) for any other allocation \( \{S^m\} \)

and allocations \( \{S^m_k\}_{m \neq k} \) for every \( k \in M \) such that
- \( S^m_k \in D^m \), for \( m \neq k \)
- \( \sum_{m \neq k} p^m(S^m_k) \geq \sum_{m \neq k} p^m(S^m) \) for any other allocation \( \{S^m\} \)

then go to step S5. Otherwise go to step S3.

S3 (Decrease Prices): For every bidder \( m \), and item \( i \), if \( \{i\} \in D^m \), then do not update \( p^m_i \), and set \( \Psi^m := \Psi^m \cup \{i\} \). Otherwise decrement \( p^m_i \) by one.

For every bidder \( m \), and edge \( (i, j) \in E \), if \( i, j \in \Psi^m \), and \( \{i, j\} \notin D^m \), then decrement \( p^m_{ij} \) by one. Otherwise do not update \( p^m_{ij} \).

S4 (Reset?): For every bidder \( m \), if (i) \( p^m_i < 0 \) for some \( i \in N \), or (ii) \( S \in D^m \) and there exists some \( i \in S \) such that \( \{i\} \notin D^m \), or \( i, j \in S \) such that \( (i, j) \in E \) and \( \{i, j\} \notin D^m \), then go to Step S1.

S5 (Terminate): Terminate, by allocating items according to \( \{S^m_0\} \), and assigning a final payment of \( \sum_{m \neq k} p^m(S^m_k) - p^m(S^m_0) \) to each bidder \( k \).

At each stage of the auction, the auctioneer offers bidder-specific graphical prices, and bidders report their demand at the given prices. The auction is initialized with high prices, at which bidders do not demand any items. In the auction, first the node prices are decreased. The edge prices are decreased only after the nodes associated with this edge are demanded (in isolation). Consider the prices \( \{p^m_i, p^m_{ij}\} \) obtained at a given stage of the auction in Table 2. Implicitly defining \( \{\pi^m, \pi^s\} \) variables as

\[
\pi^m = \max_S v^m(S) - \sum_{i \in S} p^m_i - \sum_{i, j \in S \mid ij \in E} p^m_{ij}, \quad \text{and} \quad \pi^s = \max_{\mu} \sum_m \sum_{i \in \mu} p^m_i + \sum_{i, j \in \mu \mid ij \in E} p^m_{ij}, \quad (4)
\]

a dual solution \( \{(p^m_i), (p^m_{ij}), (\pi^m), \pi^s\} \) of D2 can be obtained. It can be checked that the construction of \( \pi^m \) and \( \pi^s \) variables guarantee that this solution is feasible in D2. Thus, prices that emerge at each stage of the auction correspond to a dual feasible solution of D2. The termination condition (in Step S2) of the auction is closely related to complementary slackness (CS) conditions in LP2/D2 (as explained below) and corresponds to checking if this feasible solution constitutes
a special optimal solution. The auction terminates once a special optimal solution is reached, and at termination the final allocation and payoffs are respectively the efficient allocation and the VCG payments suggested by this solution. We next explain Steps S2–S5 of the auction in more detail.

**Step S2:** This step checks if the current prices/demand sets satisfy two sets of termination conditions. The first set of termination conditions in Step S2 (those involving $S^m_0$) are equivalent to checking if there exists a primal feasible solution in LP2 satisfying complementary slackness (CS) conditions with the given dual feasible solution of D2. More precisely, assume that bidders reveal their demand truthfully and this termination condition holds. Consider the following primal feasible solution associated with allocation $\{S^m_0\}$: $x^m(S^m_0) = 1$ for all $m$, $\delta(\{S^m_0\}) = 1$, and $x^m(S) = \delta(\mu) = 0$ for remaining $\mu$ and $(m, S)$. The construction of $\pi^m$, $\pi^s$ variables (as specified in (4)), together with conditions in Step S2 imply that $\pi^m = v^m(S^m_0) - \sum_{i \in S^m_0} p^m_i - \sum_{ij \in S^m_0} p^m_{ij} \geq v^m(S^m) - \sum_{i \in S^m} p^m_i - \sum_{ij \in S^m} p^m_{ij}$ (recall that $S^m_0 \in D^m_0$), and $\pi^s = \sum_m p^m(S^m_0) \geq \sum_m p^m(S^m)$ for any allocation $\{S^m\}$. Thus, these primal and dual feasible solutions satisfy complementary slackness, and hence are optimal in LP2/D2. Conversely, if complementary slackness (CS) conditions hold, then by Theorem 3.1 an integral feasible solution of LP2 satisfies these conditions with the given dual feasible solution. It can be seen that if $\{S^m_0\}$ is chosen according to this allocation, then the complementary slackness conditions imply the first set of termination conditions in Step S2.

Similarly, the second set of termination conditions in Step S2 (those involving $S^m_k$) correspond to complementary slackness conditions in formulations of D2 with bidders $\mathcal{M} - \{k\}$ for all $k \in \mathcal{M}$. Since when the complementary slackness conditions hold, the aforementioned dual feasible solutions (in a formulation of D2 with bidders $\mathcal{M}$ and $\mathcal{M} - \{k\}$ for all $k \in \mathcal{M}$) are optimal, we conclude that the termination conditions in Step S2 hold when a special optimal solution of D2 is identified. Moreover, since the efficient allocation and optimal prices in D2 correspond to a pricing equilibrium with a bidder-specific pricing rule (Theorem 3.1), it follows that the iterative auction in Table 2 terminates when a pricing equilibrium is reached (or a market clearance condition holds), and the final allocation obtained in the auction is efficient.

**Step S3:** The auctioneer decrements the prices of items that are not demanded in isolation (i.e., $\{i\} \notin D^k$), until bidders start demanding them. Once bidder $k$ demands such an item $i$, it is added to $\Psi^k$, the set of items that are demanded at an earlier stage of the auction. If two end points of an edge $(i, j)$ belong to this set (they are demanded in isolation at an earlier stage), and the bundle $\{i, j\}$ is not demanded, then the auctioneer decreases the price associated with this edge. Since the auction is initialized with high prices, initially no item is demanded in isolation, and the auctioneer only decrements the prices associated with the items. Once a bidder starts demanding items in isolation, then the auctioneer starts decreasing the prices of edges between demanded items (nodes). The updates of node/edge prices stop if the relevant node is demanded.
in isolation, or the end points of an edge are jointly demanded. This structure guarantees that \( p^m_i \geq w^m_i \) and \( p^m_{ij} \geq w^m_{ij} \) throughout the auction, provided that bidders truthfully report their demand. This suggests that for demanded bundles the price is equal to the value of the relevant bundle to the bidder. Since both valuations and prices have a graphical structure this implies that if set \( S \) is demanded by bidder \( m \), then \( p^m_i = w^m_i \) and \( p^m_{ij} = w^m_{ij} \) for \( i, j \in S \). Hence, when bidders truthfully report their demand, demanded bundles exhibit monotonicity, i.e., if bidder \( m \) demands bundle \( S \), then she also demands any \( S' \subset S \).

Note that in Step S3 the prices of items (or edges) that do not belong to a demand set are decreased. Hence, if bidders truthfully reveal their demand, and the auction does not terminate prior to that, eventually the bundle that consists of all items is demanded. Since for a demanded bundle \( S \) we have \( p^m_i = w^m_i \) and \( p^m_{ij} = w^m_{ij} \) for \( i, j \in S \), it follows that when such a bundle is demanded, the termination conditions in Step S2 hold. Thus, we conclude that when bidders truthfully report their demand (throughout the auction), the auction eventually terminates.

**Step S4:** If bidders report their demand truthfully throughout the auction then, as discussed in Step S3, \( p^m_i \geq w^m_i \geq 0 \) and \( p^m_{ij} \geq w^m_{ij} \). Additionally, if some bundle \( S \) is demanded by bidder \( m \) (since the inequalities hold with equality for demanded sets), it follows that \( \{i\} \in D^m \), \( \{i, j\} \in D^m \) for \( i, j \in S \). This implies that the conditions in Step S4 do not hold when bidders bid truthfully. On the other hand, when bidders do not bid truthfully, and these conditions hold, this step allows the auctioneer to reset the prices to their initial levels. Without this step, after a nontruthful history even if bidders truthfully report their demand in all subsequent stages, the auction need not terminate. This step guarantees that in such cases prices are reset, and the auction eventually terminates. This feature of the auction suggests that the first condition of Theorem 3.2 holds for the truthful bidding strategy (recall that Step S2 identifies a pricing equilibrium, and hence the allocation in this step is efficient).

It can be shown by contradiction that the conditions of Step S4 guarantee that the auction terminates after any history (provided that bidders truthfully report their demand following this history). In order to see that assume that after some history bidders truthfully reveal their demand but the auction does not terminate. This implies that Step S4 never holds (as otherwise prices are reset, and due to truthful bidding the auction terminates). Additionally, since the auction does not terminate node/edge prices decrease at each stage. Note that node prices cannot decrease below zero, as otherwise condition of Step S4 holds, and the prices are reset. Thus, eventually the price updates for nodes cease, and edge prices keep decreasing. On the other hand, if the edge prices unboundedly decrease, then bidders start demanding larger bundles of items and they do not demand items in isolation. However, in this case condition of Step S4 holds, prices are reset, and the auction terminates.

---

7Assume that after any history, if bidders use truthful bidding either the auction eventually terminates, or a condition \( C \) holds. Any such condition \( C \) could be used in Step S4 of the auction for resetting the prices after a nontruthful history.
Step S5: If the termination condition holds, then the auction terminates with a final allocation and the final payments that are only a function of the prices that emerge in the last stage of the auction. If bidders bid truthfully in Step S2, a special optimal solution is identified, and the final payments correspond to the VCG payments of agents (as described in Section 3.3.2). Note that if the auction does not terminate, the auctioneer does not allocate any items, and bidders do not make any payments. Hence, in this case bidders receive a payoff of zero.

We conclude that when bidders truthfully reveal their demand, our auction format identifies dual feasible solutions, checks complementary slackness conditions, and updates prices (or dual variables) in a similar fashion to primal-dual algorithms. On the other hand, the auction is slightly different than a primal-dual algorithm in two aspects. First, even after an optimal solution is found the auction may not terminate. This can be seen by observing that if the first set of conditions in Step S2 (hence complementary slackness) hold, but the second set of conditions do not, the price updates continue. This feature of the auction guarantees that price updates terminate only when a special optimal solution is found (and possibly after an optimal solution is found). Secondly, in primal-dual algorithms, after dual update, a solution with a strictly improved objective value is identified. In contrast, in our auction the dual updates need not strictly improve the objective value of D2. This can be seen by noting that even after an optimal solution is found, price updates may continue.

Theorem 3.3 shows that when bidders truthfully reveal their demand, the iterative auction presented in Table 2 converges to a special optimal solution of D2. Additionally, when final allocation and payments are chosen as in Step S5, it is an equilibrium for bidders to reveal their demand truthfully, and this equilibrium implements the efficient allocation.

Theorem 3.3. (i) Let \( \{v^m\} \) denote the valuations of bidders. Assume that the iterative auction in Table 2, terminates with prices (\( \{p^m_i\}, \{p^m_{ij}\} \)), and at the last stage of the auction, bidders’ demand reports are truthful. The solution of D2 obtained by setting \( \pi^m = \max_S v^m(S) - \sum_{i \in S} p^m_i - \sum_{i,j \in S | ij \in E} p^m_{ij} \), and \( \pi^* = \max_{\mu} \sum_m \sum_{i \in \mu} p^m_i + \sum_{i,j \in \mu | ij \in E} p^m_{ij} \), is a special optimal solution.

(ii) In this auction, it is an ex-post perfect equilibrium for bidders to reveal their demand truthfully. Moreover, the corresponding final allocation is efficient, and payments are the associated VCG payments.

The proof (given in the appendix) formally establishes some of the observations related to different steps of the auction given in this section (e.g., when bidders truthfully report their demand \( p^m_i \geq w^m_i \) and \( p^m_{ij} \geq w^m_{ij} \)). Additionally, by exploiting these observations it shows that truthful bidding satisfies the conditions of Theorem 3.2, and hence is an ex-post perfect equilibrium.

The high-level idea behind primal-dual algorithms is to start with a dual feasible solution, and check if there exists a primal feasible solution that satisfies the complementary slackness condition with the given dual solution. If this is the case, optimal solutions to both problems are found and the algorithm can terminate. Otherwise, the dual solution can be updated to another feasible solution with an improved objective value. We refer the reader to Papadimitriou and Steiglitz (1998), Bertsimas and Tsitsiklis (1997) for a general overview of primal-dual algorithms, and to Vohra (2011) for a discussion of their applications in mechanism design.
of our auction. In particular, it establishes that after any history, truthful bidding by all agents guarantees termination. Since prices cannot unboundedly decrease, either the conditions of Step S4 hold, and they are reset or the auction eventually terminates. If Step S4 holds and prices are reset, then at all subsequent updates $p_i^m \geq w_i^m$ and $p_i^{m_j} \geq w_i^{m_j}$ and decreasing price updates eventually guarantee termination. Moreover, the final allocation is efficient (since the termination conditions in Step S2 correspond to complementary slackness conditions and identify an efficient allocation). Similarly, the proof establishes that if opponents of some bidder $m$ bid truthfully and the termination condition in Step S2 holds, then the payment of that bidder corresponds to her VCG payment, by exploiting the observation that termination condition holds when a special optimal solution is identified and the VCG payment for bidder $m$ can be computed from this solution. This payment structure guarantees that bidders have no incentive to misreport their demand and the efficient allocation is implemented at an ex-post perfect equilibrium (see Theorem 3.2).

Our results indicate that for graphical valuations it is possible to implement the efficient outcome using a simple iterative auction format that relies on bidder-specific graphical prices. In the next section we establish that this observation holds more generally, and for more general classes of valuations it is possible to implement the efficient outcome using an auction format that employs a pricing rule with a similar structure to valuations.

4 An Alternative LP formulation and Generalizations

In the previous section, we showed that D2 has special optimal solutions that can be used to compute the VCG payments, and designed an efficient iterative auction that relies on convergence of prices (and the remaining dual variables) to a special optimal solution of D2. In Section 4.1, we provide an alternative LP formulation of the efficient allocation problem, in which all dual optimal solutions can be used to find the VCG payments. This formulation can be used with any primal-dual algorithm to systematically develop novel efficient iterative auction formats that rely on a bidder-specific graphical pricing rule. In Section 4.2, we extend this formulation to more general classes of value functions that are additively decomposable over subsets of items. Our results indicate that even for classes of valuations that are more general than graphical valuations, it is possible to implement the efficient outcome by using iterative auction formats that rely on prices that have a similar structure to the valuations of the bidders.

4.1 An Alternative LP Formulation of the Efficient Allocation Problem

In this section, we provide an alternative LP formulation of the efficient allocation problem, for which any dual optimal solution can be used to compute the VCG payments. The definition of the VCG payments suggest that the payment of some bidder $m$, can be obtained by computing the difference in the welfare of bidder $m$’s opponents when items are (efficiently) allocated to bidders
The main idea behind our approach in this section is to reformulate LP2/D2 in a way that jointly identifies the efficient allocation for such sets of bidders.

In D2, optimal solutions satisfy \( \pi^s = \max_\mu \sum_m \left( \sum_{i \in \mu^m} p_i^m + \sum_{i,j \in \mu^m | ij \in E} p_{ij}^m \right) \). Complementary slackness conditions require that at optimal integral solutions of LP2 if \( \delta(\hat{\mu}) = 1 \) then \( \hat{\mu} \) should be a maximizer of the right hand side of this equation. Moreover, Assumption 2.1 implies that LP2 has an optimal integral solution that satisfies \( x_m(\hat{\mu}) = 1 \), and \( x_m(S) = 0 \) for \( S \neq \mu^m \).

These observations suggest that the constraint \( \pi^s \geq \sum_m \left( \sum_{i \in \mu} p_i^m + \sum_{i,j \in \mu | ij \in E} p_{ij}^m \right) \) holds with equality when \( \mu \) corresponds to the efficient allocation, and hence this constraint can be used to identify the efficient allocation for set of bidders \( \mathcal{M} \).

We next reformulate D2 by adding similar constraints that can be associated with an efficient allocation of items to bidders \( \mathcal{M} - \{m\} \) for all \( m \in \mathcal{M} \). In particular, for each \( m \in \mathcal{M} \) we define variables \( \pi^s_m \) and add a constraint \( \pi^s_m \geq \sum_{k \neq m} \left( \sum_{i \in \mu^k} p_i^k + \sum_{i,j \in \mu^k | ij \in E} p_{ij}^k \right) \) for \( \mu \) such that \( \mu^m = \emptyset \). Thus, intuitively each such constraint focuses on allocations of items to bidders \( k \neq m \).

After modifying the objective function to incorporate the new variables, we obtain the following optimization formulation:

\[
\begin{align*}
\min & \quad \left( \pi^s + \sum_m \pi^s_m \right) + \sum_m \left( \pi^s_m + \sum_{k \neq m} \pi^s_k \right) \\
\text{s.t.} & \quad \pi^m \geq v^m(S) - \sum_{i \in S} p_i^m - \sum_{i,j \in S | ij \in E} p_{ij}^m \quad \forall S, m \\
(D3) & \quad \pi^s \geq \sum_m \left( \sum_{i \in \mu^m} p_i^m + \sum_{i,j \in \mu^m | ij \in E} p_{ij}^m \right) \quad \forall \mu \\
& \quad \pi^s_m \geq \sum_{k \neq m} \left( \sum_{i \in \mu^k} p_i^k + \sum_{i,j \in \mu^k | ij \in E} p_{ij}^k \right) \quad \forall m, \mu | \mu^m = \emptyset \\
& \quad \pi^m, \pi^s, \pi^s_m \geq 0.
\end{align*}
\]

Similar to our interpretation of \( \pi^s \) in D2, in dual problem D3 the variable \( \pi^s_m \) can be interpreted as the maximum revenue of the auctioneer at an allocation where bidder \( m \) does not receive any items. The interpretation of the remaining variables/constraints is exactly the same as in D2. The objective function of this problem involves the sum of auctioneers revenue and bidders surpluses not only for the market with bidders \( \mathcal{M} \), but also for the markets with bidders \( \mathcal{M} - \{k\} \) for all \( k \in \mathcal{M} \).

We next present the corresponding primal optimization problem. This problem involves new variables corresponding to the third constraint of D3. Associating the variables \( \{\delta^k(\mu)\}_{k, \mu | \mu^k = \emptyset} \)
with the relevant constraints, the corresponding primal can be given as follows:

\[
\text{max } \sum_m \sum_S x^m(S) v^m(S) \\
\text{s.t. } \sum_S x^m(S) \leq M \quad \forall m \\
\sum_\mu \delta(\mu) \leq 1 \\
\sum_{\mu|\mu^m=\emptyset} \delta^m(\mu) \leq 1 \quad \forall m \\
\sum_{S|i \in S} x^m(S) = \sum_{\mu|i \in \mu^m} \delta(\mu) + \sum_{k \neq m, \mu|\mu^k=\emptyset, i \in \mu^m} \delta^k(\mu) \quad \forall i, m \\
\sum_{S|ij \in S} x^m(S) = \sum_{\mu|ij \in \mu^m} \delta(\mu) + \sum_{k \neq m, \mu|\mu^k=\emptyset, i,j \in \mu^m} \delta^k(\mu) \quad \forall ij \in E, m \\
\delta^k(\mu), \delta(\mu), x^m(S) \geq 0.
\]

In this optimization problem, we jointly solve for the efficient allocation for the set of bidder \(\mathcal{M}\), as well as the sets of bidders \(\mathcal{M} - \{k\}\) for all \(k \in \mathcal{M}\). We could formulate separate optimization problems to compute these efficient allocations: (i) for the case involving all players in \(\mathcal{M}\), we could use variables \((x^m(S), \delta(\mu))\) and formulate LP2, (ii) for bidders in \(\mathcal{M} - \{k\}\) we could use variables \((x^m(S), \delta^k(\mu))\) and appropriately reformulate LP2 for this set of bidders. Instead, LP3 solves all of these optimization problems jointly, by aggregating their constraints. For instance, as opposed to imposing \(\sum_S x^m(S) \leq 1\) as in the \(\mathcal{M}\) separate formulations of LP2 involving bidder \(m\) (as described above), LP2 imposes a single constraint \(\sum_S x^m(S) \leq M\). Similarly, the fourth and fifth constraint in LP3 respectively aggregate the constraints in these formulations of LP2 for different items/edges. Consequently, in LP3 we have a single constraint for each bidder-item or bidder-edge pair, and its dual D3, still suggests a bidder-specific graphical pricing rule.

Our next result formally establishes that for graphical valuations, solving LP3/D3 (as opposed to LP2/D2) allows for jointly identifying the efficient outcome in problem instances with bidders \(\mathcal{M}\) and \(\mathcal{M} - \{k\}\) (for all \(k \in \mathcal{M}\)). Additionally, any dual optimal solution of D3 can be used to identify the VCG payments.

**Theorem 4.1.** Assume that bidders have graphical valuations. Let \(\{S^m\}_m\) and \(\{S^m_k\}_{m \neq k}\) (for all \(k \in \mathcal{M}\)) denote the efficient allocation for set of bidders \(\mathcal{M}\) and \(\mathcal{M} - \{k\}\) respectively.

(i) LP3 always has an optimal solution that is integral. In this solution, we have \(\delta(\{S^m\}) = \delta^k(\{S^m_k\}) = 1\) for all \(k \in \mathcal{M}\), and \(\delta(\mu) = \delta^m(\mu) = 0\) for the remaining \(\mu, m\); and \(x^m(S) = \lfloor (k|S = S^m_k) \rfloor + 1_{S=S^m}\) for all \(m, S\), where \(1_{S=S^m}\) is an indicator variable that is equal to 1 if \(S = S^m\), and 0 otherwise.
(ii) At any dual optimal solution of D3, for any bidders \( m, k \), we have
\[
\pi^m = v^m(S^m) - \sum_{i \in S^m} p^m_i - \sum_{i,j \in S^m | ij \in E} p^m_{ij}, \quad \pi^s = v^m(S^m) - \sum_{i \in S^m} p^m_i - \sum_{i,j \in S^m | ij \in E} p^m_{ij}, \quad \pi^s_k = \max \left( \sum_{m \neq k} \left( \sum_{i \in S^m} p^m_i + \sum_{i,j \in S^m | ij \in E} p^m_{ij} \right) - \left( \sum_{i \in S^m} p^m_i + \sum_{i,j \in S^m | ij \in E} p^m_{ij} \right) \right)
\]

(iii) At any dual optimal solution, for any bidder \( m \), the quantity
\[
\sum_{k \neq m} \left( \left( \sum_{i \in S^k} p^k_i + \sum_{i,j \in S^k | ij \in E} p^k_{ij} \right) - \left( \sum_{i \in S^k} p^k_i + \sum_{i,j \in S^k | ij \in E} p^k_{ij} \right) \right)
\]
is equal to the VCG payment of bidder \( m \), for acquiring bundle \( S^m \) of items.

This theorem suggests that solutions of LP3/D3 via iterative algorithms (such as primal-dual algorithm) can be used to identify the efficient outcome, and VCG payments at the same time. Moreover, these algorithms lead to efficient iterative auction formats that rely on the bidder-specific graphical pricing rule. The auction provided in Table 2 can be viewed as an example of such an iterative auction. Despite the fact that this auction was developed by focusing on iterative solutions of LP2/D2, by setting \( \pi^m = \max_S v^m(S) - \sum_{i \in S} p^m_i - \sum_{i,j \in S | ij \in E} p^m_{ij}, \pi^s = \max_{\mu} \sum_{m} \left( \sum_{i \in \mu} p^m_i + \sum_{i,j \in \mu | ij \in E} p^m_{ij} \right) \), and \( \pi^s_k = \max_{\mu | \mu = \emptyset} \sum_{m} \left( \sum_{i \in \mu} p^m_i + \sum_{i,j \in \mu | ij \in E} p^m_{ij} \right) \), a dual feasible solution of D3 can be obtained using the prices at any stage of the auction. Thus prices that emerge at each stage of this auction can be associated with a dual feasible solution of D3. Additionally, it can be shown that the termination condition (Step S2) in this auction, corresponds to checking whether the given dual feasible solution satisfies complementary slackness condition with the primal optimal solution of LP3 given in Theorem [4.1]. Hence, when bidders reveal their demand truthfully, the price (or dual variable) updates in this auction, correspond to a solution of LP3/D3 with an iterative algorithm that is similar to a primal-dual algorithm. On the other hand, since unlike in Section [3.3], any optimal solution of the dual problem can be used to compute VCG prices, any primal-dual algorithm that solves LP3/D3 (by potentially relying on a different price update rule than the one in Table 2), can be used to develop an iterative auction (after appropriately modifying Step S3 of the auction in Table 2).\cite{Vohra2011}

\footnote{Such dual variable updates can be systematically made by solving the restricted primal/dual problems (Papadimitriou and Steiglitz, 1998; Vohra, 2011).} The details are omitted, since these auctions are identical to the one in Table 2, except for the price updates in Step S3, and their analysis follows the same approach as in the previous section.
4.2 Generalization to Additively Decomposable Valuations

In this section, we focus on additively decomposable valuations, which is a class of value functions that generalizes graphical valuations. The valuations in this class additively decompose over subsets of items (which potentially have cardinality larger than two), thereby allowing for complementarity/substitutability not only for pairs, but also for larger sets of items. We establish that in order to find the efficient outcome and VCG payments for additively decomposable value functions, it is sufficient to solve a generalization of LP3/D3 of Section 4.1. Additionally, this LP formulation suggests efficient iterative auction formats that rely on prices that also exhibit the same additively decomposable structure as the value functions.

We start our analysis by formally defining additively decomposable valuations. Consider a collection of subset of items $B$, i.e., $B \in B$ is such that $B \subseteq N$. Assume that the valuations of bidders can be additively decomposed over these subsets as follows:

$$v^m(S) = \sum_{B \in B} w^m_B(S \cap B),$$  \hspace{1cm} (7)

where $w^m_B : 2^B \rightarrow \mathbb{R}$, captures the component of the valuation of bidder $m$ associated with subset $B$. We refer to such valuations as *additively decomposable valuations* with collection $B$.

We note that any value function can be represented using additively decomposable valuations by considering a collection $B$ that contains set $N$. On the other hand, if $B$ consists of few sets of small cardinality, then the value functions can be compactly represented by specifying their components $\{w^m_B\}$. For instance, graphical valuations are a special class of additively decomposable valuations, where

- $B$ consists of singletons, and pairs of items that correspond to the edges of the underlying value graph,
- $w^m_i(S) = 0$ if $S \neq \{i\}$, and it equals to the weight associated with node $i$ otherwise,
- $w^m_{ij}(S) = 0$ if $S \neq \{i,j\}$, and it equals to the edge weight for edge $(i,j)$ otherwise.

We next provide a generalization of LP3 and its dual D3 (denoted by $LP3G$ and $D3G$ respectively) that, as we establish subsequently, allow for finding the efficient allocation and VCG
payments for additively decomposable valuations:

\[
\begin{align*}
\text{max} & \quad \sum_m \sum_S x^m(S)v^m(S) \\
\text{s.t.} & \quad \sum_S x^m(S) \leq M \quad \forall m \\
& \quad \sum_{\mu} \delta(\mu) \leq 1 \\
\end{align*}
\]

(LP3G)

\[
\begin{align*}
\sum_{S|S'=S \cap B} x^m(S) &= \sum_{\mu|\mu^m \cap B = S'} \delta(\mu) + \sum_{k \neq m, m|\mu^k = \emptyset, \mu^m \cap B = S'} \delta^k(\mu) \quad \forall m, S' \subset B, B \in \mathcal{B} \\
\delta^k(\mu), \delta(\mu), x^m(S) &\geq 0. \quad \forall m, S, \mu.
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad \pi^s + \sum_m \pi^m_s + M \sum_m \pi^m \\
\text{s.t.} & \quad \pi^m \geq v^m(S) - \sum_B p^m_B(S \cap B) \quad \forall m, S \\
\end{align*}
\]

(D3G)

\[
\begin{align*}
\pi^s &\geq \sum_m \sum_B p^m_B(\mu^m \cap B) \quad \forall \mu \\
\pi^s &\geq \sum_{k \neq m} \sum_B p^k_B(\mu^k \cap B) \quad \forall m, \mu|\mu^m = \emptyset \\
\pi^m, \pi^s, \pi^s &\geq 0. \quad \forall m,
\end{align*}
\]

It can be checked that for graphical valuations, by focusing on a collection \( \mathcal{B} \) that consists only of singletons, and pairs of items that are connected with an edge in the underlying value graph, LP3G and D3G reduce to LP3 and D3.

Analogous to the results of Theorem 4.1, we next show that this primal-dual LP pair can be used to find the efficient allocation, and the VCG payments for additively decomposable valuations with a given collection \( \mathcal{B} \).

**Theorem 4.2.** Assume that bidders have additively decomposable valuations with a collection \( \mathcal{B} \). Let \( \{S^m\}_m \) and \( \{S^m_k\}_{m \neq k} \) (for all \( k \in \mathcal{M} \)) denote the efficient allocation for set of bidders \( \mathcal{M} \) and \( \mathcal{M} - \{k\} \) respectively.

(i) LP3G always has an optimal solution that is integral. In this solution, we have \( \delta(\{S^m\}) = \delta^k(\{S^m_k\}) = 1 \) for all \( k \in \mathcal{M} \), and \( \delta(\mu) = \delta^m(\mu) = 0 \) for the remaining \( \mu, m \); and \( x^m(S) = \left|\{k|S = S^m_k\}\right| + 1_{S = S^m} \) for all \( m, S \), where \( 1_{S = S^m} \) is an indicator variable that is equal to 1 if \( S = S^m \), and 0 otherwise.
(ii) At any dual optimal solution of $D3G$, for any bidders $m, k$, we have

\[
\pi^m = v^m(S^m) - \sum_B p^m_B(S^m \cap B) = v^m(S^m_k) - \sum_B p^m_B(S^m_k \cap B),
\]

\[
\pi^s = \sum_m \sum_B p^m_B(S^m \cap B),
\]

\[
\pi^s_k = \sum_{m \neq k} \sum_B p^m_B(S^m_k \cap B).
\]

(iii) At any dual optimal solution, for any bidder $m$, the quantity

\[
\sum_{k \neq m} \left( \sum_B p^k_B(S^m_k \cap B) - \sum_B p^k_B(S^k \cap B) \right)
\]

is equal to the VCG payment of bidder $m$, for acquiring bundle $S^m$ of items.

**Proof.** The proof of this theorem is identical to that of Theorem 4.1 and obtained following the same steps, after replacing $p^m_i, p^m_{ij}$ by $p^m_B$, and constructing a feasible dual solution $p^m_B = w^m_B$. \qed

This result suggests that if the valuations are additively decomposable over certain subsets of items, then the auctioneer can find the efficient outcome and VCG payments by using an LP formulation whose dual suggests a pricing rule that also decomposes over these subsets. Moreover, using the complementary slackness conditions, it can be shown that the efficient allocation and dual optimal prices constitute a pricing equilibrium (with the aforementioned pricing rule). Using primal-dual algorithms with these LP formulations, it is also possible to develop iterative auction formats that implement the efficient outcome (and VCG payments) for general additively decomposable valuations. These auctions rely on bidder-specific prices that decouple over the underlying collection of sets, i.e., $\{p^m_B\}$, and terminate when a pricing equilibrium with this pricing rule is identified. Thus, when valuations are additively decomposable over a few sets with small cardinality (as in the case of graphical valuations), a simple pricing rule can be used for iterative auction design.

5 Conclusions

In this work, we design a novel iterative auction format that guarantees efficiency for graphical valuations. This iterative auction relies on bidder-specific graphical prices, and terminates when a pricing equilibrium with this pricing rule is identified. Before we obtain our auction format we establish that a pricing equilibrium with bidder-specific graphical prices exists for all graphical valuations. We also provide an LP formulation of the efficient allocation problem, whose primal/dual optimal solutions can be used to identify such a pricing equilibrium. In this formulation, the dual problem has a special optimal solution that can be used to compute the VCG
payments. Using this result with iterative algorithms that can be employed to solve the aforementioned LP formulation, we develop an iterative auction format that relies on bidder-specific graphical pricing, terminates when such an optimal solution (and hence a pricing equilibrium) is found, and implements the efficient outcome for all graphical valuations at an ex-post perfect equilibrium. Finally, we provide an alternative LP formulation of the efficient allocation problem for general graphical valuations. This formulation still suggests employing bidder-specific graphical prices, and its solution simultaneously reveals the efficient outcome for (i) all players, as well as (ii) all players but one. We establish that any dual optimal solution of this LP formulation (as opposed to a special one) can be used to compute the VCG payments. Thus, iterative solutions of this LP with any primal-dual algorithm leads to iterative auctions that implement the efficient outcome for graphical valuations. Moreover, this formulation generalizes to settings with additively decomposable valuations, thereby providing a systematic framework for developing simple iterative auction formats that guarantee efficiency beyond graphical valuations. The results of this paper imply that when value functions of bidders exhibit some structure (such as the additively decomposable structure, or the graphical structure), it is possible to develop efficient iterative auction formats that rely on pricing rules which have a similar structure. Thus, in practice it may be possible to develop iterative auction formats that rely on simple pricing rules, by first identifying the structure of valuations of bidders, and then following the framework provided here to exploit this special structure.

Our results also suggest a number of interesting future directions, as outlined below:

**Robustness of iterative auctions:** The results provided in this paper rely on the assumption that the valuations of bidders can be modeled by graphical valuations. However, in practical settings we expect graphical valuations to be only approximations of reality. How sensitive are the results presented in this paper to the deviations from the graphical valuation assumption? For instance, do the auction formats we provide lead to inefficiency, if the true valuations of bidders are not graphical valuations but are approximated by graphical valuations? If so, is it possible to provide bounds on the resulting inefficiency? What are the qualitative properties of value functions for which the auctions we provide achieve approximate efficiency?

**Limitations and scope of anonymous prices:** In this paper, we obtain our results by focusing on a bidder-specific graphical pricing rule. In some settings it may be desirable to have auction formats that rely on anonymous pricing rules, where all bidders are offered same prices. Anonymous item prices do not allow for implementing the efficient outcome except for special settings (i.e., settings where a Walrasian equilibrium exists such as the setting of gross substitutes). It is interesting to understand if anonymous graphical pricing (or offering pairwise discounts) allows for implementation in more general settings.
**Interdependent valuations:** It is known that in single-item settings where valuations of bidders are interdependent, iterative auctions have interesting revenue properties. In particular, for such value functions, single-item iterative auctions may lead to higher revenues for the seller than the sealed bid alternatives. Do similar conclusions hold for multi-item auctions? A simple class of value functions for which this question can be studied, is the class of graphical valuation. It is an interesting future direction to understand revenue properties of various iterative auction formats, in settings where valuations of bidders are graphical.

**Empirical work:** For what type of auction environments are graphical valuations a good approximation of the true valuations of bidders? How do the auctions, introduced in this paper, compare with other multi-item auctions employed in such environments? These questions require a rigorous empirical analysis. We believe that this is an interesting direction for future work.

**References**


A Proofs

Proof of Theorem 3.1. The first part of the claim can immediately be verified from LP2. Below we establish parts (ii) and (iii).

(ii) First assume that LP2 has an optimal solution that is integral. Primal feasibility of this solution suggests that for each bidder \( m \in \mathcal{M} \), there exists at most one bundle \( \hat{S}^m \) such that \( x^m(\hat{S}^m) = 1 \), and \( \delta(\hat{\mu}) = 1 \) for an allocation \( \hat{\mu} \) such that \( \hat{S}^m \subset \hat{\mu}^m \). Moreover, by Assumption 2.1 it follows that another optimal solution of LP2 can be obtained by setting \( x^m(\hat{\mu}^m) = \delta(\hat{\mu}) = 1 \) for every \( m \in \mathcal{M} \), and \( x^m(S) = \delta(\mu) = 0 \) for remaining \( m, \mu, S \). By part (i) it follows that \( \hat{\mu} \) leads to a higher objective value than any feasible allocation. Hence it follows that \( \hat{\mu} \) is efficient.

Observe that the complementary slackness conditions suggest that the corresponding dual optimal solutions are such that \( \pi^m = v^m(\hat{\mu}^m) - \sum_{i \in \hat{\mu}^m} p^m_i - \sum_{i, j \in \hat{\mu}^m | i, j \in E} p^m_{ij} \), and

\[
\pi^s = \sum_{m} \left( \sum_{i \in \hat{\mu}^m} p^m_i + \sum_{i, j \in \hat{\mu}^m | i, j \in E} p^m_{ij} \right).
\]

Thus, together with dual feasibility conditions, Definition 2.3 implies that \( \hat{\mu} \) and prices \( \{ p^m(S) = \sum_{m} \left( \sum_{i \in S} p^m_i + \sum_{i, j \in S | i, j \in E} p^m_{ij} \right) \} \) constitute a pricing equilibrium with bidder-specific graphical prices.

Conversely, assume that a pricing equilibrium with bidder-specific graphical pricing exists. Denote the associated allocation by \( \hat{\mu} \), and prices by \( \{ p^m(S) = \sum_{m} \left( \sum_{i \in S} p^m_i + \sum_{i, j \in S | i, j \in E} p^m_{ij} \right) \} \). Definition 2.3 immediately implies that choosing \( \pi^m = v^m(\hat{\mu}^m) - \sum_{i \in \hat{\mu}^m} p^m_i - \sum_{i, j \in \hat{\mu}^m | i, j \in E} p^m_{ij} \), and \( \pi^s = \sum_{m} \left( \sum_{i \in \hat{\mu}^m} p^m_i + \sum_{i, j \in \hat{\mu}^m | i, j \in E} p^m_{ij} \right) \) with these prices, leads to a feasible solution of D2. Moreover, it can be checked that this solution satisfies complementary slackness conditions with \( x^m(\hat{\mu}^m) = \delta(\hat{\mu}) = 1 \), and \( x^m(S) = \delta(\mu) = 0 \) for remaining \( m, S, \) and \( \mu \). Moreover, the latter solution is feasible in LP2, and hence is optimal, and the claim follows.

(iii) Assume that \( \hat{\mu} \) is an efficient allocation. Consider the corresponding feasible integral solution of LP2 that is obtained by choosing \( \delta(\hat{\mu}) = 1 \), and \( x^m(\hat{\mu}^m) = 1 \) for all \( m \), and setting \( \delta(\mu) = x^m(S) = 0 \) for \( \mu \neq \hat{\mu}, S \neq \hat{\mu}^m \). Observe that the objective value of LP2 at this solution is equal to the total value associated with the efficient allocation \( \hat{\mu} \). Denote this quantity by \( W^* \).

Consider the following dual solution: \( p^m_i = w^m_i, p^m_{ij} = w^m_{ij}, \pi^m = 0, \) and \( \pi^s = W^* \). By construction this solution suggests \( v^m(S) - \sum_{i \in S} p^m_i - \sum_{i, j \in S | i, j \in E} p^m_{ij} = 0 \), thus the first constraint of D2 is immediately satisfied. Additionally, the construction also implies that \( W^* \geq \sum_{m} \left( \sum_{i \in \mu^m} p^m_i + \sum_{i, j \in \mu^m | i, j \in E} p^m_{ij} \right) \) for all \( \mu \). Thus, we conclude that \( \pi^s = W^* \) satisfies the second constraint, and feasibility of the suggested solution follows.

Observe that the objective value associated with the feasible dual solution we constructed is \( W^* \). However, as we established earlier, the optimal objective value of LP2 is at least \( W^* \). Thus, it follows that the dual feasible solution we constructed is optimal, and the feasible integral solution of LP2 that corresponds to the efficient allocation is optimal in the primal problem. The
existence of pricing equilibria follows from part (ii).

Proof of Theorem 3.2. Let \( \{\hat{S}^k\} \) denote the efficient allocation associated with valuations \( \{v^k\} \). From the first and second conditions of the theorem it follows that for agent \( m \), the payoff of the strategy profile \( \{s^k(v^k)\} \) leads to (after history \( H_t \)), is given by

\[
\begin{align*}
u^m(s^m(v^m), s^m(v^{-m})|H_t, v^m) &= v^m(\hat{S}^m) - \gamma^m(\hat{S}^m, \{v^k\}_{k \neq m}) \\
&= v^m(\hat{S}^m) + \max_{\{Z^k\}|Z^k \cap Z^l = \emptyset, Z^k \cap \hat{S}^m = \emptyset} \sum_{k \neq m} v^k(Z^k) - \max_{\{Z^k\}|Z^k \cap Z^l = \emptyset} \sum_{k \neq m} v^k(Z^k) \\
&= \sum_k v^k(\hat{S}^k) - \max_{\{Z^k\}|Z^k \cap Z^l = \emptyset} \sum_{k \neq m} v^k(Z^k). 
\end{align*}
\]

(9)

Since \( \{\hat{S}^k\} \) is the efficient allocation we have \( \sum_k v^k(\hat{S}^k) - \max_{\{Z^k\}|Z^k \cap Z^l = \emptyset} \sum_{k \neq m} v^k(Z^k) \geq 0 \), and hence (9) implies that \( u^m(s^m(v^m), s^m(v^{-m})|H_t, v^m) \geq 0 \). Thus, after history \( H_t \), bidder \( m \) has no incentive to deviate to a strategy that prevents the auction from terminating.

Assume that after history \( H_t \) bidder \( m \) can use a strategy \( z^m \) so that the auction terminates with allocation \( \{S^m\} \). The second condition of the theorem together with (9) implies that

\[
\begin{align*}
u^m(s^m(v^m), s^m(v^{-m})|H_t, v^m) &= \sum_k v^k(\hat{S}^k) - \max_{\{Z^k\}|Z^k \cap Z^l = \emptyset} \sum_{k \neq m} v^k(Z^k) \\
&\geq v^m(S^m) + \max_{\{Z^k\}|Z^k \cap Z^l = \emptyset, Z^k \cap S^m = \emptyset} \sum_{k \neq m} v^k(Z^k) - \max_{\{Z^k\}|Z^k \cap Z^l = \emptyset} \sum_{k \neq m} v^k(Z^k) \\
&= v^m(S^m) - \gamma^m(S^m, \{v^k\}_{k \neq m}) = u^m(z^m, s^m(v^{-m})|H_t, v^m),
\end{align*}
\]

where the inequality follows from the fact that \( \{\hat{S}^k\} \) is the efficient allocation, and the last equality follows from the second condition of the theorem. Thus, we conclude that bidder \( m \) cannot improve her payoff by deviating from \( s^m(v^m) \) to another strategy that leads the auction to terminate. Since bidder \( m \), history \( H_t \), strategy \( z^m \), and payoff realization \( \{v^k\} \) are arbitrary, the claim follows from Definition 3.2.

Proof of Lemma 3.1. Denote an efficient allocation by \( \{\hat{S}^m\} \). Consider a solution of D2, where \( p^m_i = w^m_i, p^m_{ij} = w^m_{ij}, \pi^m = 0 \), for \( m \in \mathcal{M} \), and \( \pi^s \) equals to the maximum total value, i.e., \( \pi^s = W^* \triangleq \max_{\{S^m\} | S^m = \emptyset} \sum_k v^k(S^k) \). It can be immediately checked that this solution is feasible in D2. Additionally, the corresponding objective value is equal to \( W^* \). However, LP2 has a feasible solution, associated with the efficient allocation, which has the same objective value (Theorem 3.1). This implies that the constructed dual feasible solution is also optimal.

Observe that after restricting this solution to a set of bidders \( \mathcal{M}_0 \), and replacing \( \pi^s \) with \( \pi^s = \max_{\{S^m\} | S^m = \emptyset} \sum_k v^k(S^k) \), we obtain a solution for a formulation of D2 with bidders \( \mathcal{M}_0 \). Feasibility of this solution can immediately be checked. Note that the objective value of D2 associated with this solution, is equal to the maximum total value that can be obtained by the bidders in \( \mathcal{M}_0 \) (i.e., \( \max_{\{S^m\} | S^m = \emptyset} \sum_k v^k(S^k) \)). On the other hand, when D2 is
formulated with set of bidders $M_0$, the corresponding formulation of LP2 has a feasible solution associated with the efficient allocation to set of bidders $M_0$. Thus, it follows that the solution we constructed, is also optimal in a formulation of D2, with set of bidders $M_0$. We conclude that the restriction of prices and bidder surpluses of the optimal solution obtained in the first part to set $M_0$, agrees with an optimal solution to D2 formulated with set of bidders $M_0$. Hence, the claim follows.

Proof of Theorem 3.3. (i) Observe that if the auction terminates, then the conditions of Step S2 hold. Let $\{S_0^m\}$, and $\{S_k^m\}$ be as defined in this step. Since in the last stage bidders report their demand truthfully, this implies that $\pi^m$, $\pi^s$ given in the theorem statement can alternatively be expressed as $\pi^m = v^m(S) - \sum_{i \in S} p_i^m - \sum_{i,j \in S, j \in E} p_{ij}^m$, for all $S \in D^m$, and $\pi^s = \sum_m p^m(S_0^m)$. This implies that $\{(p_i^m, p_{ij}^m, \pi^m, \pi^s)\}$ is feasible in D2. Additionally, it can be checked that this solution satisfies complementary slackness conditions with a primal feasible solution of LP2 $\{(x^m(S), \delta(\mu))\}$, such that for every $m$, $x^m(S_0^m) = 1$, $x^m(S) = 0$ for $S \neq S_0^m$, and $\delta(\{S_0^m\}) = 1$, $\delta(\mu) = 0$ for $\mu \neq \{S_0^m\}$. Thus, it follows that $\{(p_i^m, p_{ij}^m, \{\pi^m\}, \{\pi^s\})\}$ is optimal in D2.

Consider a formulation of D2 with bidders $M - \{k\}$. Observe that the restriction of prices and bidder surpluses of $\{(p_i^m, p_{ij}^m, \{\pi^m\}, \{\pi^s\})\}$ to $M - \{k\}$ satisfies constraints of D2 involving $\pi^m$ variables, since this solution satisfies same constraints in a formulation of D2 with bidders $M$. On the other hand, Step S2 of the auction implies that $\pi^s = \sum_{m \neq k} p^m(S_k^m)$ for any other complete allocation $\{\hat{S}_k^m\}_{m \neq k}$ of items to bidders. Thus, it follows that $\{(p_i^m, p_{ij}^m, \pi^m, \pi^s)\}_{m} \in M - \{k\}$ is feasible in D2, with bidders $m \in M - \{k\}$. In addition, it can be checked that this solution satisfies complementary slackness condition with the primal feasible solution $\{(x^m(S), \delta(\mu))\}_{m \in M - \{k\}}$ such that for every $m \neq k$, $x^m(S_0^m) = 1$, $x^m(S) = 0$ for $S \neq S_0^m$, and $\delta(\{S_0^m\}) = 1$, $\delta(\mu) = 0$ for $\mu \neq \{S_0^m\}$. This implies that the aforementioned solution is also optimal in a formulation of D2 with bidders $m \in M - \{k\}$.

Since the restriction of prices and bidder surpluses of $\{(p_i^m, p_{ij}^m, \{\pi^m\}, \pi^s)\}$ to bidders $m \in M - \{k\}$, agrees with an optimal solution of a formulation of D2 with bidders $m \in M - \{k\}$, and $k$ is arbitrary, we conclude that this dual solution is a special optimal solution of D2.

(ii) We establish the ex-post perfect equilibrium result by showing that for the truthful bidding strategy, after any history $H_t$, the conditions of Theorem 3.2 hold. In obtaining a proof of this part, we make use of some auxiliary lemmas (given below). The proofs of these lemmas can be found after the proof of the theorem.

We first show that if after $H_t$ bidder $m$ bids truthfully, then the conditions of Step S4 can hold at most once (after history $H_i$) for bidder $m$. In order to establish this result, we make use of the following intermediate lemma:

Lemma A.1. Assume that at time $\hat{t}$ one of the conditions of Step S4 holds, and $p_i^m, p_{ij}^m, \Psi^m$ are updated accordingly. If bidder $m$ bids truthfully after $\hat{t}$, then at all times $t > \hat{t}$ we have $p_i^m \geq w_i^m$ for all $i$, and $p_{ij}^m \geq w_{ij}^m$ for all $(i, j) \in E$. 

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This lemma implies that \( p^m_i \geq w^m_i > 0 \) after \( \hat{t} \). Thus, it follows that if at some time \( r > \hat{t} \) we have \( S \in D^m \), then it should be the case that \( p^m_i = w^m_i \) and \( p^m_{ij} = w^m_{ij} \) for \( i, j \in S \). On the other hand, this implies that \( S' \in D^m \) for all \( S' \subset S \) (since all subsets lead to a surplus of zero, i.e., \( v^m(S') - p^m(S') = 0 \)). These suggest that conditions of Step S4 cannot hold after \( \hat{t} \). Thus, we conclude that if after \( H_t \) bidder \( m \) bids truthfully, then the condition of Step S4 can hold at most once for \( r > t \) and bidder \( m \). Since this is true for all bidders, it follows that conditions of Step S4 do not hold for any bidder after some \( \hat{t} \geq \hat{t} \).

Second, we show that if all bidders bid truthfully, and conditions of Step S4 do not hold after time \( \hat{t} \) for any bidder, then at some \( r \geq \hat{t} \), the conditions of Step S2 hold.

**Lemma A.2.** Assume that after \( \hat{t} \) conditions of Step S4 do not hold for any bidder, and all bidders bid truthfully. Then, at some time \( r \geq \hat{t} \), the conditions of Step S2 hold.

These results imply that if after any history \( H_t \) all bidders bid truthfully, then eventually conditions of Step S2 hold, and the auction terminates. On the other hand, part (i) implies that the solution that emerges when the auction terminates is a special optimal solution. It can be checked that the primal feasible solution \( x^m(S_0^m) = 1, \delta(\{S_0^m\}) = 1 \), and \( x^m(S) = \delta(\mu) = 0 \) for remaining \( m, S, \mu \); satisfies complementary slackness conditions with this dual solution. Thus, the allocation that the auction obtains at the end, \( \{S_0^m\} \), is the efficient allocation. Hence, the first condition of Theorem 3.2 holds.

Next assume that after history \( H_t \) bidders \( k \neq m \) bid truthfully, bidder \( m \) uses strategy \( z^m \), and the auction terminates with a final allocation \( \{S_k^m\} \) (that is potentially inefficient). Since bidders \( k \neq m \) bid truthfully, we obtain \( S_m^k, S_0^k \in D^k \) at the last stage of the auction, and hence \( p^k(S_m^k) - p^k(S_0^k) = v^k(S_m^k) - v^k(S_0^k) \). It also follows from \( S_m^k \in D^k \) that \( v^k(S_m^k) - p^k(S_m^k) \geq v^k(S) - p^k(S) \) for all sets \( S \). Additionally, the termination conditions in Step S2 imply that \( \sum_{k \neq m} p^k(S_m^k) \geq \sum_{k \neq m} p^k(S_k^m) \). These inequalities imply that

\[
\sum_{k \neq m} v^k(S_m^k) \geq \sum_{k \neq m} v^k(S_k^m) + \sum_{k \neq m} p^k(S_m^k) - \sum_{k \neq m} p^k(S_k^m) \geq \sum_{k \neq m} v^k(S_k^m),
\]

for any allocation \( \{S_k^m\}_{k \neq m} \). This suggests that \( \{S_m^k\}_{k \neq m} \) is an efficient allocation of set of items \( N \) to bidders \( k \neq m \). Similarly, the termination conditions in Step S2 imply that \( \sum_{k \neq m} p^k(S_0^k) \geq \sum_{k \neq m} p^k(S_k^m) \) where \( \{S_k^m\}_{k \neq m} \) is an allocation of items \( N \) - \( S_0^m \) to bidders \( k \neq m \). Hence, using this inequality together with \( S_0^k \in D^k \) for \( k \neq m \), a similar inequality to (10) (that replaces sets \( \{S_m^k\}_{k \neq m} \) with sets \( \{S_0^k\}_{k \neq m} \)) can be obtained, and it follows that \( \{S_k^m\}_{k \neq m} \) is an efficient allocation of items \( N \) - \( S_0^m \) to bidders \( k \neq m \). Thus, bidder \( m \)'s final payment is equal to \( \sum_{k \neq m} p^k(S_m^k) - p^k(S_0^k) = \sum_{k \neq m} v^k(S_m^k) - v^k(S_0^m) = \gamma(S_m^k, \{v_k\}_{k \neq m}) \). This implies that the second condition of Theorem 3.2 holds after any history \( H_t \).

Finally, by construction bidders receive a payoff of zero if the auction does not terminate. Hence, the last condition of Theorem 3.2 also holds.

Thus, it follows from Theorem 3.2 that truthful bidding is an ex-post perfect equilibrium.
strategy. Additionally, as shown above, when bidders bid truthfully the final allocation \( \{S_0^m\} \) is efficient, and payments \( \{\gamma(S_0^m, \{v^k\}_{k\neq m})\} \) correspond to VCG payments. Hence, the claim follows.

\[\square\]

Proof of Lemma A.1. Observe that the claim holds at \( t + 1 \), since at \( t \) prices are updated as in Step S4. Assume that it holds until time \( r \geq t \). We will provide an inductive proof for the claim by showing that the claim holds at \( r + 1 \) as well.

At time \( r \) the prices are updated either as suggested in Step S3, or as reseted following Step S4. If it is the latter, it immediately follows that at \( r + 1 \) we have \( p_i^m \geq w_i^m \), for all \( i \), and \( p_{ij}^m \geq w_{ij}^m \) for all \( (i,j) \in E \).

Assume that at \( r \) prices are updated as suggested in Step S3. Observe that if \( p_i^m > w_i^m \) then after price update \( p_i^m \geq w_i^m \). On the other hand if \( p_i^m = w_i^m \), then \( \{i\} \in D^m \), since \( p^m(S) \geq v^m(S) \) at time \( r \) and hence \( p_i^m \) is not updated. Thus, we conclude that for node prices \( p_i^m \geq w_i^m \) at \( r + 1 \).

Consider an edge \((i,j)\) such that \( i,j \in \Psi^m \), and \( \{i,j\} \notin D^m \) (note that the prices for the remaining edges are not updated in the auction). Since after \( t \) bidder \( m \) bids truthfully, and \( i,j \in \Psi^m \), it follows that for some \( \hat{t} \) such that \( \hat{t} < \hat{t} \leq r \), \( \{i\} \in D^m \). Since until time \( r \) the claim holds \( \{p^m(S) \geq v^m(S)\} \), and prices are nonincreasing, it follows that \( p_i^m = w_i^m \), \( p_j^m = w_j^m \) and \( \{i\}, \{j\} \in D^m \) after \( \hat{t} \). Since, \( \{i,j\} \notin D^m \), this implies that \( p_{ij}^m > w_{ij}^m \). Thus, after the price update \( p_{ij}^m \geq w_{ij}^m \) for edge prices as well.

Hence, it follows that the inequalities \( p_i^m \geq w_i^m \) for all \( i \), and \( p_{ij}^m \geq w_{ij}^m \) for all \( (i,j) \in E \) hold at \( r + 1 \) as well. By induction, the claim follows.

\[\square\]

Proof of Lemma A.2. Consider some bidder \( m \), and observe that at \( r > t \) if there exists some \( i \) such that \( \{i\} \notin D^m \), then \( p_i^m \) is decreased. Conversely assume that at time \( r \), for all \( i \) we have \( \{i\} \in D^m \). Then at time \( r + 1 \), we have \( \Psi^m = \mathcal{N} \). Consider any edge \((i,j)\) after \( r + 1 \).

If \( \{i,j\} \notin D^m \), then \( p_{ij}^m \) decreases. Since prices are lower bounded by weights, as suggested by Lemma A.1, it follows that if conditions of Step S2 do not hold, then eventually, we have \( p_i^m = w_i^m \), and \( p_{ij}^m = w_{ij}^m \) for all \( i \in \mathcal{N} \), and \( (i,j) \in E \). Note that in this case we have \( D^m = 2\mathcal{N} \).

Since this is true for all bidders, it follows that if conditions of Step S2 do not hold, eventually \( D^m = 2\mathcal{N} \), for all bidders. On the other hand, it can be easily checked that the conditions of Step S2 hold in this case. Hence, the claim follows.

\[\square\]

Proof of Theorem 4.1. (i) Denote by \( W^* \) and \( W_k^* \) the total value generated by the allocations \( \{S^m\} \) and \( \{S'^m\} \) respectively. It can be immediately checked that the solution specified in the theorem statement is feasible, and the associated objective value is equal to \( W^* + \sum_k W_k^* \). This suggests that the optimal solution of LP3 is lower bounded by \( W^* + \sum_k W_k^* \).
Consider the dual solution $p_i^m = w_i^m$, $p_{ij}^m = w_{ij}^m$, $\pi^m = 0$, for all $m, i \in \mathcal{N}$, $(i, j) \in E$, and

\[
\pi^s = \max_{\mu} \sum_m \left( \sum_{i \in \mu} p_i^m + \sum_{i, j \in \mu | ij \in E} p_{ij}^m \right),
\]

\[
\pi^s_m = \max_{\mu | \mu k = \emptyset} \sum_{m \neq k} \left( \sum_{i \in \mu} p_i^m + \sum_{i, j \in \mu | ij \in E} p_{ij}^m \right).
\]

It follows from D3 that this solution is feasible. Moreover, by construction the corresponding dual objective value is equal to $W^* + \sum_k W_k^*$. Thus, the optimal objective of the dual problem is upper bounded by $W^* + \sum_k W_k^*$.

By strong duality it follows that the primal and dual feasible solutions we construct above are optimal for the LP3 and D3 respectively. Hence, the first part of the theorem follows.

(ii) Consider the optimal solution to LP3 given in part (i) of the theorem. Complementary slackness suggests that any dual optimal solution satisfies the conditions in the second part of the theorem. Hence, the claim immediately follows.

(iii) It follows from part (ii) that \( \left( \sum_{i \in S_k^m} p_i^k + \sum_{i, j \in S_k^m | ij \in E} p_{ij}^k \right) = v^k(S_k^m) - \pi_k \), and similarly \( \left( \sum_{i \in S_k^m} p_i^k + \sum_{i, j \in S_k^m | ij \in S_k^m} p_{ij}^k \right) = v^k(S_k^m) - \pi_k \). Thus, the quantity in (6) is equivalent to

\[
\sum_{k \neq m} (v^k(S_k^m) - \pi_k) - (v^k(S_k^m) - \pi_k) = \sum_{k \neq m} (v^k(S_k^m) - v^k(S_k^m)).
\]

Since allocations \( \{S^m\} \) and \( \{S_k^m\} \) are efficient (for set of bidders \( \mathcal{M} \) and \( \mathcal{M} - \{k\} \) respectively), the quantity in the right hand side is equal to the VCG payment of bidder \( m \), and the claim follows.

\[\square\]